ON COMPLETENESS OF DYNAMIC TOPOLOGICAL LOGIC

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ABSTRACT. A classical result on topological semantics of modal logic due to McKinsey and Tarski (often called Tarski theorem) states that the logic $S4$ is complete with respect to interpretations in $\mathbb{R}^n$ for each $n$. Recently several authors have considered dynamic topological logics, which are interpreted in dynamic spaces (abstract dynamic systems). A dynamic space is a topological space together with a continuous function on it. In [4] a bimodal logic $S4C$ was introduced and proven to be complete with respect to the class of all dynamic spaces. A number of polymodal logics for dynamic topological systems were considered in [7], [8], [9]. It was shown by the author in [15] that the analogue of Tarski theorem does not hold for $S4C$; this result has also been established independently from the author by P. Kremer and later by J. van Benthem (private communication). In this paper we show that a certain generalization of Tarski theorem applies in the dynamic case. We prove that for any formula $\phi$ undervivable in $S4C$ there exists a countermodel in $\mathbb{R}^n$ for $n$ sufficiently large. We give also an upper bound on the dimension of a refuting model. It remains an open question whether our upper bound is exact.

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1. Introduction

The language of classical propositional logic consists of propositional symbols and Boolean connectives $\&$, $\to$, $\neg$, $\bot$, $\ldots$ (respectively ‘and’, ‘implies’, ‘not’, ‘falsity’ etc.). Mathematically, classical formulas are interpreted as variables taking values in a Boolean algebra, the most natural example of such algebra being the two-element algebra (true, false). A more general example of a Boolean algebra is the algebra of subsets of some fixed set, in which case logical connectives correspond to set-theoretic operations of intersection, union, complement etc. Classical logic turns out to be the logic of Boolean algebras and, in particular, the logic of algebras of subsets.

In modal logic one enriches the classical language with some new operations corresponding to modalities. There may be numerous motivations for introduction of modalities. Modalities may come from natural language, from abstract philosophical considerations, from applied problems etc. Again, mathematically, modal...
formulas are interpreted in a Boolean algebra, but now this algebra has also some additional structure.

Consider the modal language \( L \Box \), which is the language of classical propositional logic enriched with the unary operator \( \Box \). In topological semantics one interprets \( \Box \) as the interior of a set. Thus formulas are interpreted in the algebra of subsets of a topological space, a topological Boolean algebra \([13]\). The relevant logic in this setting is propositional \( \textbf{S4} \); this system is complete with respect to topological semantics.

Although the topological completeness of \( \textbf{S4} \) has been well-known for a long time, until recently this semantics was considered as some exotic curiosity. The only interesting result in the field beyond the completeness of the system with respect to interpretations in the class of all topological spaces was established by McKinsey and Tarski in 1940-ies. McKinsey and Tarski showed in \([10]\) that \( \textbf{S4} \) is complete with respect to interpretations in the real space \( \mathbb{R}^n \) (even stronger, \( \mathbb{R}^n \) in the above sentence can be replaced with any sufficiently nice space, for example with any dense in itself metric space). However this non-trivial theorem apparently did not lead to further developments and the whole field of topological semantics remained rather marginal.

It was only somewhere in 1990-ies that the work of McKinsey and Tarski drew serious attention of sufficiently many researchers and investigations in this field were renewed. Using modern language of Kripke models Mints proposed a simplification of the proof of McKinsey and Tarski result for the case of the Cantor set (rather than \( \mathbb{R}^n \)). Bezhanishvili and Gehrke considered the case of \( \mathbb{R}^n \) \([5]\); they applied a technique similar to that of \([11]\) to the original proof of McKinsey and Tarski and produced a shorter proof. A different proof of the same result and a different technique was proposed by Aiello, van Benthem and Bezhanishvili \([2]\). These works dealt with a problem, which had already been solved, namely with the problem of completeness of \( \textbf{S4} \) with respect to interpretations in \( \mathbb{R}^n \) (or the Cantor set). A more interesting question, which stimulated the research in the field of topological semantics, consists in the following: what spatial structures may be characterized by means of modal logic? What is the logic of space? How to encode in modal logic different geometric relations? There are many questions of this sort and only very little is known at the moment. For example Shehtman in \([14]\) introduced an extension \( \textbf{S4UC} \) of \( \textbf{S4} \), which turns out to be the the logic of connected metric spaces. Aiello and van Benthem in in their “modal walk through space” considered different topological and geometric structures (connectedness, affine structure, convexity etc) and proposed a number of languages extending \( L \Box \) in order to describe these structures \([1]\). At present time the field of “spatial reasoning” is growing rapidly and the classical work of McKinsey and Tarski seen in this light turns out to be much more significant than mere curiosity.

In this work we investigate the Dynamic Topological Logic \( \textbf{S4C} \). Dynamic Topological Logics correspond to interpretations of the language in dynamic spaces (abstract dynamic systems). A dynamic space is a pair consisting of a topological space \( X \) and a continuous function \( f: X \to X \). Dynamic Topological Logics are formulated in the language \( L_{\Box, a} \), which is the extension of \( L_{\Box} \) with a new modality \( [a] \), “next”. The latter modality \( [a] \) is interpreted in a dynamic space \((X, f)\)
as the inverse image of a set under \( f \). The logic \( \mathbf{S4C} \), which is the logic of all dynamic spaces, was introduced and proven to be complete with respect to the intended semantics by Artemov, Davoren and Nerode in [4] (and we follow notations of this paper). Dynamic Topological Logics were considered also by Kremer, Mints and Rybakov (independently from [4]); in particular these authors axiomatized the logic of homeomorphisms and certain fragments of \( \mathbf{S4C} \) and considered a further extension of the language with a modality for the iterated application of a function [7], [8], [9]. Dynamic Topological Logics were developed further and applied to the study of hybrid control systems ([6]).

But what about completeness of \( \mathbf{S4C} \) with respect to interpretations in \( \mathbb{R}^n \)? One would like to see whether the result of McKinsey and Tarski corresponding to the “static” case of \( \mathbf{S4} \) lifts to the dynamic case. This question seems quite natural, in particular because natural examples of dynamic spaces, namely dynamic systems, come from real topology. It was shown in [9] that Dynamic Topological Logic corresponding to homeomorphisms is indeed complete with respect to the real space. Mints and Zhang showed in [12] that \( \mathbf{S4C} \) is complete with respect to the topology of the Cantor set. However these results do not generalize to the case of \( \mathbf{S4C} \) and \( \mathbb{R} \) in a straightforward way; this was independently shown by several researchers. P. Kremer (private communication), J. van Benthem on the basis of P. Kremer’s work (private communication) and the author [15] provided examples of formulas in the language \( L_{\mathbf{S4}} \) valid in \( \mathbb{R} \) and undervivable in \( \mathbf{S4C} \).

