RIGIDITY THEOREMS FOR GENERIC HOLOMORPHIC GERMS
OF DICRITIC FOLIATIONS AND VECTOR FIELDS IN \((\mathbb{C}^2, 0)\)

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To Yu. S. Il’yanenko on his 60th birthday

Abstract. We consider the class \(V_{n+1}^d\) of dicritical germs of holomorphic vector fields in \((\mathbb{C}^2, 0)\) with vanishing \(n\)-jet at the origin for \(n \geq 1\). We prove, under some genericity assumptions, that the formal equivalence of two generic germs implies their analytic equivalence. A similar result is also established for orbital equivalence. Moreover, we give formal, orbitally formal, and orbitally analytic classifications of generic germs in \(V_{n+1}^d\) up to a change of coordinates with identity linear part.

Key words and phrases. Dicritic foliations, dicritic vector fields, rigidity, formal equivalence, analytic equivalence.

0. Introduction

0.1. About rigidity theorems. It is well known that the analytic and formal classifications of holomorphic vector fields at generic singular points coincide (see [A]). The failure of the genericity assumptions (in the case of saddle resonant and saddle node singularities, for example) leads to a relatively simple formal classification, which differs from the analytic one in that the analytic classification has functional moduli (see [B], [VG], [VM], [Te]). The same is true for orbital equivalence (analytic and formal) of holomorphic vector fields (see [MR2], [MR1], [I2], [I1]), and for mappings (see [A], [I2], [E], [V1]). For higher codimensions, the remarkable phenomenon of “rigidity” occurs; namely, the formal and analytic classifications coincide, although even the formal classification is generally boundless and, as rule, has functional moduli.

The first rigidity theorem (where rigidity is understood as above) was proved for the analytic classification of nonsolvable finitely-generated groups of germs of holomorphisms on the complex line (see [CMo], [R], [EISV]) (its analogue for topological rigidity, which says that topological equivalence implies analytic equivalence, was proved in [Shc]). The rigidity phenomena in the classification of foliations in
terms of holomorphic germs of vector fields, which are known as orbital rigidity, were discovered for degenerate singular points having nilpotent Jordan cells in its linearization (see [CMo], [EISV]). A rigidity theorem for foliations in the class \( \mathcal{V}_{n+1} \) (next in complexity), which consists of germs having vanishing \( n \)-jets at their singular points, was proved (for generic non-dicritic germs) in [V2]. A similar result for the rigidity of vector fields was obtained in [ORV]. Finally, orbital rigidity for families of germs with fixed geometry (under weak genericity assumptions) was obtained in [L] and can be derived from the results of [MS]. In the present paper, a rigidity theorem (in the case of foliations and vector fields) is proved for generic dicritic germs in the class \( \mathcal{V}_{n+1} \). Moreover, a formal classification of the corresponding germs (for foliations and vector fields) and a complete list of invariants for the analytic classification of foliations are given. A close result (for families of foliations) was obtained in [M]; a topological classification of dicritic foliations was considered in [Kl], [Kl2].

0.2. Statement of results

Definition 0.1. We consider the class \( \mathcal{V}_{n+1} \) of germs of holomorphic vector fields in \((\mathbb{C}^2, 0)\) with vanishing \( n \)-jet at zero for \( n \geq 1 \). As usual, two germs from \( \mathcal{V}_{n+1} \) are said to be \textit{analytically orbitally equivalent} if there exists a germ of holomorphic change of coordinates in \((\mathbb{C}^2, 0)\), which maps the phase curves of one germ to the phase curves of the other. Two germs from \( \mathcal{V}_{n+1} \) are \textit{formally orbitally equivalent} if there exists a formal change of coordinates in \((\mathbb{C}^2, 0)\) transforming one germ into the other multiplied by some formal power series with non-zero constant term. If the linear part of the change of coordinates is the identity map, then the equivalence is called \textit{strict}.

Let \( v \in \mathcal{V}_{n+1} \), and let \((P_{n+1}, Q_{n+1})\) be its term of order \( n+1 \) in the standard coordinates \((x, y)\) in \( \mathbb{C}^2 \). The germ \( v \) is said to be \textit{dicritical} if the polynomial \( xQ_{n+1} - yP_{n+1} \) vanishes identically. We denote the class of dicritical germs in \( \mathcal{V}_{n+1} \) by \( \mathcal{V}_{n+1}^d \).

Theorem 1 (rigidity of dicritic foliations). \textit{The formal orbital equivalence of generic germs of the class} \( \mathcal{V}_{n+1}^d \) \textit{implies their analytic orbital equivalence}.

Definition 0.2. As usual, two germs \( v_1 \) and \( v_2 \) are said to be \textit{analytically (formally) equivalent} if there exists a germ \( H \) of holomorphic (formal) change of coordinates in \((\mathbb{C}^2, 0)\), conjugating these germs, i.e., such that \( H \cdot v_1 = v_2 \circ H \).

Theorem 2 (rigidity of dicritic vector fields). \textit{The formal equivalence of generic germs of vector fields in the class} \( \mathcal{V}_{n+1}^d \) \textit{implies their analytic equivalence}.

This theorem is not a direct corollary to Theorem 1. Indeed, even the formal classification of germs analytically orbitally equivalent to a given one has functional moduli (see Sections 3 and 4). Nevertheless, Theorem 1 is used in the proof of Theorem 2.

Similar rigidity theorems were obtained previously in [V2] and [ORV] for germs in \( \mathcal{V}_{n+1} \); their genericity conditions included the non-dicriticality of the germs.

Let \( \mathcal{R}(x, y) = \prod_{j=1}^{n} (y - u_jx) \) be a homogeneous polynomial of degree \( n \), where \( u_j \in \mathbb{C} \) with \( j = 1, \ldots, n \), are non-zero pairwise distinct complex numbers.
\( \mathcal{VR} \) denote the class of germs \( v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} \), whose \((n+1)\)-jets at zero can be written as
\[
{j^0_n}^{n+1} v = \mathcal{R}(x, y) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right). \tag{0.1}
\]

In what follows, we give a formal classification of germs in \( \mathcal{VR} \) up to strict equivalence (see Definition 0.1).

We also prove, under some additional conditions, the uniqueness of the formal normal form constructed below.

**Theorem 3.** (i) Any germ in \( \mathcal{VR} \) is strictly formally equivalent to a germ
\[
v_{a,b,\delta}(x, y) = x \mathcal{R}(x, y) \left[ 1 + \sum_{j=1}^{n} \frac{a_j(x)}{y - u_j x} \frac{\partial}{\partial x} \right. \\
+ y \mathcal{R}(x, y) \left[ 1 + \sum_{j=1}^{n} \frac{b_j(x)}{y - u_j x} + \delta_n(x, y) \right] \frac{\partial}{\partial y} \right] \tag{0.2}
\]
where \( a_j(x), b_j(x) \) with \( j = 1, \ldots, n \) are formal series without linear and constant terms and \( \delta_n \) is a homogeneous polynomial of degree \( n \).

(ii) For generic germs \( v \in \mathcal{VR} \), the coefficients \( a_j(x) = \sum_{s=2}^{\infty} a_s^j x^s \) and \( b_j(x) = \sum_{s=2}^{\infty} b_s^j x^s \) in (0.2) satisfy the relations
\[
a_n^{n+2} = b_n^{n+2} \text{ for } j = 1, \ldots, n. \tag{0.3}
\]

The formal normal form in (0.2) normalized by (0.3) is unique.

**Remark 0.1.** The genericity assumptions in the second assertion of Theorem 3 are expressed as
\[
c_j^2 := b_j^2 - a_j^2 \neq 0, \quad j = 1, \ldots, n. \tag{0.4}
\]
Hence the second assertion in Theorem 3 says that two formal normal forms (0.2) satisfying normalization conditions (0.3) and genericity assumptions (0.4) are strictly formally equivalent if and only if they coincide.

**Theorem 4.** Any generic germ \( v \in \mathcal{VR} \) is strictly formally orbitally equivalent to some normal form (0.2) satisfying (0.4) and such that
\[
\delta_n = 0 \quad \text{and} \quad a_s^j = b_s^j \quad \text{for any } s, j, \quad 3 \leq s, 1 \leq j \leq n, \quad s \geq j + 1. \tag{0.5}
\]
Moreover, the normal form (0.2) satisfying (0.5) is unique.

**Remark 0.2.** 1. Condition (0.3) is contained in (0.5).

2. Any formal normal form (0.2) satisfying (0.5) can be written as
\[
v_{a,c}(x, y) = \mathcal{R}(x, y) \left[ 1 + \sum_{j=1}^{n} \frac{a_j(x)}{y - u_j x} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \right. \\
+ y \mathcal{R}(x, y) \left[ \frac{c_j^2 x^s}{y - u_j x} + \frac{c_j^2 x^2}{y - u_1 x} \right] \frac{\partial}{\partial y} \right] \tag{0.6}
\]
where \( c_j^2 = b_j^2 - a_j^2 \) and \( a_j^2, b_j^2 \) are the coefficients in Theorem 3.
Open questions. 1. Are the formal normal forms mentioned in Theorems 3 and 4 analytic? Or, equivalently, is it true that any dicritic germ is (orbitally) analytically equivalent to its (orbital) formal normal form?

Remark 0.3. For the (orbital) analytic classification of singular points with nilpotent linear part, a rigidity theorem was proved in [CMa] (see also [EISV]) and a simple normal form was constructed (this is the Takens normal form; see [T2]). In this case, a question similar to Question 1, has a positive answer; namely, the Takens normal forms are analytic [SZ].

2. What are “good” (that is, “very simple and informative”) analytic normal forms for dicritic germs?

Even if the formal normal forms in Theorem 3 and 4 are analytic, they are too complicated. The following two theorems may be useful in constructing such normal forms.

Definition 0.3. Let \( v \) be a generic germ of class \( \mathcal{VR} \), and let \( v_{a,c} \) be its formal normal form (0.6) (in the sense of strict formal equivalence). The collection \( c = \{ c_s \} \) of coefficients of the normal form \( v_{a,c} \), where \( c \in \mathbb{C}^k \) and \( k = \frac{n(n-1)}{2} + 1 \), is called the \( c \)-invariant of the germ \( v \), and denoted by \( c_v \).

Note that the \( c \)-invariant \( c_v \) of a generic germ \( v \) satisfies the genericity condition (0.4). Let \( \mathcal{C}^k \) denote the class of collections \( c = \{ c_s \} \) defined in Remark 0.2 and satisfying the genericity condition (0.4).

Remark 0.4. The \( c \)-invariant is well-defined, since the formal normal form \( v_{a,c} \) is uniquely determined by \( v \).

It can be shown (see Section 1.4) that any phase curve of a generic germ \( v \in \mathcal{VR} \), crossing at the singular point \( O \) with tangent direction \( y = ux \), close enough to the singular direction \( y = u_j x \) returns to the origin with tangent direction \( y = u^* x \), where \( u^* \neq u \) (see Fig. 0.1). Therefore, in a neighborhood of the point \( u_j \in \mathbb{C} \), the involution \( I^*_v: u \mapsto u^* \) (with fixed point \( u_j \)) is well-defined. The set of \( n \) involutions with fixed points \( u_j, j = 1, \ldots, n \), is called the standard collection of involutions, and the standard collection \( \{ I^*_v \} \) of the germ \( v \) is denoted by \( I_v \).

Theorem 5. Two generic germs in \( \mathcal{VR} \) are strictly analytically orbitally equivalent if and only if their collections of standard involutions and their \( c \)-invariants coincide.

Theorem 6 (realization of invariants). For any standard collection of involutions \( I \) and any \( c \in \mathcal{C}^k \), there exists a germ \( v \in \mathcal{VR} \) such that \( I = I_v \) and \( c = c_v \).

Remark 0.5. As follows from Theorems 5 and 6, the dimension of the moduli space in the problem of strict orbital analytic classification of germs of dicritic foliations with given standard involutions is equal to \( k = \frac{n(n-1)}{2} + 1 \). This coincides with the analogous (local) result obtained by Mattei in [M]. In the case under consideration, his result implies that the dimension of the base of versal deformations of germs of dicritic foliations in the class of germs with fixed involutions is equal to \( k - 1 \) (the dimension differs by one because we use strict equivalence in this paper).
0.3. Contents (steps and organization of the paper). In Section 1.1, we introduce some of the notions and notations used in this paper. The genericity assumptions used in Theorem 1 are discussed in Section 1.2. A scheme of the proof of Theorem 1 is given in Section 1.3, and the detailed proof in Sections 1.4–1.8.

The proof of Theorem 2 is given, basically, in Section 2.1, in which we construct a biholomorphism conjugating two given formally equivalent germs in a generic family denoted by $\mathcal{V}_{n+1}$. In Section 2.2 some supplementary characteristics of germs in $\mathcal{V}_{n+1}$ are considered. We prove that these characteristics are, in some sense, invariants for the analytic and formal equivalence of germs in $\mathcal{V}_{n+1}$. The end of the proof of Theorem 2 is given in Section 2.3; it consists in verifying that the holomorphism constructed in Section 2.1 is well-defined.

The formal and formal orbital classification theorems (Theorem 3 and Section 4) are proved in 3.1 and 3.2. Section 4.1 is devoted to the proof of the equivalence and equimodality of $c$-invariants (Theorem 5); the realization theorem (Theorem 6) is proved in Section 4.2.

1. Proof of the Rigidity Theorem for Generic Dicritic Foliations

In this section, we prove the so-called rigidity theorem for generic degenerated dicritic singular points of holomorphic foliations in $(\mathbb{C}^2, 0)$.

1.1. Blow-up. Consider the natural map of $\mathbb{C}^2 \setminus \{0\}$ to $\mathbb{C}P^1$ which associates each point of $\mathbb{C}^2 \setminus \{0\}$ to the straight line generated by this point. Let $M$ be the graph of this map; its closure $\mathcal{M} := M \cup \mathcal{L}$, where $\mathcal{L} = \{0\} \times \mathbb{C}P^1$, in the direct product $\mathbb{C}^2 \times \mathbb{C}P^1$ is a complex manifold. We denote the restriction to $\mathcal{M}$ of the projection from $\mathbb{C}^2 \times \mathbb{C}P^1$ to $\mathbb{C}^2$ along the second factor by $\pi$. The sphere $\mathcal{L} = \pi^{-1}(0)$ is called the pasted sphere. The map $\pi$ is holomorphic and $\pi(\mathcal{M}) = \mathbb{C}^2$; its restriction to $\mathcal{M} \setminus \mathcal{L}$ is a biholomorphism onto $\mathbb{C}^2 \setminus \{0\}$, and the inverse map $\sigma$ of this restriction is called the $\sigma$-process.
Let \( U \subset \mathbb{C}^2 \) be a neighborhood of the origin; the inverse image \( \tilde{U} = \pi^{-1}(U) \) is called the blow-up of \( U \).

Choose a coordinate system \((x, y)\) on \( U \): in a neighborhood of the pasted sphere \( L \) on the blow-up \( \tilde{U} \), we use the standard charts \((x, u)\) and \((v, y)\) with transition functions \( v = u^{-1} \) and \( y = ux (u \neq 0) \). In these charts the projection \( \pi \) is given by the relations:

\[
\pi: (x, u) \mapsto (x, y), \quad y = xu, \quad \pi: (v, y) \mapsto (x, y), \quad x = yv;
\]

moreover, the pasted sphere is given by the equations \( \{x = 0\}, \{y = 0\} \).

Suppose that \( v \) is a holomorphic vector field on \( U \), \( 0 \in U \) is its unique singular point, and \( k \) is the degree of the first non-zero term. The lifting \( \mathbf{v} \) of the vector field \( v \) to \( \tilde{U} \) is given by \( \hat{v} = \sigma_v \mathbf{v} \); it has an analytic extension, at zero, to the pasted sphere \( L \). There exist (see [8]) vector fields \( \mathbf{v}_+ \) and \( \mathbf{v}_- \) with a finite number of points on \( L \) such that, in the domain of definition of the standard charts, they satisfy the equalities \( \hat{v} = x^l \mathbf{v}_+ \) and \( \hat{v} = y^l \mathbf{v}_- \) for some positive integer \( l \geq k \). The vector fields \( \mathbf{v}_\pm \) generate a field of directions \( \mathbf{v} \) with a finite number of singular points on the pasted sphere. This field of directions \( \mathbf{v} \) (and the pair of generators \( \mathbf{v}_\pm \)) is called the blow-up of \( v \). The projection \( \pi \) transforms the field of directions \( \mathbf{v} \) into the field of directions in \( U \setminus \{0\} \) generated by the vector field \( v \). We define the blow-up \( \mathbf{v} \) of a germ \( v \in \mathcal{V}_{n+1} \) as the blow-up of any of its representatives.

We denote the foliation of a small neighborhood of the origin by the phase curves of some representative \( v \) by \( \mathcal{F}_v \) and the foliation associated to the blow-up of this neighborhood by the phase curves of the blow-up \( \mathbf{v} \) by \( \mathcal{F}_{\mathbf{v}} \). We call the foliation \( \mathcal{F}_{\mathbf{v}} \) the blow-up of the foliation \( \mathcal{F}_v \).

