LOWER BOUNDS FOR TRANSVERSAL COMPLEXITY OF TORUS BUNDLES OVER THE CIRCLE

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To V. A. Vassiliev, on occasion of his 50th birthday, with sincere gratitude

Abstract. For a 3-dimensional manifold $M^3$, its complexity $c(M^3)$, introduced by S. Matveev, is the minimal number of vertices of an almost simple spine of $M^3$; in many cases it is equal to the minimal number of tetrahedra in a singular triangulation of $M^3$. Usually it is straightforward to give an upper bound for $c(M^3)$, but obtaining lower bounds remains very difficult. We consider manifolds fibered by tori over the circle, introduce transversal complexity $tc(M^3)$ for such manifolds, and give a lower bound for $tc(M^3)$ in terms of the monodromy of the fiber bundle; this estimate involves a very geometric study of the modular group action on the Farey tessellation of hyperbolic plane. As a byproduct, we construct pseudominimal spines of the manifolds fibered by tori over $S^1$. Finally, we discuss some potential applications of these ideas to other 3-manifolds.

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1. Introduction

The notion of complexity of three-dimensional manifolds was introduced by S. Matveev in 1990, see [14], [17]. This complexity is a natural “filtration” on the set of compact 3-manifolds. It is additive with respect to taking the connected sum of manifolds, and for any $k \in \mathbb{Z}$ there are only finitely many compact prime 3-manifolds of complexity at most $k$; they can be enumerated by a simple algorithm. For any compact prime 3-manifold $M$ different from $S^3$, $\mathbb{RP}^3$, $L_{3,1}$, and $S^2 \times S^1$, the complexity $c(M)$ is nothing but the minimal possible number of tetrahedra in a singular triangulation of $M$. On the other hand, these four manifolds are the only closed prime manifolds of complexity 0.
The problem of evaluating the complexity of 3-manifolds is unresolved and appears to be very difficult. The only manifolds of known complexity are those with complexity less than or equal to 11. The lists in [16] and [24] (up to complexity 6 and 7, respectively) were obtained by enumeration of all special spines of closed orientable 3-manifolds of small complexity (by the algorithm mentioned above), followed by determining which of the spines obtained are equivalent (that is, are spines of the same manifold); this “equivalence problem” is difficult. The lists of manifolds of higher complexity (so far up to 11) require more advanced technique, see the survey [18] and references therein.

Obviously, any almost simple spine (or singular triangulation) of a manifold $M$ provides an upper bound for $c(M)$. There is an algorithm for simplification of a given spine, see [15]; for all manifolds from [16] and [24], this algorithm is efficient, that is, stops at a minimal spine of a manifold. There is no proof of efficiency of this algorithm in the general case, although one can, of course, use it to find quite reasonable upper bounds for $c(M)$.

Much less is known about lower bounds. Clearly, $c(M) > 7$ whenever $M$ is not homeomorphic to any manifold from tables in [16], [24], and $c(M) > 11$ provided that $M$ does not appear in the list of manifolds up to complexity 11. Also, one can easily show that $c(M) \geq b_1(M, G) - 1$ for any commutative group $G$; here $b_1$ is the first Betti number. However, in most cases these estimates are very inadequate. Until recently, the only known way to prove that $c(M) = k$ for some $k > 0$, where $M$ is a closed prime three-manifold, was to construct a special spine of $M$ with $k$ vertices (or a singular triangulation of $M$ with $k$ tetrahedra) and verify that $M$ is not homeomorphic to any manifold of lower complexity. See [2], [9] for recent examples of infinite series of 3-manifolds of known complexity in non-compact and compact cases, respectively, and [20] for lower bounds for $c(M)$ in terms of cardinality of $\text{Tor}(H_1(M, \mathbb{Z}))$. There are also some lower bounds for other functions similar to complexity $c(M)$ (like “block complexity” $c_b(M)$ introduced in a series of preprints by the authors of [9]), but the only known relationship between $c_b(M)$ and $c(M)$ is the obvious inequality $c_b(M) \geq c(M)$.

Here we study 3-manifolds that can be fibered over the circle with torus fiber. We present “reasonable” special spines of these manifolds and discuss different ways of obtaining lower bounds for their complexity. The construction is based on encoding isotopy classes of $\theta$-curves on the torus by triangles of the Farey tesselation. The conjecture that arised is very similar to S. Matveev’s Conjecture 1 in Section 3 on the complexity of the lens spaces. Furthermore, we introduce another function related to $c(M)$, namely, transversal complexity $tc(M)$, see Definition 11, and give a lower bound for $tc(M(A))$ in terms of the monodromy $A$ of a fiber bundle $M(A) \xrightarrow{T^2} S^1$.

The paper is organized as follows: in Section 2, we give necessary definitions (following mainly [16]). In Section 4, we construct pseudominimal spines with small number of vertices for the total spaces of torus bundles over the circle; as a byproduct, in Section 5 we obtain another description of pseudominimal spines of lens spaces. Both constructions are based on the study of $\theta$-curves in tori, $\text{SL}(2, \mathbb{Z})$-action on the set $\Gamma$ of their isotopy classes (which is represented by the
set of triangles of the Farey tesselation of hyperbolic plane), flips, and the distance $d$ on $\Gamma$ defined by flips, see Section 3.

We conjecture that the spines constructed in Section 4 and Section 5 are minimal, that is, the complexity of the corresponding manifolds is equal to the number of vertices in those spines, and the rest of the paper is devoted to this conjecture. In Sections 6–9 we restrict ourselves to the spines of these spaces that are transversal to the fibers and prove that the number of vertices of any special spine with this property is at least one fifth of its conjectured value, see Theorem 8 (a similar statement holds for all Stallings manifolds, not for torus bundles only, see Section 11).

Till the end of Section 9, everything relies heavily on the study of isotopy classes of $\theta$-curves, which form an infinite binary tree $\Gamma$. A generalization of this construction is presented in Section 10; potentially, it can give a good estimate or even the exact value of $c(M)$, but this requires to solve a problem about 2-chains similar to the problem discussed in [31, Section 4]; also see [11]. Finally, in Section 11 we discuss 3-manifolds different from the total spaces of torus bundles.

Apart from study of $\theta$-curves, another approach deserves to be mentioned. For a 3-manifold $M$, consider its Turaev–Viro invariants, see [34]. The examples of these invariants constructed in [34] involve a root of unity $q$, $q^n = 1$, of arbitrary degree $n$. By construction, $TV_q(M)$ is a certain sum of products over all vertices of $P$ of some polynomial expressions in $q$ assigned to the vertices of $P$, where $P$ is a special spine of $M^3$. Obviously, the degree of $TV_q(M)$ in $q$ (or in $\sigma = q + q^{-1}$) does not exceed the maximal degree $d(q)$ (which depends on $q$ only) of an expression corresponding to a vertex of $P$ times the number of vertices of $P$, where a spine $P$ can be assumed to have the minimal possible number of vertices. So it is possible, in principle, to estimate the complexity of $M$ by dividing the degree of $TV_q(M)$ (in $q$ or in $\sigma$) over $d(q)$; since $q$ is an $n$th root of unity, it makes sense to consider the limit of that ratio as $n$ tends to infinity. We do not elaborate this idea in the present paper; note that evaluating the invariants $TV_q(M)$ is a problem difficult enough by itself, and spines constructed in Section 4 can be used in the calculations.

