LIVSIĆ THEORY FOR COMPACT GROUP EXTENSIONS OF HYPERBOLIC SYSTEMS

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Dedicated to Yu. S. Ilyashenko on the occasion of his 60th birthday

Abstract. We prove Livsić-type theorems for rapidly mixing compact group extensions of Anosov diffeomorphisms.


Key words and phrases. Cocycle equation, transfer operator, partial hyperbolicity, small divisors.

1. Introduction

In recent years, several new phenomena in dynamics were discovered by looking at small perturbations of compact group extensions of hyperbolic systems [8], [14]. In view of this, it is desirable to develop a general theory of perturbations of such systems. The first step towards this goal is to understand infinitesimal perturbations, that is, study homological equations over such systems. In this paper, we study the regularity of solutions to cocycle equations. Regularity theory plays an important role in rigidity theory. Two of the most studied cases are translations of \( \mathbb{T}^d \) and Anosov diffeomorphisms (see [7], [11], [15] for the analysis of some other systems). The systems considered in this paper exhibit a mixture of hyperbolic and elliptic behaviors.

Let \( M \) be a compact \( C^\infty \) Riemannian manifold, and let \( f : M \to M \) be an Anosov diffeomorphism. Suppose that \( G \) is a compact connected Lie group, \( H \) a Lie subgroup of \( G \), \( Y = G/H \), and \( \tau \in C^\infty(M, G) \). Let \( N = M \times Y \). We define \( F : N \to N \) by \( F(x, y) = (fx, \tau(x)y) \). A function \( A \) on \( N \) is called a coboundary if

\[
A = B - B \circ F.
\]  

(1)

If \( B \) is bounded, Hölder, smooth, etc., we say that the coboundary \( A \) is bounded, Hölder, smooth, etc. Take \( \phi \in C^\infty(M) \); let \( \mu_\phi \) be the Gibbs state with potential \( \phi \), and let \( d\nu_\phi = d\mu_\phi dy \).

In this paper, we prove the following theorem. Take \( z_0 \in N \).
Let $F$ be rapidly mixing. If $A \in C^\infty(N)$ is a coboundary in $L^2(\nu_\phi)$ for some Hölder function $\phi$, then $A$ is a $C^\infty$ coboundary. In particular, if $A$ is a bounded coboundary, then it is $C^\infty$. Moreover, there exists a $k_0$ such that, if $A$ belongs to $C^k(N)$, then $B$ belongs to $C^{k-k_0}(N)$. If $B$ satisfies the normalization condition
\[ B(z_0) = 0, \]
then
\[ \|B\|_{k-k_0} \leq \text{Const} \|A\|_k. \] (2)

We refer the reader to the next section for the definition of rapid mixing. Recall [4] that a generic extension is rapidly mixing.

Observe that Theorem 1 implies, in particular, that the set of coboundaries is closed in $C^k$ for $k > k_0$.

We also present versions of this theorem for extensions of subshifts of finite type. Our results are also true for relative coboundaries. Let $A_0(x) = \int A(x, y) dy$. We say that $A$ is a relative coboundary if
\[ A = A_0 + B - B \circ F. \] (3)

Theorem 2. The assertions of Theorem 1 are valid for relative coboundaries.

2. Preliminaries

2.1. Subshifts of finite type. Here we present some results about subshifts of finite type and their compact extensions. The proofs can be found in [13, Chapters 3 and 8]. For a geometric interpretation of the results about extensions, see, e.g., [4, Section 2]. Let $a$ be a finite alphabet; by $A$ we denote a $\text{Card}(a) \times \text{Card}(a)$-matrix whose entries are zeroes and ones. Suppose that $\Sigma = \Sigma_A$ is the associated (two-sided) subshift of finite type, that is, $\Sigma = \{\{\omega_i\}_{i=-\infty}^{\infty} : \omega_i \in a \text{ and } A_{\omega_i, \omega_{i+1}} = 1\}$. Let $\sigma$ be the shift defined by $\sigma(\omega)_i = \omega_{i+1}$. Given $\theta < 1$, we consider the metric $d_0$ on $\Sigma$ given by $d_0(\omega', \omega'') = \theta^j$, where $j = \max(k: \omega'_i = \omega''_i \text{ for } |i| < k)$. Let $C_\theta(\Sigma)$ denote the set of $d_0$-Lipschitz functions. Given $\phi \in C_\theta(\Sigma)$, we denote the Gibbs measure with potential $\phi$ by $\mu_\phi$, that is,
\[ h_{\mu_\phi} + \phi(\phi) = \sup_\mu (h_\mu + \mu(\phi)), \]

