ANDERSON–BERNOULLI MODELS

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Dedicated to Ya. Sinai

ABSTRACT. We prove the exponential localization of the eigenfunctions of the Anderson model in \( \mathbb{R}^d \) in the regime of large coupling constant for the random potentials which values are independent and Bernoulli distributed.

2000 Math. Subj. Class. 82B44 (60H25, 81Q10, 82B10).

Key words and phrases. Anderson localization, random Bernoulli potential.

INTRODUCTION

The main focus of this report is recent work of the author (joint with C. Kenig) on the higher dimensional Anderson–Bernoulli model in the continuum setting (see [BK]). In particular we prove localization and dynamical localization at the lower edge of the spectrum. These seem to be the first results for Anderson–Bernoulli when \( D > 1 \). We recall, that localization is the phenomena when the eigenfunctions of the Hamiltonian are exponentially localized. Compared with the situation when there is a continuum site distribution, many difficulties appear when one tries to carry out the usual Frohlich–Spencer scheme and they require a set of new ideas. In particular, crucial use is made of quantitative versions of the unique continuation principle (this is the place where we rely on the \( \mathbb{R}^d \)-setting with the continuum Laplacian rather than the lattice setting).

Let us first recall the general setting of the Anderson model with random potential.

Lattice version. In the lattice case one considers the Hamiltonian of the form \( \Delta + \sum_{n \in \mathbb{Z}^d} V_n \delta_{n'n'} \), where the disorder variables \( \{ V_n : n \in \mathbb{Z}^d \} \) are independent identically distributed, and

\[
\Delta(n, n') = \begin{cases} 
1 & \text{if } |n - n'| = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

is lattice Laplacian.

Received July 4, 2005.
Continuum version. Here one similarly considers the operator

\[-\Delta + \sum_{n \in \mathbb{Z}^d} V_n \varphi(x - n) \quad (x \in \mathbb{R}^d),\]

where \(\varphi\) is some fixed ‘bumpfunction’.

Rough spectral structure of these Hamiltonians is the following:

In dimension \(d = 1\) one has Anderson localization (AL) almost surely; namely,

The spectrum is pure point; the eigenstates are exponentially decaying.

In \(d > 1\) in the lattice version one has AL for large disorder or at edge of spectrum.

In the continuum version one has AL at the bottom of the spectrum.

One of the most important open questions of the field is the presence of absolutely continuous spectrum in dimension \(d \geq 3\).

1. The Anderson–Bernoulli Model

This is the situation where the \(V_n\) are discrete valued, say

\[V_n \in \{0, \lambda\}.\]

These models are significantly harder to analyze, due to the fact that certain basic techniques in this field do depend on a continuous distribution (for instance, eigenvalue variation arguments).

2. Anderson–Bernoulli in Dimension One

In dimension 1 theory is reasonably understood. There are several proofs of AL. The proof due to Carmona–Klein–Martinelli [CKM], uses transfer matrix plus Furstenberg–Lepage method. The one by Shubin–Vakilian–Wolff [SVW], is based on super symmetric approach. The continuum model is treated by Damanik–Sims–Stolz [DSS].

Problem. The key question here is the behavior of the density of states of the operator

\[H_\lambda = \Delta + \lambda \sum_{n \in \mathbb{Z}} \varepsilon_n \delta_{nn'},\]

for small \(\lambda\).

Denote by \(N(E)\) the IDS (Integrated density of states) related to the Lyapounov exponent by Thouless formula

\[L(E) = \int \log |E - E'| dN(E'),\]

\[L(E) = \lim_{N \to \infty} L_N(E),\]

\[L_N(E) = \frac{1}{N} \int \log \|M_N(E; \varepsilon)\| d\varepsilon\]
where $M_N(E; \varepsilon)$ denotes the ‘transfer-matrix’, in our case

$$M_N(E; \varepsilon) = \prod_{n=N}^1 \left( \lambda \varepsilon_n - E - 1 \right).$$

It is known that the IDS for one-dimensional Bernoulli-model is Hölder continuous

$$|N(E) - N(E')} \leq C|E - E'|^\alpha$$

for some exponent $\alpha = \alpha(\lambda) > 0$ (see [SVW] for instance), and $\alpha(\lambda) \leq 2 \log 2 \arccosh(1 + |\lambda|^2)$ (Simon–Taylor–Halperin).

The author showed that $\alpha(\lambda) > 1/5 - O(\lambda)$ for $\lambda$ small and $E$ inside $]-2, 2[$, see [B], using the Figotin–Pastur linearization method.

3. Anderson–Bernoulli in $D \geq 2$ (Continuum)

The following theorem was recently obtained:

**Theorem** (J. Bourgain and C. Kenig [BK]). Consider the Hamiltonian

$$H_\varepsilon = -\Delta + V,$$

where

$$V = V_\varepsilon(x) = \sum_{n \in \mathbb{Z}^d} \varepsilon_n \varphi(x - n)$$

with $\varepsilon \in \{0, 1\}^{\mathbb{Z}^d}$ being Bernoulli variables, while the function $\varphi$ satisfies

$$0 \leq \varphi \leq 1, \quad \varphi \text{ smooth}, \quad \text{supp } \varphi \subset B\left(0, \frac{1}{10}\right).$$

Then at energies near the bottom of the spectrum ($E > 0, E \approx 0$), $H_\varepsilon$ displays Anderson localization for almost all realisations of $\varepsilon$’s.