This shows that the bimodal language \( L_{\mathbf{S4}} \) has greater expressive power than the usual \( L_{\mathbf{S4}} \), the last one being unable to distinguish the real line from any other reasonable space. However let us ask a more general question: does the dynamic topological logic distinguish between Euclidean spaces of higher dimensions? The counterexamples which have been proposed are based on one-dimensionality of \( \mathbb{R} \) and already do not work for \( \mathbb{R}^2 \).

In this paper we attempt to answer this more general question. In particular we show that the following version of McKinsey and Tarski theorem holds for the dynamic topological logic: if a formula \( \phi \) is not derivable in \( \mathbf{S4C} \) then there exists a countermodel for \( \phi \) in a Euclidean space of sufficiently high dimension. In fact we show that the dimension of a refuting model can be made equal to one plus the depth of occurrences of the “next” modality in \( \phi \), the one-dimensional space corresponds then to the dynamic-free fragment \( \mathbf{S4} \). It remains an open question whether there exists a better upper bound.

2. Topological Semantics

2.1. Topological semantics for the modal logic

**Definition 1.** Language \( L_{\mathbf{S4}} \) consists of a countable set of propositional symbols, a constant \( \bot \), binary connective \( \to \), and unary operator \( \Box \).

Other Boolean connectives are definable in the usual way.

Let \( X \) be a topological space.

**Definition 2.** A (static) topological model \( \langle X, \parallel \parallel \rangle \) in a topological space \( X \) consists of a valuation \( \parallel \parallel \) which assigns to any formula \( p \in L_{\mathbf{S4}} \) a subset \( \parallel p \parallel \subseteq X \) and
satisfies the following conditions:

\[ \| \perp \| = \emptyset; \quad \| A \to B \| = -\| A \| \cup \| B \|; \quad \| \Box A \| = \text{Int} \| A \|. \]

We will also use the notation \( x \models p \) for \( x \in \| p \| \).

We say that a formula \( \phi \in L_\Box \) is true in a model if \( \| \phi \| = X \). We say that \( \phi \) is valid in \( X \) if it is satisfied in any topological model in \( X \). At last we call a formula topologically valid if it is valid in any topological space.

The axiomatic system corresponding to this semantics is propositional \( \textbf{S4} \).

**Definition 3.** Logic \( \textbf{S4} \) contains following schemes:

- **CP:** axioms of classical propositional logic in \( L_\Box \),
- **\( \Box T \):** \( \Box \phi \to \phi \),
- **\( \Box K \):** \( \Box(\phi \to \psi) \to (\Box \phi \to \Box \psi) \),
- **\( \Box 4 \):** \( \Box \phi \to \Box \Box \phi \)

and inference rules:

- **modus ponens:** \( \phi, \phi \to \psi \models \psi \),
- **\( \Box \)-necessitation:** \( \phi \models \Box \phi \).

Connections between this system and topological semantics are summarized in the following well-known completeness theorem (see [10]):

**Theorem 1.** For any formula \( \phi \in L_\Box \) the following are equivalent:

1. \( \textbf{S4} \vdash \phi \);
2. \( \phi \) is topologically valid;
3. \( \phi \) is true in any finite topological space.

One may argue that finite topological spaces are rather exotic and wonder if this completeness result may be extended to some habitual spaces such as \( \mathbb{R}^n \). Indeed, a result of McKinsey and Tarski (often called Tarski theorem) gives a positive answer to this question.

**Theorem 2** (Tarski theorem, [10]). Let \( X \) be a metric space that is dense in itself. If a formula \( \phi \in L_\Box \) is not derivable in \( \textbf{S4} \) then there exists a topological model in \( X \) refuting \( \phi \).

Note that the real space is dense in itself and it follows that \( \textbf{S4} \) is complete over \( \mathbb{R}^n \) for all \( n \).

### 2.2. Topological Dynamic Logic \( \textbf{S4C} \)

In this section we recall the logic \( \textbf{S4C} \), an extension of \( \textbf{S4} \) introduced and proven to be complete by Artemov, Davoren, and Nerode in [4]. We follow the notation of [4].

The language \( L_{\Box_\cdot a} \) is the language of \( \textbf{S4} \) enriched with the modal operator \( [a] \). Let \( X \) be a topological space. Assume that \( f : X \to X \) is a continuous function. The pair \( \langle X, f \rangle \) is called a dynamic space (over the space \( X \)).

**Definition 4.** A dynamic topological model on \( X \) consists of a dynamic space \( \langle X, f \rangle \) over \( X \) and a valuation \( \| \| \) which assigns to any formula \( p \in L_{\Box_\cdot a} \) a subset
∥p∥ ⊆ X and satisfies all conditions of Definition 2 in addition to the following condition:

∥[a]A∥ = f^−1(∥A∥).

All terms valid, true, etc. are defined in the same way as for S4.

We will often omit the adjectives “static” and “dynamic” when speaking about topological models if this does not lead to a confusion.

**Definition 5.** Logic S4C contains axioms and rules of S4 in \(L\Box\), a and the following schemes:

- \([a]K\): \([a](\phi \rightarrow \psi) \rightarrow ([a]\phi \rightarrow [a]\psi)\),
- \([a]¬\): \([a]¬\phi \leftrightarrow ¬[a]\phi\),
- \(\text{Cont}: [a]\Box\phi \leftrightarrow \Box[a]\Box\phi\)

and the inference rule

\([a]\)-necessitation: \[\frac{\phi}{[a]\phi}\].

The axiom \(\text{Cont}\) expresses continuity; it says precisely that the inverse image of an open set is open.

One can easily show the following:

**Note 1.** In S4C the connective \([a]\) commutes with all Boolean connectives.

As in S4, we have the following theorem:

**Theorem 3** [4]. For any formula \(\phi \in L\Box\) the following are equivalent:

(i) S4C ⊨ \(\phi\);
(ii) \(\phi\) is topologically valid;
(iii) \(\phi\) is true in any finite topological space.

### 2.3. Functorial behavior of topological models.

In this section we will see how topological models behave under maps between topological spaces. Results of this section concerning interpretation of the language \(L\Box\) (i.e. the static topological semantics) are well known, see [13].

At first let us define a useful notion of the \([a]\)-degree of a formula.

**Definition 6.** For a formula \(A \in L\Box\) the \([a]\)-degree of \(A\) is defined inductively:

\[
\begin{align*}
\deg_a(\bot) &= 0; \\
\deg_a(p) &= 0, \text{ where } p \text{ is a propositional symbol}; \\
\deg_a(p \rightarrow q) &= \max(\deg_a(p), \deg_a(q)); \\
\deg_a(\Box p) &= \deg_a(p); \\
\deg_a([a]p) &= \deg_a(p) + 1.
\end{align*}
\]

Our first observation is that topological models are “local” in the following sense:

**Lemma 1.** Let \(X\) be a topological space and let \(U \subseteq X\) be open. Assume that there is a static topological model \(\|\|\) in \(U\) and let us define a valuation \(\|\|’\) in \(X\) by saying that \(x \in \|p\|’\) if and only if \(x \in \|p\|\), where \(p\) is a propositional symbol, and by extending \(\|\|’\) inductively to all formulas. Then the two valuations coincide on \(U\): for any \(x \in U\) and any formula \(A \in L\Box\) it holds that \(x \in \|A\|\) if and only if \(x \in \|A\|’\).
Proof. Since the truth-value of an \([a]\)-free formula at a given point depends only on the truth-values of its subformulas in an open neighborhood of the point the statement is obvious.