where the supremum is over all $\sigma$-invariant probability measures. We shall use the fact that homologous functions have the same Gibbs measures. Let $G$ be a compact connected Lie group, and let $\tau \in C_\theta(\Sigma, G)$. We set $\tilde{N} = \Sigma \times Y$ and define $\tilde{F}: \tilde{N} \to \tilde{N}$ by $\tilde{F}(\omega, y) = (\sigma \omega, \tau(x)y)$. Consider the measure $\nu_\phi$ given by $d\nu_\phi = d\mu_\phi \, dg$. Let $C_{k,\theta}(\tilde{N}) = C_\theta(\Sigma, C^k(G))$. We say that $\tilde{F}$ is rapidly mixing if, for any $\phi$ and $N$, there exists a $k$ such that
\[ |\nu_\phi(A(\omega, y)B(\tilde{F}^n(\omega, y))) - \nu_\phi(A)\nu_\phi(B)| \leq \text{Const} \|A\|_{k,\theta} \|B\|_{k,\theta} \|n^\infty - N \]
for all $A$, $B \in C_{k,\theta}$. It was shown in [4, Theorem 4.3] that rapid mixing is generic among compact group extensions of subshifts of finite type.

Let $\Sigma^+$ be the associated one-sided subshift, which is defined similarly to $\Sigma$ but for the one-sided sequence $\omega = \{\omega_i\}_{i=0}^\infty$. The set $C_\theta(\Sigma^+)$, Gibbs states, rapid
mixing, etc. are defined for one-sided shifts as for two-sided shifts. Suppose that $\tilde{F}: \hat{N} \to \hat{N}$ is the skew extension defined by $\tau \in C_0(\Sigma, G)$, $\phi \in C_0(\Sigma)$ is a potential, and $A \in C_{k, \rho}$ is an observable. Then there are $\tau^* \in C_{\imath}(\Sigma, G)$, $M \in C_{\imath}(\Sigma, G)$, $\phi^* \in C_{\imath}(\Sigma^+)$, $\psi \in C_\imath(\Sigma)$, $A^* \in C_{k, \imath}(\Sigma^+)$, and $K \in C_{k, \imath}(\Sigma)$ such that

$$
\tau^* = (M \circ \sigma) \tau M^{-1}, \quad \phi^* = \phi + \psi - (\psi \circ \sigma), \quad A^* = A + K - K \circ \tilde{F}.
$$

Moreover, $\phi^*$ can be chosen so that

$$
\sum_{\sigma \omega = \omega} e^{\phi^*(\omega)} = 1 \quad \text{for any } \omega. \quad (4)
$$

The skew products defined by $\tau$ and $\tau^*$ are conjugate, $\phi$ and $\phi^*$ have the same Gibbs measure, and $A$ is a coboundary if and only if $A^*$ is a coboundary.

Let $N^+ = \Sigma^+ \times Y$. Suppose that $\tilde{F}$ is the skew extension determined by some $\tau \in C_0(\Sigma^+, G)$, $\Delta$ is a $G$-invariant Laplacian on $G$, and

$$
H_\lambda = \{ \varphi: \Delta \varphi = \lambda \varphi \}.
$$

We endow $H_\lambda$ with the $L^2$-norm. We set $C_{\lambda, \bar{\theta}}(\Sigma^+) = C_\theta(\Sigma^+, H_\lambda)$. Let $\phi$ be any Hölder function on $\Sigma^+$ such that

$$
\sum_{\sigma \omega = \omega} e^{\phi(\omega)} = 1 \quad \text{for any } \omega, \quad (5)
$$

and let $\mu_\phi$ be the Gibbs measure for $\phi$. Consider the transfer operator

$$
(L(h))(\omega, g) = \sum_{\sigma \omega = \omega} e^{\phi(\sigma)} h(\omega, \tau^{-1}(\sigma)g). \quad (6)
$$

It preserves $C_{\lambda, \bar{\theta}}(\Sigma^+)$. Let $L_\lambda$ denote the restriction of $L$ to $C_{\lambda, \bar{\theta}}(\Sigma^+)$.\n
**Proposition 1** [4, Proposition 4.4]. If $\tilde{F}$ is rapidly mixing, then there exist a $C$ and an $s$ such that

$$
\|L_\lambda^n\| \leq C\lambda^s \left(1 - \frac{1}{C\lambda^s}\right)^n. \quad (7)
$$

### 2.2. Anosov diffeomorphisms.

Recall that a diffeomorphism $F: M \to M$ is said to be Anosov if there exist an $f$-invariant splitting

$$
TM = E^s \oplus E^u
$$

and constants $C, \rho < 1$ such that

$$
\|df^n v\| \leq C \rho^n \|v\| \quad \text{for any } v \in E^s \quad \text{and} \quad \|df^{-n} v\| \leq C \rho^n \|v\| \quad \text{for any } v \in E^u.
$$