The essential role in the proof is played by ‘unique continuation principle’, for which no discrete counterpart seems to be known.

There are two main steps in the proof.

- Green’s function estimates with fixed energy.
- Elimination of the energy.

This is the ‘usual’ scheme, but each step requires novel ingredients.

4. Fixed energy: The Wegner inequality

**Proposition A.** Denote $\Lambda_\ell \subset \mathbb{R}^d$ an $\ell$-cube.

There is a subset $\Omega \subset \{0, 1\}^{\Lambda_\ell \cap \mathbb{Z}^d}$ such that

$$|\Omega| > 1 - \ell^{-\rho}$$

(for any $\rho < \frac{3}{8}d$) such that for $\varepsilon \in \Omega$, the resolvent $R_\Lambda (E$ fixed$)$ satisfies

$$\|R_\Lambda\| < e^{\ell^{-1}},$$

$$\|\chi_x R_\Lambda \chi_{x'}\| < e^{-cd} \quad \text{for } |x - x'| > \ell/10.$$
Remarks. (i) The estimate (4.1) gives a weaker estimate on the exceptional set than what is usually appearing in a multi-scale analysis. It also complicates matters in the energy elimination step that will require a different argument.

(ii) The usual Wegner estimate refers only to the resolvent bound (4.2) and is gotten from first order eigenvalue variation without relying on multiscale arguments. In the [BK] paper, we were unable to prove a Wegner estimate directly and had to rely on a multi-scale process involving both conditions (4.2), (4.3) already at this stage.

Discussion of the proof. (i) In order to perform eigenvalue variation, we need to consider the natural extension of $V_\varepsilon(x)$ to $\varepsilon \in [0, 1]^Z_d$. First order eigenvalue variation gives then that

$$\partial_n E = \int_{\mathbb{R}^d} \xi(x)^2 \varphi(x-n) dx$$

with $\xi = \xi_{E,\varepsilon}$ the corresponding normalized eigenfunction, i.e., $H \xi = E \xi$, $\|\xi\|_2 = 1$ (more precisely we performed a finite scale restriction, replacing $H$ by $H^{(N)}$ which is the restriction of $H$ to the box $[-N - \frac{1}{2}, N + \frac{1}{2}]^d$, imposing Dirichlet conditions.)

The ‘influence’ $I_n$ on $E$, defined as

$$I_n(\varepsilon) = E(\varepsilon_n = 1; (\varepsilon_{n'})_{n' \neq n}) - E(\varepsilon_n = 0; (\varepsilon_{n'})_{n' \neq n})$$

$$= \int_0^1 (\partial_n E) d\varepsilon_n$$

$$= \int_0^1 \int \xi_\varepsilon(x)^2 \varphi(x-n) dx d\varepsilon_n > 0.$$  \hspace{1cm} (4.5)

There are 2 issues: upper bounds and lower bounds for $|n| \to \infty$.

Upper bounds are gotten from Green’s function estimates, the usual way

$$|G^{(N)}(x_0, x)| < e^{cN} \quad \text{for} \quad |x_0 - x| \sim N$$

and $G^{(N)}$ denoting the Green’s function corresponding to $H^{(N)}$.

In view of the preceding, we will need estimates on $G^{(N)}(x_0, x; \varepsilon)$ not only for $\varepsilon \in \{0, 1\}^{(2^d\cdot [-N,N]^d)}$ but also allowing the range $\varepsilon_n \in [0, 1]$ for a sufficiently large collection of sites $n$, which will be called ‘free sites’.

The need for these free sites complicates considerably matters at the probabilistic level, when formulating Wegner estimates. Proposition A is actually replaced by a more technical statement verified by induction on the scale $\ell$. A more precise description of the set $\Omega$ is needed and we assume $\Omega$ to be a disjoint union of ‘cylinders’ of the form

$$C = \{0, 1\}^S \times \prod_{n \in \Lambda \setminus S} \{\varepsilon_n\}$$

where $S \subset \Lambda \cap \mathbb{Z}^d$ and $(\varepsilon_n)_{n \in \Lambda \setminus S} \in \{0, 1\}^{\Lambda \setminus S}$ depend on $C$. The set $S$ will then provide ‘free sites’ and estimates (4.2), (4.3) remain true extending the range $\varepsilon_n \in [0, 1]$ for $n \in S$.

(ii) Returning to (4.5), the main difficulty is to obtain a lower bound on $I_n$. We only know to resolve that issue in the continuum. By (4.5), we need to insure
certain lower bounds on
\[
\max_{|x-n| \leq 1} |\xi(x)|, \quad H \xi = E \xi.
\]

These lower bounds are independent of the randomness \(\varepsilon\) and obtained from Carleman inequalities (only available in the continuum case).