Let \( v \in \mathcal{V}_{n+1}^d \). In the \((x, y)\)-coordinates, \( v \) has the expression

\[
v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}, \quad P = \sum_{m=n+1}^{\infty} P_m, \quad Q = \sum_{m=n+1}^{\infty} Q_m,
\]

where \( P \) and \( Q \) are holomorphic functions and \( P_m \) and \( Q_m \) are homogeneous polynomials in \( x, y \) of degree \( m \geq n + 1 \) corresponding to the terms of order \( m \) in its Taylor expansion at the origin. Consider the difference \( R_m(x, y) = xQ_m(x, y) - yP_m(x, y) \). The blow-up \( \mathbf{v} \) of \( v \) is given, in the standard charts, by the equations

\[
\frac{du}{dx} = \frac{xQ(x, ux) - uxP(x, ux)}{x^2 P(x, ux)}, \quad \frac{dv}{dy} = \frac{yP(vy, y) - vyQ(vy, y)}{y^2 Q(vy, y)}.
\]

The condition \( R_{n+1} \equiv 0 \) for \( v \) to be dicritic implies

\[
xQ_{n+1}(x, y) = yP_{n+1}(x, y);
\]

hence, \( Q_{n+1}(x, y) = y\mathcal{R}(x, y) \) and \( P_{n+1}(x, y) = x\mathcal{R}(x, y) \), where \( \mathcal{R} \) is a homogeneous polynomial of degree \( n \). Therefore, the blow-up \( \mathbf{v} \) of a dicritic germ \( v \in \mathcal{V}_{n+1}^d \) is given by its generators \( \mathbf{v}_+ = B(x, u) \frac{\partial}{\partial x} + A(x, u) \frac{\partial}{\partial u} \) in the \((x, u)\)-charts and \( \mathbf{v}_- = A(v, y) \frac{\partial}{\partial v} + B(v, y) \frac{\partial}{\partial y} \) in the \((v, y)\)-charts, where

\[
\begin{align*}
(B(x, u), A(x, u)) &= (\mathcal{R}(1, u) + O(x), Q_{n+2}(1, u) - uP_{n+2}(1, u) + O(x)), \\
(A(v, y), B(v, y)) &= (P_{n+2}(v, 1) - vQ_{n+2}(v, 1) + O(y), \mathcal{R}(v, 1) + O(y)).
\end{align*}
\]
1.2. Genericty assumptions and class $\mathcal{V}^d_{n+1}$. The genericity assumptions for a germ $v \in \mathcal{V}^d_{n+1}$ in Theorem 1 are as follows.

(1) The blow-up $\tilde{\nu}$ of the germ $v$ has no singular points on the pasted sphere.

(2) The field of directions $\tilde{\nu}$ has exactly $n$ (distinct) points of tangency with the pasted sphere.

These assumptions on the germ $v \in \mathcal{V}^d_{n+1}$ of the form (1.1) are expressed, in the notation of Section 1.1, as follows.

(a) In the case $R(0, 1) \neq 0$, the polynomials $r(u) := R(1, u)$ and $\rho(u) := Q_{n+2}(1, u) - uP_{n+2}(1, u)$ have no common zeroes; the polynomial $r(u)$, whose degree is exactly $n$, has exactly $n$ distinct zeroes.

(b) In the case $R(0, 1) = 0$, the polynomials $r(u)$ and $\rho(u)$ have no common zeroes; $P_{n+2}(0, 1) \neq 0$; the polynomial $r(u)$, whose degree is exactly $n - 1$, has exactly $n - 1$ zeroes.

Thus, the genericity assumptions 1 and 2 are inequality conditions on the coefficients of the $(n + 2)$-jet of the dicritic vector field $v$. Hence these assumptions hold for generic germs in $\mathcal{V}^d_{n+1}$.

Remark 1.1. By a linear change of coordinates, we can always exclude the case (b); in what follows, we assume that the genericity assumption is as in case (a).

Remark 1.2. The conditions that $\deg r = n$ and $r$ has exactly $n$ distinct zeroes imply that all zeroes of $r$ are simple.

Let $u_1, \ldots, u_n$ be the zeroes of the polynomial $r$; to be more precise, suppose that $r(u_j) = 0$ and $r'(u_j) \neq 0$ for $j = 1, \ldots, n$. For $j = 1, \ldots, n$, we denote the points with coordinates $x = 0, u = u_j$ by $p_j$; these points are called the points of tangency of the field $\tilde{\nu}$ with the pasted sphere $\mathcal{L}$.

We use the notation $\mathcal{V}^d_{n+1}$ for the class of germs in $\mathcal{V}^d_{n+1}$ satisfying the genericity assumptions (a).

1.3. Sketch of the proof of the rigidity theorem

1.3.1. A scheme based on an auxiliary foliation. In the simplest cases, a map that conjugates two given foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ can be constructed as follows. Let $\mathcal{F}_1$ be a foliation of a domain $U$, and let $\mathcal{F}_1^\top$ be a foliation of the same domain transversal to $\mathcal{F}_1$. Suppose that the leaves of both foliations have some natural index (parametrization). Then the pair $(\mathcal{F}_1, \mathcal{F}_1^\top)$ defines a coordinate system on $U$: the coordinates of the point $\xi \in U$ are given by $(p, q)$, where $p$ is the index of the leaf of $\mathcal{F}_1$ passing through the point $\xi$ and $q$ is the corresponding index of the leaf of $\mathcal{F}_1^\top$ passing through the same point. Similarly, if $\mathcal{F}_2$ and $\mathcal{F}_2^\top$ are foliations on a domain $V$, then the pair $(\mathcal{F}_2, \mathcal{F}_2^\top)$ determines a coordinate system on the domain $V$. Consider the map $H: U \to V$ which transforms each point on $U$ to the corresponding point on $V$ (to the point with the same coordinates). This map clearly conjugates the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, because it transforms the leaves of $\mathcal{F}_1$ to the leaves of $\mathcal{F}_2$ (it also conjugates $\mathcal{F}_1^\top$ and $\mathcal{F}_2^\top$).

However, in solving a concrete problem, this scheme usually needs to be refined. Often, it is possible to neither define an univaluated parametrization of the leaves
of the foliation nor construct a foliation everywhere transversal to the given one. Because of this, the coordinate systems constructed above may be multivaluated. To give a correct definition of the conjugating map \( H \) (which must also be analytic at the points of tangency), we must require that both coordinate systems have the same kind of multivaluation. This suggests the scheme of the proof presented below.

Let \( v_1 \) and \( v_2 \) be two formally orbitally equivalent germs of vector fields in \( \mathcal{V}_{n+1} \), and let \( \tilde{v}_1 \) and \( \tilde{v}_2 \) be their corresponding blow-ups. Let \( \tilde{F}_1 \) and \( \tilde{F}_2 \) be the foliations by phase curves of the fields of directions \( \tilde{v}_1 \) and \( \tilde{v}_2 \), respectively, of a neighborhood of the pasted sphere \( \mathcal{L} \). The leaves of \( \tilde{F}_j \), where \( j = 1, 2 \), are almost everywhere transversal to \( \mathcal{L} \). As a parameter of a leaf of the foliation we take the \( u \)-coordinate of the intersection of the leaf with the pasted sphere. In a neighborhood of the points of tangency, this parametrization leads to a multivaluation problem: a non-unique choice of values of the parameter; in Section 1.4, we discuss the nature of such non-uniqueness and its coincidence for formally orbitally equivalent germs.

Further, as auxiliary foliations \( \mathcal{F}_j^T \) and \( \mathcal{F}_j^F \) we take first the standard foliation \( \mathcal{F}_0 := \{ x = \text{const} \} \) (the same for both foliations); such a choice is motivated by the fact that \( y = 0, v = 0 \) is the unique singular point of \( \mathcal{F}_0 \) (in the \((v, y)\)-charts) and, therefore, its monodromy group (see [ORV] for a definition) is trivial (this gives the simplest parametrization of the leaves: the parameter for the leaf \( \{ x = c \} \) is the number \( c \)). Thus, each pair of foliations \( (\mathcal{F}_j, \mathcal{F}_0) \), where \( j = 1, 2 \), corresponds to a collection of values characterizing it: the curves of tangency, their standard parametrization, and the transition functions between the curves of tangency. These invariants are constructed in Section 1.5.

However, to find a well-defined diffeomorphism \( H \) giving coordinate concordance, as described above, the coincidence of the above invariants in both foliations is necessary. This condition leads us to modify the foliations. Namely, instead of the foliations \( \mathcal{F}_j \), we take the foliations \( \mathcal{F}_w_j \) associated to germs \( w_j \) analytically orbitally equivalent to the original germs \( v_j \) and such that all the invariants of the pairs \( (\mathcal{F}_w_j, \mathcal{F}_0) \) coincide. The analytic orbital equivalence of the germs \( w_1 \) and \( w_2 \) implies the orbital analytic equivalence of the initial germs. This step is described in Section 1.6.

Finally, in Section 1.7, we prove that the diffeomorphism \( H \) giving the coordinate concordance can be analytically continued to a neighborhood of the pasted sphere \( \mathcal{L} \), being the identity at \( \mathcal{L} \). The diffeomorphism of the neighborhood of the origin in \( \mathbb{C}^2 \) given by the projection of \( H \) conjugates, as needed, the germs \( w_1 \) and \( w_2 \).

1.4. The structure of dicritic foliations in a neighborhood of the points of tangency. First integrals and involutions. Let \( v \in \mathcal{V}_{n+1} \) be a germ of the form (1.1), and let \( \tilde{v} \) be its blow-up. In the notations of Sections 1.1 and 1.2, its generator \( \tilde{v}_+ := B \frac{\partial}{\partial y} + A \frac{\partial}{\partial u} \) is given by (1.3). Let \( p_j = (0, u_j) \) be one of the points of tangency of \( \tilde{F}_v \) with the pasted sphere \( \mathcal{L} \): \( B(0, u_j) = 0 \). It follows, by virtue of the genericity assumptions, that \( A(0, u_j) \neq 0 \). Hence, as a consequence of the rectifying theorem, \( \tilde{v}_+ \) has a holomorphic first integral in a neighborhood of \( p_j \); that is, there exists a function \( J \) holomorphic at \( (0, u_j) \) such that \( J(0, u_j) = 0 \), \( \nabla J(0, u_j) \neq 0 \), and it is constant along the phase curves of \( \tilde{v}_+ \), i.e., \( B \frac{\partial J}{\partial y} + A \frac{\partial J}{\partial u} = 0 \).
The coincidence of the points of tangency is almost evident: strict formally orbitally equivalent germs

For strictly formally orbitally equivalent germs

The normalized first integral is uniquely determined by the germ of

called the

standard involutions

moreover,

functions in terms of the other; i.e., there exists a holomorphic function

that

Rewriting, if necessary, the function \( J_u \) by the composition \( \varphi^{-1} \circ J \), we obtain

\( J(x, u_j) = x \). Due to this relation, the first integral \( J \) is uniquely determined; moreover, \( J''_{uv}(0, u_j) \neq 0 \), because \( u = u_j \) is a simple zero of the polynomial \( r(u) = B(0, u) \) (see Remark 1.2 of Section 1.2). Hence the function \( \psi(u) := J(0, u) \) has a fold at the point \( u = u_j \). For this reason, for sufficiently small \( c \), the equation \( \psi = c \) has (in a neighborhood of \( u_j \)) exactly two solutions (\( u_j \) has multiplicity 2).

Therefore, the holomorphic involution \( I_j \) permuting such solutions is well-defined.

Let \( g(u) = (u_j - u)(I_j(u) - u_j) \); then \( g \circ I_j = g \), \( g(u_j) = g'(u_j) = 0 \), and \( g''(u_j) \neq 0 \). Since function \( \psi \) also has these properties, we can express one of these functions in terms of the other; i.e., there exists a holomorphic function \( h \) such that \( h(0) = 0 \), \( h'(0) \neq 0 \), and \( g = h \circ \psi \).

Now, consider the function \( J_{\nu_j} = h \circ I_j \). This function is also a first integral for \( \nu \); in what follows, we call it the 

normalized first integral for \( \nu \) at \( (0, u_j) \).

Remark 1.3. The normalized first integral is uniquely determined by the germ of vector field \( \nu \); hence any of its finite jets (at the point of tangency) is defined by a finite jet (at zero) of the germ \( \nu \). The restriction of \( J_{\nu_j} \) to the pasted sphere \((J_{\nu_j})|_{\mathcal{S}} = g(u))\) is uniquely determined by \( I_j \).

Thus, for each point of tangency \( p_j \), we have constructed a holomorphic involution \( I_j \) permuting the intersection points of the level curves of the first integral (the phase curves of \( \nu \)) with the pasted sphere; the involution \( I_j \) is uniquely determined by \( \nu \). Let us denote the collection \( \{p_j\}_{j=1}^{n+1} \) of points of tangency by \( \mathcal{P}_q \) and the collection \( \{I_j\}_{j=1}^{n+1} \) of the corresponding involutions by \( I_\nu \); these involutions will be called the standard involutions.

Lemma 1.1. For strictly formally orbitally equivalent germs \( \nu_1, \nu_2 \in \hat{V}_{n+1}^d \) the collections of the points of tangency and the standard involutions coincide: \( \mathcal{P}_{\nu_1} = \mathcal{P}_{\nu_2} \) and \( I_{\nu_1} = I_{\nu_2} \).

Proof. The coincidence of the points of tangency is almost evident: strict formally orbitally equivalent germs from \( V_{n+1}^d \) have proportional \((n+1)\)-jets. The coincidence of the standard involutions for strict orbitally analytic equivalent germs is also evident.

Now, consider two strictly formally orbitally equivalent germs \( \nu_1, \nu_2 \in \hat{V}_{n+1}^d \). If needed, we change the germs \( \nu_1 \) and \( \nu_2 \) for analytically equivalent germs whose jets at zero of order as high as necessary coincide. It follows from the above considerations that this does not change the collections of standard involutions.

However, any finite jet of the standard involutions at the point of tangency is uniquely determined by a finite jet (at the same point) of the corresponding first integral. At the same time, any finite jet of this first integral is uniquely determined by a finite jet at zero of the initial germ. Therefore, we may assume that the standard involutions associated to the germs \( \nu_1 \) and \( \nu_2 \) (at the corresponding points of tangency) have the same jets at zero of order as high as needed. Clearly, this means the coincidence of such germs of involutions, which completes the proof of Lemma 1.1. □
1.5. Invariants of the pair of foliations \((\tilde{\mathcal{F}}_v, \tilde{\mathcal{F}}_0)\): curves of tangency and their standard parametrizations. Let \(\mathcal{F} = \mathcal{F}_v\) for \(v \in \mathbb{V}_{n+1}^d\), and let \(\mathcal{F}_0\) be the standard foliation on straight lines \(\{x = \text{const}\}\). We denote their corresponding blow-ups by \(\tilde{\mathcal{F}}\) and \(\tilde{\mathcal{F}}_0\). In this section, we analyze the invariants related to the pair of foliations \((\tilde{\mathcal{F}}, \tilde{\mathcal{F}}_0)\).

1.5.1. Curves of tangency. First, consider the points of tangency between the foliations \(\tilde{\mathcal{F}}\) and \(\tilde{\mathcal{F}}_0\).

In the \((v, y)\)-charts, the foliation \(\tilde{\mathcal{F}}_0 = \{x = vy = \text{const}\}\) is the foliation by phase curves of the field \(\tilde{v}_0 = v \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\); \(\tilde{v}_0\) has a unique singular point at \((0, 0)\). Foliation \(\tilde{\mathcal{F}}\) is defined in the \((v, y)\)-chart by the vector field \(\tilde{v}_- = A \frac{\partial}{\partial v} + B \frac{\partial}{\partial y}\) (see (1.4)).

Since \(\tilde{B}(0, 0) \neq 0\), the leaf \(\Gamma_0\) of \(\tilde{\mathcal{F}}\) passing through the point \(v = y = 0\) is a smooth curve. We can rectify its projection \(\gamma_0 = \pi(\Gamma_0)\) by a local holomorphism \(H\) with identical linear part. Changing the initial germ \(v\) by its image under the action of \(H\), we reduce the problem to the case where the set of tangency of \(\tilde{\mathcal{F}}\) and \(\tilde{\mathcal{F}}_0\) near the point \(v = y = 0\) consists of the points of the line \(\Gamma_0 = \{v = 0\}\).

In the \((x, u)\)-charts, the foliation \(\tilde{\mathcal{F}}\) is defined by the vector field \(\tilde{v}_+ = B \frac{\partial}{\partial x} + A \frac{\partial}{\partial u}\) (see Section 1.3) and the foliation \(\tilde{\mathcal{F}}_0\), by the constant vector field \(\frac{\partial}{\partial u}\). Hence, the set \(\mathcal{F}_+\) of points \((x, u)\) for which the leaves of these foliations are tangent is given by the equation \(B(x, u) = 0\). By the genericity assumptions (see Section 1.2), the zeroes of the polynomial \(r(u) = B(0, u)\) are simple. Therefore, by the implicit function theorem, the set \(\mathcal{F}_+\) consists of \(n\) non-singular analytic curves \(\Gamma_j\), where \(j = 1, \ldots, n\); the curve \(\Gamma_j\) passes through the point of tangency \(p_j\) and is defined in a neighborhood of this point by the equation \(u = \beta_j(x)\), where \(\beta_j\) is an analytic function in \((\mathbb{C}, 0)\), \(\beta_j(0) = u_j\), and \(\beta_j'(0) \neq 0\).