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2. Definitions

In this section we follow [16], [14]. By $K$ denote the 1-dimensional skeleton of the tetrahedron, which is nothing but the clique (that is, the complete graph) with 4 vertices. Note that $K$ is homeomorphic to a circle with three radii.

**Definition 1** [6]. A compact 2-dimensional polyhedron is said to be almost simple if the link of each of its points can be embedded in $K$. An almost simple polyhedron $P$ is said to be simple if the link of each point of $P$ is homeomorphic to either a circle or a circle with a diameter or the whole graph $K$. A point of an almost simple polyhedron is non-singular if its link is homeomorphic to a circle, it is called
a *triple point* if its link is homeomorphic to a circle with a diameter, and it is called a *vertex* if its link is homeomorphic to $K$. The set of singular points of a simple polyhedron $P$ (i.e., the union of the vertices and the triple lines) is called its *singular graph* and is denoted by $SP$.

It is easy to see that any compact subpolyhedron of an almost simple polyhedron is almost simple as well. Neighborhoods of non-singular and triple points of a simple polyhedron are shown in Fig. 1 (a,b); Fig. 1 (c–f) represent four equivalent ways of looking at vertices.

![Figure 1](image.png)

**Figure 1.** Nonsingular (a) and triple (b) points; ways of looking at vertices (c–f)

**Definition 2.** A simple polyhedron $P$ with at least one vertex is said to be *special* if it contains no closed triple lines (without vertices) and every connected component of $P \setminus SP$ is a 2-dimensional cell.

**Definition 3.** A polyhedron $P \subset \text{Int} M$ is called a *spine* of a compact 3-dimensional manifold $M$ if $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$ if $\partial M \neq 0$ or to an open 3-cell if $\partial M = 0$. In the other words, $P$ is a spine of $M$ if a manifold $M$ with boundary (or punctured at one point closed manifold $M$) can be collapsed onto $P$. A spine $P$ of a 3-manifold $M$ is said to be *almost simple*, *simple*, or *special* if it is an almost simple, simple, or special polyhedron, respectively.

**Definition 4.** The *complexity* $c(M)$ of a compact 3-manifold $M$ is the minimal possible number of vertices of an almost simple spine of $M$. An almost simple spine with the minimal possible number of vertices is said to be a *minimal* spine.

**Theorem 1** [14]. Any compact 3-manifold has a special spine.

**Theorem 2** [14]. Let $M$ be a compact orientable prime 3-manifold with incompressible (or empty) boundary and without essential annuli. If $c(M) > 0$ (that is, if $M$ is different from (possibly punctured) $S^3$, $\mathbb{RP}^3$, $L_{3,1}$, and $S^2 \times S^1$), then any minimal almost simple spine of $M$ is special.
Recall that a 3-manifold $M$ is said to be prime if it cannot be represented as a connected sum $M = M_1 \# M_2$ with $M_1$, $M_2$ both different from $S^3$.

**Remark 1.** In this paper, we consider the total spaces of torus bundles over the circle and lens manifolds $L_{p,q}$, $q > 3$. All these manifolds satisfy the assumptions of Theorem 2.

**Remark 2.** Starting from a special spine $P$ of a manifold $M$, one can triangulate $M$ into $n$ tetrahedra, where $n$ is the number of vertices of $P$. This singular triangulation has the only vertex somewhere inside $M \setminus P$, its edges are dual to the 2-cells of $P$, and triangles are dual to the edges of $P$. On the other hand, given a singular triangulation of $M$ containing $n$ tetrahedra, one can easily obtain a special spine of the manifold $M$ punctured at all vertices of the triangulation. It was shown in [14] that puncturing does not affect the complexity. Thus for a manifold $M$ satisfying assumptions of Theorem 2 (in particular, for any prime manifold without boundary), its complexity $c(M)$ is equal to the minimal possible number of tetrahedra in a singular triangulation of $M$, provided that $c(M) > 0$.

**Remark 3.** Let a special spine $P$ of a manifold $M$ without boundary have $n$ vertices. Since each vertex of the graph $SP$ has degree 4, $P$ contains $2n$ edges. Since the Euler characteristic of any 3-manifold equals zero and $M \setminus P$ is a 3-cell, we have the equality $n - 2n + f - 1 = 0$, which implies $f = n + 1$, where $f$ stands for the number of 2-dimensional “faces” of $P$. It follows from the construction of Remark 2 that the groups $\pi_1(M)$ and $H_1(M)$ have at most $f$ generators. Therefore, $f - 1 = c(M) \geq b_1(M) - 1$.

### 3. $\theta$-Curves and Farey Tessellation

**Definition 5.** A $\theta$-curve $L \subset T^2$ is a graph with two vertices and three edges (not loops) connecting these vertices, embedded in $T^2$ in such a way that the edges are pairwise non-homotopic; this is equivalent to the condition that the complement $T^2 \setminus L$ is a 2-dimensional cell.

![Figure 2. A $\theta$-curve](image)

Up to isotopy, any two $\theta$-curves can be taken to one another by a linear automorphism of the torus, see [3]. Another way to change the isotopy class of a $\theta$-curve is to apply a sequence of flips.

**Definition 6.** A flip along an edge of a trivalent graph embedded in a surface is an invertible restructuring of the graph that acts on a neighborhood of this edge as shown in Fig. 3. A flip does not change the graph outside of this neighborhood.
For any two θ-curves \(L_1, L_2\), there exists a sequence of flips (and isotopies) that takes \(L_1\) to \(L_2\), see [1], [3]. Now let us explain how one can find the minimal number of flips required for such a sequence. We follow here some ideas of [3] but use a very different language.

Fix a basis in the lattice \(Z^2 = \pi_1(T^2)\), i.e., choose a parallel \(σ\) and a meridian \(µ\) on the torus. Any pair of edges of a θ-curve defines a 1-cycle \(±(nσ + mµ)\) with the slope \(\frac{m}{n}\) ∈ \(Q \cup \{∞\}\); note that the fraction \(m/n\) is in lowest terms because the cycle formed by two edges is a simple curve on the torus. We have assigned a set of three slopes to any isotopy class of θ-curves. One can easily see that the set of three slopes defines a θ-curve up to isotopy.

If \(m/n\) and \(p/q\) are two slopes defined by a θ-curve, then \(mq - np = ±1\), since this expression gives the intersection index of two corresponding cycles. Furthermore, as the sum of three cycles in a θ-curve (with proper orientations) is zero, given two slopes \(m/n\) and \(p/q\), the two only possibilities for the third slope of a θ-curve are \(\frac{m+p}{n+q}\) and \(\frac{m-p}{n-q}\).

The natural action of \(SL(2, Z)\) on the set of bases of \(\pi_1(T^2)\) induces the following action of the modular group \(SL(2, Z)/\{±I\}\) on the set of slopes:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{m}{n} \mapsto \frac{am + bn}{cm + dn} = a(m/n) + b \frac{c(m/n) + d}{c(m/n) + d}.
\]

(1)

This action is transitive on the set of triples of slopes. Indeed, one can map the first slope to \(∞\) and then two other slopes to 0 and 1, so any triple of slopes can be taken to \(\{0, 1, ∞\}\) by a modular map.