Clearly
\[
|\Delta \xi| \leq C |\xi|.
\]

Proposition [BK]. If \(\xi(0) = 1, |\xi| \leq C\) and (4.7) then \(\forall x, |x| > 10,\)
\[
\max_{|x-x'| \leq 1} |\xi(x')| > c' \exp(-c'(| \log |x| |)^{4/3}).
\]

(4.8)

We will discuss inequality (4.8) later.

It clearly yields the lower bound (assuming \(\xi(0) = 1\))
\[
I_n > e^{-c|n|^{4/3} \log |n|}.
\]

(4.9)

Let us point out that (4.8) is the best result one can derive from the Carleman method and the \(\frac{4}{3}\)-exponent is essential in the present application.

(iii) A basic difficulty with the Bernoulli model is that, unlike the case of a continuous site distribution, small probability of events can not be established by variation of \(V\) on a single site but requires many sites.

Our basic tool is the following immediate consequence of Sperner’s lemma on sets of incomparable elements of \(\{0, 1\}^M\).

Lemma. Let \(E = E(\varepsilon_1, \ldots, \varepsilon_M)\) be a function on \(\{0, 1\}^M\) such that for all \(j = 1, \ldots, M\)
\[
I_j > k > 0.
\]

Then, for all \(E_0 \in \mathbb{R}\)
\[
\text{mes}_{\{0, 1\}^M} \left[ |E - E_0| < K \right] < M^{-1/2}.
\]

(4.10)

In the application, \(\varepsilon_1, \ldots, \varepsilon_M\) will be chosen among the free sites. Since at scale \(N\), we certainly need
\[
k > e^{-N^{1-}}
\]

it follows from (4.9) that moreover these sites \(n\) need to be chosen within
\[
[n] < N^\frac{2}{\rho} \]

and hence their total number is at most \(N^\frac{2d-}{\rho} = M\).

The resulting estimate in the lemma becomes
\[
N^{-\frac{2d-}{\rho}}
\]
from where \(\rho = \frac{4}{8}d-\) in Proposition A.

(iv) The inductive verification of Proposition A.

As usual, one verifies Proposition A first at an initial large scale \(\ell_0\). The argument is perturbative and uses the assumption \(E \approx 0\) (it is only used here).

The Bernoulli aspect is irrelevant here.
We sketch the inductive step. Let $\Lambda$ be a size $\ell$-box and take $\ell_0 \sim \ell^\alpha$, $\alpha = \frac{3}{4} - $. Assume Proposition A valid at scale $\ell_0$. Take a cover of $\Lambda$ by $\sim \left( \frac{\ell}{\ell_0} \right)^d$ boxes $\Lambda_0$ of size $\ell_0$. Call $\Lambda_0$ ‘good’ if (4.2), (4.3) hold at scale $\ell_0$ and ‘bad’ otherwise.

We want to ensure that in previous cover, there is only a bounded number of ‘bad’ $\ell_0$-boxes. This will be achieved provided
\[
\left( \frac{\ell}{\ell_0} \right)^d \ell_0^{-\rho} < \ell^{-\delta} \quad \text{(for some $\delta > 0$)}
\]
hence
\[
(1 - \alpha)d < \rho \alpha \quad \text{(4.11)}
\]
(the number of bad boxes will depend on $\delta$).

Assume there is only one bad box $\Lambda_0$.

Use ‘free sites’ at distance $\sim \ell_0$ from $\Lambda_0$ and denote $I_n$ the influence of free site $n$ for an eigenvalue function $E = E_\tau$
\[
|E_\tau - E_0| < e^{-\ell_0}.
\]
(4.12)

Since the corresponding eigenfunction $\xi$ satisfies
\[
|\xi(x)| < e^{-c \text{dist}(x, \Lambda_0)} \quad \text{for dist}(x, \Lambda_0) > \ell_0
\]
we have
\[
I_n < e^{-c \text{dist}(n, \Lambda_0)} < e^{-\ell_0}. \quad \text{(4.13)}
\]
(4.13) ensures in particular that (4.12) remains essentially preserved if $\varepsilon_n \in [0, 1]$ varies.

Also in view of (4.9)
\[
I_n > e^{-\text{dist}(n, \Lambda_0) \frac{3}{4} + } > e^{-\ell_0^\frac{3}{4} + } \quad \text{(4.14)}
\]
which forces us to impose the condition
\[
\ell_0^\frac{3}{4} + = \ell_0^\frac{3}{4} + < \ell^{1 -}
\]
satisfied by the choice of $\alpha$.