Let \(J_j = J_{\tilde{v}_+, j}\) be the normalized first integral of the field of directions \(\tilde{v}\) defined in a neighborhood of the point \(p_j\) (see Section 1.4). Since \(\frac{\partial J_j}{\partial u}(0, u_j) \neq 0\) and \(\frac{\partial J_j}{\partial u}(0, u_j) = 0\), its restriction to \(\Gamma_j\), \(z_j := J_j|_{\Gamma_j}\), is a local parameter on \(\Gamma_j\), and \(z_j(p_j) = 0\). The collection \(\mathcal{G}_v\) of curves of tangency \(\Gamma_j\) with the parametrizations \(z_j\), where \(j = 1, \ldots, n\), is called the collection of curves of tangency for the pair of foliations \((\tilde{\mathcal{F}}, \tilde{\mathcal{F}}_0)\), and the set \(\Gamma_v \cup \Gamma_0\) is the extended collection of curves of tangency for the pair of foliations.

Lemma 1.2. Let \(v_1\) and \(v_2\) be strictly formally orbitally equivalent germs in \(\mathbb{V}_{n+1}^d\). Then there exists a germ \(w_2 \in \mathbb{V}_{n+1}^d\) such that it is strictly analytically orbitally equivalent to \(v_2\) and the pair of foliations \((\tilde{\mathcal{F}}_{v_1}, \tilde{\mathcal{F}}_0)\) and \((\tilde{\mathcal{F}}_{w_2}, \tilde{\mathcal{F}}_0)\) have the same collection of curves of tangency.

Proof. Without loss of generality, we may suppose that the germs \(v_1\) and \(v_2\) are vertical at \(\{x = 0\}\). Since \(v_1\) and \(v_2\) are strictly formally orbitally equivalent germs in \(\mathbb{V}_{n+1}^d\), there exists a formal change of coordinates \(\hat{H} = \text{Id} + \ldots\) and a formal series \(\hat{K}\) with nonzero constant term such that

\[
\hat{K} \cdot (\hat{H}^t v_2) \circ \hat{H}^{-1} = v_1,
\]
where \( \hat{H} \) contains \( x \) as the factor of its first component. Let \( N \) be a large-enough natural number, and let \( H_N \) and \( K_N \) be the \( N \)-partial sums of the series \( \hat{H} \) and \( K \), respectively. Then the vector field

\[
v_3 = K_N \cdot (H_N'v_2) \circ H_N^{-1}
\]

is holomorphic in \((\mathbb{C}^2, 0)\) and strictly analytically orbitally equivalent to \( v_2 \). The \( N \)-jets at zero of the germs \( v_1 \) and \( v_3 \) coincide \( (\hat{j}_i^N v_3 = \hat{j}_i^N v_1) \) and \( v_3 \) is vertical at \( \{x = 0\} \).

Recall that the finite-order jets at the points of tangency of all the germs constructed above (involution, normalized first integrals, curves of tangency, and their parametrizations) are completely determined by finite jets of the initial germ of vector field. Therefore, we may assume that the corresponding germs constructed for the vector fields \( v_1 \) and \( v_3 \) coincide up to a high-order jet (at the respective points of tangency).

Let \( \Gamma_j \) be the curve of tangency of the pair of foliations \((\tilde{\mathcal{F}}_{01}, \tilde{\mathcal{F}}_0)\) passing through the point \( p_j \in \mathcal{L} \), and let \( \{u = \beta_j(x)\} \), where \( u_j = \beta(0) \), be its local expression. Let \( J_j = J_{01,j} \) and \( \tilde{\beta}_j = J_{01,j}^{-1} \) be the normalized first integrals of the germs \( v_1 \) and \( v_3 \), respectively, at the point \( p_j \). The restrictions \( j_j = J_j|\gamma_j \), \( \tilde{\gamma}_j = \tilde{\beta}_j|\gamma_j \) are local biholomorphisms from \((\Gamma_j, p_j)\) to \((\mathbb{C}, 0)\) and, therefore, the composition \( \varphi_j = z_j^{-1} \circ \gamma_j \circ (\tilde{\beta}_j|\gamma_j) \) is a local biholomorphism. Let \( \gamma_j = \pi(\Gamma_j) \) be the projection of the curve \( \Gamma_j \) to \((\mathbb{C}^2, 0)\); then \( F_j = \pi \circ \varphi_j \circ (\pi|\gamma_j)^{-1} \) is the corresponding local biholomorphism from the curve \( \gamma_j \) to itself.

**Remark 1.4.** The curves \( \gamma_j \) are not necessarily the curves of tangency of the pair of foliations \((\mathcal{F}_{03}, \mathcal{F}_0)\). The following lemma provides a vector field analytically equivalent to \( v_3 \) whose foliation is tangent to \( \mathcal{F}_0 \) at \( \gamma_j \).

**Lemma 1.3.** Let \( \gamma_0 = \{x = 0\} \), and let \( \gamma_j = \{y = x \beta_j(x)\} \), where \( j = 1, \ldots, n \), be non-singular analytic germs of curves in \((\mathbb{C}^2, 0)\) transversally intersecting at zero. Suppose that \( v_1 = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} \) is vertical at each \( \gamma_j \), i.e., \( P(x, x \beta_j(x)) = 0 \). If \( F_j: \gamma_j \to \gamma_j \), where \( j = 1, \ldots, n \), are near to the identity germs of biholomorphisms and the germ \( v_1 \) coincides with the germ \( v_3 \) up to a high-order jet at zero, then there exists a local holomorphism \( H: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) near to the identity such that \( H(\gamma_j) = \gamma_j \) for \( j = 0, \ldots, n \), \( H|\gamma_j = F_j \), and \( H \) transforms the field \( v_3 \) into a field \( w_2 \) vertical at \( \gamma_j \) for \( j = 1, \ldots, n \).

**Lemma 1.2** follows from Lemma 1.3.

Indeed, by Lemma 1.3, we may assume that the vector field \( w_2 \) is tangent to \( \gamma_j \) to the foliation \( \mathcal{F}_0 \). This gives the coincidence of the collections \( \Gamma_{v_1} \) and \( \Gamma_{w_2} \).

To prove the coincidence of the standard parametrizations, let \( \hat{H} = \pi^{-1} \circ H \circ \pi \) be the lifting of the holomorphism \( H \). The function \( \hat{J}_j = \tilde{\beta}_j \circ \hat{H}^{-1} \) is a normalized first integral for the germ \( \hat{w}_2 \) at the point \( p_j \); this follows from the definition of the normalized first integral and from the fact that the lifting holomorphism \( \hat{H} \) is the identity on the pasted sphere. Hence, the parametrization \( \hat{z}_j = J_j|\gamma_j \) coincides with the standard parametrization \( z_j \). Indeed, since

\[
\hat{z}_j = \hat{J}_j|\gamma_j \circ \tilde{\beta}_j \circ \hat{H}^{-1}|\gamma_j.
\]
it follows from from the definition of $\tilde{H}$ and the equality $H|_{\gamma_j} = F_j$ that
\[
\hat{z}_j = j_j \circ \pi^{-1} \circ (H|_{\gamma_j})^{-1} \circ \pi|_{\Gamma_j} = j_j \circ \pi^{-1} \circ F_j^{-1} \circ \pi|_{\Gamma_j}.
\]
Finally, the definition of $\varphi_j$ implies
\[
\hat{z}_j = j_j \circ \varphi_j^{-1} = j_j \circ \zeta_j^{-1} \circ z_j = z_j.
\]
This proves Lemma 1.2.

\[\square\]

Proof of Lemma 1.3. The proof of Lemma 1.3 is similar to the proof of Lemma 12 in [V2]. Namely, let $F_j$ be the biholomorphisms mapping the points $(x, y)$ of curve $\gamma_j$ ($y = x\beta_j(x)$) to the points $(\tilde{x}, \tilde{y})$, where $\tilde{x} = f_j(x)$ and $\tilde{y} = \tilde{x}\beta_j(\tilde{x})$. Since the functions $F_j$ are near to the identity, we have $f_j(x) = x + o(x^N)$ as $x \to 0$ for $N$ large enough and $j = 1, \ldots, n$.

Let $H^1_j(y)$ be the Lagrange interpolation polynomial of degree $n - 1$ which takes the value $f_j(x) - x$ at the point $x = \beta_j(x)$ for $j = 1, \ldots, n$. Let $H^2_j(y)$ be the Lagrange interpolation polynomial of degree $n - 1$ which takes the value $f_j(x)\beta_j(f_j(x)) - x\beta_j(x)$ at the point $x\beta_j(x)$ for $j = 1, \ldots, n$. We define $H(x, y) := (x, y) + (H^1_j(y), H^2_j(y))$ for $x \neq 0$ and $H(0, y) = (0, y)$. Then $H(x, y) = (x, y) + o(x^{N-n+1})$ (here we use the transversality assumption: all the values $u_j = \beta_j(0)$
are distinct), and $H$ coincides with $F_j$ on $\gamma_j$:

$$H(x, x\beta_j(x)) = (f_j(x), f_j(x)\beta_j(f_j(x))).$$

Moreover, the holomorphism $H$ takes the vector field $v_3$ to a field $\tilde{v}_3$ near enough to $v_1$. Thus, the proof reduces to the case where all the maps $F_j$ are the identity. In this case, $H$ can be expressed as

$$H(x, y) = (x + R(x, y)h(x, y), y),$$

where $R(x, y) = \prod_{j=1}^{n}(y - x\beta_j(x))$, and $h$ is an unknown function. By definition, $H|\gamma_j = \text{Id}$.

Let $v_3 = \tilde{P}\frac{\partial}{\partial x} + \tilde{Q}\frac{\partial}{\partial y}$. Suppose that the map $H$ transforming the field $v_3$ into $w_2$ has expression (1.5). The condition that $w_2$ is vertical at $\gamma_j$ means

$$\tilde{P} + h(\tilde{P}R_x + \tilde{Q}R_y) = 0 \quad \text{on} \quad \gamma_j$$

(here we used the fact that $R = 0$ on $\gamma_j$). Therefore, the unknown function $h$ must satisfy the equality

$$h = \frac{-\tilde{P}}{\tilde{P}R_x + \tilde{Q}R_y} \quad \text{on} \quad \gamma_j.$$  

To determine the order of the function $h_j(x) = h|_{\gamma_j}(x, x\beta_j(x))$ as $x \to 0$, we first observe that, since $P = 0$ at $\gamma_j$ and $j_0\tilde{P} = j_0\tilde{P}$, it follows that $\tilde{P}(x, x\beta_j(x)) = o(x^N)$. Moreover, since all the points $u_j = \beta_j(0)$ are distinct and $u_j \neq 0$, we have $R_j(x, x\beta_j(x)) = c_j x^{n-1} + \ldots$, where $c_j \neq 0$. The equality (1.6) implies that, since $\tilde{PR}_x$ and $R_y$ are of orders $N + n - 1$ and $n - 1$, respectively, in $x$ at the points $(x, x\beta_j(x))$ (as $x$ tends to zero), it remains to determine the order of $\tilde{Q}$ at these points.

The genericity assumption 1 (see Section 1.2, assumption (a)) implies that the difference $\Delta(x, y) = xQ(x, y) - yP(x, y)$ has order exactly $n + 3$ at the points of the curve $y = x\beta_j(x)$, where $\beta_j(0) = u_j$; $\Delta(x, x\beta_j(x)) = xQ(x, x\beta_j(x)) = Cx^{n+3} + o(x^{n+3}) (C \neq 0)$ as $x \to 0$. This implies that $\tilde{Q}R_y(x, x\beta_j(x)) = uC x^{2n+1} + o(x^{2n+2})$. Hence, for $N > 2n + 2$, the values of the function $h_j(x)$ defined by (1.7) at the points $(x, x\beta_j(x))$ of the curve $\gamma_j$ have order no smaller than $N - 2n - 2$:

$$h_j(x) = o(x^{N-2n-2}), \quad x \to 0.$$  

For each fixed $x$, let $h(x, y) = h_x(y)$ be the Lagrange interpolation polynomial taking the values $h_j(x)$ at the points $y = x\beta_j(x)$ for $j = 1, \ldots, n$. Estimate (1.8) implies that, for $N > 3n + 1$, the function $h(x, y)$ admits an analytic continuation to the point $x = 0$, and $h(x, y) = o(x^{N-3n-1})$ as $x \to 0$. We have defined the unknown function $h$ satisfying (1.7) and, thereby, the coordinate change $H$. □

**Remark 1.5.** We stress that the biholomorphism constructed in Lemma 1.3 is the identity on the straight line $\{x = 0\}$; thus, the germs $w_2$ and $v_1$ have coinciding extended collections of curves of tangency (and, by Lemma 1.1, their collections of points of tangency and involutions coincide).

### 1.6. Normalization of a pair of foliations at the points of tangency.

In this section, we prove that, in a neighborhood of each point of tangency with the pasted sphere, the dicritic foliation can be transformed, by means of a local
holomorphism, into a normal form depending only on the curve of tangency, on its standard parametrization, and on the restriction of the normalized first integral to the pasted sphere (i.e., on involutions).

**Lemma 1.4.** Let \( \mathcal{F} \) be a foliation of a neighborhood of the point \( p_j = (0, u_j) \) by level curves of the holomorphic function \( J : (\mathbb{C}^2, p_j) \to (\mathbb{C}, 0) \), where \( J(0, u_j) = 0 \), \( J'_u(0, u_j) \neq 0 \), \( J'_u(0, u_j) = 0 \), and \( J''_{uu}(0, u_j) \neq 0 \). Let \( u = \alpha(x) \), where \( \alpha(0) = u_j \), be the holomorphic solution to the implicit equation \( J'_u(x, u_j) = 0 \). Suppose that \( z = z(x) = J(x, \alpha(x)) \) is the restriction of the function \( J \) to the curve \( \{u = \alpha(x)\} \) and \( g(u) = J(0, u) \) is the restriction of \( J \) to the straight line \( \{x = 0\} \). Then there exists a local holomorphism \( H : (\mathbb{C}^2, p_j) \to (\mathbb{C}^2, p_j) \) preserving each straight line \( \{x = c\} \), acting identically on \( \{x = 0\} \), and transforming the foliation \( \mathcal{F} \) into the standard foliation \( \mathcal{F}_{z, g} := \{(x, u) : z(x) + g(u) = \text{const}\} \).

**Proof.** We seek a change of coordinates \( H^{-1} \) transforming the standard foliation \( \mathcal{F}_{z, g} \) into the foliation \( \mathcal{F} \) in the form \( (x, u) \to (x, \zeta(x, u) + \alpha(x)) \), where \( \zeta \) is an unknown function. Since the holomorphism is required to be the identity at \( \{x = 0\} \), the following condition must hold:

\[
\zeta(0, u) = u - u_j. \tag{1.9}
\]

Moreover, as the curve \( \{z(x) + g(u) = c\} \) needs to be transformed into the curve \( \{J(x, u) = c\} \) with the same value \( c \), we have

\[
J(x, \alpha(x)) + J(0, u) = J(x, \zeta + \alpha(x)), \quad \zeta = \zeta(x, u). \tag{1.10}
\]

By assumption, \( g(u_j) = g'(u_j) = 0 \) and \( g''(u_j) \neq 0 \); so, there exists a holomorphic function \( f(u) \) such that \( f(u_j) = 0 \), \( f'(u_j) \neq 0 \), and \( g(u) = (f(u))^2 \) in \( (\mathbb{C}, u_j) \).

Let \( \phi(x, \zeta) = J(x, \zeta + \alpha(x)) - J(x, \alpha(x)) \); then we have \( \phi(x, 0) = 0 \), \( \phi'_u(x, 0) = J'_u(x, \alpha(x)) = 0 \), and \( \phi''_{uu}(0, 0) = J''_{uu}(0, u_j) \neq 0 \). Hence there exists a holomorphic function \( \phi \) on \( (\mathbb{C}^2, 0) \) such that \( \phi(0, 0) \neq 0 \) and \( \phi(x, \zeta) = \zeta^2 \tilde{\phi}(x, \zeta) \). Let \( \psi(x, \zeta) = \sqrt{\phi(x, \zeta) \text{ and } F(x, \zeta) = \zeta \psi(x, \zeta) \text{, so that } \phi = F^2 \text{; we choose the branch of } \sqrt{\phi(0, \zeta) \text{ such that the function } F(0, \zeta) = \sqrt{\phi(0, \zeta) = \sqrt{J}(0, \zeta + u_j) = \sqrt{g(\zeta + u_j) \text{ coincides with the function } f(\zeta + u_j)}}.}

Equation (1.10) is then equivalent to \( \phi(x, \zeta) = g(u) \), and it holds if \( F(x, \zeta) = f(u) \). This equality is satisfied at \( x = 0 \), \( \zeta = 0 \), \( u = u_j \) (and even at all the points \( x = 0 \), \( \zeta = u - u_j \), as \( F(0, \zeta) = f(\zeta + u_j) \)). Since \( F''(0, 0) \neq 0 \), by the implicit function theorem, this equation (and, hence, (1.10)) has a unique holomorphic solution \( \zeta = \zeta(x, u) \) such that \( \zeta(0, u_j) = 0 \) in \( (\mathbb{C}^2, 0) \). But then (1.9) also holds, which proves Lemma 1.4.

