ON HOLOMORPHIC FOLIATIONS TRANSVERSE TO SPHERES

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Dedicated to Professor Tatsuo Suwa on his sixtieth birthday

ABSTRACT. We study the problem of existence and classification of holomorphic foliations transverse to a real submanifold in the complex affine space. In particular we investigate the existence of a codimension one holomorphic foliation transverse to a sphere in $\mathbb{C}^n$ for $n \geq 3$.


Key words and phrases. Holomorphic foliation, transverse section, foliation with singularities.

1. Introduction

In this paper we address the following question: Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold $M^n$ transverse to the boundary $\partial \Omega$ of some simply-connected regular domain $\Omega \subset M$. Then what can be said about $\mathcal{F}$?

Several are our motivations for this. For instance, in [15] it is proven that if a holomorphic vector field $Z$ in a neighborhood of the closed ball $B^{2n}(R) = \{ z \in \mathbb{C}^n; |z| \leq R \}$, is transverse to the sphere $S^{2n-1}(R) = \partial B^{2n}(R) = \{ z \in \mathbb{C}^n; |z| = R \}$, then such vector field exhibits only one singularity $o \in B^{2n}(R)$, which is in the Poincaré-domain. In the sake of generalizations of this result, we consider in Section 3 the following situation: $\mathcal{F}$ is a codimension-one foliation on a neighborhood $U$ of the closed ball $B^{2n}(1) \subset \mathbb{C}^n$; and we investigate the transversality of $\mathcal{F}$ with the sphere $S^{2n-1}(1)$. We do not know whether we may have $\mathcal{F} \pitchfork S^{2n-1}(1)$ with $n \geq 3$; nevertheless non-transversality results are proven for the cases $\mathcal{F}$ is linear (cf. Theorem 3.1) or given by a homogeneous one form (Section 3.1). We also prove (Section 3.2):

Theorem 1.1. Let $\mathcal{F}$ be a codimension one holomorphic foliation in $U \subset \mathbb{C}^n$, $n \geq 3$, exhibiting a separatrix $\Lambda \subset \mathbb{C}^n$ which is contained in some “piece of” hyperplane $\{ x_n = 0 \}$ for some local chart $(x_1, \ldots, x_n) \in U$. Let $M \subset U$ be the sphere $M = \{ (x_1, \ldots, x_n) \in U; \sum_{j=1}^n |x_j|^2 = R^2 \}$. Then $\mathcal{F}$ is not transverse to $M$.

For the case $n = 2m + 1$ we obtain a general result (Section 3.3):

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Theorem 1.2. Let $F$ be a codimension one holomorphic foliation in a neighborhood of the sphere $S^{4m+1}(1) \subset \mathbb{C}^{2m+1}$. The sphere is not transverse to $F$.

In Section 4 we consider the case of dimension one (holomorphic) foliations; if $U$ is a (neighborhood of a) polydisc in $\mathbb{C}^n$ then we may assume that the foliation is given by a holomorphic vector field $Z$. As a generalization of [2] and [15] we prove:

Theorem 1.3. Let $Z$ be a holomorphic vector field defined in a neighborhood of the domain $\Omega \subset \mathbb{C}^n$ where $\Omega$ is either $B_{2}^{n} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n; \sum_{j=1}^{n} |x_j|^p \leq 1 \}$ (for $p \geq 2$) or the closed polydisc $\Delta^n \subset \mathbb{C}^n$. Assume $Z$ is transverse to the boundary $\partial \Omega$. Then $Z$ exhibits a unique singular point $o \in \Omega$, all the orbits of $Z$ tend to the singular point $o$, the germ of $Z$ at $o$ is a singularity in the Poincaré domain.

Section 5 is dedicated to some applications of our preceding results. As a kind of Haefliger obstruction result (see Haefliger’s Theorem in [10]) we prove:

Corollary 1.1. Let $F$ be a non-singular one dimensional foliation on $M^n$. There exists no holomorphic embedding $\varphi: \Omega \rightarrow M$ of a neighborhood $\Omega$ of the polydisc $\Delta^n \subset \mathbb{C}^n$ such that $F$ is transverse to the boundary $\varphi(\partial \Delta^n)$.

We also prove the following characterization of linear hyperbolic foliations on $\mathbb{C}P(2)$.

Corollary 1.2. Let $F$ be a polynomial foliation on $\mathbb{C}^2$. Assume that:

(i) There exists a sequence of spheres on $S^3(p_n, r_n) \subset \mathbb{C}^2$ transverse to $F$ with $\lim_{n} r_n = \infty$.

(ii) The singularities of the corresponding foliation $\mathcal{F}$ induced on $\mathbb{C}P(2)$ are hyperbolic.

Then $F$ is a linear logarithmic hyperbolic foliation $\mathcal{F}: xdy - \lambda ydx = 0$, $\lambda \in \mathbb{C} - \mathbb{R}$ in some affine chart $(x, y) \in \mathbb{C}^2$.

These are sharp results as showed by Example 5.1 and Remark 5.2. In Section 5.3 we investigate the transversality of codimension one foliations with trivially embedded products of spheres, proving the following

Corollary 1.3. There exists no codimension one holomorphic foliation $F$ in a neighborhood of the polydisc $\Omega^3$ in $\mathbb{C}^4$ with the property that $F$ is transverse to the product of spheres $S^1_1(1) \times S^2_2(1) = \{(x, y, z, w) \in \mathbb{C}^4; |x|^2 + |y|^2 = 1, |z|^2 + |w|^2 = 1 \}$.

The structural stability for vector fields transverse to the boundary of the ball $B^{2n}(1) \subset \mathbb{C}^n$ is characterized as follows:

Corollary 1.4. Let $X$ be a holomorphic vector field defined in a neighborhood $U$ of the closed ball $B(p; R)$ on $\mathbb{C}^n$ and denote by $\mathcal{F}$ the corresponding foliation on $U$. If $X$ is transverse to $S^{2n-1}(p; R)$ then there exists a neighborhood $B(p; R) \subset V \subset U$ such that $\mathcal{F}|_V$ is structurally stable if, and only if, $\mathcal{F}|_{B(p; R)}$ exhibits only hyperbolic singularities.

In the last section, we give a definition of transverse holomorphic rank for closed real hypersurfaces of complex manifolds. In the same spirit of [18] we prove
The transverse holomorphic rank of any sphere $S^{2n-1}(p; R) \subset \mathbb{C}^n$ is one.

2. Foundational Material

In this section we shall state some foundational material for the remaining of the paper. Let $F$ be a codimension one holomorphic foliation in a neighborhood $U \subset \mathbb{C}^n$ of the closed ball $\overline{B}^{2n}(R) := \{(x_1, \ldots, x_n) \in \mathbb{C}^n; \sum_{j=1}^n |x_j|^2 \leq R^2\}$, $n \geq 2$. The sphere $S^{2n-1}(R) = \partial \overline{B}^{2n}(R)$ is given by $\sum_{j=1}^n |x_j|^2 = R^2$. We may assume that $F$ is given by a holomorphic vector field $Z (x_1, \ldots, x_n) dx_j$, satisfying the integrability condition $\omega = \sum_{j=1}^n A_j (x_1, \ldots, x_n) dx_j$, of the bidisc $\Delta^2 \subset \mathbb{C}^2$, of the closed ball $\overline{B}^{2n}(R)$, such that $\omega$ is accumulated by each orbit of $Z$ in $\overline{B}^{2n}(R)$, which is linearizable in the following sense. Moreover $\phi$ is injective as a map between leaf spaces.

Theorem 2.1 [15]. Let $Z$ be a holomorphic vector field defined in a neighborhood $U$ of the closed ball $\overline{B}^{2n}(R) = \{z \in \mathbb{C}^n; |z| \leq R\}$ in $\mathbb{C}^n$. Suppose that $Z$ is transverse to the sphere $S^{2n-1}(R) = \partial \overline{B}^{2n}(R) = \{z \in \mathbb{C}^n; |z| = R\}$, then $Z$ has only one singularity $o \in B^{2n}(R)$, which is accumulated by each orbit of $Z$ in $\overline{B}^{2n}(R)$, and also the germ of $Z$ at $o$ is either linearizable in the Poincaré-domain, or it is of Poincaré–Dulac normal form type. For $n = 2$ we have, in suitable local coordinates around $o$, $Z(x, y) = \lambda x (\partial/\partial x) + \mu y (\partial/\partial y)$, $\lambda/\mu \in \mathbb{C} - \mathbb{R}_-$, or $Z(x, y) = x (\partial/\partial x) + (ny + x^n) (\partial/\partial y)$, $n \in \mathbb{N},$ respectively.

A holomorphic vector field $Z$ defined in some neighborhood of $\Delta^2$ is transverse to the boundary $\partial \Delta^2$ if it is transverse to $S^1 \times \mathbb{D}$, $\mathbb{D} \times S^1$ and $S^1 \times S^1$. This situation is described by the following result of Sad and Brunella:

Theorem 2.2 [2]. Let $Z$ be a holomorphic vector field transverse to the boundary of the bidisc $\Delta^2 \subset \mathbb{C}^2$. Then $Z$ is linearizable in the following sense: there exist a linear hyperbolic vector field $Z_\lambda$ on $\mathbb{C}^2$ and a locally injective holomorphic map $\varphi: U \to \mathbb{C}^2$, defined in some neighborhood $U$ of $\partial \mathbb{D}$ in $\mathbb{C}^2$, such that the foliation $\mathcal{F}(Z)$ induced by $Z$ on $U$ is given by $F(z) = \varphi^*(\mathcal{F}(Z_\lambda))$. Moreover $\varphi$ is injective as a map between leaf spaces.

2.2. Malgrange’s Theorem. For codimension one foliations in dimension $n \geq 3$ the following Theorem of Malgrange will be useful:

Theorem 2.3 [19]. Let $p \in \text{sing}(\mathcal{F})$ and assume that $\text{codim sing}(\mathcal{F}) \geq 3$ (which means that for any irreducible component $\Lambda \subset \text{sing}(\mathcal{F})$ with $p \in \Lambda$ we have codim $\Lambda \geq 3$). Then there exists an open neighborhood $p \in W \subset U$ such that $\mathcal{F}|W$ admits a holomorphic first integral $f: W \to \mathbb{C}$.
2.3. Kupka singularities. We recall that the Kupka set of a codimension one foliation \( F = F_\omega \) given by an integrable holomorphic one-form \( \omega \) as above is the set \( K(F) = \{ p \in \text{sing}(F), d\omega(p) \neq 0 \} \). According to [7], [17] we have the following:

**Theorem 2.4.** The Kupka set \( K(F) \) is a locally closed codimension two subvariety such that given any point \( p \in K(F) \) there exist a holomorphic submersion \( \varphi : W \rightarrow \mathbb{C}^2 \), of some open neighborhood \( W \) of \( p \), and a germ of holomorphic foliation \( F_p \) at \( 0 \in \mathbb{C}^2 \), called the transverse type of \( F \) at \( p \), with the following properties:

(a) \( F|_W = \varphi^{-1}(F_p) \);  
(b) \( K(F) \cap W = \varphi^{-1}(0) \);  
(c) if \( K \subset K(F) \) is a connected component and \( K \ni p, q \), then \( F_p = F_q \), that is, the transverse type is constant along “connected components of the Kupka set” (called Kupka components).