The number of free sites available is $\sim \ell_0^d$. 
From the probabilistic lemma

$$P[|E_\tau - E_0| < e^{-\ell^\delta}] \lesssim \ell_0^{-\frac{d}{2}} \sim \ell^{-\alpha\frac{d}{2}}.$$ (4.15)

This estimate (4.15) has to be multiplied with the number of eigenvalues $E_\tau$ satisfying (4.12). Using a separate argument, this number may be bounded by $\sim \ell^\delta$, for any $\delta > 0$ (by showing that moreover $\xi$ is localized on at most $C_\delta$ subcubes $\Lambda'$ of $\Lambda_0$ of size $\ell^\delta$).

In conclusion

$$P[\|R_{\Lambda_0}(E_0)\| > e^{\ell^\delta}] \lesssim \ell^{-\alpha\frac{d}{2}+\delta}$$ (4.16)

and we may take

$$\rho = \frac{\alpha}{2} = \frac{3}{8}d^-.$$

Condition (4.11) becomes

$$\frac{1}{4}d < \left(\frac{3}{8}d^-ight)\frac{3}{4}$$ (4.17)

which is barely satisfied.

5. The Unique Continuation Result

The key inequality (4.8) results from the following Carleman type inequality.

**Proposition.** There are constants $C_1$, $C_2$, $C_3$ depending only on $d$ (the dimension $d \geq 2$ arbitrary) and an increasing function $w = w(r)$ for $0 < r < 10$ such that

$$\frac{1}{C_1} < \frac{w(r)}{r} < C_1$$

and for all functions $f \in C_0^\infty(B_1 \setminus \{0\})$, $\alpha > C_2$, we have

$$\alpha^3 \int w^{-1-2\alpha} f^2 \leq C_3 \int w^{2-2\alpha} (\Delta f)^2.$$ (5.1)

It is essentially contained in Escauriaza–Vessela [EV] and is going back to Hörmander [11].

Application of (5.1) to deduce (4.8) is standard.

Assume $|\Delta \xi| \leq C|\xi|$, $\xi$ bounded and

$$\xi(a) = 1 \quad \text{where } |a| = R \text{ (large)}. $$

Our aim is to obtain a lower bound on $\max_{|x| \leq 1} |\xi(x)|$.

First, rescale the problem, defining

$$u(x) = \xi(Rx).$$

This $u(R) = 1$ where $|R| = 1$ and we estimate $\max_{|x| \leq \frac{1}{R}} |u(x)|$ from below.

Localize to complement of $\frac{1}{R}$-neighborhood of 0

$$f(x) = u(x)\psi(x)$$
\[ \Delta f = \Delta u.\psi + O(|\nabla u| |\nabla \psi| + |u| |\Delta \psi|) = O(R^2 |f|) + O(R^2 |u| + R|\nabla u|) \chi_{\frac{2\pi}{\pi} < |x| < \frac{\pi}{2}} + O(R). \]

Contribution of first term in (5.1)

\[ CR^4 \int u^{2-2\alpha} f^2 \]

may be absorbed in left side of (5.1) taking

\[ \alpha \sim R^{4/3}, \]

hence

\[ \max_{|x| \leq \frac{\pi}{2}} |u(x)| + |\nabla u(x)| \gtrsim \left( \frac{C}{R} \right)^{2\alpha} > e^{-CR^{4/3} \log R}. \]

This gives (4.8).

**Remark.** The exponent \( \frac{4}{3} \) is the limitation of the Carleman method.

**Proposition** (V. Meshkov [M]).

(i) Assume \( |\Delta u| \leq C|u| \) on \( \mathbb{R}^d \) and \( |u(x)| \) decays faster than

\[ \exp(-a|x|^{4/3}) \quad \text{for } |x| \to \infty, \text{ for any } a > 0. \]

Then \( u \equiv 0. \)

(ii) There is an example of complex-valued \( u \) such that \( |\Delta u| \leq C|u| \) on \( \mathbb{R}^2, \)
\( u \neq 0, \) and

\[ |u(x)| \leq C \exp(-c|x|^{4/3}), \quad \forall x \in \mathbb{R}^2. \]

Notice that the Carleman method does not distinguish between real and complex case. However, the following problem remains unanswered.

**Problem.** Assume \( u \) real-valued and \( |\Delta u| \leq C|u| \) on \( \mathbb{R}^d. \) What may be said about

\[ \lim_{|x| \to \infty} \frac{\log \log \frac{1}{|u(x)|}}{\log |x|} \quad \text{and} \quad \lim_{|x| \to \infty} \min_{|x-x'| \leq 1} \frac{\log \log \frac{1}{|u(x)|}}{\log |x|}? \]

**Remark.** Is there a discrete analogue of the theorem for the lattice model

\[ H_x = \Delta + \lambda \sum_{n \in \mathbb{Z}^d} \varepsilon_n \delta_{nn'} \quad (d \geq 2)? \]

No Carleman-type inequalities for the discrete Laplacian seem known and there is little understanding of ‘unique continuation’ on the lattice, when \( d \geq 2 \) (the case \( d = 1 \) is immediate from the transfer-matrix formulation).
6. Elimination of the Energy

A different argument is needed because of the weak probabilistic estimate in Proposition A.