**Remark 1.6.** We shall apply this lemma to the case where \( J \) is a normalized first integral; in this case, \( \{u = \alpha(x)\} \) is a curve of tangency and \( z = z(x) \) is its standard parametrization. A similar normalization lemma was proved in [V2].

**1.7. Normalization of a pair of foliations outside the points of tangency.**

In this section, we construct local rectifying mappings for the pair of foliations \( (\mathcal{F}, \mathcal{F}_0) \) at non-tangency points between the foliation \( \mathcal{F} \) and the pasted sphere.
As in Section 1.4, \( v \) is a germ in \( \mathcal{V}_{n+1}^0 \) of type (1.1), \( \tilde{v}_+ = B \frac{\partial}{\partial v} + A \frac{\partial}{\partial u} \) is its blow-up defined by (1.3), and \( p_0 = (0, u_0) \) is a point on the pasted sphere different from the points of tangency \( p_j \), where \( j = 1, \ldots, n \). Let \( J \) be a holomorphic (at a neighborhood of \( p_0 \)) first integral of \( \tilde{v} \), that is, \( J(0, u_0) = 0, \nabla J(0, u_0) \neq 0 \), and \( A \frac{\partial}{\partial u} + B \frac{\partial}{\partial v} = 0 \) in a neighborhood of \( p_0 \). The inequality \( B(0, u_0) \neq 0 \) implies that \( J'(0, u_0) \neq 0 \). Therefore, the function \( \phi(u) = J(0, u) \) is invertible in a neighborhood of \( u_0 \). Changing, if needed, the function \( J \) for the composition \( \phi^{-1} \circ J \) (which is also a first integral), we ensure that \( J(0, u) \equiv u \). Thus, the map \( H_{p_0} : (x, u) \mapsto (x, J(x, u)) \) is a local change of coordinates in a neighborhood of \( p_0 \). This map is the identity at the pasted sphere: \( H_{p_0} : (0, u) \mapsto (0, u) \). Moreover, \( H_{p_0} \) leaves the foliation \( \mathcal{F}_0 = \{ x = \text{const} \} \) invariant and transforms the level curves of \( J \) (i.e., the phase curves of the field \( \tilde{v}_+ \), or leaves of the foliation \( \mathcal{F} = \mathcal{F}_0 \)) into the straight lines \( \{ u = \text{const} \} \). We call this map the rectifying map for a pair of foliations \( (\mathcal{F}, \mathcal{F}_0) \) at \( p_0 \).

Now, consider the foliation \( \mathcal{F} \) of a neighborhood of the point \( p_\infty \), where \( p_\infty \), is defined in charts \( (v, y) \) by \( y = 0, v = 0 \). As above, we construct a first integral \( \tilde{J} = \tilde{J}(v, y) \) for the field \( \tilde{v} \) (\( \tilde{J}(v, y) = \tilde{A} \frac{\partial}{\partial v} + \tilde{B} \frac{\partial}{\partial y} \) (see Section 1.4) normalized by \( \tilde{J}(v, 0) = v \). Suppose that the straight line \( \{ v = 0 \} \) is a phase curve of the field \( \tilde{v}_- : \tilde{A}(0, y) = 0 \). Then \( \tilde{J}(0, y) = 0 \), so that \( \tilde{J}(v, y) = v\tilde{F}(v, y) \) for some holomorphic function on \( (\mathbb{C}^2, 0) \) for which \( \tilde{F}(v, 0) = 1 \) (hence \( \tilde{F}(v, y) \neq 0 \) in a neighborhood of zero). Let \( H_\infty : (v, y) \mapsto (v\tilde{F}(v, y), \frac{y}{\tilde{F}(v, y)}) \). The map \( H_\infty \) is a local change of coordinates in \( (\mathbb{C}^2, 0) : H_\infty'(0, 0) = E \), where \( E \) is the identity matrix. Moreover, \( H_\infty \) is the identity on the pasted sphere, \( H_\infty(v, 0) = (v, 0) \). Finally, each leaf \( \{ vy = c \} \) of the foliation \( \mathcal{F}_0 \) is transformed into itself. We call the map \( H_\infty \) the rectifying map for the pair of foliations \( (\mathcal{F}, \mathcal{F}_0) \) at \( p_\infty \).

**Remark 1.7.** The rectifying maps for pairs of foliations are uniquely determined.

**1.8. End of the proof of Theorem 1.** Let \( v_1 \) and \( v_2 \) be two strictly orbitally formally equivalent germs in \( \mathcal{V}_{n+1}^0 \). As was shown in Section 1.5, we may assume that the straight line \( \{ (0, y) : y \in (\mathbb{C}, 0) \} \) is a common leaf of their corresponding blow-ups \( \mathcal{F}_1 = \mathcal{F}_{v_1} \) and \( \mathcal{F}_2 = \mathcal{F}_{v_2} \). By Lemma 1.1 in Section 1.4, the set of points of tangency and the set of the standard involutions associated to these foliations coincide. By Lemma 1.2 and the remark to it, we may assume that the pair of foliations \( (\mathcal{F}_1, \mathcal{F}_0) \) and \( (\mathcal{F}_2, \mathcal{F}_0) \) have the same set of curves of tangency (and their corresponding standard parametrizations). For each point \( p_\ast \) in the pasted sphere, we construct local holomorphisms \( H_{p_\ast}^1 \) and \( H_{p_\ast}^2 \) defined in a neighborhood of \( p_\ast \) as follows: for each point \( p_\ast = p_j = (0, u_j) \), let \( H_{p_j}^k \) be the normalizing holomorphism of the foliation \( \mathcal{F}_k \), \( k = 1, 2 \), defined in Lemma 1.4; for the other points, let \( H_{p_\ast}^1, k = 1, 2 \), be the rectifying holomorphism from Section 1.7. We recall that all these holomorphisms leave invariant the foliation \( \mathcal{F}_0 \), and their restrictions to the pasted sphere are the identity. For this reason, for each point \( p_\ast \) the composition \( H_{p_\ast} = (H_{p_\ast}^2)^{-1} \circ H_{p_\ast}^1 \) is well-defined. Moreover, each \( H_{p_\ast} \) leaves invariant the foliation \( \mathcal{F}_0 \), transforms the leaves of foliation \( \mathcal{F}_1 \) into leaves of foliation \( \mathcal{F}_2 \), and its restriction to the pasted sphere is the identity map. We choose a finite number of
holomorphisms $H_p$, such that their domains of definition form a finite covering of the pasted sphere (this can be done by compactness). According to Remark 1.7, we may restrict, if needed, the domains of definition of the finitely many holomorphisms $H_p$ so that they coincide on their overlapping domains. Therefore, the collection of these local holomorphisms defines a holomorphism from a neighborhood of the pasted sphere to a neighborhood of the pasted sphere such that the foliation $\mathcal{F}_1$ is transformed to the foliation $\mathcal{F}_2$ and the foliation $\mathcal{F}_0$ is invariant. As this holomorphism is the identity on the pasted sphere, it is, in fact, the lifting of a holomorphism $H: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. This holomorphism transforms the foliation $\mathcal{F}_{v_1}$ into $\mathcal{F}_{v_2}$ as required. This completes the proof of Theorem 1.

Remark 1.8. In the proof of Theorem 1, we used the coincidence, up to higher-order terms, of the jets at zero of the germs $v_1$ and $v_2$. It follows from the proof that, under these conditions, the germ of the holomorphism conjugating the corresponding foliations has identity jet at zero up to a higher-order term.

2. Rigidity of Formally Equivalent Dicritic Vector Fields (Flows)

In this section, we prove that the formal equivalence of two generic germs in $\mathcal{V}_{n+1}$ implies their analytic equivalence.

2.1. Construction of the conjugating holomorphism. Let $v_1$ and $v_2$ be two formally equivalent germs in $\mathcal{V}_{n+1}$. Without loss of generality (see Section 1.5), we assume that, for $N$ large enough (namely, for $N \geq 3n$), the $N$-jets of both germs coincide:

$$j^N_0 v_1 = j^N_0 v_2. \quad (2.1)$$

Theorem 1 implies that the germs $v_1$, $v_2$ are analytically orbitally equivalent. Therefore, we may assume that their foliations coincide, i.e., $\mathcal{F} = \mathcal{F}_{v_1} = \mathcal{F}_{v_2}$, and that the vector fields $v_1$ and $v_2$ are proportional, i.e., $v_1 = kv_2$, where $k$ is an holomorphic function in $(\mathbb{C}^2, 0)$ such that $k(0, 0) \neq 0$ (this can be done by changing the germ $v_2$ for its image under the action of the conjugating holomorphism constructed in Theorem 1). Moreover, by (2.1), we may also assume that $j^N_0 k \equiv 1$.

Let $\tilde{\mathcal{F}}$ be the blow-up of the foliation $\mathcal{F}$. By $\gamma_p$ we shall denote the leaf of $\tilde{\mathcal{F}}$ passing through the point $p \in L$.

Let $\tilde{v}_1$ and $\tilde{v}_2$ be the liftings (not the blow-ups) of the vector fields $v_1$ and $v_2$, respectively, to a neighborhood of the pasted sphere. Without loss of generality (up to a linear change of coordinates), we can assume that $v_1(x, y) = (xR + P_{n+2} + \ldots) \frac{\partial}{\partial x} + (yR + Q_{n+2} + \ldots) \frac{\partial}{\partial y}$, where $R = \prod_{j=1}^n (y - u_k x)$ is a homogeneous polynomial of degree $n$, $P_{n+2}$ and $Q_{n+2}$ are homogeneous polynomials of degree $n + 2$, and the dots denote terms of order higher than $n + 2$. Then, in the $(x, u)$-charts, where $u = y/x$, the field $\tilde{v}_1$ is defined as

$$\tilde{v}_1(x, u) = x^{n+1}(R(1, u) + xP_{n+2}(1, u) + o(x)) \frac{\partial}{\partial x} + x^{n+1}(R_{n+2}(1, u) + O(x)) \frac{\partial}{\partial u} \quad \text{as } x \to 0, \quad (2.2)$$

where $R_{n+2}(x, y) = xQ_{n+2}(x, y) - yP_{n+2}(x, y)$. 

In the \((v, y)\)-charts, where \(v = x/y\), the vector field \(\hat{v}_1\) has the expression
\[
\hat{v}_1(v, y) = y^{n+1}(\Re(v, 1) + yQ_{n+2}(v, 1) + o(y)) \frac{\partial}{\partial y} \\
+ y^{n+1}(-R_{n+2}(v, 1) + O(y)) \frac{\partial}{\partial v} \quad \text{as } y \to 0, \quad (2.3)
\]
Recall that, by the genericity assumption (a) (see Section 1.2), \(R_{n+2}(1, u_j) \neq 0\) for every \(j\), all \(u_j\) are pairwise distinct, and \(u_j \neq 0\). Relation (2.1) implies the following asymptotic relations for the corresponding expressions for \(\hat{v}_2\):
\[
\hat{v}_2(x, u) = \hat{v}_1(x, u) + O(x^N), \quad x \to 0, \quad (2.4)
\]
\[
\hat{v}_2(v, y) = \hat{v}_1(v, y) + O(y^N), \quad y \to 0. \quad (2.5)
\]

The foliation \(\mathcal{F}\) is the foliation by phase curves of the holomorphic field of directions defined, in the \((x, u)\)-charts, by the relation \(\hat{v}_1(x, u) = x^{-n-1} \hat{v}_1(x, u)\):
\[
\hat{v}_1(x, u) = B \frac{\partial}{\partial x} + A \frac{\partial}{\partial u} \\
:= (\Re(1) + O(x)) \frac{\partial}{\partial x} + (R_{n+2}(1, u) + O(x)) \frac{\partial}{\partial u}; \quad (2.6)
\]
in the domain of the \((v, y)\)-charts, it is defined by \(\hat{v}_1(v, y) = y^{-n-1} \hat{v}_1(v, y)\):
\[
\hat{v}_1(v, y) = \hat{B} \frac{\partial}{\partial v} + \hat{A} \frac{\partial}{\partial v} \\
:= (-\Re(v, 1) + O(y)) \frac{\partial}{\partial v} + (R_{n+2}(1, v) + O(y)) \frac{\partial}{\partial v}. \quad (2.7)
\]

All the points of the pasted sphere \(\mathcal{L}\) are singular for the liftings \(\check{v}_1\) and \(\check{v}_2\) (non-singular for the fields of directions \(\hat{v}_1\) and \(\hat{v}_2\)). These vector fields are tangent to the leaf \(\gamma_p\) at each of its points. We denote the restriction of the vector field \(\check{v}_j\) to \(\gamma_p\) by \(v_j^p\) for \(j = 1, 2\); clearly, \(v_j^p(p) = 0\).

We will construct the biholomorphism \(H\) conjugating the vector fields \(v_1\) and \(v_2\) (more precisely, its lifting \(\hat{H}\) to a neighborhood of the pasted sphere) as a conjugation of the vector fields \(v_1^p\) and \(v_2^p\) for each point \(p \in \mathcal{L}\). Namely, for each \(p_0 \in \mathcal{L}\), we construct a biholomorphism \(H_{p_0}\) from a neighborhood \(U_{p_0}\) of this point to a neighborhood \(\hat{U}_{p_0}\) of the same point in such a way that \(H_{p_0}\) is the identity at the intersection \(U_{p_0} \cap \mathcal{L}\), it transforms each leaf \(\gamma_p \cap U_{p_0}\) to the leaf \(\gamma_{\hat{p}} \cap \hat{U}_{p_0}\), and its restriction \(H_{p_0}|_{\gamma_p}\) conjugates the vector fields \(v_1^p\) and \(v_2^p\). After proving that the constructed biholomorphisms \(H_{p_0}\) coincide in the intersections of their domains, we cover the sphere \(\mathcal{L}\) by a finite number of neighborhoods \(U_{p_0}\) and define the required biholomorphism \(\hat{H}\).

2.1.1. Parabolic singular points of analytic vector fields on the line. A singular point of a holomorphic vector field \(\omega(z) \frac{d}{dz}\) on \((\mathbb{C}, 0)\) is called parabolic if the function \(\omega(z)\) has order higher than one at that point. As is known (see [11]), necessary and sufficient conditions for the (local) analytic equivalence of two holomorphic vector fields \(\omega(z) \frac{d}{dz}\) and \(\tilde{\omega}(z) \frac{d}{dz}\) having a parabolic singular point at zero are the coincidence of the orders of their respective singular points and the coincidence
of their corresponding residues: $\text{Res}_0 \frac{1}{\omega(z)} = \text{Res}_0 \frac{1}{\tilde{\omega}(z)}$. In particular, if the order of the singular point 0 of each vector field is equal to $n + 1$, then, for analytic equivalence, the coincidence of their $(2n + 1)$-jets at zero is sufficient:
\[
\tilde{J}_0^{2n+1} = \tilde{J}_0^{2n+1} \omega.
\] (2.8)

The holomorphism $z = h(\tilde{z})$ conjugating the vector fields $\omega(z)$ and $\tilde{\omega}(z)$ is found by solving the equation
\[
\int \frac{dz}{\omega(z)} = \int \frac{d\tilde{z}}{\tilde{\omega}(\tilde{z})} + C.
\] (2.9)

Indeed, the mappings on the left- and right-hand sides of (2.9) are rectifying mappings for the vector fields $\omega$ and $\tilde{\omega}$, respectively: they both transform these fields into the standard field $\frac{d}{dt}$.

The conjugating holomorphism $h$ is determined up to the constant $C$ of (2.9).

**Remark 2.1.** In particular, for vector fields satisfying condition (2.8) the conjugating holomorphism normalized by the condition $\tilde{J}_0^{n+1}h = \text{Id}$ exists and is unique.

The singular point $p$ of the vector fields $\nu_1^p$ and $\nu_2^p$ is parabolic. We shall find a conjugating holomorphism $h_p$ for them by solving the corresponding equation (2.9).

Since these holomorphisms must analytically depend on the parameter $p \in \mathcal{L}$ and uniformly (with respect to the parameter) depend on their domains to definition, the existence of a local solution of (2.9) is not sufficient for our purposes. We shall obtain these additional properties by reducing (2.9) to the implicit function theorem.

Generally speaking, for the points $p$ near to the points of tangency, the vector fields $v^p_j$, $j = 1, 2$, have two singular points. To prove the analytic equivalence of such vector fields, the results of versal deformations of parabolic singular points can be used (see [K]). Nevertheless, for our purposes, it is more convenient to reduce (2.9) to the implicit function theorem.