Consider the Poincaré model of hyperbolic plane \(H^2\) in the upper halfplane \(\text{Im} z > 0\). The modular group acts on \(H^2\) by the same rule (1): \(z \mapsto \frac{az+b}{cz+d}\).

**Definition 7.** (Compare with [28].) The Farey tessellation is the tessellation of hyperbolic plane (in the Poincaré model) into the images of the triangle \(∆_0 = (0, 1, ∞)\) under the modular group action.

Equivalently, the Farey tessellation can be defined as follows. Consider the \(SL(2, Z)/\{±I\}\)-orbit of the point \(e^{πi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i\). Then the Voronoi diagram of this orbit (that is, the set of points \(x \in H^2\) having more than one nearest \(^1\) point of the orbit) is the set of edges of Farey triangles and Voronoi domains coincide with Farey triangles. Yet another equivalent way to construct the Farey tessellation is to start with the triangle \(∆_0\) and reflect it repeatedly in its sides, sides of its reflections, and so on; see Fig. 4.

It follows from the definitions that the set of Farey vertices is \(Q \cup \{∞\}\) (same as the set of slopes in \(T^2\)); the modular group acts transitively on this set and can

\(^1\)With respect to hyperbolic distance.
take 0 anywhere. Further, \(m/n\) and \(p/q\) are connected by a Farey edge if and only if \(mq - np = \pm 1\) (check it for \(m/n = 1/0\) and combine transitivity of the group action with the invariance of \(mq - np\) under that action), which means that Farey edges represent pairs of slopes appearing in \(\theta\)-curves.

**Theorem 3.** Fixing a basis in \(\pi_1(T^2)\) gives a one-to-one correspondence between the isotopy classes of \(\theta\)-curves in \(T^2\) and the Farey triangles. Furthermore, \(\theta\)-curves related by a flip correspond to neighboring Farey triangles.

**Proof.** The first statement follows from the above. The second statement is a consequence of the fact that a flip preserves two out of three slopes. \(\square\)

Consider the graph \(\Gamma\) dual to the Farey tessellation. This graph is shown in Fig. 4 by dotted lines. It is an infinite binary tree (there are no cycles, because for any edge of \(\Gamma\) there exists a Farey edge that cuts \(H^2\) into two pieces and intersects \(\Gamma\) only once). By Theorem 3, vertices of \(\Gamma\) represent the isotopy classes of \(\theta\)-curves and edges of \(\Gamma\) correspond to flips. Note that \(\Gamma\) is invariant under the modular transformations.

**Definition 8.** Let \(A\) be a modular transformation. Its **complexity** \(c(A)\) is the minimal distance in \(\Gamma\) between a vertex \(x \in \Gamma\) and its image \(A(x) \in \Gamma\). In other words, \(c(A)\) is the minimal number of flips required to convert a \(\theta\)-curve \(L\) to its image \(\tilde{A}(L)\) under an \(\text{SL}(2, \mathbb{Z})\) transformation \(\tilde{A}\) corresponding to \(A\); note that the minimum is taken over all \(\theta\)-curves \(L\). Note also that \(c(A)\) depends on the conjugacy class of \(A\) only and \(c(A) = c(A^{-1})\). For an operator \(\tilde{A} \in \text{SL}(2, \mathbb{Z})\), we define its complexity \(c(\tilde{A})\) as \(c(A)\), where \(A\) is the corresponding modular transformation.

Modular transformations are isometries of \(H^2\). A modular transformation different from identity has either one fixed point in \(H^2\) and no fixed points in \(\partial H^2\), or no fixed points in \(H^2\) and one or two fixed points in \(\partial H^2\). In these cases, it is called **elliptic**, **parabolic** or **hyperbolic**, respectively. Let \(\tilde{A} \in \text{SL}(2, \mathbb{Z})\) be a matrix representing a modular transformation \(A\) (given a basis, \(\tilde{A}\) is defined up to a sign). Then \(A\) is elliptic if \(|\text{tr} \tilde{A}| < 2\), parabolic if \(|\text{tr} \tilde{A}| = 2\), and hyperbolic if \(|\text{tr} \tilde{A}| > 2\), see [7], [29], [30]. The complexity \(c(A)\) can now be determined as follows.

If \(A\) is elliptic, then \(c(A) = 0\) or \(c(A) = 1\). Indeed, the fixed point \(x\) of \(A\) lies either in a Farey triangle \(\Delta\) or on a Farey edge. In the first case, the triangle \(\Delta\) must
be taken to itself by $A$, which means that $x$ is the center of $\Delta$, that is, the vertex of $\Gamma$ that lies in $\Delta$, and $c(A) = 0$. In the second case, $A$ is the symmetry in the midpoint of the edge (the intersection point of the Farey edge and the corresponding edge of $\Gamma$), and $c(A) = 1$.

If $A$ is parabolic, the equation for its fixed point is linear with integer coefficients; it has an element of $\mathbb{Q} \cup \{\infty\}$ as the solution. Therefore, the fixed point $y \in \partial H^2$ of a parabolic modular transform is a Farey vertex. Consider the set of Farey triangles with a vertex $y$. The subgraph $\Gamma_A \subset \Gamma$ whose vertices are dual to these triangles is an infinite chain mimicking an oricycle. All vertices $x \in \Gamma_A$ get moved by the same distance $c(A)$ by $A$. Any vertex $x \in \Gamma \setminus \Gamma_A$ gets moved by a greater distance $c(A) + 2d$, where $d$ is the distance from $x$ to $\Gamma_A$, because $\Gamma$ is a tree: the only way from $x$ to $A(x)$ consists of the segment from $x$ to the point $x_0 \in \Gamma_A$ nearest to $x$, the segment from $x_0$ to $A(x_0)$, and the segment from $A(x_0)$ to $A(x)$. To find $c(A)$ in a less geometrical way, conjugate $A$ so that its fixed point is $\infty$; then $A$ takes the form $z \mapsto z + c$ with $c \in \mathbb{Z} \setminus \{0\}$, and $c(A) = |c|$.

If $A$ is hyperbolic, its fixed points $y, z \in \partial H^2$ are conjugate quadratic irrationals. The geometric recipe to compute $c(A)$ is as follows. Draw the geodesic line connecting $y$ to $z$; it is $A$-invariant. Consider the set of Farey triangles crossed by that line; this set is $A$-invariant, too. Let $\Gamma_A$ be the subgraph of $\Gamma$ whose vertices are centers of the triangles that have nonempty intersection with the line $yz$. As in the previous case, $\Gamma_A$ is an infinite chain, and any vertex $x \in \Gamma_A$ gets moved by the same distance $c(A)$, while any other vertex $x \in \Gamma \setminus \Gamma_A$ gets moved by a greater distance $c(A) + 2d$, where $d$ is the distance from $x$ to $\Gamma_A$.