**Lemma 6.1.** Let $\xi$ be an extended state of $H_\varepsilon$ with energy $E$ and $\xi(0) = 1$. Then for $\ell$ sufficiently large

$$\text{dist}(E, \text{Spec} H_{\Lambda(0,\ell)}) < e^{-c\ell}. \quad (6.2)$$

**Amplifications.** (1) Instead of assuming $\xi(0) = 1$, it suffices to assume $\xi(0) > e^{-\ell^{1/2}}$ (and $\log |\xi(x)| \lesssim \log |x|$).

(2) The event in Lemma 6.1 holds with probability at least $\ell^{-C}$ where $C(c)$ (c = constant in (6.2)) can be made large letting $c$ be small enough.

Proof of Lemma 6.1 is based on bootstrap reasoning using a Peierls type argument. We perform a sequence of approximations of $E$ up to $e^{-\ell^{1/2}}$, $e^{-\ell^{4/5} + \delta}$, etc. until achieving (6.2).

Lemma 6.1 permits us to approximate $E$ up to $e^{-c\ell}$ by an element of $\text{Spec} H_{\Lambda(0,\ell)}$ having $\lesssim \ell^d$ elements and only dependent on $(\varepsilon_n)_{n \in \Lambda(0,\ell) \cap \mathbb{Z}^d}$. However, because of the weak measure bound in Proposition A, we can’t deal with $\text{Spec} H_{\Lambda(0,\ell)}$ which is too large. We need to reduce in (6.2) $\text{Spec} H_{\Lambda(0,\ell)}$ to a subset $S$ satisfying

$$|S| < \ell^d \quad \text{where } \rho < \frac{3}{8}. \quad (6.3)$$

In fact, we will prove the following property:

**Lemma 6.4.** Let $0 < \delta \leq 1$ and $\ell$ large enough. There is a subset $S \subset \text{Spec} H_{\Lambda(0,\ell)}$ such that

$$|S| < \ell^{\delta d} \quad \text{where } \rho < \frac{3}{8}. \quad (6.5)$$

and whenever $E$ and $\xi$ are as in Lemma 6.1, then

$$\text{dist}(E, S) < e^{-c\ell}. \quad (6.6)$$

Let further $0 < \gamma < 1$ (arbitrary and fixed). There is moreover a collection $\mathcal{F}$ of at most $C(\gamma)$ subcubes $Q \subset \Lambda(0, \ell)$ of size $\ell^\gamma$, such that $\Lambda(0, \ell) \setminus \bigcup_{Q \in \mathcal{F}} Q$ may be covered by ‘good’ subcubes $\Lambda_1 \subset \Lambda(0, \ell)$ of size $\ell_1 \in [\ell^\gamma, \ell^{4/5}]$ (and depending on the cube $\Lambda_1$).

The event stated in Lemma 6.4 fails with small $\varepsilon$-probability. This will be made more precise later on.

Observe that on $\Lambda(0, \ell)$ $\xi$ decays exponentially fast away from $\bigcup_{Q \in \mathcal{F}} Q$.

To prove Lemma 6.4, we will use the following independent statement (it is also used in proving Proposition A).

**Claim 6.7.** Fix an energy $E$. Let $0 < \gamma < \frac{4}{5}$ and $A > 1$. Let $\Lambda_0$ be an $\ell$-cube in $\mathbb{R}^d$. Then there is a collection of at most $C = C(\gamma, A)$ cubes $Q \subset \Lambda_0$ of size $\ell^{\gamma}$ such that $\Lambda_0 \setminus \bigcup_{Q \in \mathcal{F}} Q$ may be covered by ‘good’ subcubes $\Lambda_1 \subset \Lambda_0$ of size $\ell_1 \in [\ell^\gamma, \ell^{4/5}]$. Moreover this statement fails with $\varepsilon$-probability at most $\ell^{-A}$. 
Proof. Let \( m \in [\ell^{\frac{5}{4}}, \ell] \) and \( \Lambda' \subset \Lambda_0 \) an \( m \)-cube.

Perform an admissible covering of \( \Lambda' \) by essentially disjoint cubes \( \Lambda'' \) of size \( m^{4/5} \). Their number is \( \sim m^{d/5} \). We estimate the probability that at least \( C_1 \) (disjoint) \( \Lambda'' \)-cubes are ‘bad’. Since the probability of \( \Lambda'' \) to be bad is at most \( m^{-\frac{d}{5} \gamma} \) by Proposition A, there is the bound

\[
(cm^{d/5} ) \lesssim m^{d(\frac{1}{2} - \frac{d}{5})} \sim m^{-\frac{d}{5} C_1} \tag{6.8}
\]

(the first factor stands for the number of \( C_1 \)-triples of \( \Lambda'' \)-cubes in the covering of \( \Lambda' \)).

We need to multiply further (6.8) with the number of \( m \)-cubes \( \Lambda' \subset \Lambda(0, \ell) \) and sum over all sizes \( m \in [\frac{5}{4}, \ell] \). This gives the measure bound

\[
\sum_{\ell^{\frac{5}{4}} \leq m \leq \ell} \ell^{d/5} m^{-\frac{d}{5} C_1} \sim \ell^{-A} \tag{6.9}
\]

taking \( C_1 = 30 \gamma^{-1} A \).