2.1.2. The conjugating holomorphism in a neighborhood of nontangency points.

Suppose that $p_0 = (0, u_0)$ (in $(x, u)$-charts) is a point in the pasted sphere $\mathcal{L}$ not being a point of tangency of the foliation $\mathcal{F}$ with the pasted sphere $\mathcal{L}$, i.e., such that $u_0 \neq u_j$ for $j = 1, \ldots, n$. Let $\tilde{v}_1$ be the vector field given in (2.6). Suppose that $J = J(x, u)$ is a holomorphic first integral of vector field $\tilde{v}_1$ (see Section 1.7) defined in a neighborhood of the point $p_0$:

\[
A \frac{\partial J}{\partial u} + B \frac{\partial J}{\partial x} = 0, \quad \nabla J(0, u_j) \neq 0.
\]

Since $B(0, u_0) \neq 0$, we have $J'_0(0, u_0) \neq 0$; therefore, without loss of generality, we may assume that $J(0, u) \equiv u - u_0$. By the implicit function theorem, the equation $J(x, u) = c$ has a holomorphic solution $u = u(x, c)$, $u(0, 0) = u_0$, uniquely determined for sufficiently small values of $x$ and $c$. Thus, for $c$ small enough, the leaf of foliation $\mathcal{F}$ passing through the point $p$ with coordinates $(0, u_0 + c)$ is given by the equation $\{u = u(x, c)\}$; hence $x$ becomes a parameter on $\gamma_p$. From (2.2) we obtain the following expression for the vector field $v_1(x, c) := v^p_1$, $p = (0, u)$, $u = u_0 + c$:

\[
v_1(x, c) = x^{n+1}f(x, c) \frac{d}{dx}.
\] (2.10)
where $f(x, c) = (\Re(1, u) + O(x))|_{u=u(x,c)}$ and $f(0, 0) \neq 0$. Therefore, (2.4) implies that, for $v_2(x, c) := v_2^0$,

$$v_2(x, c) - v_1(x, c) = O(x^N) \frac{d}{dx}, \quad x \to 0.$$ 

Hence, for $N > 2n + 1$, the difference

$$\Delta(x, c) := 1 \frac{v_2(x, c)}{v_1(x, c)} - 1$$

is holomorphic at the point $x = 0$ for sufficiently small values of $c$. Equation (2.9) (with suitable constant $C$) for the holomorphism $\tilde{x} = h(x, c)$ transforming the vector field $v_2(x, c)$ to the vector field $v_1(x, c)$ can be written as

$$\int_x^{\tilde{x}} \frac{dz}{v_1(x, c)} = \int_x^0 \Delta(x, c)dz. \quad (2.11)$$

It is easy to solve (2.11) under conditions (2.10). Nevertheless, for our purposes, it is more convenient to give a more general construction.

We seek a solution $\tilde{x} = h(x, c)$ to (2.11) in the form

$$\tilde{x} = x + \varphi v_1(x, c), \quad (2.12)$$

where $\varphi = \varphi(x, c)$ is an unknown function such that $\varphi(0, 0) = 0$. To this aim, we prove the following lemma, which enables us to use the implicit function theorem for finding the holomorphic function $\varphi$.

**Lemma 2.1.** Let $v(x, c)$ be a holomorphic function on the polydisk $D = D_{r, \epsilon} := \{|x| < r\} \times \{|c| < \epsilon\}$, such that, for any fixed $c$ such that $|c| < \epsilon$, $v(x, c)$ has at most finitely many zeroes. Suppose that $v$ satisfies the bounds

$$|v| < \delta, \quad \left| \frac{\partial v}{\partial x} \right| < 1/2 \quad \text{on} \quad D, \quad 0 < \delta < r, \quad (2.13)$$

and let

$$F(x, c, \varphi) = \int_{v(x, c)}^{v(x, \varphi)} \frac{dz}{v(x, c)}, \quad \ell_{x, c, \varphi} = [x, x + \varphi v(x, c)], \quad (2.14)$$

where $|x| < r - \delta$, $|c| < \epsilon$, $|\varphi| < 1$, and $v(x, c) \neq 0$. Then the function $F(x, c, \varphi)$ has an analytic extension (which we also denote by $F$) to the polydisk $\tilde{D} := \{|x| < r - \epsilon\} \times \{|c| < \epsilon\} \times \{|\varphi| < 1\}$ such that

$$F(x, c, 0) = 0, \quad F'_{\varphi}(x, c, 0) = 1 \quad \text{for} \quad (x, c, 0) \in \tilde{D}. \quad (2.15)$$

**Proof.** Note that $v(z, c) \neq 0$ for any $z$ such that $|x - z| < |v(x, c)|$, $|x| < r - \delta$. Indeed, if $v(z, c) = 0$, then

$$|x - z| < |v(x, c)| = |v(x, c) - v(z, c)| \leq |x - z| \sup_{\xi \in [x, z]} \left| \frac{\partial v}{\partial x}(\xi, c) \right|;$$

hence, (2.13) implies $|x - z| \leq \frac{1}{2}|x - z|$, whence $x = z$, which is impossible. So, the quotient $\frac{1}{v(x, c)}$ is a function having no singular points on the disk of radius $|v(x, c)|$ centered at $x$ and containing $\ell_{x, c, \varphi}$; therefore, expression (2.14) indeed defines a
holomorphic function on the domain \( D^o := D \setminus \{ v(x, c) = 0 \} \). Moreover, (2.13) implies \(|v(x, c) - v(z, c)| \leq \frac{1}{2}|x - z|\); hence, for \( z \in [x, x + \varphi v(x, c)] \) and \(|\varphi| < 1\),

\[
|v(x, c) - v(z, c)| \leq \frac{1}{2}|v(x, c)|.
\]

This implies \(|v(z, c)| \geq \frac{1}{2}|v(x, c)|\) for \( z \in \ell_{x, c, \varphi}\). Since the length of the interval \( \ell_{x, c, \varphi} \) is less than \(|v(x, c)|\), we obtain the uniform bound \(|F(x, \varphi)| \leq 2\) on \( D^o\). By the theorem on the removability of a singular point ([Sha]), all the singular points of function \( F \) (more precisely, the points where \( v(x, c) = 0\)) are removable. Let us continue the function \( F \) to these points by a limit process. Direct computations show that if \( x_0 \) is a pole of order higher than one for the function \( \frac{1}{v(x, c)} \), then

\[
\lim_{x \to x_0} F(x, c, \varphi) = \varphi. \quad \text{Moreover, if } x_0 \text{ is a pole of first order of } \frac{1}{v(x, c)} \text{ and } a = \text{Res}_{x_0} \frac{1}{v(x, c)}, \text{ then } \lim_{x \to x_0} F(x, c, \varphi) = \frac{1}{a} \ln(1 + a\varphi) \quad \text{(here we consider the branch of the logarithm with } \ln(1) = 0).\]

Now, equalities (2.15) for points at which \( v(x, c) \neq 0 \) follow from the definition of the function \( F \); for the other points, the equalities follow from the above expressions for its continuation. This completes the proof of Lemma 2.1. \( \square \)

By using equalities (2.12) and (2.14), we express equation (2.11) as

\[
F(x, c, \varphi) = \int_0^x \Delta(z, c) \, dz. \tag{2.16}
\]

Conditions (2.10) imply bounds (2.13) for \( x \) and \( c \) small enough. By Lemma 2.1, conditions (2.15) hold and, by the implicit function theorem, a solution \( \varphi \) of (2.16) such that

\[
\varphi = \varphi(x, c), \quad \varphi(0, 0) = 0, \tag{2.17}
\]

exists, and \( h(x, c) = x + \varphi(x, c)v_1(x, c) \) is the conjugating holomorphism.

**Remark 2.2.** From (2.10), (2.12) and (2.17) it follows that the conjugating holomorphism \( h = h(x, c) \) (for any fixed \( c \)) has \((n + 1)\)-jet at zero equivalent to the identity: \( j_{0}^{n+1} h(x, c) \equiv x \).

**2.1.3. The conjugating holomorphism in a neighborhood of \( u = \infty \).** The conjugating holomorphism \( H_\infty \) acting in a neighborhood of the point \( v = y = 0 \) (in the \((v, y)\)-charts) is constructed similarly, by using expressions (2.3), (2.7), and (2.5), instead of (2.2), (2.6), and (2.4), respectively.

**Remark 2.3.** As in Section 2.1.2, the \((n+1)\)-jet of the restriction of the constructed holomorphism to each leaf \( \gamma_p \), where \( p \in \mathcal{L} \) and \( p \neq p_j \) (and at the leaf \( \gamma_\infty \) passing through \( y = 0, v = 0 \)) at the singular point \( p \) of the vector fields \( v_1^p \) and \( v_2^p \) is equal to the identity.

**2.1.4. The conjugating holomorphism in a neighborhood of the point of tangency.** Here and in what follows, we apply all the notations, definitions and results obtained in Section 1 for the vector field \( v \) to the vector field \( v_1 \).

Let \( p_j = (0, u_j) \) (in the \((x, u)\)-charts) be one of the points of tangency of the foliations \( \mathcal{F} = \mathcal{F}_v = \mathcal{F}_{v_1} \) with the pasted sphere \( \mathcal{L} \). Let \( J \) be the normalized first integral for the vector field (2.6) in a neighborhood of the point \( p_j \) (see Section 1.4).
Since \( J(0, u_j) = 0 \) and \( J'_p(0, u_j) \neq 0 \), the implicit function theorem implies the existence of a unique holomorphic solution \( x = x(u, u_0) \), \( x(u_0, u_0) = 0 \), to the equation \( J(x, u) = J(0, u_0) \) (it is uniquely determined for any \( u \) and \( u_0 \) near enough to \( u_j \)). Hence, for any point \( p = (0, u_0) \) (near to \( p_j \)), the leaf \( \gamma_p \) of the foliation \( \overline{F} \) passing through \( p \) is given by \( \{ x = x(u, u_0) \} \). Thus, \( u \) is a parameter on \( \gamma_p \). The above considerations and (2.2) imply the following expression for the vector field \( v^\rho_1 = v_1(u, u_0) \):

\[
v_1(u, u_0) = x^{n+1}(R_{n+2}(u) + O(x))|_{x = x(u, u_0)} \frac{d}{du}.
\] (2.18)

The vector field \( v^\rho_2 := v_2(u, u_0) \) has a similar expression; thus, by (2.4),

\[
v_2(u, u_0) = v_1(u, u_0) + O(x^N), \quad x = x(u, u_0) \quad \text{as} \quad x = x(u, u_0) \to 0.
\] (2.19)

By the genericity assumptions, \( R_{n+2}(u_j) \neq 0 \), so (2.18) and (2.19) give the following asymptotic equality for \( \Delta(u, u_0) = \frac{1}{v_2(u, u_0)} - \frac{1}{v_1(u, u_0)} \):

\[
\Delta(u, u_0) = O(x^{N-2(n+1)}) \quad \text{as} \quad x = x(u, u_0) \to 0.
\] (2.20)

The theorem on the removability of a singular point (see [Sha]) implies that, for \( N > 2(n+1) \), all the singular points \((u, u_0)\) of the function \( \Delta(u, u_0) \) (namely, those where \( x(u, u_0) = 0 \)) are removable. Therefore, the function \( \Delta = \Delta(u, u_0) \) has a holomorphic extension to a neighborhood of the points \((u_j, u_j)\) and satisfies (2.20) there.

Now, we proceed as in Section 2.1.2. Namely, we look for a holomorphism \( \tilde{u} = h(u, u_0) \) conjugating the vector field \( v_2(u, u_0) \) to the vector field \( v_1(u, u_0) \); so, we rewrite equation (2.9), for a suitable value of \( C \), as

\[
\int_u^{\tilde{u}} \frac{dz}{v_1(z, u_0)} = \int_{u_0}^{u} \Delta(z, u_0) dz.
\] (2.21)

As above, we seek a solution to (2.21) in the form

\[
\tilde{u} = u + \varphi v_1(u, u_0),
\] (2.22)

where \( \varphi = \varphi(u, u_0) \) is an unknown function satisfying

\[
\varphi(u_0, u_0) = 0.
\] (2.23)

Setting

\[
F(u, u_0, \varphi) = \int_u^{u + \varphi v_1(u, u_0)} \frac{dz}{v_1(z, u_0)}
\]

and using (2.22), we reduce equation (2.21) to the form

\[
F(u, u_0, \varphi) = \int_{u_0}^{u} \Delta(z, u_0) dz.
\]

By Lemma 2.1 and the implicit function theorem, there exists a unique holomorphic solution \( \varphi = \varphi(u, u_0) \) to equation (2.21) satisfying (2.23).

**Remark 2.4.** As above, equalities (2.22) and (2.23) imply that \((n+1)\)-jet at \( u_0 \) of the holomorphism \( h \) is equal to the identity: \( j^{n+1}_{u_0} h(u, u_0) \equiv u \). Moreover, for \( u_0 = u_j \), we have \( j^{2n+1}_{u_j} v_1(u, u_j) = 0 \), so that \( j^{2n+1}_{u_j} h(u, u_j) \equiv u \).
2.1.5. Construction of the conjugating holomorphism in a neighborhood of the pasted sphere \( \mathcal{L} \). Remarks 2.2–2.4 imply that, for any \( p_0 \in \mathcal{L} \), the restriction of the conjugating holomorphism \( H_{p_0} \) to any of the leaves \( \gamma_p \), \( p \in \mathcal{L} \) (\( p \) near to \( p_0 \)), has \((n + 1)\)-jet at \( p \) equal to the identity. As we saw in Section 2.1.2, the point \( p \) is parabolic for vector fields \( v_1^p, v_2^p \) of order \( n + 1 \) (if \( p \) is not a point of tangency of the foliation \( \tilde{T} \) with the sphere \( \mathcal{L} \)) or of order \( 2n + 1 \) (if \( p \) is a point of tangency for \( \tilde{T} \) and \( \mathcal{L} \)). Remark 2.1 implies that if \( p \neq p_j \) is a common point of the domains \( U_{p_0} \) and \( U_{p_0}' \), corresponding to the conjugating holomorphisms \( H_{p_0} \) and \( H_{p_0}' \), then the restrictions of both holomorphisms to the leaf \( \gamma_p \) coincide (specifically, in the connected component of the intersection \( U_{p_0} \cap U_{p_0}' \)).

Let \( \{H_{p_j}\} \) be the collection of conjugating holomorphisms for the points of tangency \( p_j \), and let \( \{H_{p_0,s}\} \) be a supplementary finite collection of conjugating holomorphisms constructed for points \( p_0,s \) not being points of tangency for \( \tilde{T} \) and \( \mathcal{L} \) such that the domains of definition of these collections form a finite covering of the sphere \( \mathcal{L} \). The coincidence of the holomorphisms in the intersection domains-mentioned above allows us to construct a global holomorphism \( H \) defined on a neighborhood of the pasted sphere. This holomorphism is the identity at \( \mathcal{L} \) and conjugates the vector fields \( v_1 \) and \( v_2 \) (these properties are common to all the local holomorphisms \( H_{p_0} \)). Hence \( H \) is the lifting to a neighborhood of the sphere \( \mathcal{L} \) of a holomorphism of \((\mathbb{C}^2,0)\) whose linear part at zero is equal to the identity, which conjugates the vector fields \( v_1 \) and \( v_2 \). The proof of Theorem 2 is almost complete.

2.2. Relative time invariants for dicritic vector fields. As mentioned, the proof given in Section 2.1 is not yet complete. Indeed, in the construction of the conjugating holomorphism, we used only the coincidence of the phase portraits and the \( N \)-jets at zero of the germs \( v_1 \) and \( v_2 \) for \( N > 2n + 1 \) (see Section 2.1). This coincidence is not sufficient even for the formal equivalence of the germs \( v_1 \) and \( v_2 \). Indeed, the leaf \( \gamma_p \) passing through a point \( p = (0, u_0) \) has a non-empty intersection with the sphere \( \mathcal{L} \) at the point \( p^* = (0, u_0^*) \), where \( u_0^* = I_j(u_0) \) (\( I_j \) is the standard involution defined in Section 1.4). For this reason, equations (2.21) and (2.22) define two generally different holomorphisms \( h(u, u_0) \) and \( h(u, u_0^*) \) in \( \gamma_p \): the first one is obtained from (2.22) and the second is obtained from the same equation (2.22) with \( u_0^* = I_j(u_0) \) instead of \( u_0 \). The identities \( v_1(u, u_0) \equiv v_1(u, u_0^*) \) and \( \Delta(u, u_0) \equiv \Delta(u, u_0^*) \) imply that, for the coincidence of \( h(u, u_0) \) and \( h(u, u_0^*) \), it is necessary and sufficient that

\[
\int_{u_0}^{u_0^*} \Delta(z, u_0)dz \equiv 0, \quad u_0^* = I_j(u_0). \tag{2.24}
\]

In what follows, we prove that equation (2.24) is a consequence of the formal equivalence of the germs \( v_1 \) and \( v_2 \).