Here is an algebraic way to compute $c(A)$ for hyperbolic $A$. First, conjugate $A$ so that its fixed points have opposite signs. This guarantees that the line $yz$ intersects the Farey edge $(0, \infty)$ and the neighboring Farey triangles, in particular, the triangle $\Delta_0 = (0, 1, \infty)$. Suppose that the negative fixed point of $A$ is repelling and the positive fixed point is attracting (replace $A$ with $A^{-1}$ if necessary). Then all vertices of the triangle $A(\Delta_0)$ are positive rationals, for otherwise one of its vertices is $0$ or $\infty$, which means that the chain $\Gamma_A$ always turns to the left or always turns to the right, and $A$ is parabolic. If $A(z) = \frac{az + b}{cz + d}$, then these vertices are $a/c, b/d$, and $(a + b)/(c + d)$, where $a, b, c, d$ are positive integers and $ad - bc = 1$.

**Definition 9.** Let $p, q$ be coprime positive integers. The *Euclid complexity* $E(p, q)$ is the number of subtractions (not divisions!) that the Euclid algorithm takes to convert the pair $(p, q)$ into the pair $(0, 1)$. It is easy to see that $E(p, q)$ equals the sum of the denominators of the continued fraction representing any of the rational numbers $p/q$ and $q/p$, that is, $E(p, q) = n_1 + \ldots + n_k$, where

$$\frac{p}{q} = n_1 \left\{ \begin{array}{c} 1 \\ n_2 \left\{ \begin{array}{c} 1 \\ \vdots \\ n_k \end{array} \right. \end{array} \right. \right.$$

A good exposition of the Euclid algorithm and continued fraction theory can be found in [8], [12], [35].
Theorem 4. Let $A(z) = \frac{az+b}{cz+d}$ be a hyperbolic modular map, where $a, b, c, d$ are positive integers. Then $c(A) = E(a, b)$.

Proof. Similar statements can be found in [3] and [28], so we give the sketch of the proof only. We want to find the length of the chain of Farey triangles $\Delta_0$, $\Delta_1$, $\ldots$, $\Delta_{c(A)} = A(\Delta_0)$. The last triangle in this chain, $\Delta_{c(A)}$, has vertices $a/c$, $b/d$, and $(a+b)/(c+d)$. Then $\Delta_{c(A)-1} = (a/c, b/d, (a-b)/(c-d))$, by the properties of the Farey tesselation discussed above; if, say, $a > b$, we can write $\Delta_{c(A)-1} = (p/q, b/d, (p+b)/(q+d))$, where $p = a-b > 0$, $q = c-d \geq 0$, and $pd - bq = ad - bc = 1$. This shows that a step back in the chain of Farey triangles corresponds to a subtraction forward in the Euclid algorithm, which recycles an unordered pair of coprime positive integers $(a, b)$ into the unordered pair $(1, 0)$. This procedure takes exactly $n_1 + \ldots + n_k$ subtractions, where the $n_i$ are the entries of the continued fraction representing $a/b$. \qed

Theorem 4 has a beautiful geometrical interpretation. As $\Gamma$ is a trivalent graph, any path in $\Gamma$ is described by initial position, initial direction, and a word in the alphabet $\{l, r\}$, which prescribes a sequence of left and right turns. It follows from the proof of Theorem 4 (see also [3], [23], [28]) that the part of the infinite chain $\Gamma_A$ between its entrance into $\Delta_0$ and its entrance into $\Delta_{c(A)}$ is encoded by $l^{n_1}r^{n_2}\ldots l^{n_k}$ or $l^{n_1}r^{n_2}\ldots r^{n_k}$, according to the parity of $k$. This word is the period of the infinite word corresponding to $\Gamma_A$, though not necessarily the shortest period.

The number-theoretic function $E(p, q)$ appears also in the following context.

Conjecture 1 [16], [14]. The complexity of the lens space $L_{p,q}$ is equal to $c(L_{p,q}) = E(p, q) - 3$.

4. Spines of Torus Bundles over the Circle

In what follows, a three-manifold $M$ is the total space of an orientable $T^2$-bundle over the circle and $A \in SL(2,\mathbb{Z})$ is the monodromy operator (which acts on the one-dimensional homology group of the fiber containing the base point of $M$) of the bundle. The operator $A$ is defined up to $GL(2,\mathbb{Z})$-conjugation. By $M(A)$ denote the manifold $M$ defined by the monodromy operator $A$.

In this section, we construct a pseudomiminal (see Definition 10 below) special spine of $M(A)$ with $\max(6, c(A) + 5)$ vertices. We use here the constructions from the previous section. By abuse of notation, we will use the same notation $A$ for $A \in SL(2,\mathbb{Z})/\pm\{I\}$ and $\tilde{A} \in SL(2,\mathbb{Z})$. First, let us note that the number $c(A)$ is well defined by the manifold $M^\tilde{A}$ because of the following statement.

Theorem 5 [32]. Let $A, B \in SL(2,\mathbb{Z})$. Suppose that 3-manifolds $M(A)$ and $M(B)$ are homeomorphic. Then $B$ is $GL(2,\mathbb{Z})$-conjugate to either $A$ or $A^{-1}$.

If $A$ and $B$ are $SL(2,\mathbb{Z})$-conjugate, then $c(A) = c(B)$. Further, $c(A^{-1}) = c(A)$. As the group $GL(2,\mathbb{Z})$ is generated by its subgroup $SL(2,\mathbb{Z})$ and the element $C = \begin{pmatrix}0 & 1 \\1 & 0 \end{pmatrix}$, it now suffices to check that conjugation by $C$ does not affect $c(A)$; however, this is clear from Definition 8.
Definition 10 [16]. A 2-dimensional component $\alpha$ of a special polyhedron has a counterpass if its boundary $\partial \alpha$ passes along some edge of $SP$ in both directions; it is called a component with short boundary if $\partial \alpha$ passes through at most 3 vertices and visits any of them only once. A special spine of a 3-dimensional manifold is said to be pseudominimal if it contains neither components with counterpasses nor components with short boundaries.

Figure 5. A simplification move (left) and the corresponding Pachner move (right)

If a special spine $P$ is not pseudominimal, it is not minimal, because one can apply simplification moves (see [16]) to $P$ and get an almost simple spine with a smaller number of vertices. For example, Figure 5 shows the effect of a simplification move applied to a special spine with a triangular component (the middle horizontal triangle in the left part of Fig. 5); it is easy to see that the neighborhood of a 2-cell with short boundary of length 3 in a special polyhedron $P$ looks like the left hand side of Fig. 5. This move does not change the spine outside of the fragment shown in Fig. 5. Note that the spine obtained is special again: the move produces neither closed triple lines nor non-cellular 2-dimensional components.

Remark. Consider the singular triangulation dual to a special spine with a triangular component. Then the simplification move shown in Fig. 5 corresponds to the three-dimensional $(3, 2)$ Pachner move [25], which replaces three tetrahedra by two tetrahedra. In the two-dimensional case, a flip (see Fig. 3) corresponds to the $(2, 2)$ Pachner move, which switches the diagonal in a quadrilateral formed by two neighboring triangles. Recall that the move shown in Fig. 5 and its inverse are sufficient to convert any two special spines of the same compact three-dimensional manifold to one another, see [13]; this fact is crucial for the construction of the Turaev–Viro invariants [34].

Figure 6 represents another simplification move, which is applicable to spines containing a component with short boundary of length 2; clearly, the neighborhood of this component looks like the left hand side of Fig. 6. This simplification move yields a simple, but not necessarily special, spine of the same manifold (provided that the move had been applied to a simple spine).