In other words, \( F \) has a local product structure around each Kupka singularity and the transverse type is fixed along any irreducible component of the Kupka set.

2.4. Logarithmic foliations. Now we pass to define the notion of linear logarithmic foliation we refer to in Corollary 1.2. A codimension one holomorphic foliation with singularities \( F \) in a complex manifold is logarithmic if it is given by a closed meromorphic one-form with simple poles \( \Omega \). According to [8] and [4] a such a one-form \( \Omega \) on \( \mathbb{C}^n \) can be written as \( \Omega = \sum_{j=1}^r \alpha_j \frac{df_j}{f_j} \) for some holomorphic functions \( f_j : \mathbb{C}^n \rightarrow \mathbb{C} \) and complex numbers \( \alpha_j \in \mathbb{C} - \{0\} \). Suppose now that \( F \) is a polynomial foliation (that is, one given by a polynomial integrable one-form on \( \mathbb{C}^n \)) and denote by \( \overline{F} \) the corresponding projective foliation with singularities on the complex projective space \( \mathbb{C}P(n) \). We say that the foliation \( F \) is logarithmic on \( \mathbb{C}P(n) \) if for \( \overline{F} = \overline{F}|_{\mathbb{C}^n} \) and \( \Omega \) as above we can choose \( f_j \) polynomial for all \( j \). Finally, the foliation \( F \) is linear logarithmic if the \( f_j \) are of the form \( f_j(z_1, \ldots, z_n) = z_j \) in suitable affine coordinates \( (z_1, \ldots, z_n) \in \mathbb{C}^n \). A linear logarithmic foliation is hyperbolic if \( \alpha_i/\alpha_j \in \mathbb{C} - \mathbb{R} \) for all \( i \neq j \). One of the main results in [4] states the following:

**Theorem 2.5** [4]. Let \( F \) be a codimension one foliation on \( \mathbb{C}P(2) \) having an algebraic invariant curve \( \Lambda \subset \mathbb{C}P(2) \) and such that:

(i) the singularities of \( F \) in \( \Lambda \) are hyperbolic;  
(ii) the holonomy group of the leaf \( \Lambda - \text{sing}(F) \) is abelian linearizable.

Then \( F \) is a linear logarithmic foliation.

Also in [4] we find the following lemma:

**Lemma 2.1.** Let \( G \subset \text{Diff}(\mathbb{C}, 0) \) be a group of germs of complex diffeomorphisms fixing the origin \( 0 \in \mathbb{C} \). Assume that:

(i) the pseudo-orbits of \( G \) are discrete outside the origin.  
(ii) \( G \) contains some hyperbolic attractor.

Then \( G \) is abelian linearizable.
3. Transversality with Spheres

Some immediate consequences of Malgrange’s Theorem and the Maximum Modulus principle are:

1. Let $M \subset \mathbb{C}^n$ be a differentiable closed hypersurface and let $\mathcal{F}$ be a holomorphic codimension one foliation defined in a neighborhood $U$ of $M$ in $\mathbb{C}^n$, $n \geq 2$. If $\mathcal{F}$ is given by a holomorphic first integral $f: U \to \mathbb{C}$ then $\mathcal{F}$ is not transverse to $M$.

2. If $n \geq 3$ then there exists $r_0 > 0$ such that if $0 < r < r_0$ then $\mathcal{F}$ is not transverse to the sphere $S^{2n-1}(r)$.

3. Let $\mathcal{F}$ be a codimension one holomorphic foliation in a neighborhood $U$ of $\overline{B^{2n}(1)}$ and transverse to the boundary $\partial B^{2n}(1)$, then $\text{sing}(\mathcal{F}) \cap \overline{B^{2n}(1)}$ is a finite set of points.

4. Let $\mathcal{F}$ be a codimension one holomorphic foliation in a neighborhood $U$ of $S^{2n-1}(1)$ in $\mathbb{C}^n$, $n \geq 3$. Assume that $\mathcal{F} \cap S^{2n-1}(1)$. If $\mathcal{F}$ is given by a meromorphic first integral, then it admits a holomorphic first integral $f: U \to \mathbb{CP}(1)$, i.e., meromorphic without base points. In particular, $\mathcal{F}$ defines a $\mathbb{CP}^1$-fibration over $S^2$ with holomorphic fibers.

Proof. First of all we remark that according to Hartogs’s Extension Theorem [13] $\mathcal{F}$ extends to $B^{2n}(1) \cup U = \tilde{U}$ which is a neighborhood of $\overline{B^{2n}(1)}$ in $\mathbb{C}^n$. We already know that $\text{sing}(\mathcal{F}) \cap \overline{B^{2n}(1)}$ is a finite set. Let $f: \tilde{U} \to \mathbb{CP}(1)$ be a meromorphic first integral for $\mathcal{F}$. We may choose $\tilde{U}$ as an open ball $B^{2n}(R)$, $R > 1$, and therefore we may write $f = \frac{E}{G}$ for some holomorphic $F, G: \tilde{U} \to \mathbb{C}$ without common irreducible components [13]. If either $\{F = 0\} = \emptyset$ or $\{G = 0\} = \emptyset$ then we may assume that $f$ is holomorphic and apply statement 1 above. Thus we may assume that $\{F = 0\} \neq \emptyset$ and $\{G = 0\} \neq \emptyset$. If there exists some point $p \in \{F = 0\} \cap \{G = 0\} \cap \overline{B^{2n}(1)}$ then $\Lambda = \{F = 0\} \cap \{G = 0\}$ is analytic and, by the same argumentation above, we must have $\Lambda \cap S^{2n-1}(1) \neq \emptyset$ what is not compatible with $\mathcal{F} \cap S^{2n-1}(1)$ (notice that $\Lambda \subset \text{sing}(\mathcal{F})$). Therefore, we obtain $f = \frac{E}{G}: \tilde{U} \to \mathbb{CP}(1)$ a holomorphic function that is, a meromorphic function without indefinite points. By Ehresmann Theorem [10] the restriction $f|\overline{S^{2n-1}(1)}$ gives a smooth fibration $S^{2n-1}(1) \to S^2(1) \cong \mathbb{CP}(1)$ with fiber of holomorphic fibers. \hfill $\square$

3.1. Linear case. The set of tangent points of a codimension one foliation $\mathcal{F}$ with $S^{2n-1}(R)$ is given by the condition $T(\mathcal{F}, S^{2n-1}(R)) := \{ (x_1, \ldots, x_n) \in S^{2n-1}(R); \bar{x}_j A_i(x_1, \ldots, x_n) = \bar{x}_i A_j(x_1, \ldots, x_n), i, j = 1, \ldots, n \}$.

Theorem 3.1. Let $\omega = \sum_{j=1}^n (\sum_{i=1}^n a_{ij}z_i)dz_j$ be a linear integrable 1-form on $\mathbb{C}^n$, $n \geq 3$, endowed with coordinates $(z_1, \ldots, z_n)$. Let $n \geq 3$, the corresponding foliation $\mathcal{F}_\omega$ on $\mathbb{C}^n$ is not transverse to the sphere $S^{2n-1}(1)$. Moreover, $\mathcal{F}_\omega$ is transverse to the sphere $S^{2n-1}(1)$ off the singular set $\text{sing}(\mathcal{F}_\omega) \cap S^{2n-1}(1)$ if and only if $\mathcal{F}_\omega$ is a product $\mathcal{L}_\lambda \times \mathbb{C}^{n-2}$ for some linear foliation $\mathcal{L}_\lambda$: $x dy - \lambda y dx = 0$, in the Poincaré domain on $\mathbb{C}^2$.

Proof. Let $A = (a_{ij})_{i,j=1}^n$ be the $n \times n$ complex matrix naturally associated to $\omega$. Then $\text{sing}(\omega) = \text{Kernel of } A$ as a linear map. We divide the proof in several steps:
Suppose $\omega$ has an isolated singularity at $0 \in \mathbb{C}^n$, then $A$ is non-singular and according to Malgrange's Theorem ([19]) there exists a holomorphic first integral $f: U \to \mathbb{C}$ for $F: \{\omega = 0\}$, in some neighborhood $U$ of $0 \in \mathbb{C}^n$. We may therefore write $\omega|_U = gdf$ for some unity $g \in \mathcal{O}(U)^*$. Write now $g = g_0 + g_1 + g_{r+1} + \cdots$, $f = f_0 + f_{r+1} + \cdots$, in sum of homogeneous forms, to obtain $gdf = g_0 df_0 + g_0 df_{r+1} + g_0 df_{r} + \cdots$, so that comparing same degree terms we conclude $\omega = g_0 df_0$, $\rho - 1 = 1$. Thus $d\omega = df_0 df_0 = 0$ (for $g_0$ is constant) and $\omega - df = 0$ for any $i$, $j$ so that $A$ is symmetric. In particular we may diagonalize $A$ in some affine chart $(u_1, \ldots, u_n) \in \mathbb{C}^n$ and write $w(u_1, \ldots, u_n) = \sum_{k=1}^n \lambda_k u_k du_k$ with $\lambda_k \neq 0$. Since $F := \{\omega = 0\}$ admits a holomorphic first integral $f: U \to \mathbb{C}$ we may assume that $\lambda_k \in \mathbb{N}$ for any $k \in \{1, \ldots, n\}$ and (since $\omega$ is linear of degree one) $\lambda_k = 1$ for any $k$. Therefore $\omega \equiv \sum_{k=1}^n u_k du_k = dF$ where $F = \frac{1}{2}(u_1^2 + \cdots + u_n^2)$.

ii) Assume now $A$ is singular but still $\omega$ has a holomorphic first integral around the origin. Again we may write conclude that $d\omega = 0$ and $A$ is symmetric diagonalizable, and write $\omega(u_1, \ldots, u_n) = \sum_{k=1}^n \lambda_k u_k du_k$ but now some $\lambda_k$ is zero. This gives us $\lambda_k \in \{0, 1\}$ for any $k \in \{1, \ldots, n\}$ and $F = \frac{1}{2}(u_1^2 + \cdots + u_k^2)$, $1 \leq k \leq n$, as a global first integral.

iii) If we only assume that $A$ is symmetric then we have one of the two cases above; indeed, if $A$ is symmetric then $\omega$ is closed, therefore $\omega = dF$ for some polynomial $F: \mathbb{C}^n \to \mathbb{C}$, thus $F$ has a local first integral around $0 \in \mathbb{C}^n$.

iv) Finally, we assume that $A$ is not symmetric. In this case $d\omega \neq 0$ everywhere.