2.2.1. A lemma on the preservation of the relative time of motion

**Lemma 2.2.** Let \( \nu \) be a holomorphic vector field defined on a convex domain \( D \subset \mathbb{C} \) and having exactly two parabolic singular points \( a \) and \( b \) of order \( n + 1 \) in \( D \). Let \( h : D \rightarrow \tilde{D} \subset \mathbb{C} \) be a conformal map on \( D \) such that it is near to the identity, has
fixed points $a$ and $b$, and
\[ j_{a}^{n+1}h = \text{Id}, \quad j_{b}^{n+1}h = \text{Id}. \]  

Let $\omega$ be the image under $h$ of the vector field $\nu$. Let $f$ be a meromorphic function on $D \cup \hat{D}$ with poles at the points $a$ and $b$ (and analytic outside $a$ and $b$) such that $\Delta_{\nu}(z) = \frac{1}{\nu(z)} - f(z)$ is analytic on $D$. Then the function $\Delta_{\omega} = \frac{1}{\omega(z)} - f(z)$ is analytic on $\hat{D}$ and
\[ \int_{a}^{b} \Delta_{\nu}(z)dz = \int_{a}^{b} \Delta_{\omega}(z)dz. \]  

**Remark 2.5.** The integral $\int_{\gamma_{p,q}} \frac{dz}{\omega(z)}$ along the path $\gamma_{p,q}$ joining the points $p$ and $q$ is the time of motion along the path $\gamma_{p,q}$ from $p$ to $q$ with velocity $\nu$. Thus, the integral on the left- (right-) hand side of (2.26) can be understood as the difference of the time of motion from $a$ to $b$ for the fields $\nu$ and $\frac{1}{f}$ (for $\omega$ and $\frac{1}{f}$, respectively). Besides, for $\nu$, $\frac{1}{f}$, and $\omega$, the time of motion from $a$ to $b$ is infinite, because the corresponding integrals diverge. For this reason, the integral expressions in (2.26) are called the relative time of motion from one singular point of the vector field to the other.

**Proof of Lemma 2.2.** Since the orders of $\nu$ and $\omega$ are equal to $n + 1$, (2.25) implies that $\Delta_{\omega}$ is holomorphic on $\hat{D}$. Let $\Delta(z) = \Delta_{\nu}(z) - \Delta_{\omega}(z)$. The function $\Delta$ is holomorphic on $D \cap \hat{D} =: D_{0}$ and
\[ \frac{1}{\nu(z)} - \frac{1}{\omega(z)} = \Delta(z), \quad z \in D_{0}, \quad z \neq a, b. \]  

Since $h$ takes the vector field $\nu$ to $\omega$, we have
\[ h'(z)\nu(z) = \omega \circ h(z), \quad z \in D. \]  

Suppose that $x_{-}$ and $x_{+}$ are points in the segment $[a, b]$ near to $a$ and $b$, respectively, $y_{\pm} = h^{-1}(x_{\pm})$, and $\gamma(x_{-}, x_{+}) = h^{-1}([x_{-}, x_{+}])$. Then, according to (2.27),
\[ \int_{[x_{-}, x_{+}]} \Delta(z)dz = \int_{x_{-}}^{x_{+}} \frac{dz}{\nu(z)} - \int_{x_{-}}^{x_{+}} \frac{dz}{\omega(z)}. \]  

Applying (2.28) and making the change $z = h(\tau)$, we obtain
\[ \int_{x_{-}}^{x_{+}} \frac{dz}{\omega(z)} = \int_{\gamma(x_{-}, x_{+})} h'(\tau) \frac{d\tau}{\omega(h(\tau))} = \int_{\gamma(x_{-}, x_{+})} \frac{1}{\nu(\tau)} d\tau. \]  

Since $h$ is near to the identity map, the following assertion holds.

**Assertion.** For points $x_{-}$ and $x_{+}$ near to $a$ and $b$, respectively, the path $\gamma(x_{-}, x_{+})$ is homotopic in $D_{0} \setminus \{a, b\}$ to the broken curve with vertices at the points $y_{-}$, $x_{-}$, $x_{+}$, and $y_{+}$ (see Fig. 2.1).

Therefore, by the Cauchy theorem, the difference on the right-hand side of (2.29) equals to
\[ \int_{[x_{-}, y_{-}]} \frac{dz}{\nu(z)} - \int_{[x_{+}, y_{+}]} \frac{dz}{\nu(z)}. \]
It immediately follows from the definition of relative time of return that

\begin{equation}
\int_{x_0}^{y_0} \gamma(x_{-}, x_{+}) \text{d}z = j_0^{2n+1} v = j_0^{2n+1} w.
\end{equation}

Condition (2.25) implies that the differences \( x_{\pm} - y_{\pm} = h(x_{\pm}) - y_{\pm} \) are small with respect to \( \nu(x_{\pm}) \) (as \( x_{-} \to a \) and \( x_{+} \to b \) respectively). Passing to the limit (as \( x_{-} \to a \) and \( x_{+} \to b \)), we obtain \( j_a^{b} \Delta(z) \text{d}z = 0 \) which implies (2.26). \( \square \)

2.2.2. Relative time of return. Suppose that \( v \in \mathcal{V}_{n+1} \), \( \{ p_j = (0, u_j) \} \) is the collection of points of tangency of the foliation \( \mathcal{F} = F_v \) with the pasted sphere, and \( \{ I_j \} \) is the collection of standard involutions. Let \( \mathcal{V}(v) \) be the class of germs \( w \in \mathcal{V}_{n+1} \) which are colinear to \( v \) and have the same jet of order \( 2n+1 \) at zero:

\begin{equation}
\int_{x_0}^{y_0} \gamma_{ij} \text{d}z = j_0^{2n+1} w.
\end{equation}

Let \( p = (0, u_0) \) be a point (in the \((x, u)\)-charts) near to the point of tangency \( p_j \), and let \( \gamma_p \) be the leaf of foliation \( \mathcal{F} \) passing through \( p \). As above, we denote the restrictions of the liftings \( \hat{v} \) and \( \hat{w} \) (of the vector fields \( v \) and \( w \)) to the leaf \( \gamma_p \) by \( v^p \) and \( w^p \). Using the \( u \)-coordinate as a parameter on \( \gamma_0 \) and applying (2.30), we see that the vector fields \( v^p = v(u, u_0) \) and \( w^p = w(u, u_0) \) can be written as

\begin{align*}
v(u, u_0) &= x^{u+1}(1, u) + o(1) \quad x = x(u, u_0) \to 0, \quad (2.31) \\
w(u, u_0) &= v(u, u_0) + O(x^{2n+2}) \quad x = x(u, u_0) \to 0, \quad (2.32)
\end{align*}

where \( x = x(u, u_0) \) is the equation of the curve \( \gamma_p \) at the point \((x, u)\).

Recall that if \( p^* = (0, u_0^*) \), where \( u_0^* = I_j(u_0) \), then \( \gamma_{p^*} = \gamma_{p^*} \), \( v(u, u_0) \equiv v(u, u_0^*) \), and \( w(u, u_0) \equiv w(u, u_0^*) \). Expressions (2.31) and (2.32) and the condition \( R_{n+2}(1, u_j) \neq 0 \) imply that the function \( \Delta_{u_0}^{v}(u, u_0) = \frac{1}{w(u, u_0)} - \frac{1}{v(u, u_0)} \) is holomorphic in the neighborhood of the point \((u_j, u_j)\). Let

\begin{equation}
\tau_{j}^{w}(u_0) = \int_{u_0}^{u_0^*} \Delta_{u_0}^{v}(z, u_0) \text{d}z, \quad \text{where } u_0^* = I_j(u_0),
\end{equation}

be defined in a neighborhood of \( u_j \) for each \( j \). The functions \( \tau_{j}^{w} \) are called the relative time of return (from the pasted sphere to the pasted sphere).

Remark 2.6. It immediately follows from the definition of relative time of return that

\begin{equation}
\tau_{j}^{w} \circ I_j = -\tau_{j}^{w}, \quad \tau_{j}^{w}(u_j) = 0.
\end{equation}

Lemma 2.3. Let \( \{ \tau_{j}^{w} \} \) be a relative time of return. Then the following assertions are valid.

1. The functions \( \tau_{j}^{w} \) are holomorphic in neighborhoods of the points of tangency \( u_j \).
2. The relative time of return for strictly analytic equivalent germs \( w, \tilde{w} \in \mathcal{V}(\mathbf{v}) \) coincide: \( \{\tau^w_j\} = \{\tau^\tilde{w}_j\} \).

3. For any \( M \), the \( M \)-jet \( j^M_0\tau^w_j \) at the point \( u_j \) of the function \( \tau^w_j \) is uniquely determined by the \((M + 2n + 1)\)-jet \( j^M_0 + 2n + 1 w \) of the vector field \( w \in \mathcal{V}(\mathbf{v}) \).

Proof. The first assertion follows from the analytic dependence of the function \( \Delta^w_0(u, u_0) \) on its variables. The second assertion follows from Lemma 2.2. Indeed, formal computations show (see Section 3) that the \((n + 1)\)-jet at zero of the holomorphism \( H \) conjugating the germs \( w, \tilde{w} \in \mathcal{V}(\mathbf{v}) \) is equal to the identity:

\[
j^2n+1_0 w = j^2n+1_0 \tilde{w}, \quad j^1_0 H = \text{Id} \implies j^{n+1}_0 H = \text{Id}.
\]

Hence the lifting of \( H, \tilde{H} : (x, u) \mapsto \tilde{H}(x, u) \), satisfies the uniform estimate

\[
\tilde{H}(x, u) = (x, u) + O(x^{n+2}), \quad x \to 0,
\]
in neighborhoods of the points \((0, u_j)\). Therefore, the restriction \( h_p : u \mapsto h_p(u, u_0) \) of \( \tilde{H} \) to the leaf \( \gamma_p \) (where \( u \) is the parameter of \( \gamma_p \) and \( p = (0, u_0) \)) satisfies also the uniform estimate

\[
h_p(u, u_0) = u + O(x^{n+2}), \quad x = x(u, u_0) \to 0.
\]

Thus, the \((n+1)\)-jet at \( u_0, u^*_0 \) of \( h_p \) is equal to the identity, and the assertion Lemma 2.2 about the conjugating holomorphism holds for \( h_p \). Therefore, by Lemma 2.2, the invariants \( \tau^w_j = \tau^\tilde{w}_j \) coincide.

The third assertion in Lemma 2.3 is almost immediate. Indeed, the coincidence of the \((M + 2n + 1)\)-jets at zero of the vector fields \( w \) and \( \tilde{w} \) implies the coincidence of the \((M + 2n + 1)\)-jets at the point \( p = (0, u_j) \) of their liftings \( \hat{w} \) and \( \hat{\tilde{w}} \). This implies the coincidence of the \((M - 1)\)-jets at \((u_j, u_j)\) of the functions \( \Delta^w_0 \) and \( \Delta^\tilde{w}_0 \). Hence the \( M \)-jets at \( u_j \) of the functions \( \tau^w_j \) and \( \tau^\tilde{w}_j \) coincide. This completes the proof of Lemma 2.3.

\[\square\]

Corollary 2.1. For formally equivalent germs \( w, \tilde{w} \in \mathcal{V}(\mathbf{v}) \), the corresponding functions of relative time of return coincide.

Proof. For any \( M \), we construct a germ \( w_M \) in the class \( \mathcal{V}(\mathbf{v}) \) such that it is analytically equivalent to the germ \( \tilde{w} \) and \( j^M_0 + 2n + 1 w_M = j^M_0 + 2n + 1 \tilde{w} \) (this can be done as in Section 2.1).

By the second assertion of Lemma 2.3, the germs \( w_M \) and \( \tilde{w} \) have the same functions of relative time of return. Moreover, the \( M \)-jets at the corresponding points \( u_j \) of the functions of relative time of return for the germs \( w \) and \( w_M \) also coincide. As \( M \) is arbitrary, this implies the coincidence of the functions of relative time of return for the germs \( w \) and \( \tilde{w} \). \[\square\]

2.3. End of the proof of Theorem 2. In the notation of Section 2.2,2. the germ \( v_2 \) belongs to the class \( \mathcal{V}(\mathbf{v}) \), where \( v = v_1 \). Clearly, the functions of relative time of return \( \tau^v_j \) for the germ \( v \) satisfy \( \tau^v_j \equiv 0 \). Since the germs \( v_1 \) and \( v_2 \) are formally equivalent, Corollary 2.1 implies that the same holds for the germ \( v_2 \): \( \tau^{v_2}_j \equiv 0 \). Thus, condition (2.24) holds. This completes the proof of Theorem 2.
3. Formal Classification of Dicritic Germs

In this section, formal and orbitally formal classifications of germs in $V_{n+1}^d$ are constructed under certain genericity assumptions.

3.1. Formal classification (Proof of Theorem 3).

Proof of Theorem 3. Suppose that $v(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ is a germ in $V_R$, $P(x, y) = \sum_{k=n+1}^{\infty} P_k(x, y)$ and $Q(x, y) = \sum_{k=n+1}^{\infty} Q_k(x, y)$, where $P_k$ and $Q_k$ are homogeneous polynomials of degree $k \geq n + 1$ such that $P_{n+1} = xR$ and $Q_{n+1} = yR$. Let $H$ be a local change of coordinates of the form

$$H(x, y) = (x + \alpha(x, y), y + \beta(x, y)),$$

where $\alpha(x, y)$ and $\beta(x, y)$ are homogeneous polynomials of degree $N \geq 2$.

The change of coordinates $H$ takes the germ $v$ to $\tilde{v} = (H, w) \circ H^{-1}$. Let us write $\tilde{v}$ as $\tilde{v} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$, where $P(x, y) = \sum P_k(x, y)$, $Q(x, y) = \sum Q_k(x, y)$, and $P_k$ and $Q_k$ are homogeneous polynomials of degree $k$. Then $P_k = \hat{P}_k$ and $Q_k = \hat{Q}_k$ for $k < n + N$ and

$$\hat{P}_{n+N} - P_{n+N} = (N - 1)\Re \alpha - x(\Re'_t \alpha + \Re'_y \beta) := \mathcal{P}_{n+N},$$

$$\hat{Q}_{n+N} - Q_{n+N} = (N - 1)\Re \beta - y(\Re'_t \alpha + \Re'_y \beta) := \mathcal{Q}_{n+N}.$$ (3.1) (3.2)

In (3.1) and (3.2), $P_{n+N}$ and $Q_{n+N}$ are known, $\alpha$ and $\beta$ are unknown, and $\hat{P}_{n+N}$ and $\hat{Q}_{n+N}$ are parameters which allow us to solve the system. Multiplying (3.1) by $y$ and (3.2) by $x$ and taking the difference of the resulting expressions, we obtain

$$y(\hat{P}_{n+N} - P_{n+N}) - x(\hat{Q}_{n+N} - Q_{n+N}) = (N - 1)\Re (ya - xb).$$ (3.3)

Therefore, instead of system (3.1), (3.2), we can consider the equivalent system (3.1), (3.3).

For any homogeneous polynomial $Z = Z(x, y)$, $z = z(u)$ denotes the polynomial $Z(1, u)$; in particular, $\mathcal{R}(1, u) := r(u)$ and $\mathcal{R}'(1, u) = r'(u)$; moreover, $x\mathcal{R}' + y\mathcal{R}' = n\mathcal{R}$ implies $\mathcal{R}'(1, u) = nr(u) - ur'(u)$. Using this notation and omitting the subscripts in (3.1) and (3.3), we obtain

$$\hat{p}(u) - p(u) = (N - n - 1)r(u)\hat{\alpha}(u) + r'(u)(u\hat{\alpha}(u) - \hat{\beta}(u)),$$ (3.4)

$$(up \hat{p}(u) - q(u)) - (up\hat{p}(u) - q(u)) = (N - 1)r(u)(u\hat{\alpha}(u) - \hat{\beta}(u));$$ (3.5)

here $p$ and $q$ are the known terms of degree at most $N + n; \deg r = n, r(u) = u^n, \ldots, r(0) \neq 0$ (equivalently, $u_j \neq 0$ for $j = 1, \ldots, n$); $\hat{\alpha}$ and $\hat{\beta}$ are the unknown terms of degree at most $N$; and $\hat{p}$ and $\hat{q}$ are polynomials whose degrees satisfy certain conditions (which ensure the existence and uniqueness of $\hat{\alpha}$, $\hat{\beta}$).