To construct a spine of $M(A)$ with not too many vertices, let us suppose that the chain $\Gamma_A \subset \Gamma$ (defined for parabolic and hyperbolic $A$ at the end of the previous
section) passes via the Farey triangle $\Delta_0 = (0, 1, \infty)$. This can be achieved by conjugation; the idea is to minimize the distance in $\Gamma$ between $\Delta_0$ and $\Delta(\Delta_0)$. The case of elliptic $A$ is simple and will be dealt with separately.

Now consider a fiber $T^2 \times \{0\}$ and choose a $\theta$-curve $L_0 \subset T^2 \times \{0\}$ in the isotopy class represented by $\Delta_0$; by doing so, we also fix a $\theta$-curve $L_1$ in $T^2 \times \{1\}$; its isotopy class is $\Delta(L_1) = \Delta(\Delta_0)$. By the results of the previous section and [3], there exists a continuous family $L_t$ transforming $L_0$ into $L_1$ by isotopy and $c(A)$ flips. Set $P_0 = \bigcup_{t \in [0,1]} L_t$; we assume that each $L_t$ is embedded in $T^2 \times \{t\}$. Note that $P_0$ is a simple polyhedron, which is a spine of some punctured torus bundle. Two-dimensional components of $P_0$ come from edges of $L_t$ as $t$ varies; similarly, one-dimensional components of $P_0$ come from vertices of $L_t$. The $c(A)$ flips correspond to the vertices of $P_0$, see Fig. 7.

**Examples.** 1. Recall that $c(A)$ equals 0 or 1 for an elliptic operator $A$. Elliptic operators $A$ with $c(A) = 1$ are conjugate to $\pm \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. This is a very interesting case. The polyhedron $P_0$ constructed above has one vertex. Consider the two-sheeted covering of the base $S^1$ of the fibering. It induces the two-sheeted covering of the total space by the manifold $M(A^2) = M(-I)$. The preimage of $P_0$ under the covering is a polyhedron in $M(-I)$ with two vertices, which can be cancelled by...
the second simplification move (in two different ways). This is the only case where $c(A^k) < |k|c(A)$ for $k \in \mathbb{Z}$, $|k| \geq 2$.

2. If $c(A) = 0$, there are no flips at all. In this case $P_0$ contains no vertices and consists of three orientable annuli and three edges if $A = I$, of three nonorientable annuli and one edge if $A = -I$, of one nonorientable annulus and one edge if $A$ is conjugate to the matrix $R_{\pi/3} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or its inverse (note that $R_{6\pi/3} = I$), and of one orientable annulus and two edges if $A$ is conjugate to $R_{2\pi/3}$ or $R_{4\pi/3}$. This exhausts the case $c(A) = 0$.

The polyhedron $P_0$ is not a spine of $M(A)$, because the fibered space $M(A)$ admits a section that does not intersect $P_0$. This section represents a nontrivial element of the group $\pi_1(M(A))$, while the complement to a spine of a closed manifold is a cell and hence cannot contain nontrivial loops. Let us put $P_1 = P_0 \cup (T^2 \times \{0\})$.

**Lemma 1.** $P_1$ is a spine of $M(A)$.

*Proof.* It is sufficient to show that $M(A) \setminus P_1$ is a 3-dimensional cell. We have $M(A) \setminus P_1 = T^2 \times \{(0, 1)\} \setminus P_0 = T^2 \times \{(0, 1)\} \setminus \bigcup_{t \in (0, 1)} L_t = \bigcup_{t \in (0, 1)} (T^2 \times \{t\} \setminus L_t)$, and the statement follows. □

Note that $P_1$ is not a simple polyhedron. Indeed, its part $T^2 \times \{0\}$ contains a singular subset $L_0$, which is more complicated than a triple line: three edges of $L_0$ yield three lines of transversal intersection of two surfaces, and any of two vertices of $L_0$ gives rise to a transversal intersection of a triple line with one extra surface.

Let us modify the previous construction by gluing $T^2 \times \{0\}$ with $T^2 \times \{1\}$ along a homeomorphism $A + \delta$ instead of $A$, where $\delta$ is a small shift of the torus in a direction transversal to the edges of $L_0$, see Fig. 8. Put $P_2 = \bigcup_{t \in [0, 1]} L_t \cup (T^2 \times \{0\})$. Again, $P_2$ is a spine of $M(A)$.

![Figure 8. $\theta$-Curve $L_0$ and its $\delta$-shift $L_1$ (shown by dashed lines) in the fiber $T^2 \times \{0\}$ of $M(A)$](https://via.placeholder.com/150)

**Lemma 2.** $P_2$ is a special spine of $M(A)$ with $c(A) + 6$ vertices.

*Proof.* It is clear from the construction that $P_2$ is a simple polyhedron. Its triple lines are the “trajectories” (as $t$ varies) of the vertices of $L_t$ and the ten segments of $L_0$ and $L_1$ shown in Fig. 8, where the torus is represented by a square with the
opposite sides to be identified. There are \( c(A) \) vertices of \( P_2 \) that correspond to \( c(A) \) flips between \( L_0 \) at \( t = 0 \) and \( L_1 \) at \( t = 1 \), and six other vertices that are drawn in Fig. 8. Two of them arise from \( T^2 \times \{0\} \) and \( L_t \), \( 0 \leq t < \varepsilon \), and their neighborhoods in \( P_2 \) look like Fig. 1(d). The last two vertices in Fig. 8 correspond to two intersection points of \( L_0 \) and \( L_1 \), and their neighborhoods look like Fig. 1(c).

Thus \( P_2 \) is a simple spine of \( M(A) \) with \( c(A) + 6 \) vertices.

It remains to prove that \( SP_2 \) (the singular part of \( P_2 \), see Definition 2) contains no closed triple lines and all connected components of \( P_2 \) are 2-dimensional cells. First group of triple lines of \( SP_2 \) is formed by ten arcs in \( T^2 \times \{0\} \) shown in Fig. 8. Obviously, they are not closed. The rest \( 2c(A) + 2 \) triple lines are swept by the vertices of the \( \theta \)-curves \( L_t \subset T^2 \times \{t\} \), \( 0 < t < 1 \). They end at vertices of \( P_2 \), too, and thus are not closed.

Connected components of \( P_2 \) also belong to two groups. Four of them, two hexagonal and two quadrilateral, lie in the fiber \( T^2 \times \{0\} \), see Fig. 8. They are cells. Any other connected component of \( P_2 \) intersects any fiber \( T^2 \times \{t\} \), \( a < t < b \) (where \( a \) is equal to either 0 or one of the flip moments, and \( b \) is either one of the flip moments or is equal to 1), along one edge of \( L_t \), and does not intersect other fibers; this implies that this component is a cell. We have proved that the polyhedron \( P_2 \) is special.

\[ \square \]

**Corollary.** \( c(M(A)) \leq c(A) + 6 \).

\[ \square \]

**Example.** Three-dimensional torus can be represented as \( M(I) \), \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Since \( c(I) = 0 \), the construction above gives a special spine of \( T^3 \) with six vertices. The manifold \( T^3 = M(I) \) is contained in Table 7 of the preprint [16] under the name 671. It is shown in [16] that all manifolds of complexity at most 5 are different from \( T^3 \). So we have \( c(T^3) = 6 \). The spine that we constructed here does not differ from the spine 671 from [16, Section 5.2], while our way of presenting spines differs significantly from the one used in [16].

\[ \square \]

**Figure 9.** The complement \( T^3 \setminus P_2 \) of the minimal spine \( P_1 \) of \( T^3 \)

The torus \( T^3 \) can be obtained from the cube by gluing its opposite faces. This yields a natural cell decomposition of \( T^3 \) with one vertex, three edges, three 2-dimensional cells and one 3-dimensional cell. The 2-dimensional skeleton \( sk_2(T^3) \) has...
singular points more complicated than triple lines and vertices of simple polyhedra. However, the minimal spine of $T^3$ can be obtained as a small perturbation of $\text{sk}_2(T^3)$.