If $T(F, S^{2n-1}(1)) = \emptyset$ then (as we have seen) $F$ must have an isolated singularity at $0 \in \mathbb{C}^n$ and a local holomorphic first integral what implies the existence of a global first integral for $F$ on $\mathbb{C}^n$. This is not compatible with the transversality $F \cap S^{2n-1}(1)$. Assume now that $T(F, S^{2n-1}(1)) \cap [S^{2n-1}(1) - \text{sing } F \cap S^{2n-1}(1)] = \emptyset$. In other words, $F$ is transverse to $S^{2n-1}(1)$ off the singular set. As we have seen, $A$ is singular. Take any codimension 2 component $\Lambda$ of $\text{sing}(F)$. Denote by $K(F)$ the Kupka set of $F$ as in Section 2.3. We have that $\Lambda \subset K(F)$ because $d\omega \neq 0$ everywhere. Also $\text{sing}(F)$ is a subspace of $\mathbb{C}^n$ so that $\text{sing}_2(F) \cap S^{2n-1}(1) = K(F) \cap S^{2n-1}(1)$ for the codimension two irreducible component $\text{sing}_2(F)$ of $\text{sing}(F)$. According to this and to the local product structure given by Theorem 2.4 in Section 2.3 (cf. [16]) we may easily conclude that for any plane $P \subset \mathbb{C}^2$ intersecting $S^{2n-1}(1)$ transversally, close enough to a point $p \in K(F) \cap S^{2n-1}(1)$, we have that $F_p : = F|_p$ is transverse to the 3-sphere $S_3^3 := P \cap S^{2n-1}(1)$. According to Douady–Ito Theorem [15] we conclude that $F_p$ is either linearizable in the Poincaré-domain or of Poincaré–Dulac normal form type; that is, $F_p : x dy - \lambda y dx = 0$ or $x dy - (ny + x^n)dx = 0$, $\lambda \notin \mathbb{R} \_-$, in some local coordinates $(x, y) \in P$. Since $\omega$ is linear and $P \subset \mathbb{C}^n$ is linearly embedded we conclude that $F_p : x dy - \lambda y dx = 0$ which gives some Jordan block of $A$. We may argue like this for each irreducible component of the singular set of $F$ and deduce from conclude that $A$ has a Jordan canonical form of $A$ that $F$ is (globally) a product foliation $\mathcal{L}_\lambda \times \mathbb{C}^{n-2}$ for some foliation $\mathcal{L}_\lambda : x dy - \lambda y dx = 0$, in the Poincaré domain.

5. Let $\omega$ be a homogeneous integrable 1-form in $\mathbb{C}^n$. The foliation $F : \{\omega = 0\}$ is not transverse to the sphere $S^{2n-1}(1)$. \qed
Proof. Let $\omega$ be a homogeneous integrable 1-form (with cod $\text{sing}(\omega) \geq 2$) on $\mathbb{C}^n$ say, $\omega$ has degree $\nu \geq 1$. If $\omega$ has some holomorphic first integral $f: U \to \mathbb{C}$ in some neighborhood of the origin $0 \in U \subset \mathbb{C}^n$ then we may write $\omega = g \, df$ for some unity $g \in \mathcal{O}(U)^*$. Write $g = g_0 + g_1 + g_2 + \cdots$, $f = f_0 + f_{\rho+1} + \cdots$ in sum of homogeneous polynomials with $f_\rho \neq 0$ and $g_0 \in \mathbb{C}^*$. We have $\omega = (g_0 + g_1 + \cdots)(df_\rho + df_{\rho+1} + \cdots) = g_0 df_\rho + (g_1 df_\rho + g_0 df_{\rho+1}) + \cdots$. Since $\omega$ is homogeneous of degree $\nu$ we must have $\omega = g_0 df_\rho$ so that $\rho - 1 = \nu$. In particular $f_\rho$ is a global polynomial first integral for $\omega$. \hfill \Box

### 3.2. Hyperplane separatrices.

In this section we prove Theorem 1.1. Let $\mathcal{F}$ be a codimension one holomorphic foliation in a neighborhood $U$ of $\mathbb{B}^{2n}(1)$ in $\mathbb{C}^n$. Assume that $0 \in \mathbb{C}^n$ is a singularity of $\mathcal{F}$. A separatrix of $\mathcal{F}$ through $0$ is a codimension one analytic subset $\Lambda$ in a neighborhood of $0$ such that $0 \in \Lambda$ and $\Lambda$ is $\mathcal{F}$-invariant. This means that $\Lambda - \text{sing}(\mathcal{F})$ is contained in a leaf of $\mathcal{F}$. The existence of a separatrix has been proved for $n = 2$ by Camacho–Sad [5] and for $n \geq 3$ by Cerveau–Cano and others authors (see [6]). Under some non-degeneracy hypothesis on $\mathcal{F}$ the separatrices are smooth so that it is natural to consider the case $\mathcal{F}$ exhibits some separatrix diffeomorphic to a “piece of” hyperplane as an open polydisc $\Delta^{n - 1} \subset \mathbb{C}^n$. First prove:

**Lemma 3.1.** Let $\mathcal{F}, \Lambda \subset \mathbb{C}^n, (x_1, \ldots, x_n) \in U$ and $M \subset U$ be as in Theorem 1.1. Assume that $\mathcal{F}$ is transverse to $M$. Then $\mathcal{F}$ admits a holomorphic (i.e., without base points) first integral $f: V \to \mathbb{C}P(1)$ in some neighborhood $V$ of $M$ in $U$.

**Proof.** We may assume that $U = \mathbb{C}^n, M = S^{2n-1}(1), \Lambda = \{x_n = 0\} \cong \mathbb{C}^{n-1}$. Denote by $\mathcal{F}_\Lambda$ the foliation induced by $\mathcal{F}$ on $S^{2n-1}(1)$. Then $\mathcal{F}_\Lambda$ is a codimension two, real foliation on $S^{2n-1}(1)$. Also $\mathcal{F}_\Lambda$ is naturally equipped with a holomorphic transverse structure inherited from $\mathcal{F}$. The intersection $\Lambda \cap M = S^{2n-1}(1) \cap \{x_n = 0\}$ is a sphere $S^{2n-3}$ which is a leaf of $\mathcal{F}_\Lambda$. Since $n \geq 3$ this a compact leaf with trivial fundamental group so that, by Lemma 3.2 below, $\mathcal{F}_\Lambda$ admits some transversely holomorphic first integral say $f: S^{2n-1}(1) \to \mathbb{C}P(1)$. Now we use the transversality of $\mathcal{F}$ with $S^{2n-1}(1)$ in order to extend $f$ to a neighborhood $V$ of $S^{2n-1}(1)$ in $\mathbb{C}^n$, this extension constant along the leaves of $\mathcal{F}$ in $V$. Choosing suitable distinguished charts for $\mathcal{F}$ in neighborhoods of points in $S^{2n-1}(1)$ we conclude that there exists an open cover of $V$ by domains $U$ of local coordinates $(x, y): U \to \mathbb{C}^{n-1} \times \mathbb{C}$ such that $\mathcal{F}|_U$ is given by $dy = 0$ and $f|_U$ is holomorphic in each variable $x$ and $y$, separately. Evoking a remarkable result of Hartogs [13] we conclude that $f|_U$ is holomorphic, so that $f: V \to \mathbb{C}P(1)$ is a holomorphic first integral for the restriction $\mathcal{F}|_V$. Hartogs’ Extension Theorem implies that $f$ extends holomorphically to a neighborhood of $\mathbb{B}^{2n}(1)$ in $\mathbb{C}^n$, what implies in particular that $\mathcal{F}$ has a holomorphic first integral, completing the proof. \hfill \Box

**Lemma 3.2.** $M$ be a compact differentiable manifold equipped with a codimension 2 real transversely holomorphic foliation $\mathcal{F}_1$. Assume that $\mathcal{F}_1$ has some compact leaf $L_0 \subset M$ with $\pi_1(L_0)$ finite. Then $\mathcal{F}_1$ is a smooth fibration $M \to Q$ of $M$ over some compact Riemann surface $Q$. In particular there exists a transversely meromorphic first integral $f: M \to \mathbb{C}$, without base points.
Proof. First we apply the Global Stability Theorem for transversely holomorphic foliations [1] to conclude that \( \mathcal{F}_1 \) is a compact foliation such that every leaf \( L \) of \( \mathcal{F}_1 \) has finite holonomy group. In particular, \( \mathcal{F}_1 \) is stable compact ([10], p. 376). Now, since \( \mathcal{F}_1 \) is stable compact it follows that the leaf space \( Q = M/\mathcal{F}_1 \) is naturally equipped with a holomorphic structure of complex dimension 1 this is done similarly to the real case using the fact that the foliation \( \mathcal{F}_1 \) above is simple (see [10], I, 1.23, p. 79). This complex space \( Q \) is also Hausdorff because the leaves of \( \mathcal{F}_1 \) are compact ([10], I, 1.2, p. 79, or Proposition 2.20, p. 103). Thus \( Q \) is a Riemann surface. Since \( M \) is compact \( Q \) is a compact Riemann surface. Again because \( \mathcal{F}_1 \) is stable the natural projection \( \pi: M \to Q \) is a submersion ([10] II, 1.23) and, as it is easy to see, \( \pi \) is actually transversely holomorphic (like \( \mathcal{F}_1 \)). By Ehresmann Theorem [10] \( \pi: M \to Q \) is a smooth fibration, which defines \( \mathcal{F}_1 \). Given any meromorphic function \( \xi: Q \to \mathbb{C}P(1) \) with connected fibers, the function \( f = \xi \circ \pi: M \to \mathbb{C}P(1) \) is a transversely meromorphic first integral for \( \mathcal{F}_1 \).

Theorem 1.1 is now an easy consequence of Lemma 3.1 and of the exact sequence of the fibration [21], that shows there exists no fibration \( S^{2n-3}(1) \to S^2(1) \) with fiber \( S^{2n-3}(1) \) for \( n \geq 3 \).

3.3. Non-transversality with spheres \( S^{4m+1} \). In this section, we prove Theorem 1.2. Let \( \omega = \sum_{k=1}^n f_k(z)dz_k \) be a holomorphic integrable 1-form in a neighborhood of the sphere in \( \mathbb{C}^n \). We define the gradient vector field of \( \omega \) as \( \text{grad}(\omega) = \sum_{k=1}^n f_k(z)\partial/\partial z_k \).