The distributions $E^s$ and $E^u$ are uniquely integrable, they are tangent to the foliations $W^s$ and $W^u$, respectively. Since $W^s$ and $W^u$ are transverse, if $x, y \in M$ are close to each other, then the intersection $W^s_{loc}(x) \cap W^u_{loc}(y)$ consists of one point, which we denote by $[x, y]$. A set $\Pi$ is called a parallelogram if $[x, y] \in \Pi$ for all $x, y \in \Pi$. A partition $\Pi = \{\Pi_1, \Pi_2, \ldots, \Pi_n\}$ is said to be Markov if, for all $x \in \text{Int}(\Pi_i)$,

$$
fW^s_{\Pi}(x) \in W^s_{\Pi}(fx), \quad f^{-1}W^u_{\Pi}(x) \in W^u_{\Pi}(f^{-1}x),
$$

where $\Pi$ is a partition is a parallelogram. The term LIVSIC THEORY FOR COMPACT GROUP EXTENSIONS 57
where $W^u_{\infty}(z) = W^u_{\infty}(z) \cap \Pi_j$ for $z \in \Pi_j$. Given a Markov partition $\Pi$, we can consider the subshift of finite type $\Sigma$ for which $a = \{1, 2 \ldots n\}$ and $A_{ij} = 1$ if and only if $f(\text{Int}P_i) \cap P_j = 0$. The map $\zeta: \Sigma \to M$ given by

$$\zeta(\omega) = \bigcap_j f^{-j} \Pi_{\omega_j}$$

defines a semiconjugacy between $\sigma$ and $f$. For a function $\tau$ from $M$ to $G$, let $\tilde{\tau} = \tau \zeta$. Then $\zeta \times \text{id}$ is a semiconjugacy between $F(x, y) = (f x, \tau(x)y)$ and $\tilde{F}(\omega, y) = (\sigma \omega, \tilde{\tau}(\omega)y)$. We shall use the fact that the skew extension $F$ is partially hyperbolic. That is, there exist an $F$-invariant splitting

$$TN = E^s_F \oplus E^c_F \oplus E^u_F$$

and constants $C, \rho < 1$ such that

$$\|dF^n v\| \leq C \rho^n \|v\| \quad \text{for any } v \in E^s_F, \quad \|dF^{-n} v\| \leq C \rho^n \|v\| \quad \text{for any } v \in E^u_F$$

and $E^c_F$ is the tangent space to the fibers. The definition of Gibbs states for $f$ is similar to that for $\sigma$. An important special case is the so-called SRB measure, which is the Gibbs measure with potential

$$\phi_{\text{SRB}} = - \ln \det(df|E^u)$$

The importance of the SRB measure comes from the fact that, if $\Phi \in C(M)$, then

$$\frac{1}{n} \sum_{j=0}^{n-1} \Phi(f^j x) \to \mu_{\text{SRB}}(\Phi) \quad \text{as } n \to +\infty$$

for Lebesgue almost all $x$.

For any $\alpha$, there exists $\theta$ such that if $\phi \in C^\alpha(M)$, then $\phi = \phi \circ \zeta \in C^\theta(\Sigma)$; $\mu_\phi$ is a Gibbs state for $f$ if and only if

$$\mu_\phi(\Omega) = \mu_\phi(\zeta(\Omega)) \quad \text{for any } \Omega \subset \Sigma.$$

We say that $F$ is rapidly mixing if for any $\phi$ and $N$, there exists $k$ such that

$$|\nu_\phi(A(x, y) B(F^n(x, y))) - \nu_\phi(A) \nu_\phi(B)| \leq \text{Const} \|A\|_{k, \nu} \|B\|_{k, \nu}^{-N}$$

for any $A, B \in C^k(M)$. The extension $F$ is rapidly mixing if and only if $\tilde{F}$ is rapidly mixing.

3. Symbolic Systems

3.1. One-sided shifts. In this subsection, $\tilde{F}: \tilde{N}^+ \to \tilde{N}^+$ is a rapidly mixing extension of the one-sided subshift. Let $C_{r, \theta}(\Sigma^+) = C_{\theta}(\Sigma^+, C^r(Y)).$

Lemma 1. Let $A \in C_{\infty, \theta}(\Sigma^+)$ be an $L^2(\nu_\phi)$-coboundary for some $\phi \in C_{\theta}(\Sigma^+)$ such that $A = B - B \circ \tilde{F}$, where $B \in L^2(\nu_\phi)$. Then $B$ has a version in $C_{\infty, \theta}(\Sigma^+)$. Moreover, there exists a $k_0$ such that, if $A \in C_{k, \theta}(\Sigma^+)$, then $B \in C_{k-k_0, \theta}(\Sigma^+)$ and

$$\|B\|_{k-k_0, \theta} \leq \text{Const}(k) \|A\|_{k, \theta}.$$
Proof. According to what was said in Section 2.1, we can assume that $\phi$ satisfies (5). Let $A = A_0 + \sum_{\lambda \neq 0} A_{\lambda}$, where $A_0(\omega) = \int A(\omega, y)dg$ and $A_{\lambda} \in H_{\lambda}$. Let $B = \sum_{\lambda} B_{\lambda}$. Since $F$ commutes with projections to $H_{\lambda}$, we have

$$A_{\lambda} = B_{\lambda} - B_{\lambda} \circ \tilde{F}. \quad (8)$$

In particular, $A_0 = B_0 - B_0 \circ \sigma$ and, according to [13], $B_0 \in C_0(\Sigma^+)$. Hence we can assume without loss of generality that $A_0 \equiv 0$. Applying $L_{\lambda}$ to (8), we obtain

$$L_{\lambda}A_{\lambda} = (L_{\lambda} - 1)B_{\lambda}.$$

Thus, $B_{\lambda} = -(1 - L_{\lambda})^{-1}L_{\lambda}B_{\lambda}$. There exists a $p = p(G)$ such that

$$\|A_{\lambda}\| \leq \frac{\text{Const}}{\lambda^{k/2-p}} \|A\|_{k, \theta}. \quad (9)$$