In order to formulate the dynamic version of the previous theorem let us make the following definition.

**Definition 7.** A stratified dynamic space is a dynamic space \(\langle U, F \rangle\) together with a finite collection \(U_1, \ldots, U_n\) of open subsets of \(U\) such that \(U = \bigcup U_k\) and \(F(U_k) \subseteq U_{k+1}\), \(k = 1, \ldots, n - 1\). The sets \(U_1, \ldots, U_n\) form a stratification of \(U\). Our notation for the stratified dynamic space will be \(\langle U, U_1, \ldots, U_n; F \rangle\).

(The term “stratified” was suggested by the anonymous referee.)

**Lemma 2.** Let \(\langle X, F \rangle\) be a dynamic space and \(U_1, \ldots, U_n \subseteq X, U = \bigcup U_k\) be open sets. Assume that \(F\) maps \(U\) to \(U\) and that the system \(\langle U_1, U_1, \ldots U_n; F \rangle\) becomes a stratified dynamic space. Assume that there is a dynamic topological model \(\| \|\) in \(U\) and let us define a valuation \(\| \|', \| p \|'\) in \(X\) by saying that \(x \in \| p \|'\) if and only if \(x \in \| p \|\), where \(p\) is a propositional symbol, and by extending \(\| \|', \| p \|'\) inductively to all formulas. Then the two valuations are related: for any \(x \in U_k\) and any formula \(A \in L_{\square, a}\) whose \([a]\)-degree is less or equal than \(n-k\) holds: \(x \in \| A \|\) if and only if \(x \in \| A \|', k = 1, \ldots, n\).

**Proof.** Use induction on the \([a]\)-degree of \(A\).

Our next observation is that interior maps between topological spaces induce “homomorphisms” between topological models.

**Definition 8.** Let \(X, Y\) be topological spaces. A continuous map \(f: X \to Y\) is interior if the image under \(f\) of any open subset of \(X\) is open in \(Y\).

**Lemma 3.** Let \(X, Y\) be topological spaces and \(f: X \to Y\) be an interior map. Assume that there is a static topological model \(\| \|_Y\) in \(Y\). Let us define a valuation \(\| \|_X, \| p \|_X\) in \(X\) by saying that \(x \in \| p \|_X\) if and only if \(f(x) \in \| p \|\), where \(p\) is a propositional symbol, and by extending \(\| \|_X\) inductively to all formulas. Then the two valuations are related: for any \(x \in X\) and any formula \(A \in L_{\square, a}\), \(x \in \| A \|_X\) if and only if \(x \in \| A \|_Y\).

**Proof by induction on \(A\).** The only interesting case is when \(A = \square A'\). So let us consider this case.

Let \(x \in X\). Assume that \(x \in \| A \|_X\). Then for some open neighborhood \(U\) of \(x\) holds \(U \subseteq \| A' \|_X\). By openness of \(f\) the set \(f(U)\) is open and by induction hypothesis \(f(U) \subseteq \| A' \|_Y\). But then \(f(x) \in \text{Int} \| A' \|_Y\) and consequently \(f(x) \in \| A \|_Y\).

Assume that \(f(x) \in \| A \|_Y\). Then for some open neighborhood \(U\) of \(f(x)\) holds \(U \subseteq \| A' \|_Y\). By continuity of \(f\) the set \(f^{-1}(U)\) is open and by induction hypothesis \(f^{-1}(U) \subseteq \| A' \|_X\). But then \(x \in \| A \|_X\).

A dynamic version of the previous lemma is as follows.

Given two stratified dynamic spaces \(\langle U, U_1, \ldots, U_n, F \rangle\) and \(\langle V, V_1, \ldots, V_n; G \rangle\), let us say that the interior map \(f: U \to V\) is a morphism between stratifications if \(f(U_k) \subseteq V_k\), \(k = 1, \ldots, n\) and \(G \circ f = f \circ F\).
Lemma 4. Let \( \langle X, F \rangle, \langle Y, G \rangle \) be dynamic spaces. Let \( U_1, \ldots, U_n \subseteq X, U = \bigcup U_k, V_1, \ldots, V_n \subseteq Y, V = \bigcup V_k \) be open sets such that \( F \) and, respectively, \( G \) map \( U \) to \( U \) and, respectively, \( V \) to \( V \). Assume that \( \langle U, U_1, \ldots, U_n; F \rangle \) and \( \langle V, V_1, \ldots, V_n; G \rangle \) become stratified dynamic spaces. Let \( f : X \to Y \) be an interior map such that the restriction of \( f \) to \( U \) induces a morphism between stratifications \( U \) and \( V \). Assume that there is a dynamic topological model \( \| \|_Y \) in \( Y \) and let us define a valuation \( \| \|_X \) in \( X \) by saying that \( x \in \| p \|_X \) if and only if \( f(x) \in \| p \|_Y \), where \( p \) is a propositional symbol and by extending \( \| \|_X \) inductively to all formulas. Then the two valuations are related: for any \( x \in U_k \) and any formula \( A \in L_{\square,a} \) whose \( [a] \)-degree is less or equal than \( n - k \) holds: \( x \in \| A \|_X \) if and only if \( f(x) \in \| A \|_Y \), \( k = 1, \ldots, n \).

Proof. Use induction on the \( [a] \)-degree of \( A \). \( \square \)

Note that Lemmas 1 and 2 are particular cases of respectively Lemma 3 and Lemma 4. The role of interior maps inducing homemorphisms of models is played by open inclusion maps.

In the next section we recall the Kripke semantics of dynamic topological logic.

2.4. Kripke semantics. A standard semantic tool for analysis of decent modal logic are Kripke models, which are based on sets of “possible worlds” and accessibility relations between worlds. Kripke semantics exists for the languages \( L_{\square} \) and \( L_{\square,a} \) as well. In fact Kripke models correspond in this setting to topological models of a special kind.

Definition 9. A static Kripke frame is a pair \( \langle W, R \rangle \) where \( W \) is a nonempty set of “worlds” and \( R \subseteq W \times W \) is a reflexive transitive relation. A dynamic Kripke frame is a triple \( \langle W, R, F \rangle \) where \( W \) and \( R \) are as above and \( F \) is a function \( F : W \to W \), which preserves the accessibility relation \( R \):

\[ w_1 R w_2 \quad \text{implies} \quad F(w_1) R F(w_2). \]

The “dynamic” and “static” versions of Kripke semantics correspond respectively to dynamic topological and purely topological logic. We will often omit these adjectives if this does not lead to a confusion.

