ON RADICALLY GRADED FINITE-DIMENSIONAL QUASI-HOPF ALGEBRAS

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Abstract. In this paper we continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. First, we completely describe the class of radically graded finite dimensional quasi-Hopf algebras over \(\mathbb{C}\), whose radical has prime codimension. As a corollary we obtain that if \(p > 2\) is a prime then any finite tensor category over \(\mathbb{C}\) with exactly \(p\) simple objects which are all invertible must have Frobenius-Perron dimension \(p^N\), \(N = 1, 2, 3, 4, 5\) or \(7\). Second, we construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra. For instance, to every finite dimensional simple Lie algebra \(\mathfrak{g}\) and a positive integer \(n\), we attach a quasi-Hopf algebra of dimension \(n^{\dim \mathfrak{g}}\).


Key words and phrases. Quasi-Hopf algebras, finite tensor categories.

1. Introduction

In [EO] it is proved that any finite tensor category over \(\mathbb{C}\) with integer Frobenius-Perron dimensions of objects is equivalent to a representation category of a finite dimensional quasi-Hopf algebra (the Frobenius-Perron dimension of a representation coincides with its dimension as a vector space). Therefore the classification of finite tensor categories with integer Frobenius-Perron dimensions of objects is equivalent to the classification of complex finite dimensional quasi-Hopf algebras. The simplest finite tensor categories to try to understand are those which have only 1-dimensional simple objects which form a cyclic group of prime order under tensor product. Equivalently, one is led to the problem of classifying finite dimensional quasi-Hopf algebras with (Jacobson) radical of prime codimension.

Let \(p\) be a prime, and let \(RG(p)\) denote the class of radically graded finite dimensional quasi-Hopf algebras over \(\mathbb{C}\), whose radical has codimension \(p\). It was shown in [EG] that any \(H \in RG(2)\) is equivalent to a Nichols Hopf algebra \(H_{2^n}\).

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n ≥ 1 [N], or to a lifting of one of the four special quasi-Hopf algebras \( H(2), H_+(8), H_-(8), H(32) \) of dimensions 2, 8, 8, and 32 (the algebra \( H(2) \) is the group algebra of \( \mathbb{Z}_2 \) with a nontrivial associator).

Later, it was shown in [G] that if \( H \in RG(p), \ p > 2, \) has a nontrivial associator and if the rank of \( H^{[1]} \) over \( H^{[0]} \) is 1, then \( H \) is equivalent to one of the quasi-Hopf algebras \( A(q) \) of dimension \( p^3 \), introduced in [G]. More precisely, the result of [G] is formulated under the assumption that \( H \) is basic (i.e., \( H/\text{Rad}(H) = \mathbb{C}[\mathbb{Z}_p] \) with some associator), but by [ENO], Corollary 8.31, this is automatic.

The purpose of this paper is to continue the structure theory of finite dimensional quasi-Hopf algebras started in [EG] and [G]. More specifically, we completely describe the class \( RG(p) \), and construct new examples of finite dimensional quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.

The structure of the paper is as follows. In Section 2 we recall the definition of the quasi-Hopf algebras \( A(q) \) and \( H_\pm(p) \).

In Section 3 we show that if \( H \in RG(p) \) has a nontrivial associator, then the rank of \( H^{[1]} \) over \( H^{[0]} \) is \( ≤ 1 \). This yields the following classification of \( H \in RG(p), p > 2, \) up to twist equivalence.

(a) Duals of pointed Hopf algebras with \( p \) grouplike elements, classified in [AS], Theorem 1.3.

(b) Group algebra of \( \mathbb{Z}_p \) with associator defined by a 3-cocycle.

(c) The algebras \( A(q) \).

This result implies, in particular, that if \( p > 2 \) is a prime then any finite tensor category over \( \mathbb{C} \) with exactly \( p \) simple objects which are all invertible must have Frobenius–Perron dimension \( p^N, N = 1, 2, 3, 4, 5 \) or 7.