By Proposition 1, there exists an $s$ such that

$$\|(1 - L_{\lambda})^{-1}\| \leq \text{Const}\lambda^s. \quad (10)$$

Hence

$$\|B_{\lambda}\| \leq \text{Const}\lambda^{2s}\|A_{\lambda}\| \leq \frac{\text{Const}}{\lambda^{k/2-(2s+p)}} \|A\|_{k, \theta}. \quad (11)$$

We have

$$\|B_{\lambda}\|_{k - k_0, \theta} \leq \text{Const}\lambda^{p+2s}(k_0/2) \|A\|_{k, \theta}. \quad (12)$$

Let $B = \sum_{\lambda} B_{\lambda}$. Then

$$\|B\|_{k - k_0, \theta} \leq \sum_{\lambda} \|B_{\lambda}\|_{k - k_0, \theta} \leq \text{Const} \lambda^{p+2s}(k_0/2) \|A\|_{k, \theta},$$

and this series converges if $k_0$ is large enough. This completes the proof. \hfill \Box

3.2. Two-sided shifts. Let $\tilde{F}: \tilde{N} \to \tilde{N}$ be an extension of the two-sided subshift of finite type.

Lemma 2. Let $A = B - B \circ \tilde{F}$. If $A \in C_{k, \theta}(\Sigma)$, then $B \in C_{k, \theta/4}(\Sigma)$. Moreover, there exists a $k_0$ such that, if $A \in C_{k, \theta}(\Sigma)$, then $B \in C_{k - k_0, \theta/4}(\Sigma)$ and

$$\|B\|_{k - k_0, \theta/4} \leq \text{Const}(k)\|A\|_{k, \theta}. \quad (13)$$

Proof. Let $\tau^* = (M \circ \sigma)\tau M^{-1}$. Then the change of variables $y^* = My$ conjugates $\tilde{F}$ and $F^*(\omega, y^*) = (\sigma\omega, \tau^*(\omega)y^*)$. Thus, $A$ is an $F$-coboundary if and only if $A^*(\omega, y^*) = A(\omega, M^{-1}y^*)$ is an $F^*$-coboundary. Let us represent $A^*$ as $A^* = A^{**} + K^* - K^* \circ F^*$, where $A^{**} \in C_{k, \theta/4}(\Sigma^+)$. Then $A^*$ is an $F^*$-coboundary if and only if $A^{**}$ is. But by Lemma 1, $A^{**} = B^{**} - B^{**} \circ \tilde{F}$, where $B^{**} \in C_{k - k_0, \theta/4}(\Sigma^+)$. Thus,

$$A = (B + K) - (B + k) \circ \tilde{F},$$

where $B(\omega, y) = B^{**}(\omega, y)$ and $K(\omega, y) = K^*(\omega, My)$. This proves the lemma. \hfill \Box

Corollary 1. If $\omega$ is a periodic orbit of $\sigma$, say $\sigma^n\omega = \omega$, then

$$\left| \sum_{j=0}^{n-1} A_{\lambda}(\tilde{F}^j(\omega, y)) \right| \leq C\lambda^n\|A_{\lambda}\|_{k, \theta_0} d(\tau_\omega(\omega)y, y).$$
Proof. We have

\[ \left| \sum_{j=0}^{n-1} A_\lambda(\bar{F}^j(\omega, y)) \right| \leq |B_\lambda(\omega, \tau_n(\omega)y) - B_\lambda(\omega, y)| \leq \sqrt{\lambda} d(\tau_n(\omega)y, y)\|B_\lambda\|, \]

and the required assertion follows from (11). \qed
4. Anosov Diffeomorphisms

4.1. Hölder continuity. We proceed to prove Theorem 1. In this section, we shall show that \( B \) has a Hölder version. Take a Markov partition \( \Pi \) of \( M \). Let \( \Sigma \) be the associated subshift of finite type, and let \( \zeta : \Sigma \to M \) be the semiconjugacy \( \zeta \circ \sigma = f \circ \zeta \).