Definition 10. A Kripke model in a frame \( \langle W, R, F \rangle \) (\( \langle W, R \rangle \)) is a valuation \( \| \| \) which assigns to any formula \( p \in L_{\square,a} (L_{\square}) \) a set \( \| p \| \subseteq W \). We often write \( w \models p \) (\( p \) is true in the world \( w \)) for \( w \in \| p \| \). The valuation should satisfy the following conditions:

\[ w \models \bot; \]
\[ w \models A \to B \quad \text{if and only if} \quad w \models B \text{ or } w \not\models B; \]
\[ w \models \square A \quad \text{if and only if} \quad \forall w' : w R w' \text{ implies } w' \models A; \]
\[ w \models [a] A \quad \text{if and only if} \quad F(w) \models A. \]

We say that a formula is true or satisfied in a model if it is true in any world of this model, and we say that a formula is valid in a frame if it is true in any Kripke model in this frame. Finally we say that a formula is Kripke valid if it is valid in any Kripke frame.
A transition from Kripke semantics to topological semantics goes as follows. Let us introduce a topology on $W$. The basis of open sets is formed by the sets $O(w) = \{w' : Rw'\}$ (cones), where $w$ ranges over elements of $W$. This is called the cone topology. One can readily check that the functions continuous in the cone topology are precisely the monotone functions i.e. those, which preserve accessibility relation.

The opposite transition exists as well. Note that the cone topology has the following property: the intersection of an arbitrary collection of open sets is open. Topological spaces enjoying this property are called Alexandroff spaces. In particular all finite topological spaces belong to this class. Now given an Alexandroff space $X$ define the relation $R$ on $X$: $xRx'$ if and only if $x$ lies in the closure of $\{x'\}$.

It is easy to verify that the two operations are inverses of each other. One can see further that valuations in the sense of topological semantics are taken to valuations in the sense of Kripke semantics and vice versa. Therefore when considering Kripke frames below we shall often simultaneously use the topological and the relational terminology. In particular since all finite topological spaces are Alexandroff spaces Theorem 3 implies the following.

**Theorem 4** [4]. For any formula $\phi \in L_{\Box, a}(L_{\Box})$ the following are equivalent:

(i) $S4C(S4) \vdash \phi$;
(ii) $\phi$ is Kripke valid;
(iii) $\phi$ is true in any finite Kripke model.

Let us say that the Kripke frame $(W, R)$ is a cone if there exists an element $w_{\text{root}}$ of $W$ such that $W = O(w_{\text{root}})$. Lemma 1 together with Theorem 4 above imply:

**Lemma 5.** If a formula $\phi \in L_{\Box}$ is not derivable in $S4$ then there exists a Kripke model in a finite cone $W = O(w_{\text{root}})$ such that $\phi$ is refuted at $w_{\text{root}}$.

We will formulate also a specific dynamic version of the above lemma.

**Lemma 6.** If a formula $\phi \in L_{\Box, a}$ is not derivable in $S4C$ then there exists a finite dynamic Kripke model $(W, R, F, || ||)$ such that $W$ is the disjoint union of finite cones $W^1, \ldots , W^n$, where $W^k = O(w_{\text{root}}^k)$, $k = 1, \ldots , n$ and $n = 1 + \deg_{\Box}(\phi)$, and $\phi$ is refuted at $w_{\text{root}}^1$. Moreover the system $(W, W^1, \ldots , W^n; F)$ is a stratified dynamic space and for all $k < n - 1$ it holds that $w_{\text{root}}^{k+1} = F(w_{\text{root}}^k)$ and $F$ maps $W^k$ to $W^{k+1}$ injectively.

**Proof.** By Theorem 4 there exists some finite Kripke model $(V, S, G, || ||_V)$ which refutes $\phi$ at some world $v$. Let $w_{\text{root}}^1 = v$. Putting $w_{\text{root}}^{k+1} = F(w_{\text{root}}^k)$, $\tilde{W}^k = O(w_{\text{root}}^k)$, $\tilde{W} = \bigsqcup_k \tilde{W}^k$, $\tilde{F} = G|_{\tilde{W}}$, $\tilde{R} = S|_{\tilde{W}}$ and restricting the original valuation to $\tilde{W}$ we obtain by Lemma 2 a new Kripke model, which satisfies all properties in the statement of the theorem except maybe that $\tilde{F}$ be injective and different cones be disjoint. In order to satisfy the last conditions we put $W^k = \tilde{W}^1 \times \ldots \times \tilde{W}^k$, $w_{\text{root}}^k = (\tilde{w}_{\text{root}}^1, \ldots , \tilde{w}_{\text{root}}^k)$. The relation $R$ is defined in a natural way: $(w_1, \ldots , w_k)R(w'_1, \ldots , w'_k)$ if and only if $w_iRw'_i$ for all $i = 1, \ldots , k$. Finally we put $F(w_1, \ldots , w_k) = (w_1, \ldots , w_k, \tilde{F}(w_k))$. Clearly $F$ is injective and continuous (i.e. $R$-monotone) since $\tilde{F}$ is continuous.
The valuation || is defined by saying that \((w_1, \ldots, w_k) \models p\) if and only if \(w_k \models p\) in \(W\). The natural projection \(W \rightarrow \tilde{W}, (w_1, \ldots, w_k) \mapsto w_k\) is an interior map so by Lemma 4 the new model still refutes \(\phi\). □

We shall call the Kripke frame \(\langle W, R, F \rangle = \langle W, W^1, \ldots, W^n; F \rangle\) of the above form a stratified dynamic Kripke frame.

In the subsequent sections we are going to construct certain continuous maps from \(\mathbb{R}^n\) to finite Kripke frames (= finite topological spaces). For that purpose it will be convenient to introduce some more structure on Kripke frames, which is not automatically determined by their topology, namely the structure of convergence. (This trick was introduced essentially by Mints in [11] although he did not use such terminology.) Given a finite Kripke frame \((W, R)\) the role of convergent sequences is played by \(R\)-monotone sequences of elements of \(W\). Since \(W\) is finite any sequence \(\{w_i\}\), where \(w_i Rw_{i+1}\), necessarily has a “limit” \(w\), which satisfies \(w_i Rw\) for all \(i\). However such a “limit” (i.e., an upper bound) is not unique in general. Therefore we have to introduce limits “by hands”.

**Definition 11.** Given a finite Kripke frame \(W = \langle W, R \rangle\) we say that limits are chosen in \(W\) if for any \(R\)-monotone sequence \(\{w_i\}\) a particular element \(w \in \{w_i\}\) such that \(w_i Rw\) for all \(i\) is fixed. We write then \(w = \lim_{i \to \infty} w_i\).

It is clear that limits can be chosen for any finite Kripke frame. In the sequel we will always assume that our Kripke frames have limits. Moreover when considering the dynamic case we would like the function \(F\) to be in the natural sense continuous with respect to the limits.

**Note 2.** If a dynamic Kripke frame \(W = \langle W, R, F \rangle\) is stratified then limits in \(W\) can be chosen to satisfy \(\\lim_{i \to \infty} F(w_i) = F(\\lim_{i \to \infty} w_i)\).