Let \( \varepsilon \) be outside this exceptional set.

By construction, in the cover of \( \Lambda_0 \) by essentially disjoint \( \ell^{4/5} \)-cubes \( \Lambda_1 \), there are at most \( C_1 \) bad ones. Considering a covering of a fixed bad \( \Lambda_1 \)-cube by \( \ell^{\frac{d}{5}} \)-cubes \( \Lambda_2 \), again at most \( C_1 \) bad \( \Lambda_2 \) cubes appear etc. Continuing until size \( \ell^{\frac{5}{4}} \) is reached the claim clearly hold with

\[
C(\delta, A) = C_1^{\log \frac{1}{\gamma}} \lesssim \left( \frac{A}{\gamma} \right)^{50 \log \frac{1}{\gamma}}. \tag{6.10}
\]

Returning to Lemma 6.4, we proceed ‘by induction’ on \( \delta \).

For \( \delta = 1 \), just take \( S = \text{Spec } H_{\Lambda(0, \ell)} \) (by Lemma 6.1).

We prove the second part of the statement. Let thus \( 0 < \gamma < 1 \).

To apply Claim 6.7, we need to decouple the energy \( E \) and the randomness \( \varepsilon \).

Take \( \ell^{\gamma} \leq m < \ell^{\frac{5}{4}} \). By Lemma 6.1

\[
\text{dist}(E, \text{Spec } H_{\Lambda(0, m)}) < e^{-cm}
\]

and \( \text{Spec } H_{\Lambda(0, m)} \) depends only on \( (\varepsilon_n)_{n \in \Lambda(0, m) \cap Z^d} \).

Cover \( \Lambda(0, m^{5/4}) \setminus \Lambda(0, 2m) \) by a bounded number of ‘cubes’ \( \Lambda_0 \) of size \( \sim m^{5/4} \).
Apply Claim 6.7 with each $E_1 \in \text{Spec } H_{\Lambda(0,m)}$ and cube $\Lambda_0$ (notice that $E_1$ and $H_{\Lambda_0}$ are independent as functions of $\varepsilon$). Thus excluding $\varepsilon$-measure at most
\begin{equation}
m^d m^{-A} < m^{A/2} \tag{6.11}
\end{equation}
we ensure that for each $E_1 \in \text{Spec } H_{\Lambda(0,m)}$ there is a collection of at most $C(\gamma, \alpha)$ cubes $Q \subset \Lambda(0, m^{5/4}) \setminus \Lambda(0, 2m)$ of size $\ell^\gamma$ such that $(\Lambda(0, m^{5/4}) \setminus \Lambda(0, 3m)) \cup Q$ admits a cover by good $\ell_1$-cubes, $\ell_1 \in [\ell^\gamma, m]$.

This property remains preserved by an $e^{-cm}$-perturbation of $E$.

Let $m$ run over a sequence $\ell^{\gamma_j}(0 \leq j \leq J)$ with $\gamma_0 = \gamma$, $\gamma_1 = \frac{\gamma}{5}$ and $\gamma_{j+1} = \frac{10}{\gamma_j} \gamma_j$. By (6.11) the excluded $\varepsilon$-measure is at most
\begin{equation}
\sum_{j=0}^{J} \ell^{-\gamma_j} \frac{\delta}{2} < \varepsilon^{-\frac{\delta}{2} d} \tag{6.12}
\end{equation}
Let $\xi, E$ be as in Lemma 6.1. Take $E_j \in \text{Spec } H_{\Lambda(0,\ell^{\gamma_j})}$ such that
\begin{equation}
|E - E_j| < e^{-c\ell^{\gamma_j}} \tag{6.13}
\end{equation}
and for which there is a collection $\mathcal{F}_j$ of at most $C(\gamma, \alpha)$ cubes $Q$ of size $\ell^\gamma$ such that $(\Lambda(0, \ell^{\gamma_j}) \setminus \Lambda(0, 3\ell^{\gamma_j})) \cup \bigcup_{Q \in \mathcal{F}_j} Q$ has the covering property. Then
\allowdisplaybreaks
\begin{equation}
\bigcup_{j \leq J} \bigcup_{Q \in \mathcal{F}_j} Q \cup \Lambda(0, 4\ell^\gamma)
\end{equation}
is covered by at most $JC(\gamma, \alpha) + 10 d < C'(\gamma, \alpha)$ cubes of size $\ell^\gamma$. It is easily checked that if $\mathcal{F}$ is this collection, then $\Lambda(0, \ell) \setminus \bigcup_{Q \in \mathcal{F}} Q$ admits a cover by good $\ell_1$-cubes with $\ell_1 \in [\ell^\gamma, m]$.

We have verified Lemma 6.4 with $\delta = 1$.