Consider \( \tilde{\tau} \) and \( \tilde{F} \) defined as in Section 2.2. We set \( \tilde{A} = A \circ \zeta, \tilde{A} = \tilde{B} \circ \tilde{F} \), and \( \tilde{B} = \sum \lambda B \). Let \( B_\lambda = B_\lambda \circ \zeta^{-1} \) and \( B = \sum \lambda B_\lambda \). Since \( \zeta^{-1} \) is discontinuous, we cannot assert that the \( B_\lambda \) are Hölder, but we can state the following lemma, which is a consequence of the periodic leaf estimates of Corollary 1. Suppose \( p = (x, y) \) has a dense orbit. We have

\[
B_\lambda(F^mp) = B_\lambda(p) - \sum_{j=0}^{n-1} A_\lambda(F^jp).
\]

**Lemma 3.** \( B_\lambda|_{\text{Orb}(x,y)} \) is uniformly Hölder continuous with Hölder constant \( C\|A_\lambda\|\lambda^s \).

*Proof.* Suppose that \( m < n \) and

\[
d(F^mp, F^np) \leq \varepsilon.
\]

We set \( k = n - m, z = f^m x \), and \( q = F^mp \). By the Anosov closing lemma, there exists an \( \tilde{x} \in M \) such that \( f^k\tilde{x} = \tilde{x} \) and

\[
d(f^jz, f^jy) \leq Cd(f^kz, z)\gamma^\rho^{\max(j,k-j)}
\]

for some \( \gamma > 0 \) and \( \rho < 1 \). Let \( u = (\tilde{x}, y) \). Then

\[
|B_\lambda(f^jq) - B_\lambda(q)| = \sum_{j=0}^{k-1} A_\lambda(F^jq) \leq \sum_{j=0}^{k-1} A_\lambda(F^ju) + \sum_{j=0}^{k-1} |A_\lambda(F^jq) - A_\lambda(F^ju)|.
\]

By Corollary 1, the first part is

\[
O(\|A_\lambda\|\lambda^sd^\alpha (p, F^np))
\]

and the second part is

\[
O(\|A_\lambda\|\sqrt{d}f^\alpha (p, F^np)),
\]

because \( A_\lambda \) is Lipschitz with constant \( \sqrt{d}\|A_\lambda\| \). \( \square \)

Since \( \text{Orb}(p) \) is dense, we can extend the \( B_\lambda \) to Hölder functions on \( N \).

**Lemma 4.** Under the conditions of Theorem 1, the restriction of \( B \) to each fiber is smooth. Moreover, there exists a \( k_0 \) such that

\[
\|B\|_{C^\alpha(M, C^{k-k_0}(G))} \leq \text{Const}(k)\|A\|_{C^\alpha(M, C^k(G))}.
\]

*Proof.* First, let us show that \( B \) has a Hölder version. By Lemma 3, each \( B_\lambda \) admits an extension from \( \text{Orb}(x, y) \) to \( N \) which is Hölder with Hölder norm at most \( \text{Const}\|A_\lambda\|\lambda^s \). By the continuity of \( A_\lambda \), this extension satisfies \( A_\lambda = B_\lambda - B_\lambda \circ F \), and

\[
\|A_\lambda\| \leq \frac{\text{Const}}{\lambda(k/2)\rho} \|A\|_{C^k(N)}.
\]
For $B = \sum_{\lambda} B_{\lambda}$, we have

$$\|B\|_{C^{1}(N)} \leq \sum_{\lambda} \|B_{\lambda}\|_{C^{1}(N)} \leq \text{Const} \left( \sum_{\lambda} \lambda^{p+s-k/2} \right) \|A\|_{C^{k}(N)},$$

and this series converges if $k$ is large enough. In other words, there exists a $k_1$ such that

$$B_{C^{1}(N)} \leq \text{Const} \|A\|_{C^{k_1}(N)}.$$  

Applying this to $\Delta^{m}A$, we obtain

$$\|B\|_{C^{1}(M,\mathcal{H}^{2}(G))} \leq \text{Const} \|\Delta^{m}B\|_{C^{1}(N)} \leq \text{Const} \|\Delta^{m}A\|_{C^{k_1}(N)} \leq \text{Const} \|A\|_{C^{k_1+2m}(N)},$$

and the application of the Sobolev embedding theorem completes the proof. □

4.2. Smoothness. In this section, we prove the smoothness of $B$ in the transverse directions.

Lemma 5. Restrictions of $B$ to the leaves of $W^{s}_{\mathcal{F}}$ and $W^{u}_{\mathcal{F}}$ are smooth.

Proof. It suffices to consider $W^{s}_{\mathcal{F}}$. We have $A(p) = B(p) - B(Fp)$. Thus, $B(p) = A(p) + B(Fp)$. Hence if $p \in W^{s}(p_0)$ then

$$B(p) - B(p_0) = \sum_{j=0}^{\infty} [A(F^{j}p) - A(F^{j}p_0)].$$

Since $F^{j}$ are contractions on $W^{s}_{\mathcal{F}}$, this series can be differentiated term by term arbitrarily many times (see [3]). □

Now, we need the following fact [10].

Proposition 2 (Journe lemma). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous transverse foliations with smooth leaves. If $B$ is a continuous function whose restrictions to the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are smooth, then $B$ is smooth. Moreover, there exists a $k_0$ such that, if the restrictions of $B$ to the leaves are $C^{k}$, then $B$ is $C^{k-k_0}$.