In Section 4 we construct new examples of finite dimensional quasi-Hopf algebras \( H \), which are not twist equivalent to a Hopf algebra. They are radically graded, and \( H/\text{Rad}(H) = \mathbb{C}[\mathbb{Z}_p^m], \) with a nontrivial associator. For instance, to every finite dimensional simple Lie algebra \( g \) and a positive integer \( n \), we attach a quasi-Hopf algebra of dimension \( n^{\dim g} \).

2. Preliminaries

All constructions in this paper are done over the field of complex numbers \( \mathbb{C} \).

We refer the reader to [D] for the definition of a quasi-Hopf algebra and a twist of a quasi-Hopf algebra.

2.1. We recall the theory of the radical filtration for finite dimensional quasi-Hopf algebras, discussed in [EG]. It is completely parallel to the classical theory of such filtration in finite dimensional Hopf algebras.

Let \( H \) be a finite dimensional quasi-Hopf algebra, and \( I \) be the radical of \( H \). Assume that \( I \) is a quasi-Hopf ideal, i.e., \( \Delta(I) \subseteq H \otimes I + I \otimes H \). In categorical terms, this means that the category of representations \( \text{Rep}(H) \) has Chevalley property, i.e., the tensor product of irreducible \( H \)-modules is completely reducible.

This is satisfied, for example, if \( H \) is basic, i.e., every irreducible \( H \)-module is 1-dimensional.
In this situation, the filtration of $H$ by powers of $I$ is a quasi-Hopf algebra filtration. Thus the associated graded algebra $\text{gr}(H)$ of $H$ under this filtration has a natural structure of a quasi-Hopf algebra.

Let now $\overline{H}$ be a finite dimensional quasi-Hopf algebra with a $\mathbb{Z}_+$-grading, i.e., $\overline{H} = \bigoplus_{m \geq 0} \overline{H}[m]$, with all structure maps of degree zero. In this case, $\overline{H}[0]$ is a quasi-Hopf algebra, $\overline{H}[i]$ is a free module over $\overline{H}[0]$ for all $i$ (by Schauenburg’s theorem [8]), and the radical $\mathcal{I}$ of $\overline{H}$ is a quasi-Hopf ideal.

One says that $\overline{H}$ is radically graded if $\overline{H} = \bigoplus_{m \geq k} \overline{H}[m]$, for $k \geq 1$. In this case, $\overline{H}[0]$ is semisimple, and $\overline{H}$ is generated by $\overline{H}[0]$ and $\overline{H}[1]$.

An example of a radically graded quasi-Hopf algebra is the algebra $\text{gr}(H)$ defined above. Moreover, $H$ is radically graded if and only if $\text{gr}(H) = H$.

Finally, we observe that if $H$ is radically graded and basic, then $H[0] = \text{Fun}(G)$ for a finite group $G$, and the associator (being of degree zero) corresponds to a class in $H^3(G, \mathbb{C}^*)$.

2.2. The following are the simplest examples of quasi-Hopf algebras not twist equivalent to a Hopf algebra.

Let $p > 2$ be a prime, and $\varepsilon = e^{2\pi i/p}$. If $z \in \mathbb{Z}$, we denote by $z'$ the projection of $z$ to $\mathbb{Z}_p$.

Let $s$ be an integer such that $1 \leq s \leq p - 1$. Let $Q = \varepsilon^{-s}$. The $p$–dimensional quasi-Hopf algebra $H(p, s)$, is generated by a grouplike element $a$ such that $a^p = 1$, with non-trivial associator

$$\Phi_s := \sum_{i,j,k=0}^{p-1} Q^{-i(j+k)-(i+k)s'} 1_i \otimes 1_j \otimes 1_k$$

where $\{1_i; 0 \leq i \leq p-1\}$ is the set of primitive idempotents of $\mathbb{Z}_p$ (i.e, $1_0 = Q^1$), distinguished elements $\alpha = a$, $\beta = 1$, and antipode $S(a) = a^{-1}$.

Let $s_0 \in \mathbb{Z}_p$ be a quadratic nonresidue. It can be shown (by considering automorphisms $a \mapsto a^m$) that for any $s$, $H(p, s)$ is isomorphic to $H_+(p) := H(p, 1)$ if $s$ is a quadratic residue, and to $H_-(p) := H(p, s_0)$ if $s$ is a non-quadratic residue. On the other hand, $H_+(p)$ and $H_-(p)$ are not equivalent.