Let the $\theta$-curves $L_t$, $t \in [0, 1]$, be very close to the bouquet of a parallel and a meridian of $T^2 \times \{t\}$, and let the shift $\delta$ involved in the construction of $P_2$ be very small. Then the 3-dimensional cell $T^3 \setminus P_2$ is very close to the 3-dimensional cube. Figure 9 represents this cell. If we identify opposite faces of this polyhedron by parallel transports (or, equivalently, tessellate $\mathbb{R}^3$ into parallel copies of this polyhedron and consider a quotient over the appropriate lattice $\mathbb{Z}^3$), we get the torus $T^3$; the image of the boundary of the polyhedron under this gluing is the minimal spine of $T^3$ close to $\text{sk}_2(T^3)$.

The same construction gives special spines with six vertices for the manifolds $M(-I) = 6_{70}$, $M((-1 -1 0)) = M((-1 -1 1)) = 6_{67}$, and $M((0 1 -1 0)) = M((0 1 -1 1)) = 6_{65}$. The spines constructed in this way are minimal spines of these manifolds, because all of them are of complexity 6; in fact, all manifolds of complexity up to 5 are quotient spaces of the sphere $S^3$, see [16].

However, in all other cases (that is, if $c(A) > 0$) the spines with $c(A) + 6$ vertices are not minimal spines of the manifolds $M(A)$. For example, the manifolds $6_{66} = M\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$, $6_{68} = M\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right)$, and $6_{69} = M\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)$ have complexity 6, while

$$c\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = c\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right) = c\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = 1,$$

and our construction gives their spines with 7 vertices.

This happens because some of the spines with $c(A) + 6$ vertices constructed above are not pseudominimal whenever $c(A) > 0$. Namely, they have a triangular component, and the first simplification move (see Fig. 5) can be applied.

Let us return to Fig. 8. Assume that the first flip in the sequence taking $L_0$ to $L_1$ involves the short edge of $L_0$, that is, the edge that does not intersect dashed lines in Fig. 8. This condition can be satisfied by an appropriate choice of the shift $\delta$ involved in the construction of $P_2$. Then the 2-dimensional cell of $P_2$ adjacent to this edge and not contained in $T^2 \times \{0\}$ is a triangle, and we can apply the first simplification move, which gives a spine of $M(A)$ with a smaller number of vertices. This spine can be described in other words as follows. Let $L'$ be the $\theta$-curve obtained after the first of $c(A)$ flips converting $L_0$ into $L_1$. Glue the square from Fig. 10 into the torus $T^2 \times \{1\} \in M(A)$ and embed $L_t$ in $T^2 \times \{t\}$ for all $t \in (0, 1)$, where the family $L_t$ contains $c(A) - 1$ flips and connects $L'$ with $L_1$. Note that the first of $c(A) - 1$ flips converting $L'$ into $L_1$ is performed along a long edge of $L'$, because
a flip along the short edge would annihilate with the flip converting $L_0$ to $L'$. The new spine $P_3$ has $6 + c(A) - 1 = c(A) + 5$ vertices. So we have $c(M(A)) \leq c(A) + 5$ whenever $c(A) > 0$.

**Figure 10.** $\theta$-Curves $L'$ and $L_1$ (dashed) in $T^2 \times \{0\}$

The proof that the spine $P_3$ is special repeats the proof of Lemma 2.

**Theorem 6.** (1) $c(M(A)) \leq \max(6, c(A) + 5)$.

(2) The spines $P_3$ constructed above are pseudominimal.

Compare the first statement with Corollary of Lemma 2.

**Proof.** If $c(A) = 0$, then $c(M(A)) = 6$. So we may suppose that $c(A) > 0$. Since $P_3$ is an almost simple (and even special) spine of $M(A)$ with $c(A) + 5$ vertices, the first statement is obvious. The argument similar to the proof of Lemma 2 shows that two-dimensional cells of $P_3$ have no counterpasses. So we only have to show that $P_3$ has no components with short boundaries. The four cells contained in $T^2 \times \{0\}$ are pentagons, see Fig. 10. The cells that have no boundary edges in $T^2 \times \{0\}$ have even numbers of edges, namely, $2k - 2$, where $k$ is the number of flips from the vertex where the cell appears to the vertex where the cell disappears (including both the first flip and the last one). The sequence of $c(A)$ flips used in the construction of $P_3$ is the shortest possible one; this implies that any two consecutive flips in it are not inverse to one another, that is, $k > 2$ for any cell considered above.

It remains to consider at most 6 two-dimensional cells that have an edge in $T^2 \times \{0\} = T^2 \times \{1\}$ (if $A$ is, up to a sign, a power of a Jordan block, there are only 5 cells of this type; otherwise, no 2-cell touches both $T^2 \times \{0\}$ along an edge of $L'$ and $T^2 \times \{1\}$ along an edge of $L_1$, so three edges of $L'$ and three edges of $L_1$ belong to six different cells of $P_3 \setminus T^2 \times \{0\}$). Two cells of $P_3 \setminus T^2 \times \{0\}$ are adjacent to the long edges of $L'$ (the edges that intersect dashed lines in Fig. 10). Each of these cells has at least 4 boundary edges: two segments of a long edge of $L'$ and two edges (transversal to fibers) that arise from the vertices of $L'$. The same argument works for two cells of $P_3 \setminus T^2 \times \{0\}$ adjacent to the long edges of $L_1$. Consider the cell of $P_3 \setminus T^2 \times \{0\}$ adjacent to the short edge of $L'$. Of course, it has at least 3 edges: the short edge of $L'$ and two edges that are trajectories of the vertices of $L'$ as $t$ varies. It has at least one more edge; otherwise, the first flip in the sequence of flips converting $L'$ to $L_1$ is performed along the short edge of $L'$, which is impossible by the construction of $P_3$, see above. For the same reason,
the last flip (which results in \( L_1 \)) cannot be performed along the short edge of \( L_1 \); indeed, it cannot be cancelled with the flip connecting \( L_0 \) with \( L' \). This means that the 2-dimensional cell of \( P_3 \setminus T^2 \times \{1\} \) adjacent to the short edge of \( L_1 \) also has more than 3 edges, and \( P_3 \) contains no components with short boundaries. The theorem is proved.

\[ \square \]

**Conjecture 2.** The pseudominimal spines of the manifolds \( M(A) \) constructed above are in fact their minimal spines, and the upper bound for complexity given in Theorem 6 is in fact its exact value: \( c(M(A)) = \max(6, c(A) + 5) \) for any monodromy operator \( A \in \text{SL}(2, \mathbb{Z}) \). In other words, any singular triangulation of \( M(A) \) involves at least \( c(A) + 5 \) tetrahedra if \( c(A) > 0 \) and 6 tetrahedra if \( c(A) = 0 \).