This proposition, together with Lemmas 4 and 5, implies that $B$ is smooth on each leaf of $W^{sc}_{\mathcal{F}}$; since it is also smooth on each leaf of $W^{u}$, we conclude that $B$ is smooth. This completes the proof of Theorem 1.

Remark. The weaker versions of the Journe lemma proven in [3], [9] are sufficient for the proof.

4.3. Relative coboundaries. Proof of Theorem 2. Apply Theorem 1 to $\Delta A$. □

4.4. A counter-example. Take $G = T^2$ and

$$F(x, t_1, t_2) = (f(x, t_1 + \alpha_1r(x), t_2 + \alpha_2r(x)).$$

Suppose that $\alpha_1/\alpha_2$ is irrational and, for all $N$, there exist $m_{1,N}, m_{2,N} \in \mathbb{Z}$ such that

$$|\alpha_1 m_{1,N} + \alpha_2 m_{2,N}| \leq m^{-N}_{2,n}. $$
Changing the indexation if necessary, we can assume that \( m_{2,N} > N^2 \). Let \( \Phi_N(x, t_1, t_2) = \exp(2\pi i(m_{1,N}t_1 + m_{2,N}t_2)) \). Then
\[
\Phi \circ F = \exp(2\pi i(m_{1,N} \alpha_1 + m_{2,N} \alpha_2) r(x)) \Phi_N.
\]
Let \( A = \sum_N ((\Phi_N - \Phi \circ F)/N^2) \). Then \( A \in C^\infty(N) \) and \( A = B - B \circ Fm \) where \( B = \sum_N \Phi_N \in C^0(N) - C^1(N) \). Considering suitable linear combinations of \( \Phi_N \), we see that \( F \) is not rapidly mixing. This shows that Theorem 1 is not valid for arbitrary extensions.

4.5. Obstructions. In this section, we give criteria for a function to be a coboundary. Most of these criteria come from other papers, however their applicability is a consequence of the fact that different notions of coboundaries coincide in the situation under consideration. Sometimes, it is easier to verify that \( A \) is a relative coboundary. It is sufficient, because it is well known when \( A_0 \) is an \( f \)-coboundary.

(i) We set
\[
D_\phi(A) = \nu_\phi(A^2) - \nu_\phi^2(A) + 2 \sum_{j=1}^\infty [\nu_\phi(A(A \circ F^j)) - \nu_\phi^2(A)].
\]

Proposition 3 [6]. \( A \) is cohomologous to a constant if and only if there exists a \( \phi \) such that \( D_\phi(A) = 0 \), or, equivalently, \( D_\phi(A) = 0 \) for any \( \psi \).

Proof. Without loss of generality, we can assume that \( \nu_\psi(A) = 0 \). Then \( D_\phi(A) = 0 \) if an only if \( A \) is an \( L^2(\nu_\phi) \)-coboundary (by the spectral theorem). The latter condition is equivalent to \( A \) being a Hölder coboundary (by Theorem 1), which is, in turn, equivalent to \( A \) being an \( L^2(\nu_\psi) \)-coboundary for each \( \psi \). \( \square \)

(ii) Let \( P = \{p_0, p_1 \ldots p_n\} \) be a chain such that \( p_{k+1} \in WF(p_k) \cup W^u(p_k) \). We say that \( P \) is closed if \( p_0 = p_n \). We set
\[
r(P) = \sum_k r(p_k, p_{k+1}),
\]
where
\[
r(p_k, p_{k+1}) = \begin{cases}
\sum_{j=0}^\infty [A(F^j p_{k+1}) - A(F^j p_k)] & \text{if } p_{k+1} \in WF p_k, \\
\sum_{j=-\infty}^{\infty} [A(F^j p_k) - A(F^j p_{k+1})] & \text{if } p_{k+1} \in WF p_k. 
\end{cases}
\]

The following statement is Corollary 3.1 from [11].

Proposition 4. If \( F \) has the accessibility property, then \( A \) is cohomologous to a constant if and only if \( r(P) = 0 \) for any closed chain \( P \).

(iii) The next result follows from the proof of Theorem 1.

Proposition 5. \( A \) is a relative coboundary if and only if \( \Delta^N A \) is a coboundary for any \( N \in \mathbb{N} \).
(iv) Let $G = T$, and let $\nu$ be the SRB measure for $F$. Consider the one-parameter family
\[
F_\varepsilon(x, z) = (f x, z + \tau(x) + \varepsilon A(x, z) + \varepsilon^2 \alpha(\varepsilon, x, z).
\]
Suppose that $\nu_\varepsilon$ is a $u$-Gibbs measure for $F_\varepsilon$, that is, the projection of $\nu_\varepsilon$ to $M$ is $\mu_{SRB}$.