Apply next Lemma 6.4 with same $\delta$ and $\ell$ replaced by $\ell^{4/5}$ (our aim is to decrease $\delta$). Let $S_1 \subset \text{Spec } H_{\Lambda(0,\ell^{4/5})}$ be the energy set. Thus
\begin{equation}
|S_1| < \ell^{\frac{4}{5}d} \tag{6.14}
\end{equation}
Notice that $S_1$ only depends on $(\ell_n)_{n \in \Lambda(0, \ell^{4/5})} \cup \mathbb{Z}$. Cover $\Lambda(0, \ell) \setminus \Lambda(0, 2\ell^{4/5})$, as before, by a bounded number of cubes $\Lambda_0$ of size $\sim \ell$. Apply Claim 6.7 to each energy $E_1 \in S_1$ and cube $\Lambda_0$, with $\gamma$ replaced by $\gamma_1 \leq \gamma$ (to be specified). Thus the property in Claim 6.7 holds for all $\Lambda_0$ and $E_1 \in S_1$ with exceptional probability at most
\begin{equation}
c|S_1| \ell^{-A} < \ell^{-A/2} \tag{6.15}
\end{equation}
Apply Lemma 6.4 with $\delta$ and $\ell$ replaced by $3\ell^{4/5}$, providing $S_2 \subset \text{Spec } H_{\Lambda(0,3\ell^{4/5})}$
\begin{equation}
|S_2| < \ell^{\frac{4}{5}d} \tag{6.16}
\end{equation}
Let now $E$ and $\xi$ be as in Lemma 6.1.

There is $E_2 \in S_2$ with $|E - E_2| < e^{-c\ell^{4/5}}$. By the second statement in Lemma 6.4, there is a collection $\mathcal{F}_2$ of $C(\gamma_1)$ subcubes $Q \subset \Lambda(0, 3\ell^{4/5})$ of size $\ell^{\gamma_1}$ such that $\Lambda(0, 3\ell^{4/5}) \setminus \bigcup_{Q \in \mathcal{F}_2} Q$ is covered by good cubes of size $\ell_1 \in [\ell^{\gamma_1}, \ell^{4/5}]$ (this property remains preserved for $e^{-c\ell^{4/5}}$-perturbation of the energy).
Notice that inside $\Lambda(0, 3\ell^{4/5})$, $\xi$ is essentially supported by $\bigcup_{Q \in \mathcal{F}} Q$. Take next $E_1 \in S_1$ with $|E - E_1| < e^{-c\ell^{4/5}}$. By construction for each $\Lambda_0$-cube constituting $\Lambda(0, \ell) \setminus \Lambda(0, 2\ell^{4/5})$, there is a collection $\mathcal{F}(\Lambda_0)$ of at most $C(\gamma_1, A) \ell^{\gamma_1}$-cubes $Q$ such that $\Lambda_0 \setminus \bigcup_{Q \in \mathcal{F}(\Lambda_0)} Q$ has the covering property.

Again it follows that on $\Lambda(0, \ell) \setminus \Lambda(0, 2\ell^{4/5})$, $\xi$ is essentially supported by $\bigcup_{\Lambda_0} \bigcup_{Q \in \mathcal{F}(\Lambda_0)} Q$.

In summary, there is a collection $\mathcal{F} = \mathcal{F}_2 \cup \bigcup_{\Lambda_0} \mathcal{F}(\Lambda_0)$ of at most $C(\gamma_1, A) \ell^{\gamma_1}$-cubes $Q$ of size $\ell^{\gamma_1}$ such that $\Lambda(0, \ell) \setminus \bigcup_{Q \in \mathcal{F}} Q$ has the covering property; on $\Lambda(0, \ell)$, $\xi$ is essentially supported by $\bigcup_{\Lambda_0} \bigcup_{Q \in \mathcal{F}(\Lambda_0)} Q$.

Observe also that by the preceding, $\mathcal{F}$ is determined by either $e^{-c\ell^{4/5}}$-approximation $E_1 \in S_1$ or $E_2 \in S_2$ to $E$. Hence, we may claim that there are at most $|S_1|$ possibilities for $\mathcal{F}$.

According to Lemma 6.1, there is $E \in \text{Spec} \, H_{\Lambda(0, \ell)}$ that

$$|E - E| < 2^{-c\ell} \quad (6.17)$$

and we denote $\zeta_E$ a corresponding eigenfunction.

By (6.17), also $\zeta_E$ is essentially supported by $\bigcup_{Q \in \mathcal{F}} Q$.

Denote $S_F \subset \text{Spec} \, H_{\Lambda(0, \ell)}$ a maximal collection of distinct energies $E$ such that $\zeta_E$ satisfies

$$\|\zeta_E(\mathcal{X}_{\bigcup_{Q \in \mathcal{F}} Q})\|_2 > \frac{1}{2}\|\zeta_E\|_2. \quad (6.18)$$

By orthogonality considerations, it is easily seen that

$$|S_F| \lesssim \text{measure} \left( \bigcup_{Q \in \mathcal{F}} Q \right) \lesssim |\mathcal{F}| e^{\gamma_1} < C(A, \gamma_1) e^{\gamma_1}. \quad (6.19)$$

Define

$$S = \bigcup_{E_1 \in S_1} S_{\mathcal{F}} \quad (6.20)$$

(recalling that $\mathcal{F}$ only depends on the approximation $E_1 \in S_1$).