5. Digression: Spines of Lens Manifolds

In [16], a construction is presented, which gives pseudominimal special spines of the lens spaces \( L_{p,q} \), \( p > 3 \), with exactly \( E(p, q) - 3 \) vertices. In that paper, spines are presented by drawing the neighborhood of the singular graph of a spine. This allows one to draw spines on the plane; however, it remains unclear how the spines are embedded into corresponding manifolds.

In this section, we construct pseudominimal special spines of \( L_{p,q} \), \( p > 3 \), with \( E(p, q) - 3 \) vertices, making use of the techniques developed in Sections 3, 4. We omit some details and proofs.

Consider two solid tori. The meridians of their boundary tori are well defined, while the parallels are defined modulo meridians only. Let \( \mu_0, \mu_1 \) be the meridians of the tori and \( \sigma_0, \sigma_1 \) be their parallels such that the pair of the oriented cycles \((\sigma_0, \mu_0)\) defines the positive orientation of the boundary of the first torus and the pair \((\sigma_1, \mu_1)\) defines the negative orientation of the boundary of the second torus. There is a unique pair of positive integer numbers \((r, s)\) such that \( r < p, s < p, \) and \( qs - pr = 1 \). Put \( A = \begin{pmatrix} s & p \\ r & q \end{pmatrix} \) and attach the solid tori to one another so that the induced homomorphism of the one-dimensional homology groups of their boundary tori has the matrix \( A \) (in the bases \((\sigma_0, \mu_0)\) and \((\sigma_1, \mu_1)\)). We get a closed orientable 3-manifold that is nothing but \( L_{p,q} \).

Note that \( A \in \text{SL}(2, \mathbb{Z}), c(A) = E(p, q) \), and the parallels \( \sigma_0 \) and \( \sigma_1 \) represent nontrivial elements of \( \pi_1(L_{p,q}) = \mathbb{Z}_p \). This implies that any spine of \( L_{p,q} \) intersects these loops. Let us shift \( \sigma_0 \) in the interior of the first solid torus and consider the tubular neighborhood \( U_0 \) of the shifted curve. Obviously, \( U_0 \) is a solid torus. Similarly, construct \( U_1 \) as a tubular neighborhood of \( \sigma_1 \) shifted inside of the interior of the second torus. We may assume \( U_0 \) and \( U_1 \) to be disjoint. Then \( L_{p,q} = U_0 \cup (T^2 \times [0, 1]) \cup U_1 \). Let \( L_i, i = 0, 1, \) be standard (that is, corresponding to the Farey triangle \( \Delta_0 = (0, 1, \infty) \) with respect to the bases \((\sigma_i, \mu_i)\) \( \theta \)-curves in the tori \( T^2 = \partial U_i \); they are defined up to isotopy. Following the construction of Section 4, consider a continuous family \( L_t \subset T^2 \times \{t\} \) connecting \( L_0 \) to \( L_1 \) by isotopy and \( c(A) \) flips. Put \( P_0 = \bigcup_{t \in [0, 1]} L_t \). Let \( D_i, i = 0, 1, \) be meridional disks of the \( U_i \) intersecting \( L_i \) transversally at one point. Put \( P_1 = D_0 \cup T^2_0 \cup P_3 \cup T^2_1 \cup D_1 \).
Lemma 3. The polyhedron $P_1$ is a special spine of $L_{p,q}$ with three punctures. It has $E(p, q) + 6$ vertices.

Proof. The complement $L_{p,q} \setminus P_1$ consists of three cells $U_0 \setminus D_0, (T^2 \times [0, 1]) \setminus P_0$, and $U_1 \setminus D_1$. There are $c(A) = E(p, q)$ vertices in the interior part of $P_0$. Further, there are 3 vertices on $T_0^3$, which correspond to two vertices of $L_0$ and the intersection point of $L_0$ and $\partial D_0$. Similarly, there are 3 vertices of $P_1$ on $T_1^3$. It remains to show that $P_1$ is a special polyhedron. This can be proven by analogy with Lemma 2. □

Below we show that one can decrease the number of vertices “inside of $P_0$” by one and the number of vertices “near each $U_i$,” by four. This gives a spine with $E(p, q) + 6 - 1 - 4 - 4 = E(p, q) - 3$ vertices.

Recall that the parallels $L_i$ are defined only modulo meridians $\mu_i$. Thus, the $\theta$-curves $L_i$ are defined only up to powers of the Dehn twists along the meridians, that is, up to transformations $\sigma_i \mapsto \sigma_i + n_i \mu_i, n_i \in \mathbb{Z}$. By varying $n_0$ and $n_1$, one can decrease the distance in the graph $\Gamma$ (see Section 3) between $B^{n_0} \Delta_0$ and $C^{n_1} A \Delta_0$ and thus decrease the number of the vertices inside of $P_0$; here $\Delta_0$ is the “base” Farey triangle, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is the matrix representing the Dehn twist along $\mu_0$, and $C = ABA^{-1}$ is the matrix corresponding to the Dehn twist along $\mu_1$.

Lemma 4. We have $\min_{n_0, n_1 \in \mathbb{Z}} d(B^{n_0} \Delta_0, C^{n_1} A \Delta_0) = E(p, q) - 1$, where $d$ is the distance in the graph $\Gamma$.

Proof. Since the graph $\Gamma$ is a tree, there are integers $m_0, m_1$ such that the path from $B^{n_0} \Delta_0$ to $C^{n_1} A \Delta_0$ for any $n_0, n_1 \in \mathbb{Z}$ consists of the following three legs: from $B^{n_0} \Delta_0$ to $B^{m_0} \Delta_0$, from $B^{m_0} \Delta_0$ to $C^{m_1} A \Delta_0$, and from $C^{m_1} A \Delta_0$ to $C^{n_1} A \Delta_0$. Now it is obvious that $\min_{n_0, n_1 \in \mathbb{Z}} d(B^{n_0} \Delta_0, C^{n_1} A \Delta_0) = d(B^{m_0} \Delta_0, C^{n_1} A \Delta_0)$. By considering the three legs of the path from $\Delta_0$ to $A \Delta_0$, one can see that $m_0 = 1$ (because $p > q > 0$), $m_1 = 0$ (because both positive and negative Dehn twists along $\mu_1$ do not affect the vertex $(p/q)$ of $A \Delta_0$ most distant from $B \Delta_0$ and thus only increase the distance to $\Delta_0$), and the length of the middle leg of this path is $d(B \Delta_0, A \Delta_0) = d(\Delta_0, A \Delta_0) - |m_0| - |m_1| = E(p, q) - 1$. □