Lemma 3.1. For $N \neq n + 1$ and polynomials $p$ and $q$ of degrees at most $n + N$, there exist unique polynomials $\hat{p}$ and $\hat{q}$ ($\hat{q}(0) = 0$) of degree at most $n - 1$ and $n$, respectively, and polynomials $\hat{\alpha}$ and $\hat{\beta}$ of degree at most $N$ satisfying system (3.4), (3.5).
Proof. Dividing the polynomials $p$ and $q$ by $r$, we obtain

$$p = rd_1 + r_1, \quad \text{where } \deg r_1 \leq n - 1,$$

$$q = rd_2 + r_2, \quad \text{where } \deg r_2 \leq n - 1;$$

(3.6) (3.7)

here $d_1$ and $d_2$ are polynomials of degrees at most $N$. Both representations are unique. Let $c = r_2(0)$ and $C = r(0)$; then the polynomial $\tilde{r}_2 = r_2 - \hat{q}r$ has degree no larger than $n$, and $\tilde{r}_2(0) = 0$. Thus, $\tilde{r}_2(u) = ur_3(u)$ for some polynomial $r_3$ of degree at most $n - 1$. Setting $d_3 = d_2 + \hat{q}$, we get

$$q = rd_3 + ur_3, \quad \text{where } \deg r_3 \leq n - 1;$$

(3.8)

this representation is also unique. Further, suppose that $\tilde{p}$ and $\hat{q}$ satisfy (3.4) and (3.5), $\deg \tilde{p} \leq n - 1$ and $\hat{q}(u) = u\hat{q}_1(u)$, where $\deg \hat{q}_1 \leq n - 1$. The substitution of (3.6) and (3.8) in (3.5) gives

$$u(\tilde{p} - \hat{q}_1 = (r_1 - r_3)) = r[(N - 1)(u\hat{\alpha} - \hat{\beta}) + (ud_1 - d_3)].$$

(3.9)

Since $r(0) \neq 0$, $\deg r = n$, and the degrees of the polynomials $\tilde{p}$, $\hat{q}$, $r_1$, and $r_2$ are at most $n - 1$, it follows that (3.9) is equivalent to the system

$$\tilde{p} - \hat{q}_1 = r_1 - r_3,$$

(3.10)

$$(N - 1)(u\hat{\alpha} - \hat{\beta}) = d_3 - ud_4.$$  

(3.11)

Equation (3.11) gives an expression for $u\hat{\alpha} - \hat{\beta}$; the substitution of this expression in (3.4) yields

$$\tilde{p} - (N - n - 1)r\hat{\alpha} = p - \frac{1}{N - 1}r'(ud_1 - d_3).$$

(3.12)

Dividing $r'(ud_1 - d_3)/(N - 1)$ by $r$, we obtain

$$\frac{1}{N - 1}r'(ud_1 - d_3) = d_4r + r_4, \quad \deg r_4 \leq n - 1.$$

(3.13)

Substituting (3.6) and (3.13) in (3.12), we obtain

$$\tilde{p} - r_1 + r_4 = r[(N - n - 1)\hat{\alpha} + d_1 - d_4].$$

(3.14)

Since the degrees of the polynomials on the left-hand side of (3.14) is less than $n$, it follows that (3.14) is equivalent to the system

$$\tilde{p} = r_4 - r_1,$$

(3.15)

$$(N - n - 1)\hat{\alpha} = -d_1 + d_4.$$  

(3.16)

It remains to observe that (3.16) gives an expression for $\hat{\alpha}$ and (3.15) gives $\tilde{p}$; hence, for $\tilde{p}$ and $\hat{\alpha}$ fixed, we can determine $\hat{q}_1$ (and then $\hat{q}$) and $\hat{\beta}$ from (3.10) and (3.11), respectively. Note that $\deg \hat{\alpha}, \hat{\beta} \leq N$ and $\deg \hat{p}, \hat{q}_1 \leq n - 1$. This proves the existence and uniqueness of the solution. \hfill \Box

Remark 3.1. For $N = n + 1$, this procedure is not valid. Indeed, in this case, the coefficient of $\hat{\alpha}$ in (3.16) is zero. So, to obtain a solution $\tilde{p}$, $\hat{q} = u\hat{q}_1$, $\hat{\alpha}$, $\hat{\beta}$, we impose a condition on the degree of $\hat{q}_1$; namely, we require that $\deg \hat{q}_1 \leq N + n - 1 = 2n$ (the conditions on the rest of the unknown functions remain unchanged). So, for $N = n + 1$, we suppose that $p(u) = r'(u)d_5(u) + r_5(u)$, where $\deg r_5 \leq n - 2$ and
\[ \text{deg } d_5 \leq n + 2 \text{ and } \tilde{p}(u) = cr'(u) + \tilde{p}_1(u), \text{ where } \text{deg } \tilde{p}_1 \leq n - 2. \text{ From (3.4) we obtain } \tilde{p}_1 - r_5 = r'(u)(u\alpha - \beta) - c + d_5(u); \text{ hence } \]
\[ \tilde{p}_1 = r_5, \quad u\alpha - \beta = c - d_5(u). \]

The substitution of these equalities in (3.5) yields
\[ \tilde{q} - q = (c - d_5)[ur' - nr]. \quad (3.17) \]

Since \( \tilde{q}(u) = u\tilde{q}_1(u) \), we have \( \tilde{q}(0) = 0 \) and \( c = d_5(0) + \frac{q(0)}{nr(0)}. \) Therefore,
\[ \tilde{q}_1(u) = [q(u) + (c - d_5(u))[ur'(u) - nr(u)]]/u. \]

The degrees of \( d_5 \) and \( ur'(u) - nr(u) \) are bounded from above by \( n + 2 \) and \( n - 1 \), respectively, this and the last equation imply that the degree of \( \tilde{q} \) is bounded above by \( 2n + 1 \), and \( \tilde{q} \) is completely and uniquely determined.

Thus, we have proved the existence and uniqueness of the polynomials \( \tilde{p} \) and \( \tilde{q} \) such that \( \text{deg } \tilde{p} \leq n - 1, \tilde{q}(0) = 0, \) and \( \text{deg } \tilde{q} \leq 2n + 1 \) for which system (3.4), (3.5) has a solution \( (\tilde{\alpha}, \tilde{\beta}) \) of degree at most \( n + 1 \). Note that, in this case, the solution is not unique. Namely, \( \tilde{\alpha} \) and \( \tilde{\beta} \) in the equality \( u\alpha - \beta = c - d_5(u) \) are determined up to a term \( (\tilde{\gamma}, u\tilde{\gamma}) \), where \( \text{deg } \tilde{\gamma} \leq n. \)

**End of the proof of Theorem 3.** As usual, we seek a normalizing formal change of variables \( H \) (for the germ \( v \)) as the limit of the composition \( H = \lim_{N \to \infty} H_N, \) where \( H_N = H_N \circ H_{N-1}, H_1 = \text{Id}, H_N = \text{Id} + (\alpha_N, \beta_N), \) and \( \alpha_N \) and \( \beta_N \) are homogeneous polynomials of degree \( N \geq 2. \)

The change of variables \( H_N \) must normalize (see (0.2)) the \( (N + n) \)-jet of the vector field \( v_{N-1} \), where \( v_{N-1} \) is the field obtained from \( v \) by the change of variables \( H_{N-1}. \) By Lemma 3.1, this can always be done. Namely, for \( N \neq n + 1, \) the homogeneous polynomials \( \tilde{P}_{n+N} \) and \( \tilde{Q}_{n+N} \) from (3.1), (3.2) and the corresponding system (3.4), (3.5) with solution \( \tilde{p}, \tilde{q} = u\tilde{q}_1 \) have degrees (with respect to \( y \)) at most \( n - 1 \) and \( n, \) respectively. Moreover, the polynomial \( \tilde{Q}_{n+N} \) is divisible by \( y. \) Hence
\[ \tilde{P}_{n+N}(x, y) = x^{n+1}R(x, y) \sum_{j=1}^{N} \frac{a_j^N}{y - u_jx}, \quad \text{where } a_j^N = \text{Res}_{u_j} \frac{\tilde{p}(u)}{r(u)}. \]
\[ \tilde{Q}_{n+N}(x, y) = yx^N R(x, y) \sum_{j=1}^{N} \frac{b_j^N}{y - u_jx}, \quad \text{where } b_j^N = \text{Res}_{u_j} \frac{\tilde{q}_1(u)}{r(u)}. \]

For \( N = n + 1, \) the same expression and the polynomial \( \tilde{q} \) is such that \( \tilde{q}(u) = u\tilde{q}_1(u), \) where \( \tilde{q}_1(u) = r(u)\tilde{\delta}(u) + r_0(u), \) deg \( r_0 \leq n - 1. \) Taking this into account, we set \( \delta_n(x, y) = x^n\tilde{\delta}(\frac{x}{y}) \) and \( b_j^{n+1} = \text{Res}_{u_j} \frac{r_0(u)}{r(u)}. \)

As the \( H_N \)-change of variables stabilizes, the infinite composition of such changes \( (N \to \infty) \) converges in the space of power formal series. This gives, in the limit, the normalizing change \( H, \) which proves the first assertion of Theorem 3.

To prove the second assertion we recall that \( H_N \) is uniquely determined for any \( N \neq n + 1. \) For \( N = n + 1, \) the parameters \( (\alpha, \beta) \) were defined up to a term \( (x\gamma, y\gamma), \) where \( \gamma \) is a homogeneous polynomial of degree \( n. \) Let us make the additional change \( H: (x, y) \mapsto (x + x\gamma, y + y\gamma), \) where deg \( \gamma = n, \) at the \( (n + 1) \)th step of
normalization. This change does not affect the \((2n+1)\)-jet of the normalization of the germ \(v\) but increases its \((2n+2)\)-term by the vector \(\langle P_{2n+2}, Q_{2n+2} \rangle\), where

\[
P_{2n+2} = -\gamma(n+1)P_{n+2} + x(\gamma_x P_{n+2} + \gamma_y Q_{n+2}),
\]
\[
Q_{2n+2} = -\gamma(n+1)Q_{n+2} + y(\gamma_x P_{n+2} + \gamma_y Q_{n+2}),
\]

and hence

\[
xQ_{2n+2} - yP_{2n+2} = -\gamma(n+1)(xQ_{n+2} - yP_{n+2}). \tag{3.18}
\]

By the genericity assumptions, the polynomial \(R_{n+2}(x, y) = xQ_{n+2} - yP_{n+2}\) does not vanish at the points \((x, y) = (1, u_j)\) for \(j = 1, \ldots, n\). So, by choosing appropriate values of the polynomial \(\hat{\gamma}(u) = \gamma(1, u)\) at the points \(u_j\), we can ensure that the right-hand side of equality (3.18) satisfy \(\hat{\gamma}(u_j)(n+1)R_{n+2}(1, u_j) = R_{2n+2}(1, u_j)\), \(j = 1, \ldots, n\), at these points. Hence, at the next step of the normalization (i.e., while normalizing the \((2n+2)\)-jet), we can assume that the difference of the polynomials \(xP_{2n+2}\) and \(xQ_{2n+2}\) in (3.3) is divisible by \(R\). Therefore, these polynomials coincide, which implies (0.3) and the possibility of reducing \(v\) to its normal form (0.2), (0.3).

Recall that the polynomial \(\hat{\gamma}(u)\) is uniquely determined by its values at the points \(u_j\), \(j = 1, \ldots, n\), up to addition of a term \(cr(u)\). This gives one free parameter in the construction of the normalizing changes. In fact, the presence of this parameter is natural: the motion \(g_u^v\) along any vector field \(v\) for a fixed time \(c\) leaves this field invariant.

This remark completes the proof of the uniqueness of the normal form (0.2), (0.3) and of Theorem 3. \(\square\)

**Remark 3.2.** We have also proved that the normalizing formal change (with linear part equal to the identity) is determined up to composition with the motion along the normalized vector field for a fixed time.

### 3.2. Formal orbital classification (proof of Theorem 4)

In what follows, we use the notation introduced in Section 3.1.

**Proof.** We shall use the normalization process given in the proof of Theorem 3. After the normalization of the \((n+N)\)-jet and before the normalization of the next jet, we make, for each \(N \geq 2\), the additional normalizing change of coordinates \(H : (x, y) \mapsto (x + x\gamma, y + y\gamma)\) and multiply the result by \(k(x, y) = 1 - \gamma(N - n - 1)\), where \(\gamma\) is a homogeneous polynomial of degree \(N - 1\). As at the end of the proof of Theorem 3, these changes do not affect the \((N + n)\)-jet of the field but increase its \((N + n + 1)\)-jet by the vector \(\langle P_{N+n+1}, Q_{N+n+1} \rangle\), where

\[
P_{N+n+1} := -N\gamma P_{n+2} + x(P_{n+2} \gamma_x + Q_{n+2} \gamma_y),
\]
\[
Q_{N+n+1} := -N\gamma Q_{n+2} + y(P_{n+2} \gamma_x + Q_{n+2} \gamma_y).
\]

Hence \(yP_{N+n+1} - xQ_{N+n+1} = -N\gamma(yP_{n+2} - xQ_{n+2})\). Appropriately choosing the values of the polynomial \(\hat{\gamma}(u)\) at the points \(u_j\), we can arbitrarily change the values (at these points) of the difference

\[
y(P_{N+n+1} + Q_{N+n+1} - x(Q_{N+n+1} + Q_{N+n+1}))_{x=1, y=u_j}, \tag{3.19}
\]
In particular, if \( \deg \tilde{\gamma} = N - 1 \geq n - 1 \) (i.e., \( N \geq n \)), then expression (3.19) can be made vanishing at all points \( u_j \); in the case of \( N < n \), the vanishing of (3.19) at the points \( u_j \) can be achieved for \( j = 1, \ldots, N \). At the next step of normalization (when the \( (N + n + 1) \)-jet is normalized), the corresponding equation (3.3) implies that the expression

\[
(y P_{N+n+1} - x Q_{N+n+1})_{(1,u)}
\]

vanishes at the points \( u = u_j \) (for all \( j \) if \( N \geq n \) and for \( j \leq N \) if \( N \leq n \)). Thus, the conditions in (0.5) are satisfied.

Finally, before the normalization of the \( (2n + 1) \)-jet of \( v \), we multiply the vector field by \( k = 1 + \eta \), where \( \eta \) is a homogeneous polynomial of degree \( n \). This allows us to use the construction of Lemma 3.1 instead of Remark 3.1 and, as a consequence, kill the terms \( \delta_n \) in (0.2). Indeed, for \( N = n + 1 \), multiplication by \( 1 + \eta \) changes the term of order \( 2n + 1 \) by \( (\eta x R, \eta y R) \). Then, making the change of coordinates \( (x + \alpha, y + \beta) \), where \( \deg \alpha, \beta = N \), we reduce equations (3.1) and (3.2) to the forms

\[
\begin{align*}
\tilde{P}_{2n+1} - P_{2n+1} - \eta x R &= n R \alpha - x(R_x' \alpha + R_y' \beta), \\
\tilde{Q}_{n+N} - Q_{n+N} - \eta y R &= (N - 1) R \beta - y(R_x' \alpha + R_y' \beta).
\end{align*}
\]

(3.20)

(3.21)

Proceeding as in Section 3.1, we obtain equations (3.4) with \( \tilde{\eta}(u) r(u) \) instead of \((N - n - 1) r(u) \tilde{\alpha}(u)\) and (3.5). Accordingly, we obtain (3.15) and \( \tilde{\eta} = -d_4 + d_4 \) instead of (3.16).

Since \( \deg p, q \leq 2n + 1 \) and \( p(u) = c_1 u^{2n+1} + \ldots \), we have \( d_4 = c_1 u^{2n+1} + \ldots \) and \( d_4 = c_1 u^{n+1} + \ldots \). Hence the polynomials \( \tilde{\eta} \) and \( \tilde{\eta} \) are uniquely determined, \( \deg \tilde{\eta} \leq n - 1 \), and \( \deg \tilde{\eta} \leq n \). This proves the reduction to the normal form (0.2), (0.5).

\[\Box\]

**Remark 3.3** (to the uniqueness of the orbital formal normal form). The orbital normal form constructed above is uniquely determined for \( N \leq n \); for \( N = n + 1 \), we can choose the value of one parameter without changing the formal normal form (this corresponds to the motion \( g_{\tilde{\gamma}}^c \), where \( c = \text{const} \)). For \( N \geq n + 1 \), in the definition of the polynomial \( \tilde{\gamma}(u) \) of degree \( N - 1 \) (by choosing its values at the points \( u_j \)), \( N - n \) free parameters appear. But the same number of free parameters arises in the consideration of the following jet in the group \( G_v \), which is generated by the motions \( g^t_{\tilde{\gamma}} \), where \( t = t(x, y) \) is a formal power series. Since all the elements of the group \( G_v \) leave the class of orbital equivalence of germ \( v \) invariant (the shift \( g^t_{\tilde{\gamma}} \) transforms \( v \) into a germ \( (1 + t, v) v \), which is formally orbitally equivalent to \( v \)), the freedom in the choice the coefficients of the normalizing change does not affect the result. This proves the uniqueness of the orbital formal normal form; at the same time, we have proved that the normalizing change is unique up to right (left) multiplication by an element of \( G_v \) (by an element of the group \( G_{v_0} \), where \( v_0 \) is the corresponding formal normal form).
4. Analytic Orbital Classification of Dicritic Foliations

In this section, we construct analytic orbital classification of generic germs of class $\mathcal{VR}$. This classification is obtained as an application of the results of Sections 1–3.

4.1. The $c$-invariants. Equivalence and equimodality (Proof of Theorem 5).

Proof of Theorem 5. It suffices to prove only the formal version of the theorem: the analytic version will follow from the orbital rigidity theorem (see Theorem 1 in Section 0.2). The ‘only if’ part is evident: the collection of standard involutions (see Lemma 1.1) and the $c$-invariants are invariants of strict formal orbital equivalence.

To prove the ‘if’ part, let $v$ be a germ in $\mathcal{V}_{n+1} \cap \mathcal{VR}$, and let $I_j$ be an involution of the set $I_v$. Suppose that $u_j$ is the fixed point for $I_j$ and $I_j(z) = -z + \sum_{k=2}^{\infty} q_{jk}^j z^k$, $z = u - u_j$.