**Proposition 6.** $A$ is a relative coboundary if and only if
\[
\lambda_c(\nu_\varepsilon) = o(\varepsilon^2).
\]
**Proof.** We use the asymptotics [5]
\[
\nu_\varepsilon(H) = \nu(H) + \varepsilon \omega(H) + o(\varepsilon),
\]
where
\[
\omega(H) = \sum_{j=1}^{\infty} \nu \left( A \circ F^{-j} \frac{dH}{dz} \right).
\]
We want to apply this to $H_\varepsilon = \ln \frac{dF_\varepsilon}{dz}$. We have
\[
\frac{dF_\varepsilon}{dz} = 1 + \varepsilon \frac{dA}{dz} + \varepsilon^2 \frac{d\alpha(0, x, z)}{dz} + o(\varepsilon^2).
\]
So
\[
\ln \frac{dF_\varepsilon}{dz} = \varepsilon \frac{dA}{dz} + \varepsilon^2 \left[ \frac{d\alpha(0, x, z)}{dz} - \frac{1}{2} \left( \frac{dA}{dz} \right)^2 \right] + o(\varepsilon^2).
\]
Hence
\[
\nu_\varepsilon \left( \ln \frac{dF_\varepsilon}{dz} \right) = \varepsilon \nu \left( \frac{dA}{dz} \right)
\]
\[+ \varepsilon^2 \left[ \nu \left( \frac{d\alpha(0, x, z)}{dz} \right) - \frac{1}{2} \nu \left( \left( \frac{dA}{dz} \right)^2 \right) + \sum_{j=1}^{\infty} \nu \left( A \circ F^{-j} \frac{d^2A}{dz^2} \right) \right] + o(\varepsilon^2).
\]
Since $d\nu = d\mu_{SRB} dg$, it follows that
\[
\nu \left( \frac{dA}{dz} \right) = \nu \left( \frac{d\alpha(0, x, z)}{dz} \right) = 0 \quad \text{and}
\]
\[
\nu \left( A \circ f^{-j} \frac{d^2A}{dz^2} \right) = -\nu \left( \left( \frac{dA}{dz} \right) \circ F^{-j} \frac{dA}{dz} \right).
\]
Hence
\[
\nu_\varepsilon \left( \ln \frac{dF_\varepsilon}{dz} \right) \sim -\varepsilon^2 \left[ \frac{1}{2} \nu \left( \left( \frac{dA}{dz} \right)^2 \right) - \sum_{j=1}^{\infty} \nu \left( \left( \frac{dA}{dz} \right) \circ F^{-j} \frac{dA}{dz} \right) \right],
\]
that is,
\[
\nu_\varepsilon \left( \ln \frac{dF_\varepsilon}{dz} \right) \sim -\varepsilon^2 D_{SRB} \frac{\frac{dA}{dz}}{2}.
\]
Therefore,
\[
\lambda_c(\nu_\varepsilon) = o(\varepsilon^2)
\]
if and only if $D_{SRB} \frac{dA}{dz} = 0$. By Proposition 3, $\frac{dA}{dz}$ is a coboundary, which means that $A$ satisfies (3). \qed
Appendix A. The Non-Mixing Case

Note that Theorem 1 may be valid for an Anosov time rotation even though it is not mixing. In this appendix, we give an extension of Theorem 1 to the non-mixing case. In order to explain the result, we recall some background. Given \( \tau \), let \( \bar{N} = M \times G \) and consider the principal extension \( \bar{F}: \bar{N} \to \bar{N} \) given by \( \bar{F}(x, g) = (f(x), \tau(x)g) \). Recall the definition of Brin groups [1], [2]. Given a partially hyperbolic diffeomorphism, we call a sequence \( P = \{p_1, p_2, \ldots, p_n\} \) a e-chain (respectively t-chain) if \( p_{j+1} \in W^u(p_j) \cup W^s(p_j) \) (respectively, \( p_{j+1} \in W^u(p_j) \cup W^s(p_j) \cup \text{Orb}(p_j) \)). Take a reference point \( x \in M \). Given any chain \( P \subset M \) with \( x_n = x_1 = x \) and any \( g_1 \in G \), there is unique chain \( P \subset N \) starting at \( (x, g_1) \) and covering \( P \). \( P \) is not closed, instead we have \( g_n = g(P)g_1 \). Let \( \Gamma_t(x) \) \((\Gamma_e(x))\) denote the set of all \( g(P) \) for all closed t-chains (respectively, e-chains) starting at \( x \).

**Proposition 7** (Brin, [1], [2]). (a) The \( \Gamma_e(x) \) are groups. The \( \Gamma_s \) of different points are conjugate, \( \Gamma_t \) is a normal subgroup of \( \Gamma_e \), and \( \Gamma_e/\Gamma_t \) is cyclic. In particular, \( \Gamma_e/\Gamma_t \) is Abelian.

(b) \((F, \nu_\phi)\) is ergodic if and only if \( \Gamma_e \) acts transitively on \( Y \).

\((F, \nu_\phi)\) is mixing if and only if \( \Gamma_t \) acts transitively on \( Y \).