Proof of Theorem 1.2. Assume that \( S^{4m+1}(1) \) is transverse to \( \mathcal{F}(\omega) \). By the projection of \( \text{grad}(\omega) \) along \( T\mathcal{F}(\omega) \) in \( T\mathbb{C}^{2m+1} \) to \( TS^{4m+1}(1) \) has a smooth 2-field. It is a contradiction to the fact that the sphere \( S^{4m+1}(1) \), \( m \geq 1 \), does not admit a continuous 2-field (see [21]).

4. Transverse Sections to Holomorphic Vector Fields

In this section we discuss transversality of real submanifolds with holomorphic vector fields. Let \( \mathcal{X} \subset \mathbb{C}^2 \) be the closed unit bidisc \( \mathcal{X} = \{(z_1, z_2) \in \mathbb{C}^2; |z_j| \leq 1, j = 1, 2 \} \) and \( Z_\lambda \) be the linear vector field \( Z_\lambda = z_1 \frac{\partial}{\partial z_1} + \lambda z_2 \frac{\partial}{\partial z_2} \), where \( \lambda \in \mathbb{C} \). We shall notice a few facts of simple proof:

6. The sphere \( S^3(1) = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1 \} \) is transverse to the vector field \( Z_\lambda \) if and only if \( \lambda \) is not real negative.

7. The torus \( T^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = 1, |z_2| = 1 \} \) is transverse to \( Z_\lambda \) if and only if \( \lambda \) is not real.

A geometrical difference between Facts 6 and 7 above is the following.

8. Let \( S \) be a closed connected real surface. If \( \lambda \) is a real number, then there exists no smooth map \( \phi \) from \( S \) to \( \mathbb{C}^2 \) such that \( \phi \) is transverse to \( Z_\lambda \).

Proof. Assume that there exists a smooth map \( \phi: S \to \mathbb{C}^2 \) such that \( \phi \) is transverse to \( Z_\lambda \). Take a rational number \( p/q \in \mathbb{Q} \) sufficiently near to \( \lambda \), then \( \phi \) is also transverse to \( Z_{p/q} \) where \( Z_{p/q} = z_1 \frac{\partial}{\partial z_1} + \frac{p}{q} \frac{\partial}{\partial w} \). In the case that \( p/q \) is negative, i.e. \( p > 0 \) and \( q < 0 \), \( Z_{p/q} \) has a first integral \( F(z, w) = z^p w^{-q} \). By the transversality
of \( \phi \) and \( \partial_\phi \), we can define a complex structure on \( S \) and the map \( F \circ \phi : S \to \mathbb{C} \) is non constant and holomorphic. Then \( F \circ \phi \) is a constant map. This is contradictory.

In the other case, if \( p/q \) is positive, we define a rational map \( G(z, w) \) from \( \mathbb{C}^2 - \{0\} \) to \( \mathbb{C} \) by \( G(z, w) = z^p/w^q \). By the same argument, the map \( G \circ \phi \) from \( S \) to \( \mathbb{C} \) is holomorphic, then the mapping degree of \( G \circ \phi \) is bigger than one. However, the map \( \phi \) is homotopic to a constant map \( \psi \) in \( \mathbb{C}^2 - \{0\} \). Therefore, the mapping degree of \( G \circ \phi \) and \( G \circ \psi \) is the same. It is a contradiction that the mapping degree of \( G \circ \psi \) is zero. 

We shall now regard the \( n \)-dimensional version of Theorem 2.2 stated in Section 2.1. Let \( \overline{\Delta} \subset \mathbb{C}^n \) be the closed unit polydisc \( \overline{\Delta} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; |z_j| \leq 1, j = 1, \ldots, n\} \). The boundary \( \partial \overline{\Delta} \) is the union of components diffeomorphic to \((S^1(1))^k \times (\mathbb{D})^{n-k}\).

**Definition 4.1.** We say that a complex vector field \( Z \) in a neighborhood of \( \partial \overline{\Delta} \) is transverse to the boundary \( \partial \overline{\Delta} \) if it is transverse to each real submanifold component of \( \partial \overline{\Delta} \) of real dimension \( \geq 2n - 2 \). If we write \( Z = X + \sqrt{-1}Y \) with \( X, Y \) real vector fields then this means that for any such component \( \Lambda \) transverse to the boundary we have \( \partial \Lambda \) is transverse to \( Z \).

Let us give an example in dimension \( n \geq 2 \):

**Example 4.1.** Let \( Z = \sum_{j=1}^n \lambda_j z_j \overline{\partial_z}_j \) be given. If \( Z \) is hyperbolic, i.e., \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) is such that \( \lambda_i/\lambda_j \notin \mathbb{R} \) for any \( i \neq j \) then \( Z \) is transverse to the boundary of \( \overline{\Delta} \). Let now \( p \geq 2 \) and \( B_{p}^{2n} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \sum_{j=1}^n |z_j|^p \leq 1\} \). Then \( B_p^{2n} \) has smooth boundary \( \partial B_p^{2n} \) and \( Z \) is also transverse to \( \partial B_p^{2n} \). Moreover \( B_p^{2n} \) is convex and therefore pseudoconvex and simply-connected. We also have \( B_p^{2n} \to \overline{\Delta} \) and \( \partial B_p^{2n} \to \partial \overline{\Delta} \) in the natural sense that we stress below for the case \( n = 2 \). For each real number \( p \geq 2 \) let \( B_p^{2} \subset \mathbb{C}^2 \) be the domain \( B_p^{2} = \{(x, y) \in \mathbb{C}^2; |x|^p + |y|^p \leq 1\} \) and \( S^3_p := \partial B_p^{2} \) is boundary. Then \( S^3_p \) is a regular boundary given by \( X_p^3 = f_p^{-1}(1) \) for \( f_p : \mathbb{R}^4 \to \mathbb{R}, f_p(x, y) = |x|^p + |y|^p \). For \( p \geq 2 \) the function \( f_p \) is convex and therefore so is \( B_p^{2} \). In particular \( \mathbb{B}_2^{2} \) is a pseudoconvex simply connected domain. Given a holomorphic vector field \( Z = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} \) is a neighborhood of \( \mathbb{B}_2^{2} \) the set of tangent points between \( Z \) and \( S^3_p = \partial B_p^{2} \) is given by \( A(x, y) \cdot \bar{x} \cdot |x|^{p-2} + B(x, y) \cdot \bar{y} \cdot |y|^{p-2} = 0 \). We may conclude in general:

9. The vector field \( Z \) is transverse to \( \partial B_p^{2n} \) if and only if \( \lambda_i/\lambda_j \notin \mathbb{C} - \mathbb{R} \) for any \( i \neq j \).

We proceed with \( n = 2 \). Notice that \( \overline{B}_p^2 \subset \overline{\Delta} \); let us estimate the distance between one point \( (x, y) \in \partial \overline{\Delta} \) and the boundary \( S_p^2 = \partial \overline{B}_p^2 \). Given any \( (x, y) \in \partial \overline{\Delta} \) we have \( (tx, ty) \in \overline{B}_p^2 \Leftrightarrow |tx|^p + |ty|^p \leq 1 \Leftrightarrow |t|^p \leq \frac{1}{|x|^p + |y|^p} \Leftrightarrow |t| \leq \frac{1}{\sqrt{1 + |x|^p + |y|^p}}. \) Assume that for instance \( |x| = 1 \). Then \( |t| \leq \frac{1}{\sqrt{1 + |y|^p}} \). Let therefore \( t := \frac{1}{\sqrt{1 + |y|^p}} \).

Then \( \text{dist}((x, y), (tx, ty))^2 = |x - tx|^2 + |y - ty|^2 = |1 - t|^2 (|x|^2 + |y|^2) \leq 2|1 - \overline{\Delta} \).
In the sake of simplicity we shall consider the case $B$.

If a holomorphic vector field $\|v\|$ to conclude that $t^2 = 2 \left( \frac{\sqrt{t+1} + |y|}{\sqrt{t+1} + |y|} \right)^2$. Since $\lim_{p \to \infty} \frac{\sqrt{t+1} + |y|}{\sqrt{t+1} + |y|} = 1$ uniformly on $y \in \mathbb{B}$ we have

$$\lim_{p \to \infty} \frac{\sqrt{t+1} + |y|}{\sqrt{t+1} + |y|} = 0$$ uniformly on $y \in \mathbb{B}$ and therefore $\overline{B}_p^2 \to \overline{\Delta}^2$ and also dist.($\partial \overline{\Delta}^2$, $\partial B_p^2$) $\to 0$, indeed, dist.($x, y$, $\partial B_p^2$) $\to 0$ uniformly on $(x, y) \in \partial \overline{\Delta}^2$. Using these ideas we may prove the following:

**Lemma 4.1.** If a holomorphic vector field $Z$ is transverse to the boundary of $\overline{\Delta}^n \subset \mathbb{C}^n$ then $Z$ is also transverse to the boundary $\partial B_p^2$ for all $p \geq 2$ large enough.

**Proof.** In the sake of simplicity we shall consider the case $n = 2$, the other cases are similar as it will be clear from the proof we give. Take any point $(x, y) \in \partial \overline{\Delta}^2$ with $|x| = 1$ and $|y| < 1$. In a neighborhood of $(x, y)$ in $\partial \overline{\Delta}^2$, the boundary $\partial \overline{\Delta}^2$ is a 3-dimensional real submanifold and the normal unitary vector to $\partial \overline{\Delta}^2$ is given by $\mathbb{N} = (x_1, x_2, 0, 0) \in \mathbb{R}^4$. Let $(\tilde{x}, \tilde{y}) \in \partial B_p$ be the point $\tilde{x} = tx$, $\tilde{y} = ty$ for $t = \frac{1}{\sqrt{t+1} + |y|}$. We know that the unitary normal to $\partial B_p$ at $(\tilde{x}, \tilde{y})$ is given by

$$\mathbb{N}_p(\tilde{x}, \tilde{y}) = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4.$$ Therefore $\lim_{p \to \infty} \mathbb{N}_p(\tilde{x}, \tilde{y}) = \mathbb{N}(x, y)$ uniformly in compact parts of $S^1 \times \mathbb{D}$. Analogously, we have $\lim_{p \to \infty} \mathbb{N}_p(\tilde{x}, \tilde{y}) = \mathbb{N}(x, y)$ uniformly in compact parts of $\mathbb{D} \times S^1$.