Thus it follows from [ENO], Corollary 8.31, that any $p$–dimensional semisimple quasi-Hopf algebra is twist equivalent either to $\mathbb{C}[\mathbb{Z}_p]$ or to $H_{\pm}(p)$.

2.3. The following are examples of $p^3$–dimensional basic quasi-Hopf algebras with radical of codimension $p$, which are not twist equivalent to a Hopf algebra.

**Theorem 2.1.** [G] Let $p$ be a prime number.

(i) There exist $p^3$–dimensional quasi-Hopf algebras $A(q)$, parametrized by primitive roots of unity $q$ of order $p^2$, which have the following structure. As algebras $A(q)$ are generated by $a, x$ with the relations $ax = q^x a x, a^p = 1, x^{p^2} = 0$. The element $a$ is grouplike, while the coproduct of $x$ is given by the formula

$$\Delta(x) = x \otimes \sum_{y=0}^{p-1} q^y 1_y + 1 \otimes (1 - 1_0)x + a^{-1} \otimes 1_0 x,$$
where \( \{1_i : 0 \leq i \leq p - 1\} \) is the set of primitive idempotents of \( \mathbb{C}[a] \) defined by the condition \( a1_i = q^{pi}1_i \), the associator is \( \Phi_s \) (where \( s \) is defined by the equation \( \varepsilon^{-s} = q^p \)), the distinguished elements are \( \alpha = a, \beta = 1 \), and the antipode is \( S(a) = a^{-1}, S(x) = -x \sum_{z=0}^{p-1} q^{p-z} 1_z \).

(ii) The quasi-Hopf algebras \( A(q) \) are pairwise non-equivalent. Any finite dimensional radically graded basic quasi-Hopf algebra \( H \) with radical of codimension \( p \) and nontrivial associator, such that \( H[1] \) is a free module of rank 1 over \( H[0] \), is equivalent to \( A(q) \) for some \( q \).

3. Quasi-Hopf Algebras with Radical of Prime Codimension

### 3.1. The main result

Let \( p > 2 \) be a prime number. Our main result in this section is the following theorem.

**Theorem 3.1.** Let \( H \) be a radically graded basic quasi-Hopf algebra with radical of codimension \( p \). If the associator of \( H \) is nontrivial, then the rank of \( H[1] \) over \( H[0] \) is \( \leq 1 \).

Theorem 3.1 is proved in the next subsection.

Theorem 3.1 and the results cited above imply the following classification result.

**Theorem 3.2.** Let \( H \) be a radically graded finite dimensional quasi-Hopf algebra with radical of codimension \( p \). Then \( H \) is one of the following quasi-Hopf algebras, up to twist equivalence:

(a) duals of pointed Hopf algebras with \( p \) grouplike elements, classified in [AS], Theorem 1.3 (including the group algebra \( \mathbb{C}[\mathbb{Z}_p] \));

(b) the algebras \( H_+(p) \) and \( H_-(p) \);

(c) the algebras \( A(q) \).

**Proof.** By Corollary 8.31 of [ENO], \( H \) is necessarily basic.

If the associator of \( H \) is trivial, then we may assume that \( H \) is a Hopf algebra. Thus \( H^* \) is a coradically graded pointed Hopf algebra with \( G(H^*) = \mathbb{Z}_p \). Such algebras are classified in [AS], Theorem 1.3, so we are in case (a).

If the associator is nontrivial, then by Theorem 3.1, the rank of \( H[1] \) over \( H[0] \) is at most 1. If the rank is 0, we are in case (b). If the rank is 1, we are in case (c) by Theorem 2.1.

We refer the reader to [EO], for the definition of a finite tensor category and the notion of its Frobenius–Perron dimension.