A quantitative version of this result was obtained in [4]. We say that a set \( S \subset G \) is Diophantine on \( Y \) if there exist constants \( K \) and \( \sigma \) such that, for any function \( h \) on \( Y \) with \( \Delta h = \lambda h \), there is an \( s \in S \) such that

\[
\|h - h \circ s\| \leq \frac{K}{\lambda^s} \|h\|_{L^2}.
\]

Let \( \Gamma_t(x, R) \) \((\Gamma_e(x, R))\) denote the set of \( g(P) \) for all chains \( P = (x_1, x_2, \ldots, x_n) \) with \( x_1 = x_n = x, n \leq R \), and \( dw \cdot (x_j, x_{j+1}) \leq R \) (if \( x_{j+1} = f^m x_j \), we require that \( |m| \leq R \)).

**Proposition 8** [4]. (a) \( S \) is Diophantine on \( Y \) if and only if \( S \) is Diophantine on \( Y/[G, G] \) and \( Y/\text{Center}(G) \).

(b) \( S \) is Diophantine on \( Y/\text{Center}(G) \) if and only if there exist no \( S \)-invariant functions, or, equivalently, \( S \) contains a finite Diophantine subset.

(c) \( F \) is rapidly mixing if and only if \( \Gamma_t(R) \) is Diophantine for large \( R \).

It was proven in [1] that there is an open dense subset of pairs \((f, \tau)\) such that \( \Gamma_t(R) = G \) for large \( R \). The goal of this appendix is to prove the following statement.

**Theorem 3.** Suppose that \( F \) is ergodic. If \( \Gamma_e(R) \) is Diophantine for large \( R \), then any solution to (1) satisfies the tame estimates (2).

**Remark.** Apparently, the above condition is also necessary for (2), but the approach of Section 4.4 (see also [4, Section 4.3]) shows only that, if \( \Gamma_e(R) \) is not Diophantine for large \( R \) and \( A = B - B \circ F \), then the norm of \( \partial_y^3 \partial_x^2 A \) cannot be bounded by the norms of \( \partial_y^3 \partial_x^2 A \). It does not eliminate the possibility that it is bounded by the norms of \( \partial_y^3 \partial_x^2 A \), although this is unlikely.
Proof. Note that the only place where we have used rapid mixing (i.e., Diophantineness of $\Gamma_t(R)$ was (10). Hence we need to show that (10) holds under the weaker condition that $\Gamma_e(R)$ is Diophantine. To this end, we estimate $(1 - L_\lambda)^{-1}$, using the series

$$(1 - L_\lambda)^{-1} = \frac{1}{2} \left(1 - \frac{1 + L_\lambda}{2}\right)^{-1} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{1 + L_\lambda}{2}\right)^j.$$ 

Thus, instead of Proposition 1, we must prove the existence of $C$ and $s$ such that

$$\|\left(\frac{1 + L_\lambda}{2}\right)^n\| \leq C\lambda^s \left(1 - \frac{1}{C\lambda^s}\right)^n.$$ (14)

The proof of (14) is similar to that of (7) which is Proposition 4.4 of [4]. Let us describe the modifications needed. Repeating the arguments from [4], p. 184, we can show that, if (14) fails, then for each $C_1$ and $\beta_4$ there exist a $\lambda$ and an $H$ such that $\|H\|_{C^0} \leq 1$, $L(H) \leq \text{Const}_\lambda$, and $\|(1 + L_\lambda)^m H\| \geq 1 - |\lambda|^{-\beta_4}$, where $m(\lambda) = C_1 \ln \lambda$ and $L(H)$ denotes the Lipschitz norm $H : \Sigma^+ \to L^2(Y)$. As in [4], this implies that

$$\|\pi_\lambda(\tau_\omega) H(\omega) - \pi_\lambda(\tau_\omega) H(\hat{\omega})\| \leq \lambda^{-\beta_5}$$

for any $\omega$ and $\hat{\omega}$, where $\beta_5 \to \infty$ as $\beta_4 \to \infty$. However, in the present setting, we also have

$$\|\pi_\lambda(\tau(\omega)) H(\omega) - H(\sigma\omega)\| \leq \lambda^{-\beta_5}$$

for all $\omega$. (15)

Indeed, the expression for $\left[(\frac{1 + L_\lambda}{2})^n H\right](\sigma\omega)$

$$\left[(\frac{1}{2})^n [H(\sigma\omega) + e^{\phi(\omega)} \pi_\lambda(\tau(\omega)) H(\omega)]\right]$$

among the other terms. These two vectors should be almost collinear in the sense of [4], p. 185, which proves (15).

Inequality (15) implies that, in our setting, Lemma 4.7 of [4] holds not only for t-chains, as in [4], but also for e-chains. Continuing as in [4], p. 186, we show that if (14) is false, then $\Gamma_e$ cannot be Diophantine. Thus, (14) holds. This proves Theorem 1 under the assumption that $\Gamma_e(R)$ is Diophantine.

\[\square\]

References


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