Let us now regard what occurs around the torus $S^1 \times S^1 \subset \partial \overline{\Delta}^2$. Let therefore $|x| = |y| = 1$. We have by hypothesis that $Z(x, y)$ is transverse to the space $\{\mathbb{N}_x, \mathbb{N}_y\perp\}$ where $\mathbb{N}_x := (x, 0)$ and $\mathbb{N}_y := (0, y)$. On the other hand we have $\mathbb{N}_p(\tilde{x}, \tilde{y}) = \mathbb{N}(\tilde{x}, \tilde{y})$.

Write $Z = X + \sqrt{-1}Y$ as usual. If $Z$ is not transverse to $\partial B_p$ at the point $(\tilde{x}, \tilde{y}) \in \partial B_p$ then we must have $X(\tilde{x}, \tilde{y}) \perp \mathbb{N}_p(\tilde{x}, \tilde{y})$ and also $Y(\tilde{x}, \tilde{y}) \perp \mathbb{N}_p(\tilde{x}, \tilde{y})$. On the other side we have by hypothesis $Z \pitchfork S^1 \times S^1$ so that $\{X(x_0, y_0), Y(x_0, y_0)\} + \{\mathbb{N}_{x_0}, \mathbb{N}_{y_0}\perp\} = \mathbb{R}^4$ (recall that $\{v, w\}$ denotes the subspace generated by the vector $v$, $w$) for all $(x_0, y_0) \in S^1 \times S^1$. This implies the existence of a $C^\infty$ real vector field $\xi = aX + bY$ in a neighborhood $W$ of $S^1 \times S^1$, where $a, b$ are $C^\infty$ real functions, with the property that $\{\xi, \mathbb{N}_x\} \geq 1$ and $\{\xi, \mathbb{N}_y\} \geq 1$ in a neighborhood $V \subset W$ of $S^1 \times S^1$ in $\mathbb{R}^4$. Write now $\mathbb{N}_p := (\mathbb{N}_{x_0}, \mathbb{N}_{y_0})$ in the obvious way. Therefore if the point $(\tilde{x}, \tilde{y})$ belongs to this neighborhood $V$ then we claim that $\{X(\tilde{x}, \tilde{y}), \mathbb{N}_p(\tilde{x}, \tilde{y})\} \neq 0$ or $Y(\tilde{x}, \tilde{y}), \mathbb{N}_p(\tilde{x}, \tilde{y})\} \neq 0$. In fact, if $X, \mathbb{N}_p = 0$ at $(\tilde{x}, \tilde{y})$ then $\{\xi(\tilde{x}, \tilde{y}), \mathbb{N}_p\} = 0$ at $(\tilde{x}, \tilde{y})$ so that $\frac{1}{\sqrt{t+1}} \{\xi(\tilde{x}, \tilde{y}) + \mathbb{N}_p\} = 0$ at $(\tilde{x}, \tilde{y})$. This shows that for $p \gg 2$ large enough we must have $(\tilde{x}, \tilde{y}) \in V$ for all $(x, y) \in S^1 \times S^1$ and therefore $Z \pitchfork \partial B_p$ at $(\tilde{x}, \tilde{y})$. □

**Proof of Theorem 1.3.** Given $Z$ with $Z \pitchfork \partial \overline{\Delta}^n$ we use Lemma 4.1 to conclude that if $p \gg 2$ is large enough, then we may also assume that sing$(Z) \cap \Delta^n \subset B_p^2$. \hfill $\Box$
Thus we may assume that $\Omega = \overline{B^n_p}$. Moreover, since in this case the result is a generalization of the main result for $p = 2$ proven in [15] (see Theorem 2.1 in Section 2.1) we shall assume that $n = 2$ and only give the basic steps:

Step 1. The transversality of $Z$ with the boundary $\partial B^4_p$ gives a $C^{\infty}$ vector field $\xi = aX + bY \neq 0$ defined in some neighborhood $V$ of $\partial B^4_p$ in $\mathbb{C}^2$, such that $\xi$ points outward $B^4_p$ and $Z = X + iY$ as usual.

Step 2. The map $f = (a, b): V \to \mathbb{R}^2 - \{0\}$ extends to a smooth map $\tilde{f} = (\tilde{a}, \tilde{b}): \tilde{V} \to \mathbb{R}^2 - \{0\}$, where $\tilde{V}$ is some neighborhood of $\overline{B^4_p}$ in $\mathbb{C}^2$. This is a consequence of the fact that $\pi_1(\partial B^4_p) = 0$. (Notice that $\partial B^4_p$ has dimension 3 and is homeomorphic to the 3-sphere $S^3(1)$.) Let $\xi = \tilde{a}X + \tilde{b}Y$ be the obtained extension of $\xi$. Then $\xi$ and $Z = X + iY$ exhibit the same singular set.

Step 3. The index of $\xi$ at any singular point $o \in \text{sing}(Z)$ is positive, because $Z$ is holomorphic and $Z = X + iY$. The Poincaré-Hopf Theorem gives us the implication: if $1 = \chi(B^4_p) = \sum_{p \in \text{sing}(Z) \cap \overline{B^4_p}} \text{Index}_p(\xi)$ then $\#(\text{sing} \xi \cap \overline{B^4_p}) = 1$ and $\text{Index}_p(\tilde{\xi}) = +1$ for the unique singular point. Therefore $Z$ has a unique singularity $o$ in $\overline{B^4_p}$, which is a simple singularity. The rest of the proof goes as in [15]. □

We point out that the techniques used in [2] do not seem to apply, in a straightforward way, to this case. This is due to the absence of an appropriate $n$-dimensional version of the Schwarz Lemma.

5. Applications

5.1. An obstruction result. In what follows we investigate the obstruction to imbedding the boundary of a polydisc transversely to a holomorphic foliation on a complex manifold $M$.

Proof of Corollary 1.1. As above let $F^* = \varphi^*F$. Take $\nabla^W \subset W @ \Omega$ as a polydisc. This a well-known fact for polidiscs in $\mathbb{C}^n$ that there exists a neighborhood $W$ of $\Delta^n$ in $\Omega$ such that $F^*|W$ is given by a holomorphic vector field $Z$, which is transverse to the boundary $\partial \Delta^n$. Now, applying Theorem 1.3 we conclude that $Z$ must have a unique singularity $o \in \Delta^n$ and this singularity is in the Poincaré domain. On the other hand, the only possible singularities for $\varphi^*F$ come from tangent points of $F$ and $\varphi(\Omega)$, so that these are singularities exhibiting local holomorphic first integrals, and are not in the Poincaré domain, contradiction. □

5.2. Global transversality. This section is dedicated to the proof of Corollary 1.2 and its counterparts.

Proof of Corollary 1.2. Since $F \pitchfork S^3(p_1, r_1)$ we have by [15] that $F \pitchfork S^3(p_1, r)$ for all $0 < r \leq r_1$, provided that we change coordinates by some Mōbius map so that $\{p_1\} = \text{sing}(F) \cap B^3(p_1, r_1)$. Now, also form [15], we know that $p_1$ is a singularity either of Poincaré–Dulac normal form type or linearizable in the Poincaré domain. Let us choose affine coordinates such that $p_1$ is the origin of $\mathbb{C}^2$. Then for each $0 < r < r_1$ the restriction $F|S^3(0, r)$ has one or two closed orbits, corresponding to the separatrices of $F$ at $o \in \mathbb{C}^2$. Let $\Lambda_1$ be one of these separatrices and denote by
$L_1$ the corresponding leaf of the foliation $\mathcal{F}$ induced by $\mathcal{F}$ on the complex projective plane $\mathbb{C} \mathbb{P}(2) = \mathbb{C}^2 \cup \mathbb{C} \mathbb{P}(1)_{\infty}$

Claim 5.1. $L_1$ is closed in $\mathbb{C}^2 - \{0\}$.

**Proof.** Let $p \in \overline{L_1}$ be any point in the closure of $L_1$ in $\mathbb{C}^2$ and assume that $p \notin L_1 \cup \{0\}$. We have $(0, p) \in B(p_{n_0}, r_{n_0})$ for some $n_0 \in \mathbb{N}$. Therefore, since $p \neq 0$, and since $\mathcal{F}$ has only one singularity in $B(p_{n_0}, r_{n_0})$, we conclude that $p$ is not a singularity of $\mathcal{F}$. Choose a Flow Box neighborhood $U$ of $p$ in $\mathbb{C}^2$. Since $L_1$ accumulates $p$, it follows that $L_1 \cap S^1(0, |p|)$ accumulates $p \in S^1(0, |p|)$ what is not possible because $L_1 \cap S^1(0; |p|)$ must be closed by Theorem 2.1 in Section 2.1 ([15]), contradiction.

Notice that the proof above also gives $\#\text{sing}(\mathcal{F}) = 1$ in $\mathbb{C}^2$. The claim shows that $(\mathbb{C}^2 \cap L_1) \cup \{0\}$ is analytic in $\mathbb{C}^2$.

Claim 5.2. $\overline{L_1} \subset \mathbb{C} \mathbb{P}(2)$ is an algebraic curve in $\mathbb{C} \mathbb{P}(2)$.

**Proof.** First we assume that $\mathbb{C} \mathbb{P}(1)_{\infty}$ is not $\overline{\mathcal{F}}$ invariant. If $L_1$ accumulates some point $q \in \mathbb{C} \mathbb{P}(1)_{\infty} - \text{sing}(\mathcal{F})$, then by the Flow Box Theorem $L_1$ is not closed in $\mathbb{C}^2 - \{0\}$, contradiction. Thus, in this case, $\overline{L_1} \subset L_1 \cup \text{sing}(\mathcal{F})$ in $\mathbb{C} \mathbb{P}(2)$.

Assume now that $\mathbb{C} \mathbb{P}(1)_{\infty}$ is $\overline{\mathcal{F}}$-invariant. In this case we must have $\text{sing}(\mathcal{F}) \cap \mathbb{C} \mathbb{P}(1)_{\infty} \neq \emptyset$. Let therefore $q_0 \in \text{sing}(\mathcal{F}) \cap \mathbb{C} \mathbb{P}(1)_{\infty}$. Since $L_1$ accumulates $\mathbb{C} \mathbb{P}(1)_{\infty}$ and since $\mathbb{C} \mathbb{P}(1)_{\infty}$ is $\overline{\mathcal{F}}$-invariant, it follows that $L_1$ accumulates $q_0$. Now, since by hypothesis $q_0$ is hyperbolic it follows that the only local leaves of $\overline{\mathcal{F}}$, in a neighborhood of $q_0$, which are closed, are the separatrices through $q_0$. Thus $\mathbb{C} \mathbb{P}(1)_{\infty}$ contains one of these and $\overline{L_1}$ contains the other separatrix. In other words, there exists a neighborhood $U$ of $q_0$ in $\mathbb{C} \mathbb{P}(2)$ such that $\overline{L_1} \cap U$ is a separatrix of $\overline{\mathcal{F}}$, transverse to $\mathbb{C} \mathbb{P}(1)_{\infty}$. The invariance of $\mathbb{C} \mathbb{P}(1)_{\infty}$ with respect to $\overline{\mathcal{F}}$ implies that $L_1$ accumulates no point $q \in \mathbb{C} \mathbb{P}(1)_{\infty} - \text{sing}(\mathcal{F})$; otherwise it would give a non-trivial accumulation of $L_1$ in $U$. Thus, once again, we have $\overline{L_1} \subset L_1 \cup \text{sing}(\mathcal{F})$ in $\mathbb{C} \mathbb{P}(2)$. Remmert–Stein Theorem shows therefore that $\overline{L_1} \subset \mathbb{C} \mathbb{P}(2)$ is analytic, and therefore it must be of pure dimension-one. By Chow’s Theorem $\overline{L_1}$ must be an algebraic curve in $\mathbb{C} \mathbb{P}(2)$.