**Corollary 3.3.** Let \( p > 2 \) be a prime. Let \( \mathcal{C} \) be a finite tensor category, which has exactly \( p \) simple objects which are all invertible. Then the possible values of the Frobenius–Perron dimension of \( \mathcal{C} \) are \( p^N, N = 1, 2, 3 \) (for all \( p \)), \( 4 \) (for \( p = 3 \) and \( p = 3k + 1 \)), \( 5 \) (for \( p = 3 \) and \( p = 4k + 1 \)) and \( 7 \) (for \( p = 3 \) and \( p = 3k + 1 \)).

**Proof.** It is clear that the Frobenius–Perron dimension of objects in \( \mathcal{C} \) are integers. Hence by [EO], there exists a quasi-Hopf algebra \( A \) such that \( \mathcal{C} = \text{Rep}(A) \). This quasi-Hopf algebra is basic, so its radical is a quasi-Hopf ideal and hence \( A \) admits a radical filtration. Let \( H := \text{gr}(A) \) (with respect to this filtration). Then Theorem 3.2 applies to \( H \), hence the result.
3.2. Proof of Theorem 3.1. Let us assume that $H[1]$ has rank $> 1$ over $H[0]$. From this we will derive a contradiction. We may assume that $H$ has the minimal possible dimension.

Let $a$ be a generator of $\mathbb{Z}_p$. We have $H[0] = \mathbb{C}[\mathbb{Z}_p]$ with associator $\Phi_z$ for some $z$.

Let us decompose $H[1]$ into a direct sum of eigenspaces of $a$: $H[1] = \bigoplus_{r=0}^{p-1} H_r[1]$, where $H_r[1]$ is the space of $x \in H[1]$ such that $axa^{-1} = Q^r x$ (we recall that $Q := \varepsilon^{a^{-1}}$). Note that $1, x = x_1, \ldots, r$ for $x \in H_r[1]$. Also, by Theorem 2.17 in [EO], $H_0[1] = 0$.

Let $\tilde{H}$ be the free algebra generated by $H[1]$ as a bimodule over $H[0]$; i.e., $\tilde{H}$ is the tensor algebra of $H[1]$ over $H[0]$. Then $\tilde{H}$ is (an infinite dimensional) quasi-Hopf algebra, and we have a surjective homomorphism $\varphi: \tilde{H} \rightarrow H$ (it is surjective since $H$ is radically graded and hence generated by $H[0]$ and $H[1]$).

Let $q$ be a number such that $q^p = Q$. Define an automorphism $\gamma$ of $\tilde{H}$ by the formula $\gamma|_{H[0]} = 1$ and $\gamma|_{H_r[1]} = q^r$. (It is well defined since $H$ is free.)

Let $L$ be the sum of all quasi-Hopf ideals in $\tilde{H}$ contained in $\bigoplus_{n \geq 2} \tilde{H}[d]$. Clearly, $\Ker \varphi \subseteq L$, so $H$ projects onto $\tilde{H}/L$. However, since $H$ has the smallest dimension, it follows that $\tilde{H}/L = H$.

Now, $\gamma(L) = L$, so $\gamma$ acts on $H$. Let us define a new quasi-Hopf algebra $\hat{H}$ generated by $H$ and a grouplike element $g$ with relations $g^p = a$, $g zg^{-1} = \gamma(z)$ for $z \in H$. Clearly, $\Ad(a) = \gamma^p$, and $g$ generates a group isomorphic to $\mathbb{Z}_p^2$.

Let $J := \sum_{i,j} c(i,j)1_i \otimes 1_j$, $c(i,j) := q^{-i(j-j')}$, where $j'$ denotes the remainder of division of $j$ by $p$, be the twist in $\mathbb{C}[\mathbb{Z}_p^2] \otimes \mathbb{C}$ defined in [G]. Define $\hat{H}$ to be the twist of $\tilde{H}$ by $J^{-1}$: $\hat{H} := \tilde{H}J^{-1}$. Since by [G], $dJ = \Phi_z$, $\hat{H}$ is a finite dimensional radically graded Hopf algebra. Since the rank of $H[1]$ over $H[0]$ is $> 1$, we have at least 2 independent over $H[0]$ skew primitive elements $x_1, x_2 \in \hat{H}[1]$ which are eigenvectors for $\Ad(g)$:

$$gx_1g^{-1} = q^{d_1}x_1, \quad \Delta(x_1) = x_1 \otimes g^{b_1} + 1 \otimes x_1$$

and

$$gx_2g^{-1} = q^{d_2}x_1, \quad \Delta(x_2) = x_2 \otimes g^{b_2} + 1 \otimes x_2.$$ 

Since $H_0[1] = 0$, $d_1, d_2$ must be relatively prime to $p$. Also, since $H$ has minimal dimension, the algebra $\hat{H}$ is generated by $g, x_1, x_2$.