**Proof.** The frame \(W\) is the disjoint union of cones \(W^1, \ldots, W^n\). Let us choose limits in \(W^1\) in an arbitrary way. Now assume that limits in \(W^k\) have been chosen and \(k < n\). For each monotone sequence \(\{w_i\}\) in \(W^k\) the sequence \(\{F(w_i)\}\) is monotone as well since \(F\) is continuous with respect to \(R\). We put by definition \(\lim_{i \to \infty} F(w_i) = F(\\lim_{i \to \infty} w_i)\). The definition is unambiguous since on \(W^k\) the function \(F\) is injective. For all other monotone sequences in \(W^{k+1}\) we choose limits in an arbitrary way. □

In the next section we recall the classical McKinsey and Tarski theorem.

### 3. A Proof of Tarski Theorem

A theorem of McKinsey and Tarski, often called Tarski theorem, states that the logic \(S4\) is complete over with respect to interpretations in any dense in itself metric space. This important result was proven first with the use of isomorphic embeddings of finite topological Boolean algebras in metric spaces (see [10], [13]). In 1990-ies Mints considered a special case of the Cantor space \(2^\omega\) and simplified the original completeness proof using a modern language of Kripke models [11]. Later Bezhanishvili and Gehrke applied a similar technique to the case of \(\mathbb{R}\) [5]. We give a proof of Tarski theorem for \(\mathbb{R}^n\), which is another variation of McKinsey &
Tarski’s construction in the same style as [11] and [5] (in fact our proof owes much to both these works). We note also that our version of the proof applies verbatim for any metric space, which is dense in itself.

Assume that \( \phi \in L_\subset \) is not derivable in \( S4 \). By Theorem 4 there exists a Kripke model \( \langle W, R, || || \rangle \), which refutes \( \phi \). Moreover by Lemma 5 we may assume that \( W \) is a cone \( O(w_{\text{root}}) \) and that \( \phi \) is refuted at the world \( w_{\text{root}} \). In view of Lemmas 3 and 1 it is sufficient to construct an interior map from an open subset of \( \mathbb{R}^n \) to \( W \) having \( w_{\text{root}} \) in its range in order to obtain a countermodel for \( \phi \) in \( \mathbb{R}^n \).

We are going to define an interior map \( \gamma \) from the unit cube \( I = [0, 1]^n \) to \( W \). The restriction of \( \gamma \) to the interior of \( I \) will solve our problem.

**Definition 12.** Given a subspace \( S \subseteq \mathbb{R}^n \) and \( \epsilon > 0 \) a subset \( \Omega \) of \( S \) is an \( \epsilon \)-net for \( S \) if for each \( x \in S \) there exists a point \( y \in \Omega \) such that \( d(x, y) \leq \epsilon \). (Here \( d(x, y) \) denotes the standard Euclidean distance.)

We make the following obvious note.

**Note 3.** If the set \( S \) in the definition above is compact then for any \( \epsilon > 0 \) there exists a finite \( \epsilon \)-net for \( S \).

Our main tool will be the following construction. We assume that we are given a point \( w_0 \in W \). Let \( w_0, w_1, \ldots, w_{r-1} \) be all elements of \( O(w_0) \) (i.e. those, which satisfy \( w_0Rw_\alpha \)). We want to define a map \( \tilde{\gamma}: I \rightarrow O(w_0) \). We define \( \tilde{\gamma} \) in a countable number of steps. At every step we choose a collection of open cubes in the ambient space and map these cubes to \( \{w_0, \ldots, w_{r-1}\} \).

**Basic Construction.** At the first step we choose \( r \) closed cubes \( A_0, \ldots, A_{r-1} \) lying in \( I \). We require also that the cubes be pairwise disjoint and be disjoint from the boundary of \( I \). We say that the cubes \( A_0, \ldots, A_{r-1} \) are terminal and we define \( \tilde{\gamma} \) on the interiors of terminal cubes. For each \( x \in \text{Int}(A_\alpha) \) we put \( \tilde{\gamma}(x) = w_\alpha \). Abusing the notation we will write \( \tilde{\gamma}(A_\alpha) = w_\alpha \). Finally we put \( I_1 = I - \bigcup_\alpha \text{Int}(A_\alpha) \).

Note that \( I_1 \) is closed and bounded hence compact.

In general at the \((i+1)\)th step we assume that we are given a compact set \( I_i \), which is homeomorphic to the closed cube with a finite number of open cubes removed from the interior. We are also given a finite list of terminal cubes whose interiors are disjoint from \( I_i \). (The union of \( I_i \) with all terminal cubes gives the original cube \( I \).) We choose for \( I_i \) a finite \( \frac{1}{2^{i+1}} \)-net \( \Omega = \{y_1, y_2, \ldots\} \). For each point \( y_\alpha \in \Omega \) we choose \( r \) closed cubes \( A_{0,\alpha}, \ldots, A_{r-1,\alpha} \) in the \( \frac{1}{2^{i+1}} \)-neighborhood of \( y_\alpha \). We add these cubes to the list of previously defined terminal cubes. We require also that the new terminal cubes be pairwise disjoint and be disjoint from the boundary of \( I_i \) (there is only a finite number of new terminal cubes so this condition can be satisfied). We define \( \tilde{\gamma} \) on the interiors of the new terminal cubes in the same way as at the first step: for \( x \in \text{Int}(A_{\beta,\alpha}) \) we put \( \tilde{\gamma}(x) = w_{\beta} \). Again for latter convenience we write simply \( \tilde{\gamma}(A_{\beta,\alpha}) = w_{\beta} \). We put

\[
I_{i+1} = I - \bigcup_{\beta,\alpha} \text{Int}(A_{\beta,\alpha}).
\]

This procedure gives us \( \tilde{\gamma} \) defined on an open dense subset of \( I \) formed by the union of interiors of all terminal cubes defined at some step. If a point \( x \) does not
belong to the interior of any terminal cube then $x$ is called an **exceptional point** and we put $\hat{\gamma}(x) = w_0$.

The following property of the Basic Construction will be crucial.

**Lemma 7.** In the notations as above let $x \in I$ be an exceptional point. Then $\hat{\gamma}(x) = w_0$ and for any $\epsilon > 0$ and $k < r$ there exists a terminal cube $T$ in the $\epsilon$-neighborhood of $x$, such that $\hat{\gamma}(T) = w_k$.

**Proof.** The point $x$ does not belong to the interior of any terminal cube. It follows from (1) that $x \in I_i$ for all $i$. So if $i$ is such that $\frac{1}{2^i} < \epsilon$ then there exists a point $y_0$ in the $\frac{1}{2^{i+1}}$-net $\Omega$ for $I_i$ such that $x$ lies in the $\frac{1}{2^i}$-neighborhood of $y_0$.