Now we analyze the holonomy group $\text{Hol}(L_1)$. According to [15] given any $R > 0$ the restriction $\mathcal{F}|_{S^3(0; R) \times \{0, 1\}}$ is $C^\infty$-equivalent to the foliation $\mathcal{F}|_{S^3(0; R) \times \{0, 1\}}$. In particular this shows that, as for the germ of $\mathcal{F}$ at 0, the leaves of $\mathcal{F}$ in $\mathbb{C}^2$ accumulate only the separatrices. This shows that $\text{Hol}(L_1)$ has pseudo-orbits accumulating only the origin and, since it contains at least some hyperbolic element (see Remark 5.1 below), $\text{Hol}(L_1)$ must be abelian linearizable (see Lemma 2.1 in Section 2.4 or [4]). The singularities of $\mathcal{F}$ in $\overline{L_1}$ are therefore linearizable and so we may conclude that $\mathcal{F}$ is a logarithmic foliation (cf. Theorem 2.5 in Section 2.4 or [4]).

**Remark 5.1.** Assume by contradiction that $\overline{L_1}$ contains no singularity of hyperbolic type. Then $\text{sing}(\mathcal{F}) \cap \overline{L_1} = \{0\}$ the origin of $\mathbb{C}^2$. Now, clearly $\mathcal{F}$ must have some other singularity $q \in \text{sing}(\mathcal{F}) \cap \mathbb{C} \mathbb{P}(1)_{\infty}$. This singularity is hyperbolic by hypothesis, so that it exhibits two local (transverse) separatrices. At least one of
these is transverse to \( \mathbb{C}P(1)_{\infty} \), and both are not contained in \( \mathbb{C}P(1)_{\infty} \), except for the case \( \mathbb{C}P(1)_{\infty} \) is \( \mathcal{F} \)-invariant.

If \( \mathbb{C}P(1)_{\infty} \) is \( \mathcal{F} \)-invariant then, since we must have \( \mathcal{L}_1 \cap \mathbb{C}P(1)_{\infty} \neq \emptyset \) by Bezout’s Theorem, it follows that \# sing(\( \mathcal{F} \)) \( \cap \mathcal{L}_1 \geq 2 \), contradiction. Thus \( \mathbb{C}P(1)_{\infty} \) is not \( \mathcal{F} \)-invariant and both separatrices above must intersect \( \mathbb{C}^2 \). These separatrices will intersect some sphere \( S^3(0; R) \) (with \( R \) large enough) in two closed orbits of the restriction \( \mathcal{F}|_{S^3(0, R)} \). This is not possible because \( \mathcal{L}_1 \cap S^3(0; R) \) must be one of the closed orbits; contradiction.

Now we conclude. We have already proved that \( \mathcal{F} \) is a logarithmic foliation on \( \mathbb{C}P(2) \). By hypothesis all the singularities of \( \mathcal{F} \) in \( \mathbb{C}P(1)_{\infty} \) are hyperbolic. Therefore, since the origin is of the form \( x \, dy - \lambda y \, dx = 0, \lambda \in \mathbb{C} - \mathbb{R} \), we may proceed as in [1] and conclude that \( \mathcal{F} \) is indeed linear logarithmic.

\textbf{Example 5.1.} Let \( Z = x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y} \) in \( \mathbb{C}^2 \) be a Poincaré-Dulac normal form; straightforward computations using real coordinates \( x = x_1 + i x_2, \; y = y_1 + iy_2 \) show that \( Z = X + Y \) for vector fields \( X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + (x_1 + y_1) \frac{\partial}{\partial y_1} + (x_2 + y_2) \frac{\partial}{\partial y_2} \) and \( Y = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + (x_2 + y_2) \frac{\partial}{\partial y_1} - (x_1 + y_1) \frac{\partial}{\partial y_2} \).

\textbf{Claim 5.3.} Let \( Z \) be as above:

1. \( Z \) is transverse to \( \mathbb{D} \times S^1 \) and to \( S^1 \times \mathbb{D} \), but \( Z \) is not transverse to \( S^1 \times S^1 \).
2. \( Z \) is transverse to the sphere \( S^3(R) = \{(x, y) \in \mathbb{C}^2; |x|^2 + |y|^2 = R \} \) for each \( 0 < R \).

\textit{Proof.} Let us prove analytically item 1 (see also 1. in Remark 5.2 below for a geometrical view). If we choose real coordinates \( x = x_1 + i x_2, \; y = y_1 + iy_2 \) in \( \mathbb{R}^4 \) then we have \( Z = X + i Y \) for vector fields

\[
\begin{align*}
X &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + (x_1 + y_1) \frac{\partial}{\partial y_1} + (x_2 + y_2) \frac{\partial}{\partial y_2}, \\
Y &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + (x_2 + y_2) \frac{\partial}{\partial y_1} - (x_1 + y_1) \frac{\partial}{\partial y_2}.
\end{align*}
\]

Let

\[
A = \begin{bmatrix}
x_1 & x_2 & x_1 + y_1 & x_2 + y_2 \\
x_2 & -x_1 & x_2 + y_1 & -(x_1 + y_1) \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
x_1 & x_2 & x_1 + y_1 & x_2 + y_2 \\
x_2 & -x_1 & x_2 + y_2 & -(x_1 + y_1) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Then rank\( (A) = 2 \) or 4 and rank\( (A) = 2 \) is equivalent to \( x_1 + y_1 = 0 = x_2 + y_2 \), therefore \( (x_1, x_2) = -(y_1, y_2) \). Thus if rank\( (A) = 2 \) and \( y \in S^1 \) then also \( x \in S^1 \).

We also have rank\( (B) = 2 \) or 4 and rank\( (B) = 2 \) is equivalent to \( x_1 = x_2 = 0 \). This proves assertion 1. Now we prove 2. If we have a point \( (x, y) \in S^3(R) \) where \( Z \) is not transverse to the sphere, then we have: \( x \bar{x} + (x + y) \bar{y} = 0, \; |x|^2 + |y|^2 = R^2 \) what is equivalent to \( x \bar{y} + R^2 = 0, \; |x|^2 + |y|^2 = R^2 \). If we write \( x = x_1 + i x_2, \; y = y_1 + iy_2 \) then we obtain the equivalent equations \( x_1 y_1 + x_2 y_2 + R^2 = 0, \; x_2 y_1 - x_1 y_2 = 0, \; x_1^2 + x_2^2 + y_1^2 + y_2^2 = R^2, \; x_1^2 + y_1^2 + 2x_1 y_1 + x_2^2 + y_2^2 + 2x_2 y_2 = R^2 - 2R^2 \) what implies \( (x_1 + y_1)^2 + (x_2 + y_2)^2 = -R^2 \), absurd. \( \Box \)
Remark 5.2. Let us make a few remarks concerning the above example.

1. Z has only one separatrix, given by \((x = 0)\), whose holonomy map is of the form \(h(x) = x + a_2x^2 + \ldots\). In particular \(h\) is an attractor in some “sectors” and is like a source in some “sectors”. This explains geometrically why \(Z\) is not transverse to \(S^3 \times S^1\). This example is a particular case of the general situation where \(Z\) is given by \(Z_n = x \frac{\partial}{\partial x} + (ny + xn) \frac{\partial}{\partial y}, n \in \mathbb{N}^*\). In this last case the holonomy \(h(x) = x + a_{n+1}x^{n+1} + \ldots\) has \(n\) attractive sectors and \(n\) repulsive sectors ([3]).

2. Regardless the global transversality of \(Z\) with the spheres \(S^3(R)\) proved above, \(\mathcal{F}_Z\) is not a linear foliation like in Theorem 1.4 above the reason is the nature of the singularities of \(\mathcal{F}\) on \(CP(1)_{\infty}\). Indeed, simple computations involving the new coordinates \((u, v) = (1/x, y/x)\) and \((r, s) = (x/y, 1/y)\) show that \(\mathcal{F}\) exhibits one unique singularity in \(CP(1)_{\infty}\) (which is invariant), and this singularity is a saddle-node (not hyperbolic therefore).

3. Let now \(R > 0\) be fixed. By Jordan–Brower Theorem the sphere \(S^3(R)\) divides \(CP(2)\) in two connected components say, \(CP(2) - S^3(R) = B^4(R) \cup M\). We have \(\partial M = S^3(R)\) and \(Z\) is transverse to the boundary \(\partial M\) so that \(M\) is a complex surface equipped with a foliation holomorphic of dimension one, transverse to its boundary \(\partial M\) diffeomorphic to the 3-sphere. However \(\mathcal{F}|_M\) exhibits one singularity which is not simple as in [15]. On the other hand \(M\) is not pseudoconvex as it contains the projective line \(CP(1)_{\infty}\).

5.3. Products of spheres. We shall motivate Corollary 1.3.

Example 5.2. Let \(\Omega = (xyzw)(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z} + \rho \frac{\partial}{\partial w})\) be defining a linear logarithmic foliation \(\mathcal{F}\) of hyperbolic type on \(\mathbb{C}^4\). Straightforward calculations show that \(\mathcal{F}\) is transverse to \(S^3(1) \times S^3(1) \subset \mathbb{C}^2 \times \mathbb{C}^2\) outside the singular set \(\text{sing}(\mathcal{F}) \cap (S^3(1) \times S^3(1))\) however \(\mathcal{F}\) is not totally transverse to the product of spheres as it follows from Corollary 1.3 proved below.