By [G], the function $c(i,j)/(i-1,j)$ is $p$-periodic in each variable. Moreover, the coproduct of $\hat{H}$ maps $x_i$ into $\hat{H} \otimes \hat{H}$; thus, similarly to [G], the function $c(b_i, b_j)/(i-1,j)q^{b_k}$ is $p$-periodic in each variable for $k = 1, 2$. Hence the function $c(b_i, b_j)/(i-1,j)q^{b_k}$ is $p$-periodic in each variable for $k = 1, 2$ (here $b_k/d_k$ is the ratio taken in $\mathbb{Z}_p^2$). We thus conclude that $b_k = d_k$ modulo $p$, for $k = 1, 2$.

Now set $g := g^{b_1}$, $\hat{g} := q^{d_1}g$, $b := b_2/b_1$ and $d := d_2/d_1$. We obtain

$$\hat{g}x_1\hat{g}^{-1} = \hat{g}x_1, \quad \Delta(x_1) = x_1 \otimes \hat{g} + 1 \otimes x_1$$

and

$$\hat{g}x_2\hat{g}^{-1} = q^dx_2, \quad \Delta(x_2) = x_2 \otimes \hat{g} + 1 \otimes x_2,$$

where $b, d \in \mathbb{Z}_p^2$ and $b = d$ modulo $p$. 
Extend $\hat{H}$ to a Hopf algebra $H'$ generated by $\hat{H}$ and two commuting grouplike elements $g_1, g_2$, with relations $g_i x_j g_i^{-1} = q^{\lambda_{ij}} x_j$, $g_i^{d_i} = 1$ for $i, j = 1, 2$, and $\bar{g} = g_1 g_2^d$. (The proof that this is possible is the same as the proof given above of the fact that $H$ can be extended by adjoining $g$.)

Let $\lambda \in \mathbb{Z}_p^2$. Let

$$T = T_\lambda := \sum_{\gamma, \beta} q^{\beta_1 \gamma_2} 1_\beta \otimes 1_\gamma \in C[\mathbb{Z}_p^2 \times \mathbb{Z}_p^2] \otimes^2,$$

where $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ and $\{1_\beta : \beta \in \mathbb{Z}_p^2 \times \mathbb{Z}_p^2\}$ is the set of primitive idempotents of $\mathbb{Z}_p^2 \times \mathbb{Z}_p^2$. This is a Hopf twist. Consider the new coproduct $\Delta_T$, obtained by twisting $\Delta$ by $T$. That is, $\Delta_T(z) = T \Delta(z) T^{-1}$.

Using the facts that $1_\beta q_{i1} = q^{\beta_1} 1_\beta$ and $1_\beta x_i = x_i 1_{\beta - \epsilon_i}$, $i = 1, 2$, where $\epsilon_1 := (1, 0)$ and $\epsilon_2 := (0, 1)$, it is straightforward to verify that

$$\Delta_T(x_1) = x_1 \otimes g_1 g_2^{\lambda + d} + 1 \otimes x_1 \quad \text{and} \quad \Delta_T(x_2) = x_2 \otimes g_1^{b - \lambda} g_2^d + g_1^\lambda \otimes x_2.$$

Therefore, if we set

$$z_1 := x_1, \quad z_2 := g_1^{-\lambda} x_2, \quad h_1 := g_1 g_2^{\lambda + d} \quad \text{and} \quad h_2 := g_1^{b - \lambda} g_2^d$$

we get

$$\Delta_T(z_1) = z_1 \otimes h_1 + 1 \otimes z_1 \quad \text{and} \quad \Delta_T(z_2) = z_2 \otimes h_2 + 1 \otimes z_2.$$ 