The statement follows then from the definition of the Basic Construction. Taking $T = A_{\alpha,k}$ we see that indeed $T$ lies in the $\frac{1}{2^i}$-neighborhood of $x$ and $\hat{\gamma}(T) = w_k$. □

Having defined the auxillary map $\hat{\gamma}$ we want to define an interior map $\gamma : I \rightarrow W$. Thus is done again in a countable number of steps by iteration of the Basic Construction. At the $i$th step we define the $i$th approximation $\gamma_i$. The desired function $\gamma$ will be the limit (in an appropriate sense) of the sequence of approximations.

The function $\gamma_i$ is defined by means of the Basic Construction taking $w_0 = w_{\text{root}}$. The **terminal cubes of $\gamma_i$** and the **exceptional points of $\gamma_i$** are respectively the terminal cubes and the exceptional points of the Basic Construction.

Assume that $\gamma_i$ has been defined and that for each terminal cube $T$ of $\gamma_i$ the interior of $T$ is mapped by $\gamma_i$ to a single element of $W$, which will be denoted by $\gamma_i(T)$. We apply the Basic Construction to the interior of each terminal cube $T$ of $\gamma_i$, where we put $w_0 = \gamma_i(T)$. This gives us $\gamma_{i+1}$ on the interiors of all terminal cubes of $\gamma_i$. We let $\gamma_{i+1}$ coincide with $\gamma_i$ on the rest of $I$. The terminal cubes of $\gamma_{i+1}$ are the terminal cubes of the Basic Construction applied to each of the terminal cubes of $\gamma_i$. The points which do not belong to the interior of any terminal cube of $\gamma_{i+1}$ are the exceptional points of $\gamma_{i+1}$.

Finally we define the desired map $\gamma$. Note that for each $x \in I$ holds

$$\gamma_i(x) R_{\gamma_{i+1}}(x). \quad (2)$$

Indeed if $\gamma_{i+1}(x) \neq \gamma(x)$ then $x$ belongs to the interior of some terminal cube $T$ of $\gamma_i$. Let $w = \gamma_i(x) = \gamma_i(T)$. The function $\gamma_{i+1}$ is defined on the interior of $T$ by means of the Basic Construction with $w_0 = w$. Hence $\gamma_{i+1}(x) \in O(w)$. Therefore for each $x \in I$ the sequence $\{\gamma_i(x)\}$ is $R$-monotone.

We define $\gamma(x) = \lim_{i \rightarrow \infty} \gamma_i(x)$.

**Lemma 8.** For any $x \in \text{Int}(I)$ the image of a sufficiently small $\epsilon$-neighborhood of $x$ is open.

**Proof.** At first note that if for some $i$ the point $x$ is exceptional for $\gamma_{i+1}$ then $\gamma_{i+1}(x) = \gamma_{i+1}(p) = \gamma(x)$ for all $p > 0$. Assume at first that this is the case and $x$ is exceptional for $\gamma_{i+1}$.

Let $T_i$ be that terminal cube of $\gamma_i$ whose interior contains $x$. (If $i = 0$ then say that $T_0 = I$.) Then $\gamma_i(T_i) = w$ for some $w \in W$ and $\gamma(x) = \gamma_{i+1}(x) = w$ as well. In virtue of (2) for all $x' \in \text{Int}(T)$ we have $w R \gamma(x')$. Thus $\gamma(\text{Int}(T)) \subseteq O(w)$.

Let $w' \in O(w)$. Let $\epsilon > 0$ be sufficiently small, so that the $\epsilon$-neighborhood $U$ of $x$ lies in $T$. By Lemma 7 there exists a terminal cube $T'$ lying in $U$ such that
\( \gamma_{i+1}(T') = w' \). Then for any point \( x' \) in \( T' \), which is exceptional for \( \gamma_{i+1} \), we have that \( \gamma(x') = \gamma_{i+1}(x) = w' \). Since such exceptional points exist it follows that \( O(w) \subseteq \gamma(U) \). Thus \( \gamma(U) = O(w) \).

Assume now that \( x \) is not exceptional for any of \( \gamma_1, \gamma_2, \ldots \). That is, for each \( i = 1, 2, \ldots \) the point \( x \) lies in the interior of some cube \( T_i \), which is a terminal cube of \( \gamma_i \). The system \( T_1, T_2, \ldots \) forms a basis of open neighborhoods of \( x \) so we may replace \( \epsilon \)-neighborhoods with these cubes. Each of \( \gamma_1, \gamma_2, \ldots \) when restricted to \( T_1, T_2, \ldots \) respectively has a constant value, say \( w_i \). The sequence \( \{w_i\} \) is monotone in virtue of (2) and for each \( i \) it holds that \( w_i R \gamma(x) \). Since \( W \) is finite it follows that starting from some \( i_0 \) we have \( w_i R \gamma(x) \). Then the image \( \gamma(T_{i_0}) \) lies in \( O(w_i) \), again in virtue of (2). The fact that actually \( \gamma(T_{i_0}) = O(w_i) \) is proven analogously to the case of an exceptional point. □

**Lemma 9.** The map \( \gamma : I \to W \) is continuous.

**Proof.** We need to show that for any \( w \in W \) the counter-image \( \gamma^{-1}(O(w)) \) of the cone \( O(w) \) is open in \( I \), i.e. if \( \gamma(x) = w \) then for some open neighborhood \( U \) of \( x \) the image \( \gamma(U) \) lies in \( O(w) \). We have shown this both for the case of \( x \) exceptional and not exceptional in the course of the proof of the previous lemma. □

Thus we have constructed the interior map \( \gamma : \text{Int}(I) \to W \) (this map in fact is defined on the boundary of \( I \) as well). It follows from remarks in the beginning of the section that we have proved Tarski theorem for Euclidean space.

In the next section we will see that Tarski theorem does not extend to the dynamic case in a straightforward way.

4. **Dynamic Topological Logic is not Complete with Respect to the Real Line: a Counterexample**

We give a simple example of a formula not derivable in \( S4C \) but valid in \( \mathbb{R} \). Recall that \( \Box \phi \) is a short for \( \neg \Box \neg \phi \). As one can easily see in the topological interpretation \( \Box \) means “the closure”.

Let the formula \( \phi \) be defined by

\[
\phi := \Box p \land \Box \neg p. \tag{3}
\]

**Note 4.** In any topological model the above formula \( \phi \) defines the boundary of the open set \( \|\Box \phi\| \).

**Note 5.** Let the nonempty set \( S \subset \mathbb{R} \) be the boundary of an open set \( U \) such that \( S \) is connected. Then \( S \) is a singleton.

**Proof.** Immediate, since any open set in \( \mathbb{R} \) is a disjoint union of open intervals. □

Now let \( \langle \mathbb{R}, f, \| \| \rangle \) be a topological model. Consider the formula

\[
\psi := \Box[a] \phi \land [a] q \land \Box[a] \neg q \tag{4}
\]

where \( \phi \) was defined by (3).