Proof of Corollary 1.3. The idea is to prove the existence of certain separatrices that will intersect in a way that the intersection (which is in the singular set of \(\mathcal{F}\)) also intersects \(S^3(1) \times S^3(1)\). Let therefore \(\mathcal{F}\) be given in a neighborhood of \(\Delta^4 \subset \mathbb{C}^4\) and with \(\mathcal{F} \cap S^3(1) \times S^3(1)\). Let us denote by \((p_1, p_2)\) the points in \(\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2\) where \(p_1 \in \mathbb{C}^2_1\) and \(p_2 \in \mathbb{C}^2_2\).

Lemma 5.1. Given any \(p_1 \in S^3(1) \subset \mathbb{C}^2_1\), the restriction \(\mathcal{F}|_{p_1 \times \mathbb{C}^2_2}\) is transverse to \(\{p_1\} \times S^3(1) \subset \{p_1\} \times \mathbb{C}^2_2\).

Proof. Given any \(p_2 \in S^3(1) \subset \mathbb{C}^2_2\) there are natural decompositions \(T_{(p_1, p_2)} \mathbb{C}^4 \simeq T_{p_1} \mathbb{C}^2_1 \times T_{p_2} \mathbb{C}^2_2\) and \(T_{(p_1, p_2)} S^3(1) \times S^3(1) \simeq T_{p_1} S^3(1) \times T_{p_2} S^3(1)\). Therefore by hypothesis we must have \(T_{(p_1, p_2)} \mathcal{F} + T_{p_1} S^3(1) \times T_{p_2} S^3(1) = T_{p_1} \mathbb{C}^2_1 \times T_{p_2} \mathbb{C}^2_2\). Therefore \(\mathcal{F}\) is transverse to \(S^3(1) \times S^3(1)\) in \(\mathbb{C}^2_1 \times \mathbb{C}^2_2\) and therefore \(\mathcal{F}|_{S^3(1) \times \mathbb{C}^2_2}\) is transverse to the fibration of \(S^3(1) \times S^3(1)\) by spheres \(\{p_1\} \times S^3(1)\) with \(p_1 \in S^3(1)\). Thus for each \(p_1 \in S^3(1)\) the restriction \(\mathcal{F}|_{\{p_1\} \times \mathbb{C}^2_2}\) is transverse to the sphere \(\{p_1\} \times S^3(1)\).
Given any \( p_1 \in S^3_1(1) \) the lemma above and [15] imply that \( \mathcal{F}|_{\{p_1\} \times \mathbb{C}^2} \) is transverse to all spheres \( \{p_1\} \times S^2_3(r_2, r_2) \) centered at some (unique) singularity of \( \mathcal{F}|_{\{p_1\} \times \mathbb{C}^2} \), for all \( 0 < r_2 \leq 1 \). Moreover the germ of \( \mathcal{F}|_{\{p_1\} \times \mathbb{C}^2} \) at \( v_2 \) is either like \( x \, dy - \lambda y \, dx = 0, \lambda \in \mathbb{C} - \mathbb{R}_- \), or like \( x \, dy - (ny + x^n) \, dx = 0, n \in \mathbb{N} \). This implies the following:

1. The restriction \( \mathcal{F}_2 := \mathcal{F}|_{S^3_2(1) \times \mathbb{C}^2} \) has as singular set a Kupka component \( K_2(\mathcal{F}_2) \) which is a product \( S^3_1(1) \times K_2 \) and whose transverse type is (fixed) one of the following (see Section 2.3):
   - (i) \( x \, dy - \lambda y \, dx = 0 \) with \( \lambda \in \mathbb{C} - \mathbb{R}_- \).
   - (ii) \( x \, dy - (ny + x^n) \, dx = 0 \) with \( n \in \mathbb{N} \).

2. \( \mathcal{F}|_{S^3_2(1) \times \mathbb{C}^2} \) exhibits one or two closed (analytic) leaves, which contain the local separatix (or separatrices) of the transverse type.

3. The analytic leaves of \( \mathcal{F}|_{S^3_2(1) \times \mathbb{C}^2} \) are transverse to the fibration \( \{p_1\} \times B^3_2(1), p_1 \in S^3_1(1), \) of \( S^3_1(1) \times B^3_2(1) \).

Denote by \( \Lambda_2 \) one of these analytic, leaves; we have \( \Lambda_2 \subset S^3_1(1) \times B^3_2(1) \). By considering smooth embeddings \( \varepsilon : S^3_2(1) \times S^3_1(1) \times (-1, 1) \to \mathbb{C}^4 \) with \( \varepsilon(S^3_2(1) \times S^3_1(1) \times \{t\}) \subset \mathbb{C}^4 \times \mathbb{C}^2 \) in a natural way for any \( t \in (-1, 1) \) and also \( \varepsilon(S^3_1(1) \times S^3_2(1) \times \{0\}) = S^3_1(1) \times S^3_2(1) \) the natural inclusion; we may apply the same argumentation above and construct \( \Lambda_2 \) in a neighborhood \( U_1 \times B^3_2(1) \) of \( S^3_1(1) \times B^3_2(1) \) in \( \mathbb{C}^4 \times \mathbb{C}^2 \).

Since \( \Lambda_2 \) is analytic we may apply Hartogs’ Extension Theorem to extend \( \Lambda_2 \) as an analytic leaf of \( \mathcal{F} \) in \( \Delta^4 \subset \mathbb{C}^4 \). Applying the same argumentation above we obtain an analytic leaf \( \Lambda_1 \) of \( \mathcal{F} \) in \( \Delta^4 \) which comes from a local separatix of a product Kupka component \( K_1(\mathcal{F}_1) = K_1 \times S^3_2(1) \) of the restriction \( \mathcal{F}_1 := \mathcal{F}|_{B^3_2(1) \times S^3_1(1)} \).

This leaf \( \Lambda_1 \) is transverse to the fibration \( \frac{S^3_2(1) \times S^3_1(1)}{B^3_2(1)} \) given by \( B^3_2(1) \times \{p_2\}, p_2 \in S^3_1(1) \). Therefore we must have \( (\Lambda_1 \cap \Lambda_2) \cap (S^3_2(1) \times S^3_1(3)) \neq \emptyset \). However \( \Lambda_1 \cap \Lambda_2 \subset \text{sing}(\mathcal{F}) \) what implies that \( \mathcal{F} \) cannot be transverse to \( S^3_2(1) \times S^3_1(2) \).

**Remark 5.3.** Let \( ((x, y), (z, w)) \in \mathbb{C}^2 \times \mathbb{C}^2 \) be affine coordinates as above and let \( X = x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, Y = z \frac{\partial}{\partial z} + \mu w \frac{\partial}{\partial w} \) be linear vector fields with \( \{\lambda, \mu\} \subset \mathbb{C} - \mathbb{R}_- \).

Then \( X \) and \( Y \) span a holomorphic foliation \( \mathcal{F} \) of complex codimension two on \( \mathbb{C}^4 \) with the property that \( \mathcal{F} \) is transverse to \( S^3_1(1) \times S^3_2(1) \). On the other hand, if \( \mathcal{F}_j \) is a \( C^\infty \) (real) codimension one real foliation on some neighborhood of \( S^3(0; 1) \) on \( \mathbb{R}^4 \) with \( \mathcal{F}_j \pitchfork S^3(0; 1) \) \( (j = 1, 2) \) then the product \( \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \) is a \( C^\infty \) real codimension two foliation in some neighborhood of \( S^3_1(1) \times S^3_2(1) \) on \( \mathbb{C}^4 \) with the property that \( \mathcal{F} \pitchfork S^3_1(1) \times S^3_2(1) \). However \( \mathcal{F} \) is never holomorphic as it follows from Hartogs’ Extension Theorem and from Corollary 1.3 above.

**5.4. Structural stability of holomorphic vector fields.** We recall that a foliation \( \mathcal{F} \) on a manifold \( M \) is structurally stable if there exists a neighborhood \( \eta \) of \( \mathcal{F} \) in the space \( \text{Fol}(M) \), of some dimensional foliations such that any foliation \( \mathcal{F}' \in \eta \) is topologically conjugate to \( \mathcal{F} \). We refer to [10] for the general notion of topology on \( \text{Fol}(M) \); however we will be dealing with the simplest case of foliations induced by vector fields.
Example 5.3. Let \( X = x \frac{\partial}{\partial x} + (ny + x^2) \frac{\partial}{\partial y} + X' \) where \( X' \) is a polynomial vector field whose components are polynomials of degree \( \geq m \) and \( m \gg n \). Then there exists \( R > 0 \) with \( X \mid S^3(0; R) \) but for suitable choice of \( X' \) we have that \( X \) is not transverse to some sphere \( S^3(0; R') \) with \( R' > R \). Let \( \mathcal{F} \) be the foliation defined by \( X \) on \( \mathbb{C}^2 \). Let now \( A \in \text{GL}(2, \mathbb{C}) \) be given and put \( Y := A \cdot X \), that is, \( Y \) is the vector given by \( Y(x, y) = A \cdot X(x, y) \) at each point \((x, y) \in \mathbb{C}^2 \). The linear part of \( Y \) at \((0, 0)\) is \( D Y(0, 0) = A \cdot D X(0, 0) = A \cdot \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \). Using the fact that the subset of hyperbolic matrices is an open dense subset of \( \text{GL}(2, \mathbb{C}) \) we obtain a sequence of matrices \( A_\nu \in \text{GL}(2, \mathbb{C}) \), \( \nu \in \mathbb{N} \) such that:

1. \( A_\nu \xrightarrow{\nu \to \infty} I_2 = \text{identity matrix} \),
2. \( B_\nu = A_\nu \cdot \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \) is hyperbolic for any \( \nu \).

Therefore, if we set \( Y_\nu := (A_\nu)_* X \) as above then the foliation \( \mathcal{F}_\nu \) defined by \( Y_\nu \) on \( \mathbb{C}^2 \) satisfies the following, for every sufficiently large \( \nu \):

1. \( \mathcal{F}_\nu|_{B(0; R')} \) is close to \( \mathcal{F}|_{B(0; R)} \),
2. \( \mathcal{F}_\nu \) is transverse to \( S^3(0; R) \),
3. \( \mathcal{F}_\nu|_{B(0; R)} \) is not topologically conjugated to \( \mathcal{F}|_{B(0; R')} \): indeed, \( \mathcal{F}_\nu \) has a hyperbolic singularity at the origin what does not occur to \( \mathcal{F} \). In few words:

Claim 5.4. \( \mathcal{F} \) is not structurally stable in \( B(0; R) \).

Now we are ready to prove:

Proof of Corollary 1.4. \((\Rightarrow)\) Let \( R' > R \) be such that \( X \) is also transverse to \( S^{2n-1}(p; R') \). There exists some neighborhood \( \eta \) of \( \mathcal{F} \) in the space of foliations in \( B(p; R') \) such that if \( \mathcal{F}' \) is a foliation in \( \eta \) then \( \mathcal{F}' \) is transverse to \( S^{2n-1}(p; R') \).