Now, the relations

$$h_1 z_1 h_1^{-1} = \bar{q} z_1, \quad h_1 z_2 h_1^{-1} = q^{\lambda + d} z_2, \quad h_2 z_1 h_2^{-1} = q^{b - \lambda} z_1 \quad \text{and} \quad h_2 z_2 h_2^{-1} = q^{bd} z_2,$$

imply that the braiding matrix $B$ of $(H')^T$ (in the sense of [AS]) is given by $b_{11} = \bar{q}$, $b_{12} = q^{\lambda + d}$, $b_{21} = q^{b - \lambda}$ and $b_{22} = q^{bd}$.

Now set $\lambda = (b-d)/2$. In this case $b_{12} = b_{21} = q^{(b+d)/2}$, so the braiding matrix is symmetric, and the corresponding Nichols algebra is of FL type in the sense of [AS].

According to [AS], the Cartan matrix $A$ corresponding to $B$ has $a_{12} = b + d$ and $a_{21} = (b + d)/bd$ (modulo $p^2$). Since in our situation $b = d$ modulo $p_2$, we get that modulo $p$, $a_{12} = 2b$ and $a_{21} = 2/b$, and hence that $a_{12} a_{21} = 4$ modulo $p$. We claim that this implies that the Cartan matrix $A$ cannot be of finite type.

Indeed, in the finite type case ($A_1 \times A_2$, $A_2$, $B_2$ and $G_2$), $a_{12} a_{21} = 0, 1, 2, 3$. Therefore if $p > 3$, $A$ cannot be of finite type. For $p = 3$, we get that $a_{12} = a_{21} = -1$ ($A_2$ case) and $b = 1$ modulo $3$. But this implies that $b^2 + b + 1 = 0$ modulo $9$, which leads to a contradiction.

Now by Theorem 1.1 (ii) in [AS], the algebra $(H')^T$ (and hence $H'$) is infinite dimensional. This gives a contradiction and completes the proof of Theorem 3.1.

4. CONSTRUCTION OFFINITE-DIMENSIONAL BASIC QUASI-HOPF ALGEBRAS

In this section we generalize the construction of $A(q)$ from [G], and construct finite dimensional basic quasi-Hopf algebras which are not twist equivalent to a Hopf algebra.
Let $n \geq 2$ be an integer, and $q$ a primitive root of 1 of order $n^2$. Let $H$ be a finite dimensional Hopf algebra generated by grouplike elements $g_i$ and skew-primitive elements $e_i$, $i = 1, \ldots, m$, such that
\[ g_i n^2 = 1, \quad g_i g_j = g_j g_i, \quad g_i e_j g_i^{-1} = q^{\delta_{ij}} e_j \]
and
\[ \Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \]
where $K_i := \prod_j g_j^{a_{ij}}$ for some $a_{ij}$ in $\mathbb{Z}_{n^2}$.

Assume that $H$ has a projection onto $\mathbb{C}[[\mathbb{Z}_{n^2}]]$, $g_i \mapsto g_i$ and $e_i \mapsto 0$, and let $B \subset H$ be the subalgebra generated by $\{e_i\}$. Then by Radford’s theorem [R], the multiplication map $\mathbb{C}[[\mathbb{Z}_{n^2}]] \otimes B \to H$ is an isomorphism of vector spaces. Therefore, $A := \mathbb{C}[[\mathbb{Z}_{n^2}]]B \subset H$ is a subalgebra of dimension $\dim(H)/n^m$. It is generated by $g_i^n$ and $e_i$.

Let $\{1_\beta|\beta = (\beta_1, \ldots, \beta_m) \in (\mathbb{Z}_{n^2})^m\}$ be the set of primitive idempotents of $\mathbb{Z}_{n^2}$, and denote by $e_i \in (\mathbb{Z}_{n^2})^m$ the vector with 1 in the $i$th place and 0 elsewhere. Note that
\[ 1_\beta g_i = q^\delta_i 1_\beta \quad \text{and} \quad 1_\beta e_i = e_i 1_\beta - e_i. \]

Let $c(z, y)$ be the coefficients of the twist $J$ as above introduced in [G]. Recall from [G] that $c(z, y) = q^{-z(y-y')}$, where $y'$ denotes the remainder of division of $y$ by $n$.