**Lemma 10.** \( \|\psi\| = \emptyset \).

**Proof.** Suppose \( x \in \|\psi\| \). The first conjunct of (4) says that for some sufficiently small open interval \( U \) containing \( x \) \( f(U) \subseteq \|\phi\| \). So by Note 4, the image \( f(U) \) is
the boundary of an open set. Now $U$ is path-connected hence the image $f(U)$ is connected as well. By Note 5 the set $f(U)$ is a singleton, i.e. $f(U) = \{f(x)\}$.

The second conjunct (4) says that $f(x) \in \|q\|$ and hence $f(U) \subseteq \|q\|$. But the third conjunct says that in any neighborhood of $x$ and, in particular, in $U$, there exists a point $y$ such that $f(y) \notin \|q\|$. This means, in particular, that $f(x) \neq f(y)$ and $f(U) \neq \{f(x)\}$. So we have a contradiction. □

Thus $\neg \psi$ is valid in $\mathbb{R}$. However it is not difficult to construct a topological model where $\|\psi\|$ is nonempty.

For example, take $\langle \mathbb{R}^2, f, \| \| \rangle$ where

$$f(x, y) = (x, 0), \quad \|p\| = \{(x, |y|): x, y \in \mathbb{R}\}, \quad \text{and} \quad \|q\| = \{(0, 0)\}.$$ 

**Note 6.** In notations as above we have

$$\|\psi\| = \{(0, y): y \in \mathbb{R}\}.$$ 

**Proof.** The formula $\phi$ defined by (3) denotes here the boundary of $\|\Box p\|$, thus $\|\phi\| = \{(x, 0): x \in \mathbb{R}\}$. The inverse image $f^{-1}(\|\phi\|)$ is then the whole space $\mathbb{R}^2$.

On the other hand the inverse image $f^{-1}(\|q\|)$ is the set $\{(0, y): y \in \mathbb{R}\}$. Further for any $y \in \mathbb{R}$ and any neighborhood $U$ of $(0, y)$ there exists a point $p' \in U$ such that $f(p') \notin \|q\|$: namely one can take $p' = (x', y)$ for $x'$ sufficiently small. Thus $\|\Box q \& \Diamond \neg \neg q\| = \{(0, y): y \in \mathbb{R}\}$, and the statement follows. □

In view of the topological soundness of $\text{S4C}$ this shows that $\neg \psi$ is not derivable in $\text{S4C}$.

Thus one cannot generalize Tarski theorem to $\text{S4C}$ and the real line. Moreover, it is clear that the same argument works for any one-dimensional topological manifold. However as we shall see in the next section one can generalize Tarski theorem to the case of $\text{S4C}$ and $\mathbb{R}^n$ assuming that the dimension $n$ is not fixed.

5. **Completeness of $\text{S4C}$ with Respect to $\mathbb{R}^n$**

At first let us recall the most elementary facts about extensions of continuous functions.

A topological space $T$ is called normal if any two disjoint closed subspaces $C_1$, $C_2$ of $T$ can be separated by open neighborhoods, i.e. there exist open subsets $U_1$, $U_2$ of $T$ with $C_i \subset U_i$, $i = 1, 2$, and $U_1 \cap U_2 = \emptyset$. One of the most important consequences of this property is the following fact:

**Theorem 5.** Assume that we are given a normal space $T$ and a closed subset $C \subset T$. Let $\tau: C \rightarrow B^n$ be a continuous function, where $B^n$ is the closed $n$-ball. Then $\tau$ extends to the whole $T$.

For a proof and a discussion see for example [3, 6.1].

Now let us return to completeness of $\text{S4C}$.

We assume that we have a formula $\phi \in L_{\text{S4C}}$ and a stratified dynamic Kripke frame as in Lemma 6 with limits chosen as in Note 2, which refutes $\phi$.

Let us spell this out. We have a Kripke frame $W = W^1 \sqcup \cdots \sqcup W^n$ (disjoint union) which consists of $n$ cones $W^k = O(w^k_{\text{root}})$ and the function $F: W^k \rightarrow W^{k+1}$
injective for all \( k < n \) and commuting with limits. The formula \( \phi \) is refuted at \( w^1 \). We want to embed this structure in the Euclidean space of sufficiently high dimension similar to the static case. In fact we are able to do this for \( \mathbb{R}^n \). Since \( n = \deg_\omega (\phi) + 1 \) this gives us an upper bound on the dimension of a refuting model.

Let \( X = I^1 \sqcup \cdots \sqcup I^n \) where \( I^k = [0, 1]^k \) and let \( f: I^k \to I^{k+1} \) be the embeddings

\[
(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \frac{1}{2}),
\]

\( k < n \). Let us collect all \( f^1, \ldots, f^{n-1} \) and define the total map \( f: X \to X \) to coincide with \( f^k \) on \( I^k \) for \( k < n \) and to be the identity on \( I^n \). We shall also consider the subspace \( \operatorname{Int} X \) of \( X \), \( \operatorname{Int} X = (0, 1) \sqcup \cdots \sqcup (0, 1)^n \). The spaces \( X = \langle X, f \rangle \) and \( \operatorname{Int} X = \langle \operatorname{Int} X, f \rangle \) are stratified dynamic spaces with stratifications given respectively by the closed and open cubes \([0, 1]^k \) and \((0, 1)^k \).

It is the space \( X \) essentially that will be the carrier of the countermodel. What we want is to define a surjective morphism \( \gamma: X \to W \), i.e. an interior map, which intertwines \( f \) and \( F \) and preserves stratifications.

By Lemma 4 the map \( \gamma \) will induce a refuting model for \( \phi \) in \( X \). We will arrange also that the restriction \( \gamma|_{\operatorname{Int} X} \) is still surjective and induces a morphism between stratifications. In order to build a refuting model in \( \mathbb{R}^n \) we choose an embedding \( e: X \to \mathbb{R}^n \). For convenience we assume that \( e \) is such that the sides of the cubes \( I^1, \ldots, I^n \) are mapped to line segments parallel to the corresponding coordinate axes. Then we define an interior map from \( \mathbb{R}^n \) to \( W \) in a closed neighborhood of \( e(X) \) by projecting a closed neighborhood of \( e(I^k) \) onto \( e(I^{k+1}) \) along the last \( n-k \) coordinate axes and composing this projection with \( \gamma \). We also get a continuous function \( g \) from a closed neighborhood of \( e(X) \) to \( e(X) \), which is the composition of the same projection with \( f \) followed by the embedding \( e \). The subspace \( e(X) \) of \( \mathbb{R}^n \) is compact therefore there exists a closed \( n \)-ball \( B \subset \mathbb{R}^n \) containing \( e(X) \). Thus \( g \) maps a closed neighborhood of \( e(X) \) to \( B \). But then, by Theorem 5, the map \( g \) extends to the whole \( \mathbb{R}^n \). These maps reduce the situation in \( \mathbb{R}^n \) to \( X \). Indeed, we have only to apply Lemma 4 to \( e(\operatorname{Int} X) \), \( W \) and \( F \). Therefore from now on we will deal with \( X \) only.