We may choose \( V = B(p; R') \) so that by the solution of Cousin Problem II any foliation \( \eta \ni \mathcal{F}' \) in \( V \) is given by a holomorphic vector field \( X' \) in \( V \). Now, according to [15] \( \mathcal{F} \) has unique singularity in \( V = B(p; R') \) so that we may assume that \( p \) is this singularity and also that \( p = 0 \) is the origin of \( \mathbb{C}^n \). The same applies to show that \( \mathcal{F}' \) has a unique singularity, say \( v' \) in \( V \). Both singularities are either Poincaré–Dulac normal form type. It is not clear in general that \( v \) and \( v' \) are singularities of the same type (see Example 5.3 above). Now, by hypothesis \( v \in \text{sing}(\mathcal{F}) \) is a hyperbolic singularity so that, since \( \mathcal{F}' \) is close to \( \mathcal{F} \), \( v' \in \text{sing}(\mathcal{F}') \) is also hyperbolic. According to [15] there are real analytic vector fields \( \xi \) (and \( \xi' \)) in \( B(0; R') \) (and \( B(0'; R') \)) such that \( \xi \) (and \( \xi' \)) points inwards \( B(0; R) \) (and \( B(0'; R') \)) and the flow gives a \( C^\infty \)-diffeomorphic equivalence of \( \mathcal{F}|_{B(0; R') \setminus \{0\}} \) (of \( \mathcal{F}'|_{B(0'; R' \setminus \{0\}} \)) with the foliation \( \mathcal{F}|_{S^{2n-1}(0, R) \times [0, 1]} \) (with \( \mathcal{F}'|_{S^{2n-1}(0', R' \times [0, 1]} \)).

Now we may proceed in two distinct ways:

**1st manner.** The restriction \( \mathcal{F}|_{S^{2n-1}(0, R)} \) is a (transverse holomorphic) real analytic flow with \( n \) periodic hyperbolic orbits. Therefore \( \mathcal{F}|_{S^{2n-1}(0, R)} \) is structurally stable [12] so that (since \( \mathcal{F}'|_{S^{2n-1}(0, R)} \) is necessarily close to \( \mathcal{F}|_{S^{2n-1}(0, R)} \)) we have a topological equivalence between \( \mathcal{F}|_{S^{2n-1}(0, R')} \) and \( \mathcal{F}'|_{S^{2n-1}(0, R')} \). This topological equivalence extends (via the flows of \( \xi \) and \( \xi' \)) to a topological equivalence between
2nd manner. Since $o \in \text{sing } F$ and $o' \in \text{sing}(F')$ are close hyperbolic singularities we may construct a topological equivalence $\psi: B(0; \varepsilon) \to B(0'; \varepsilon')$ between the (germs) $F|_{B(0,\varepsilon)}$ and $F'|_{B(0';\varepsilon')}$. Using now the flows of $\xi$ and $\xi'$ we may “extend” this equivalence to a topological equivalence of $F|_{S(0,R')}$ and $F'|_{S(0,R')}$. Finally, trivially we may construct equivalences for $F|_{S(0,R') \times (0,1)}$ and $F'|_{S(0,R') \times (0,1)}$ and therefore for $F|_{B(0,R')-(0)}$ and $F'|_{B(0,R')-(0)}$ so that $F|_{B(0,R')}$ and $F'|_{B(0,R')}$ are topologically equivalent. This proves the if part. The only if part follows from well-known facts and Example 5.3 above. □

5.5. Transverse holomorphic rank of real hypersurfaces. According to J. Milnor [20] the rank of a real closed manifold is the maximum number of pairwise commutative continuously differentiable vector fields, linearly independent at each point, that are supported by the manifold. This is a non-homotopic invariant which may be used to classify the manifold. In [18] Elon Lima showed that the rank of the three sphere $S^3$ is one. E. Lima’s result holds for any compact simply-connected three-manifold. Later on Rosenberg and Roussarie classified compact three-manifolds of rank two. Let us now consider the following situation: $M$ is a complex manifold and $N \subset M$ is a real compact submanifold of codimension one, i.e., $N \subset M_{\mathbb{R}}$ is a compact hypersurface (with empty boundary). We propose the following definition:

**Definition 5.1.** The transverse holomorphic rank of $N$ is the maximum number $k$ of everywhere linearly independent holomorphic vector fields $X_1, \ldots, X_k$ that are defined in some neighborhood $V$ of $N$ in $M$ and such that:

(i) $[X_i, X_j] = 0$ (for any $i, j$) in $V$ and

(ii) $X_i$ is transverse to $N$ for any $i$.

Of immediate proof is that given $M_j$ ($j = 1, 2$) complex manifolds and $N_j \subset M_j$ a compact real hypersurfaces ($j = 1, 2$) with a holomorphic diffeomorphism $\phi: M_1 \to M_2$ with $\phi(N_1) = N_2$ then $N_1$ and $N_2$ exhibit the same holomorphic rank.

**Proof of Theorem 1.4.** We may assume $S^{2n-1}(p; R) = S^{2n-1}(0; 1)$. Let now $X$, $Y$ be two commutative holomorphic vector fields defined in a neighborhood $V$ of $S^{2n-1}(0; 1)$ in $\mathbb{C}^n$, and both transverse to $S^{2n-1}(0; 1)$. According to Hartogs’ Theorem [13] $X$ and $Y$ extend to holomorphic commutative vector fields in a neighborhood $V$ of $B^{2n}(0; 1)$ in $\mathbb{C}^n$. Now, [15] applies to show that $X$ has some singularity say $o_X \in B^{2n}(0; 1)$ and so does $Y$, say, $o_Y \in B^{2n}(0; 1)$. We may assume that $o_X$ is the origin of $\mathbb{C}^n$ for simplicity of notation. Since $[X, Y] = 0$ it follows that also $Y$ has a singularity at $o_X$ and therefore since, $o_Y$ is unique, we have $o_Y = o_X = 0 \in \mathbb{C}^n$. The linear parts $DX(0)$ and $DY(0)$ also commute as a consequence of $[X, Y] = 0$. Therefore these linear mappings have the same eigenvectors in $\mathbb{C}^n$. This implies that we have the same number of separatrices for $X$ and $Y$ at the origin.

**Lemma 5.2.** $X$ and $Y$ have the same separatrices at 0.
Proof. For simplicity we assume $n = 2$. First we suppose that $X$ is linearizable at the origin say, $X(u, v) = u \frac{\partial}{\partial u} + \lambda v \frac{\partial}{\partial v}$ in some local chart $(u, v)$ at $0 \in \mathbb{C}^2$. Since $X$ is transverse to $S^3(0; 1)$ we have $\lambda \notin \mathbb{R}$. Now we write $Y = A(u, v) \frac{\partial}{\partial u} + B(u, v) \frac{\partial}{\partial v}$. Since $[X, Y] = 0$ we obtain

Claim 5.5. Write $A(u, v) = \sum_{i, j \in \mathbb{N}} a_{ij} u^i v^j$, $B(u, v) = \sum_{i, j \in \mathbb{N}} b_{ij} u^i v^j$ in power series, then:

1. If $\lambda \notin \mathbb{R}$ then $Y(u, v) = a_{10} u \frac{\partial}{\partial u} + b_{01} v \frac{\partial}{\partial v}$.

2. Let $\lambda \in \mathbb{R}$. We have the following possible cases:
   
   (i) $\lambda = 1$; $Y(u, v) = (a_{10} u + a_{01} v) \frac{\partial}{\partial u} + (b_{10} u + b_{01} v) \frac{\partial}{\partial v}$,
   
   (ii) $\lambda = \frac{1}{n_0} < n_0 \geq 1$; $Y(u, v) = (a_{10} u + a_{1n_0} v) \frac{\partial}{\partial u} + b_{01} v \frac{\partial}{\partial v}$,
   
   (iii) $\lambda \neq 1$ and $\frac{1}{\lambda} \notin \mathbb{N}$; $Y(u, v) = a_{10} u \frac{\partial}{\partial u} + b_{01} v \frac{\partial}{\partial v}$.

Clearly $X(u, v)$ and $Y(u, v)$ show some common separatrix except, perhaps, in the case

$$\begin{align*}
Y(u, v) &= (a_{10} u + a_{01} v) \frac{\partial}{\partial u} + (b_{10} u + b_{01} v) \frac{\partial}{\partial v}, \\
X(u, v) &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}.
\end{align*}$$

Let us consider this remaining case. The matrix \(\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}\) has some non-zero eigenvector so that we may perform a linear change of coordinates in the variables $(u, v)$ in such a way that in the new variables $Y$ exhibits same separatrix which is a line though the origin. The radial vector field $X$ also exhibits this line as a separatrix. This proves the lemma in this first case. Let us now assume that $X$ is not linearizable at the origin. By symmetry we may also assume that $Y$ is not linearizable at the origin. We must have therefore $X(u, v) = u \frac{\partial}{\partial u} + (nu + u^n) \frac{\partial}{\partial v}$ in some local chart $(u, v)$ at the origin, with $n \in \mathbb{N} - \{0\}$. Write $Y(u, v) = A(u, v) \frac{\partial}{\partial u} + B(u, v) \frac{\partial}{\partial v}$ as before to obtain now $A(u, v) = u A_u(u, v) + (nu + u^n) A_u(u, v)$ and $nu^{n-1} A(u, v) + n B(u, v) = u B_u(u, v) + (nu + u^n) B_u(u, v)$. Using power series $A(u, v) = \sum_{i, j \in \mathbb{N}} a_{ij} u^i v^j$ we obtain $\sum a_{ij} u^i v^j = \sum a_{ij} (i + nj) u^i v^j + u^n \sum a_{ij} u^i v^j$. For $i = 0$ this implies $\sum_{j \in \mathbb{N}} a_{0j} (1 - nj) v^j = \sum_{j \in \mathbb{N}} a_{0j} u^n v^j$. Therefore, $a_{0j} = 0$ for any $j$ and we conclude that $u$ divides $A(u, v)$ in $O(u, v)$. This implies that $\{u = 0\}$ is a common separatrix for $X(u, v)$ and $Y(u, v)$. Lemma 5.2 is now proved.

The local separatrices of $X$ and $Y$ at $0 \in \mathbb{C}^2$ are contained in orbits which intersect $S^3(0; 1)$. We have shown that $X$ and $Y$ are linearly dependent along these separatrices, therefore they are linearly dependent at some point in $S^3(0; 1)$. This shows that $k \leq 1$. Clearly $k \geq 1$ (take $X$ as the radial vector field) so that $S^3(0; 1)$ has transverse holomorphic rank equal to one.

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