Let
\[ J := \sum_{\beta, \gamma, \delta \in (\mathbb{Z}_{n^2})^m} \prod_{i, j = 1}^m c(\beta_i, \gamma_j)^{a_{ij}} 1_\beta \otimes 1_\gamma. \]

It is clear that it is invertible and $(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1$. Define a new coproduct $\Delta_J(z) := J \Delta(z) J^{-1}$.

**Lemma 4.1.** The elements $\Delta_J(e_i)$ belong to $A \otimes A$.

**Proof.** This lemma for $m = 1$ was proved in [G]. The general case follows from the case $m = 1$ by a straightforward computation. \[\square\]

**Lemma 4.2.** The associator $\Phi := dJ$ obtained by twisting the trivial associator by $J$ is given by the formula
\[ \Phi = \sum_{\beta, \gamma, \delta \in (\mathbb{Z}_{n^2})^m} \left( \prod_{i, j = 1}^m q^{a_{ij}\beta_i((\gamma_j + \delta_j') - \gamma_j - \delta_j)} \right) 1_\beta \otimes 1_\gamma \otimes 1_\delta, \]
where $\{1_\beta\}$ are the primitive idempotents of $\mathbb{Z}_{n^2}$, $1_\beta = q^{\rho_\beta} 1_\beta$, and we regard the components of $\beta, \gamma, \delta$ as elements of $\mathbb{Z}$. Thus $\Phi$ belongs to $A \otimes A \otimes A$.

**Proof.** One has
\[ \Phi = \sum_{\beta, \gamma, \delta \in (\mathbb{Z}_{n^2})^m} \prod_{i, j = 1}^m \left( c(\beta_i + \gamma_i, \delta_j) c(\beta_i, \gamma_j) \right)^{a_{ij}} 1_\beta \otimes 1_\gamma \otimes 1_\delta. \]

Substituting the expression of $c(z, y)$, similarly to [G] we get the statement. \[\square\]

Thus we get our second main result.
Theorem 4.3. The algebra $A$ is a quasi-Hopf subalgebra of $H^3$, which has coproduct $\Delta_3$ and associator $\Phi$.

Proof. We have shown that $\Delta_3: A \rightarrow A \otimes A$ and $\Phi \in A \otimes A \otimes A$. It is also straightforward to show that $S_3: A \rightarrow A$ and $\alpha \in A$ if $\beta$ is gauged to be 1 (where $S_3, \alpha, \beta$ are the antipode and the distinguished elements of $H^3$). Thus $A$ is a quasi-Hopf subalgebra of $H^3$. □

This yields many examples of finite dimensional basic quasi-Hopf algebras $A$. For instance, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, and $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Then we can take $H$ to be the Frobenius–Lusztig kernel $u_q(\mathfrak{b})$. In this case, $A$ is a quasi-Hopf algebra of dimension $n^{\dim \mathfrak{g}}$. Another example is obtained from $H = gr(u_q(\mathfrak{g}))$ (with respect to the coradical filtration).

Remark 4.4. If for some $i$, $a_{ii} \neq 0$ modulo $n$, then $A$ is not twist equivalent to a Hopf algebra. Indeed, the associator $\Phi$ is non-trivial since the 3-cocycle corresponding to $\Phi$ restricts to a non-trivial 3-cocycle on the cyclic group $\mathbb{Z}_n$ consisting of all tuples whose coordinates equal 0, except for the $i$th coordinate. Since $A$ projects onto $(\mathbb{C}[\mathbb{Z}_n^m], \Phi)$ with non-trivial $\Phi$, $A$ is not twist equivalent to a Hopf algebra.

For instance, this is the case in the above two examples obtained from $u_q(\mathfrak{b})$ and $u_q(\mathfrak{g})$.

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