Then by (6.14), (6.19), (6.20) and for appropriate $\gamma_1 = \min(\frac{4}{15}, \gamma)$, we have

$$|S| \lesssim |S_1| e^{\gamma_1} < e^{\delta(\frac{4}{15} + \gamma_1)} < \ell^{2\delta}. \quad (6.21)$$

From the preceding, $S$ satisfies (6.5), (6.6) with $\delta$ replaced by $\frac{\delta}{2}$. This is the first part of Lemma 6.4.

We obtained above that $\Lambda(0, \ell) \setminus \bigcup_{Q \in \mathcal{F}} Q$ admits a cover by ‘good’ cubes of size $\ell_1 \in [\ell^{\gamma_1}, \ell^{4/5}]$. Taking $\gamma_1 = \gamma$, we get part 2 of Lemma 6.4. This completes the proof. □

**Amplification.** Examination of the above argument shows that again the claim in Lemma 6.4 fails with probability at most $\ell^{-C}$, where $C$ can be made arbitrarily large by decreasing the constant $c$ in (6.5) and increasing the number $C(\gamma)$ of elements of $\mathcal{F}$.

By Lemma 6.4, we may in particular meet condition (6.3). This enables us to ensure that at energy $E$ ($\xi$, $E$ as in Lemma 6.1) for all $L \geq \ell$. $\Lambda(0, 2L) \setminus \Lambda(0, L)$
may be covered by good boxes of size $\sim L$ (with probability at least $1 - L^d \rho$, $\rho < \frac{3}{8}$).

Thus

$$R_{R^d \Lambda(0, \ell)}(E + i0; x, y) < e^{-c|x-y|}$$

for $\ell \leq |x| \leq |y|$ and $|x-y| \sim |y|$.

It implies exponential decay of $\xi$ away from $\Lambda(0, \ell)$. In particular $\xi$ is a proper state. Thus we proved that any extended state is a proper state with exponential decay at infinity, hence AL.

Returning to previous argument, it follows that if $\xi$ is an eigenfunction of $H_\varepsilon$ for the energy $E$ and $1 = \xi(0) = \|\xi\|_\infty$ (i.e., $\xi$ has center at 0), then for any other eigenfunction $\xi'$ with same eigenvalue $E$, $|\xi'| \leq 1$, we have

$$|\xi'(x)| < e^{-c|x|} \quad \text{for } |x| \geq \ell$$

(independently of the localization of the center of $\xi'$).

Further, according to Lemma 6.4, inside $\Lambda(0, 2\ell)$ there is localization of $\xi'$ on a union of at most $C(\gamma)$ cubes of size $\ell^\gamma$, where $\gamma > 0$ can be made arbitrary small. Thus $\xi'$ is essentially supported by $C(\gamma)$ cubes of size $\ell^\gamma$. This implies that $E$ has multiplicity at most $C(\gamma)\ell^\gamma$, for any fixed $\gamma > 0$ (again by orthogonality considerations) and this statement holds with probability at least $1 - \ell^{-\rho d}$, $\rho < \frac{3}{8}$.

Recall that we assumed that at least one eigenfunction with eigenvalue $E$ is centered at 0. From the probabilistic estimate above, we can assume instead that there is a center inside the box $\Lambda(0, \ell^{\frac{5}{4}})$, say. Consequently, in addition to AL, we obtain a finite multiplicity estimate in the following quantitative form.

**Proposition.** The following holds with $\varepsilon$-probability at least $1 - \tau$. If $E$ is an eigenvalue of $H_\varepsilon$ and there is an eigenfunction $\xi$ of $H_\varepsilon$ with energy $E$ and center $x \in \mathbb{R}^d$, then $E$ is of multiplicity at most $C(\tau, \gamma)|x|^{\gamma}$, where $\gamma > 0$ may be chosen arbitrary small.

Using the analogues developed above, one may also establish dynamical localization. Thus

**Theorem'.** Let $H = -\Delta + V$ be as in the theorem. Let $\xi \in L^2(\mathbb{R}^d)$ have sufficiently fast decay at infinity and have $H$-spectrum near the bottom (as in the theorem). Then

$$\sup_t \left[ \int (1 + |x|^2)(e^{itH}\xi)(x)^2 \right] < \infty$$

(no dispersion).

One may also derive a log-Hölder continuity property for the IDS (the weak form is due to the weakness of the Wegner estimate).

**Proposition.** With $H$ as above, we have

$$N(E) - N(E') \ll \left( \log \frac{1}{|E - E'|} \right)^{-\rho} \quad \text{for } \rho < \frac{11}{8}d$$

near the bottom of the spectrum.
References