So let us proceed to the definition of an interior map \( \gamma: X \to W \). More specifically the map \( \gamma \) will take \( I^k \) to \( W^k \), \( k = 1, \ldots, n \). The construction runs in complete parallel with the static case; however everything is repeated \( n \) times — one time for each pair \( (I^k, W^k) \).

**Dynamic Basic Construction.** We assume that we are given the points \( w^k_0 \in W^k \), \( k = 1, \ldots, n \), such that \( F(w^k_0) = w^{k+1}_0 \). We define the map \( \tilde{\gamma}: I^k \to O(w^k_0) \) satisfying

\[
\tilde{\gamma}(f(x)) = F(\tilde{\gamma}(x)) \quad \text{for all } k < n.
\]

On the first cube \( I^1 \) the map \( \tilde{\gamma} \) is defined by the static Basic Construction with \( w_0 = w^1_1 \).

Assume that \( \tilde{\gamma} \) is defined on \( I^k \) and \( k < n \). We are going to apply the Basic Construction to \( I^{k+1} \) with \( w_0 = w^{k+1}_0 \), but we need to take care of the equality (6). Therefore we make a tiny modification of the static case. Let \( w^k_0, \ldots, w^{k+1}_{k+1} \) be all elements of \( O(w^{k+1}_0) \). At the \( i \)th step we have a finite list of terminal cubes in
I^{k+1}, which have already been defined, and a compact set \( I_i^{k+1} \) homeomorphic to a \((k+1)\)-cube with holes. We choose a finite \( \frac{1}{2^n} \)-net \( \Omega \) for \( I_i^{k+1} \) and in the \( \frac{1}{2^n} \)-neighborhood of each point \( y \in \Omega \) we choose \( r^{k+1} \) new terminal cubes corresponding to the elements of \( O(w_0^{k+1}) \). In the static case we require that new terminal cubes be pairwise disjoint and be disjoint from the boundary of \( I_i^{k+1} \). In the dynamic case we require in addition that they be disjoint from the image \( f(I^k) \) of \( I^k \) in \( I^{k+1} \). Since the collection of new terminal cubes is finite there exists \( \epsilon > 0 \) such that the closed \( \epsilon \)-neighborhood of \( f(I^k) \) is disjoint from all of them. Now for each terminal cube \( T \) of \( I^k \) defined at the \( i \)th step let \( T' \) be the closed \( \epsilon \)-neighborhood of \( f(T) \). We add \( T' \) to the list of terminal cubes of \( I^{k+1} \) and define \( \tilde{\gamma} \) on the interior of \( T' \) to be identically \( F(\tilde{\gamma}(T)) \). This makes (6) hold. We treat the rest of terminal cubes as in the static case. The set \( I_{i+1}^{k+1} \) is \( I^{k+1} \) with the interiors of new terminal cubes removed.

The exceptional points, i.e. those which do not belong to the interior of any terminal cube, are also treated as in the static case. If \( x \in I^k \) is exceptional we put \( \tilde{\gamma}(x) = w_0^k \). Note that (6) holds for exceptional points as well since exceptional points of \( I^k \) are taken by \( f \) to exceptional points of \( I^{k+1} \) and by assumption \( F(w_0^k) = w_0^{k+1} \).

As in the static case we use the Basic Construction in order to define a sequence \( \{\gamma_i\} \) of maps to \( W \), which gives the interior map \( \gamma \) in the limit.

The map \( \gamma_1 \) is \( \tilde{\gamma} \) of the Dynamic Basic Construction with \( w_0^k = w_{\text{root}}^k \), \( k = 1, \ldots, n \).

Now assume that \( \gamma_i \) has already been defined and the following holds:

(i) \( \gamma_i \circ f = F \circ \gamma_i \);

(ii) for any terminal cube \( T \) of \( I^k \) the function \( f \) maps \( T \) to some terminal cube \( T' \) of \( I^{k+1} \) by means of the embedding induced by the standard embedding \( I^k \to I^{k+1} \) (5).

Recall that in order to define \( \gamma_{i+1} \) in the static case we apply the Basic Construction for each terminal cube \( T \). Namely we construct a map \( \tilde{\gamma}_T: T \to O(\gamma_i(T)) \) and then paste different \( \tilde{\gamma}_T \)'s along the boundaries, where \( \gamma_{i+1} \) coincides with \( \gamma_i \). We repeat this for the dynamic case. For each terminal cube \( T \) of \( I^1 \) let \( T = T_i \) and let \( T^{k+1} \) be the terminal cube of \( I^{k+1} \) containing \( f(T) \), \( k = 1, \ldots, n - 1 \). We apply the Dynamic Basic Construction for \( T^1, \ldots, T^n \) with \( w_0^k = \gamma_i(T^k) \). Thus the Dynamic Basic Construction gives us \( \gamma_{i+1} \) on the interiors of all terminal cubes of \( \gamma_i \). We put \( \gamma_{i+1} \) to coincide with \( \gamma_i \) on the rest of \( X \).

Finally, for each \( x \in X \) we put \( \gamma(x) = \lim_{i \to \infty} \gamma_i(x) \). Note that all considerations of the static case apply and the map is well defined and interior. The equality (6) in Dynamic Basic Construction implies that the analogous equality is satisfied by all of \( \gamma_1, \gamma_2, \ldots \). Since, by assumption, \( F \) commutes with limits it follows that \( \lim_{i \to \infty} \gamma_i = \gamma \) satisfies the analogue of (6) as well. Hence \( \gamma \) is a morphism of stratifications. In view of remarks in the beginning of the Section this gives us the following dynamic version of Tarski theorem.

Theorem 6 (Dynamic Tarski theorem). A formula \( \phi \in L_{C,\alpha} \) is derivable in \( S4C \) if and only if \( \phi \) is true for any interpretation in \( \mathbb{R}^n \) where \( n = \deg_{\alpha}(\phi) + 1 \).
6. Some Concluding Remarks

We have proved a generalization of Tarski theorem for the Dynamic Topological Logic. However this does not yet completely settle the question of completeness of $S4C$ with respect to the Euclidean topology. In fact we showed that if a formula $\phi$ is not derivable in $S4C$ then there exists a countermodel for $\phi$ in $\mathbb{R}^n$ for $n$ sufficiently large. We gave an upper bound on $n$ but as for lower bound we managed only to show that in general $n > 1$. It remains an open question whether there exists a better estimate for $n$. In fact we believe that $n = \text{deg}_a(\phi) + 1$ is a lower bound as well.

Note that our counterexample for $n = 1$ is based on the fact that any boundary subset of $\mathbb{R}$ is disconnected, i.e. 0-dimensional. It would be very instructive to find some kind of induction on dimension and to lift this counterexample to larger values of $n$. Since the inductive topological dimension of a space is defined in terms of dimensions of its boundary subspaces this idea seems quite natural. However we did not manage to implement this idea although some conjectures were very plausible.

We can ask a more general question in this connection: is there an extension of $S4$ (or may be of some other modal logic), which is capable to characterize dimension? Is $S4C$ a suitable candidate?

References


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