To determine the coefficients $q_{jk}^j$, let $v_{a,c}$ be the formal orbital normal form of the germ $v$ such that $c = c_v$. The coefficients of the involution are invariants of the strict formal orbital equivalence (see Lemma 1.1), hence the coefficients $q_{jk}^j$ are uniquely determined by the parameters $a$ and $c$ of the formal normal form $v_{a,c}$. For odd $k$, the coefficients $q_{jk}^j$ are defined by the coefficients $q_{k'}^j$ with $k' < k$. For even $k$ ($k = 2s$), a direct calculation gives

$$q_{2s}^j = a_{js}^s + c_{sj} + \ldots, \quad c_{sj} = (-1)^{s-1} \frac{s!}{2(2s + 1)!!} (u_j c_j^j)^s \neq 0; \quad (4.2)$$

the dots denote the terms depending on the coefficients $a_{m'}^j$, $m' < m$, of the collection $a$ and on the coefficients of the collection $c$, and $(2s+1)!! = (2s+1) \cdot \ldots \cdot 5 \cdot 3 \cdot 1$. Therefore, for given $c$, the coefficients of the collection $a$ are uniquely determined by the coefficients of the involutions. So, any two germs with the same $c$-invariants and the same collection of involutions have formal normal forms with coinciding $a$-components. This implies the formal equivalence of such germs. The proof of Theorem 5 follows now from Theorem 1. $\square$

4.2. Proof of Realization Theorem (Theorem 6). In this section, we prove the independence of the constructed invariants $I_v$ and $c_v$ for the germs $v \in \mathcal{V}_{n+1} \cap \mathcal{VR}$. We recall that the standard collection of involutions $\{I_j\}$ consists of involutions with fixed points $u_j$ and collection $c \in \mathbb{C}^k$ ($c = c_j^j$) satisfies the inequalities $c_j^j \neq 0$.

Proof of Theorem 6. Let $N$ be a large enough natural number. We construct the collection $a \{a_j^s\}$ by setting $a_j^s = 0$ for $s > N$ and defining (for given $c \in \mathbb{C}^k$) the remaining coefficients $a_j^s$ in equation (4.2) by the coefficients of the $2N$-jet of the involutions $I_j$, as in Section 4.1.

Let $w = v_{a,c}$ be the (polynomial) formal normal form (0.6) corresponding to the collections $a$ and $c$. In accordance with Sections 1.4, 1.5, and 1.6, for the germ
\( \mathbf{u} \), we construct the standard involutions \( \tilde{I}_j \) (and their corresponding functions \( \tilde{g}_j \)), the curves of tangency (and their standard parametrizations \( z_j = z_j(x) \)), and the standard foliations \( \tilde{F}_j = \{ z_j(x) + \tilde{g}_j(u) = \text{const} \} \). Further, let \( \tilde{H}_0: (x, u) \mapsto (x, \tilde{J}(x, u)) \) be the rectifying transformation for the germ \( \mathbf{u} \) defined outside the points of tangency (see Section 1.7), and let \( \tilde{H}_j: (x, u) \mapsto (x, \tilde{\psi}_j(x, u)) \) be the normalizing transformation from Lemma 1.4 (Section 1.6). Consider the mapping \( \Phi_j: (x, u) \mapsto (x, \varphi_j(x, u)) \), where the function \( \varphi_j \) is defined by the equality

\[
\varphi_j(x, u) = \tilde{F}_j(x, u) = \tilde{g}_j(u). \tag{4.3}
\]

The function \( \tilde{g}_j \) can be written as \( \tilde{g}_j(u) = (\tilde{f}_j(u))^2 \) for some holomorphic (in a neighborhood of \( u_j \)) function \( \tilde{f}_j \), such that \( \tilde{f}_j(u_j) = 0 \) and \( \tilde{f}'_j(u_j) \neq 0 \); so, \( \varphi_j \) has the expression

\[
\varphi_j(x, u) = \tilde{f}_j^{-1}(\tilde{g}_j(u)) = \left( \frac{1 - z_j(x)}{(\tilde{f}_j(u))^2} \right). \tag{4.4}
\]

Since \( z_j(0) = 0 \) and \( \tilde{f}_j(u_j) \neq 0 \), the function \( \varphi_j \) is analytic on the annular domain

\[
V_j = \{ |x| < \epsilon, \epsilon_1 < |u - u_j| < \epsilon_2 \} \tag{4.5}
\]

for small enough positive values of \( \epsilon, \epsilon_1 \), and \( \epsilon_2 \) such that \( \epsilon_1 < \epsilon_2 \).

Equality (4.3) shows that the mapping \( \tilde{\Phi}_j \) transforms each vertical line \( \{ u = c_0 \} \) into the leaf \( \{ z_j(x) + \tilde{g}_j(u) = c_0 \} \) of the foliation \( \tilde{F}_j \), and it is the identity at \( \{ x = 0 \} \).

From the above construction follows that the mappings \( \tilde{H}_0, \tilde{H}_j, \) and \( \tilde{\Phi}_j \) are related by the equality

\[
\tilde{\Phi}_j = \tilde{H}_j \circ \tilde{H}_0^{-1}. \tag{4.6}
\]

For appropriate domains \( U_0, U_j, \) and \( V_j \) corresponding to the mappings \( \tilde{H}_0, \tilde{H}_j, \) and \( \tilde{\Phi}_j \), a neighborhood of the pasted sphere is obtained from the domains \( U_0 \) and \( U_j \) glued together by the maps \( \tilde{\Phi}_j \) (and the pasted sphere is obtained from its intersection with \( U_0 \) and \( U_j \)), and the foliation \( \mathcal{F}_u \) is obtained by gluing together the standard foliation \( \{ u = \text{const} \} \) in \( U_0 \) with the standard foliations \( \tilde{F}_j \) of the domains \( U_j \).

Finally, let \( \tilde{\mathbf{w}} \) be the lifting of the vector field \( \mathbf{w} \) to a neighborhood of the pasted sphere (see Section 1.1); suppose that the maps \( \tilde{H}_0 \) and \( \tilde{H}_j \) transform the fields \( \tilde{\mathbf{w}} \) into the fields \( \tilde{\mathbf{w}}_0 \) and \( \tilde{\mathbf{w}}_j \), respectively. Then the vector field \( \tilde{\mathbf{w}} \) is obtained from the mentioned gluing of the vector fields \( \tilde{\mathbf{w}}_0 \) and \( \tilde{\mathbf{w}}_j \). In what follows, we slightly modify this construction. Namely, we let \( I = \{ I_j \} \) and consider the function \( g_j \) for each involution \( I_j \) (see Section 1.4). By construction, \( j_{u_j}^{2N} I_j = j_{u_j}^{2N+1} \tilde{g}_j \); hence \( j_{u_j}^{2N+1} g_j = j_{u_j}^{2N+1} \tilde{g}_j \). Further, consider the standard foliation \( \tilde{F}_j \) by the level curves of the function \( z_j(x) + g_j(u) \); let us construct the corresponding “rectifying” map \( \Phi_j: (x, u) \mapsto (x, \varphi_j(x, u)) \), where the function \( \varphi_j \) is defined by the equality

\[
z_j(x) + g_j(u) = g_j(u). \tag{4.7}
\]

Let \( \tilde{M} \) be the complex manifold obtained from the domains \( U_0 \) and \( U_j \) glued together by the mappings \( \Phi_j \). All the gluings are the identity at the straight line \( \{ x = 0 \} \); hence, gluing together the parts of this line lying on \( U_0 \) and \( U_j \) we obtain a complex manifold \( L \) biholomorphically equivalent to \( \mathbb{C}P^1 \). As the maps \( \Phi_j \) are near to \( \tilde{\Phi}_j \),
the self-intersection index of \( L \) in \( \tilde{M} \) is equal to \(-1\). Therefore, by Grauert’s theorem \([G]\), the manifold \( \tilde{M} \) is biholomorphically equivalent to the blow-up of some neighborhood of the origin, and the sphere \( L \) is the pasted sphere of this blow-up. This implies the existence of holomorphic mappings \( H_0 \) and \( H_j \) for the factorization problem

\[
\Phi_j = H_j \circ (H_0)^{-1}. \tag{4.8}
\]

The maps \( H_j \) are defined and holomorphic in a neighborhood \( U_j \) of \((0, u_j)\), and \( H_0 \) is defined and holomorphic in a neighborhood of the sphere with holes \( L_0 := L \setminus \bigcup_{j=1}^n U_j \). All the holomorphisms \( H_0 \) and \( H_j \) are the identity on \( \{x = 0\} \), and equality (4.8) is satisfied in an annular domain of the form (4.6). The maps \( \Phi_j \) transform the standard foliation \( \mathcal{F}_0 = \{u = \text{const}\} \) into the standard foliation \( \mathcal{F}_j \); hence, on \( \tilde{M} \), some foliation \( \mathcal{F} \) is defined; this foliation is obtained from \( \mathcal{F}_0 \) and \( \mathcal{F}_j \) by gluing. We have to prove the following assertions:

A. The foliation \( \mathcal{F} \) is the blowing-up of the foliation \( \mathcal{F}_v \) by the phase curves of a germ \( v \in \mathbb{V}_{n+1} \cap \mathbb{R} \);

B. The germ \( v \) has the given invariants \( I \) and \( c \).

1. We may assume that the \( x \)-component of the mappings \( H_j \) and \( H_0 \) is the identity. Indeed, the gluings \( \Phi_j \) have this property, and on \( M \) a non-dicritic foliation \( \mathcal{F}_x \) is defined, which is obtained from the standard foliation \( \mathcal{F}_x^0 = \{x = \text{const}\} \) by gluing. The projection of this foliation is a non-singular foliation of \((\mathbb{C}^2, 0)\). This foliation can be rectified (transformed into the standard foliation \( \{x = \text{const}\} \)) by a local holomorphism of \((\mathbb{C}^2, 0)\). The blow-up \( \tilde{H} \) of this holomorphism transforms the foliation \( \mathcal{F}_x \) into \( \mathcal{F}_x^0 \). Replacing the mappings \( H_j \) and \( H_0 \) in (4.8) by the compositions \( H_j \circ \tilde{H} \) and \( H_0 \circ \tilde{H} \), we obtain the desired property.

2. Let \( H_j : (x, u) \mapsto (x, x + h_j(x, u)) \) and \( H_0^{-1} : (x, u) \mapsto (x, x - h_0(x, u)) \). Suppose that the coefficients of the Taylor expansions in \( x \) of the functions \( h_j \), \( h_0 \), \( \varphi_j \), and \( \varphi_0 \) are \( h_j^k(u) \), \( h_0^k(u) \), \( \varphi_j^k(u) \), and \( \varphi_0^k(u) \), respectively. Equating the coefficients of the same power of \( x \) in (4.8), we obtain the infinite system of equations

\[
h_j^k(u) - h_0^k(u) = P^k_j[h] + \varphi_j^k =: \Psi_j^k, \quad j = 1, \ldots, n, \tag{4.9}
\]

where \( P^k_j[h] \) is a polynomial in the variables \( h_j^k \) and \( h_0^k \) (and in their derivatives of order at most \( k - s \)) for \( s < k \) and the functions \( \varphi_j^k \) and \( \Psi_j^k \) are holomorphic in the annulus \( \{\epsilon_1 < |u - u_j| < \epsilon_2\} \); the unknown functions \( h_j \) and \( h_0 \) are holomorphic, respectively, inside the disk \( \{|u - u_j| < \epsilon_2\} \) and outside the disk \( |u - u_j| \leq \epsilon_1 \), and the functions \( h_0^k \) have poles of orders at most \( k \) at infinity (see \([V2]\)). In the case where exactly one of the functions \( \Psi_j^k \) (for \( j = j_0 \)) is different from zero, system (4.9) has the following elementary solution: \( h_{j_0}^k \) is the part of the Laurent expansion of \( \Psi_{j_0}^k \) with non-negative powers and the remaining functions \( h_j^k \) \((0 \leq j \leq n, j \neq j_0)\) form the principal part of this expansion. In the general case, the solution of system (4.9) can be represented as a sum of elementary solutions; the solution obtained in such a way is said to be normalized.

Finally, the general solution to (4.9) is obtained from the normalized solution by adding an arbitrary polynomial of degree at most \( k \) to all of its components (this polynomial is called the correction term). The solution to (4.8) has precisely the
same degree of arbitrariness: multiply the components $H_{j}$ and $H_{0}$ of any of the solutions on the right by the maps $\tilde{H}_{k}: (x, u) \mapsto (x, u + x^{k}P_{k})$, we obtain another solution to (4.8). Since equation (4.8) has a holomorphic solution, it follows that, for any solution $h_{j}^{k}$ of the equation (4.9) with $k \leq 2n$, there exists a holomorphic solution of equation (4.8) with exactly the same coefficients $h_{j}^{k}$ of the terms having degree $k \leq 2n$ in $x$.

3. Let us prove that, without loss of generality, we may assume that the solutions $(\tilde{H}_{j}, \tilde{H}_{0})$, $\tilde{H}_{0}$ of equation (4.6) and (4.8) satisfy the asymptotic relations

$$\tilde{H}_{0}(x, y) - H(x, y) = o(x^{2n}), \quad x \to 0. \quad (4.10)$$

To this aim, let us find explicit expressions for the coefficients $\varphi_{j}^{k}$. By taking the $k$th derivatives in (4.7) with respect to the $x$-variable and letting $x = 0$, we obtain

$$\varphi_{j}^{k} = \frac{A_{k}[g_{j}]}{(g'_{j})^{k+1}},$$

where $A_{k}[g_{j}]$ is a polynomial in $g_{j}$ and its derivatives of order at most $k$ and in the functions $\varphi_{j}^{s}$ for $s < k$. Since $g_{j}(u_{j}) = g'_{j}(u_{j}) = 0$, $g''_{j}(u_{j}) \neq 0$, and the outer radius $\epsilon_{2}$ of the annulus $K_{j}$ is small enough, we can show by induction that each function $\varphi_{j}^{k}$ admits an analytic continuation to a punctured neighborhood of the point $u_{j}$ and has a pole at $u_{j}$: the order of the pole is no larger than a number depending on $k$. We can prove by induction that, for all $k$, the principal part of the Laurent expansion of the function on the right-hand side of (4.9) (in $K_{j}$) has a finite number of terms; each of the coefficients in the Laurent series is defined by a finite number of derivatives of the functions $g_{j}$ (and by the corrections defined above). Therefore, if the $N$-jets of the functions $g_{j}$ and $\tilde{g}_{j}$ at the points $u_{j}$ coincide and $N$ is large enough, then, for any collection of solutions $\tilde{h}_{j}^{k}$, $\tilde{h}_{0}^{k}$ ($j = 1, \ldots, n, k \leq 2n$) to the system

$$\tilde{h}_{j}^{k} - h_{j}^{k} = P_{j}[\tilde{h}] + \varphi_{j}^{k} \quad (4.11)$$

which is similar to (4.9), it is possible to find solutions $h_{j}^{k}$, $h_{0}^{k}$ to system (4.9) such that $h_{0}^{k} \equiv \tilde{h}_{0}^{k}$ for $k \leq 2n$: it is sufficient to make the same corrections at each step in solving these systems. Hence, taking into account the results obtained at stage 2, we obtain the asymptotic equality (4.10).

4. **Construction of the vector field $v$ (end of the proof).** Let $w_{0}$ and $w_{j}$ be the vector fields constructed at the beginning of this section. Since the vector field $w_{j} = (w_{j}^{x}, w_{j}^{y})$ is tangent to the level line of the function $z_{j}(x) + g_{j}(u)$, the vector field $v_{j} = \left( w_{j}^{x}, \frac{g'_{j}(u)}{g_{j}(u)}w_{j}^{y} \right)$ is tangent to the level curves of the function $z_{j}(x) + g_{j}(u)$.

Since the map $\Phi_{j}$ transforms the vector field $w_{0}$ into the vector field $w_{j}$ and the $x$-components of the mappings $\Phi_{j}$ and $\Phi_{j}$ are the identity, the map $\tilde{\Phi}_{j}$ transforms the vector field $w_{0}$ into the field $v_{j}$. Therefore, the maps $H_{0}$ and $\tilde{H}_{j}$ transform the vector fields $w_{0}$ and $v_{j}$ to the same vector field $\tilde{v}$, which is tangent at each point to the leaves of the foliation $\mathcal{F}$. Let $v$ be the vector field obtained from $\tilde{v}$ by projecting $\tilde{M}$ to $(\mathbb{C}^{2}, 0)$. The vector fields $\tilde{w}$ and $\tilde{v}$ were obtained from the same vector field $w_{0}$ by using near (in the sense of (4.10)) mappings $\tilde{H}_{0}$ and $H_{0}$: hence the vector fields $w$ and $v$ are also near: their $2n$-jets at zero coincide. This implies
Finally, since the $c$-invariant is determined by the $(2n+1)$-jet of the field and $c = c_{v}$, it follows that $c = c_{v}$. By construction, $F = F_{v}$, whence $I = I_{v}$. This completes the construction of the germ $v \in \mathcal{V} \mathcal{R}$ with invariants $c$ and $I$ and the proof of Theorem 6.

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References