By virtue of Lemma 4, we can decrease by one the number of the vertices inside of $P_1$ by another choice of a $\theta$-curve $L_0$. Now we have a special spine of $L_{p,q}$ with three...
punctures having \( E(p, q) + 5 \) vertices. The disks \( D_0 \) and \( D_1 \) are components with short boundaries. By \( \tau_i \) denote the edge of \( L_i \) that intersects \( \partial D_i \). Note that two other edges of \( L_i \) form the meridian of \( T_i \), the first flip in the sequence connecting \( L_0 \) with \( L_1 \) is performed along \( \tau_0 \) while the last flip in this sequence is performed along \( \tau_i \) (flips along other edges are equivalent to meridional Dehn twists and thus do not lead out of the sets \( \Gamma_B = \{ B^n \Delta_0; n_0 \in \mathbb{Z} \} \) and \( \Gamma_C = \{ C^n \Delta_3; n_3 \in \mathbb{Z} \} \), while the path between \( L_0 \) and \( L_1 \) (that is, between \( B \Delta_0 \) and \( A \Delta_0 \)) is the shortest path that connects \( \Gamma_B \) and \( \Gamma_C \). We can apply the following simplification move in the neighborhood of \( D_i \). First, add a parallel copy \( D_i' \) of \( D_i \). Second, delete the lateral surface of the cylinder bounded by \( D_i, D_i' \), and a thin strip of \( T_i^2 \). Finally, delete the cell of \( P_1 \) adjacent to \( \tau_i \); this cell is triangular, because \( \tau_i \) is the edge involved in the flip in \( P_0 \) closest to \( T_i^2 \), see Fig. 11. So, the first step adds one vertex on each \( T_i^2 \), the second step kills two vertices on each \( T_i^2 \), and the last step kills three vertices near each of \( T_i^2 \). By \( P_1 \) denote the polyhedron obtained by the construction above. One can easily see that it has \( E(p, q) - 3 \) vertices. Further, one can see that two remaining edges of \( L_i \) (which differ from \( \tau_i \)) form a closed triple line \( S_i^1 \) and the complement \( L_{p, q} \setminus P_1 \) still consists of three cells, two of which are bounded by the spheres \( S_i^2 \) obtained from the torus \( T_i^2 \) and their meridional disks \( D_i, D_i' \) by deleting the thin strip bounded by \( \partial D_i \) and \( \partial D_i' \) from \( T_i^2 \). The circles \( S_i^1 \) divide the spheres \( S_i^2 \) into two disks each; one of the disks contains \( D_i \), the other contains \( D_i' \). Delete from \( P_1 \) the disks of the \( S_i^2 \) that contain \( D_i' \). This yields a polyhedron \( P \) with \( E(p, q) - 3 \) vertices such that \( L_{p, q} \setminus P \) is a cell.

**Theorem 7.** The polyhedron \( P \) is a pseudominimal special spine of \( L_{p, q} \) with \( E(p, q) - 3 \) vertices. It coincides with the spine of \( L_{p, q} \) described in [16].

It was shown in [16] that the spines constructed there are pseudominimal. So it suffices to prove only the second statement; we leave it to the reader.

### 6. The Main Result

**Definition 11.** Let \( M \) be a Stallings manifold, that is, a 3-manifold that fibers over \( S^1 \). Its **transversal complexity** \( tc(M) \) is the minimal possible number of vertices of a special spine of \( M \) that is transversal to all fibers. Spines are stratified sets in the sense of [10], and the transversality is also understood in the sense of [10] here: all edges and 2-cells of the spine should be transversal to all fibers of \( M \). To deal with transversality, we fix a smooth structure on \( M \) and require all edges and 2-cells of the spine to be at least \( C^1 \)-smooth.

In Sections 7–9 we get a lower bound for transversal complexity of three-manifolds \( M(A) \) fibered by tori over \( S^1 \) with parabolic or hyperbolic monodromy \( A \). We are not interested in the case of elliptic \( A \), because then the exact values of complexity are known: \( c(M(A)) = 6 \) for all 6 manifolds with elliptic monodromy \( A \) (see also the examples in Section 4). Theorem 8 below holds for this case, too, but only gives \( tc(M(A)) \geq 3 \) or \( tc(M(A)) \geq 2 \), depending on the monodromy \( A \), which is useless because of the obvious inequality \( tc(M) \geq c(M) = 6 \).

The main result of this article is the following statement.
Theorem 8. Transversal complexity of a torus bundle over the circle is bounded from below in terms of the complexity of its monodromy operator as follows:

$$tc(M(A)) \geq \frac{1}{5}c(A) + 2.$$ 

We postpone the proof until Section 9 for we need first the results of Sections 7, 8. It follows from the transversality condition that the sections of a spine $P$ by fibers are isotopic to each other as long as the fiber does not reach the next vertex of $P$. The bifurcations that occur at vertices are studied in Section 7 and are reduced to at most 5 flips each in Section 8.

7. Morse Transformations in Simple Polyhedra

Projection $p: M(A) \rightarrow S^1$ defines an $S^1$-valued function on $M(A)$. Let us study its restriction $p|_P$ to a simple spine $P$ of $M(A)$. The polyhedron $P$ is a stratified space; its strata are the vertices, the edges, and the two-dimensional components of $P$. We may assume that $p|_\sigma$ has an everywhere continuous derivative for any edge or 2-component $\sigma$: a $C^0$-small deformation of the embedding mapping $i: P \hookrightarrow M^3$ is sufficient. Since the space of Morse functions on a manifold is $C^1$-dense in the space of all $C^1$-smooth functions on it \[21\], we may assume that $p|_\sigma$ is a Morse function for any edge or 2-component $\sigma$ of $P$. In this case $p$ is called a Morse function on $P$.

Consider a fiber $F_t = p^{-1}(t)$ of the fibering $p: M(A) \rightarrow S^1$, where $t$ is a local coordinate on $S^1$. If $t$ is a nonsingular value of $p$ (that is, $F_t$ contains neither vertices of $P$ nor critical points of the restrictions of $p$ to the edges and 2-components of $P$), then $F_t$ is transversal to all strata of $P$ and the intersection $K_t = F_t \cap P$ is a trivalent graph, possibly disconnected. We will explore how $K_t$ changes as $t$ varies.

For much more comprehensive exposition of Morse transformations in stratified spaces, see \[10\].

Without loss of generality, it can be assumed that any singular fiber (that is, a singular level of $p$) contains only one singular point of $p|_P$ (including vertices), and there are only finite number of singular fibers. Also, we may assume that the restrictions $p|_\sigma$ to the closures of triple lines and two-dimensional components of $P$ have no boundary critical points. Let two nonsingular fibers $F_-$ and $F_+$ be the preimages of two close points $t_-, t_+ \in S^1$. If the interval $(t_-, t_+)$ contains no singular values of $p$, then the graphs $K_-$ and $K_+$ are isotopic, that is, there is a continuous family of embeddings $i_t: K \hookrightarrow F_t, \ t \in [t_-, t_+]$, of the same graph $K$ to the tori $F_t$ such that $K_- = i_{t_-}(K)$ and $K_+ = i_{t_+}(K)$.

\[\text{Figure 12. Minimax (left) and saddle (right) Morse transformations}\]

Now suppose that there is exactly one singular fiber $F_0$ between $F_-$ and $F_+$; by $t_0$ denote the corresponding critical value. Thus $F_0$ contains either a singular
point of the restriction of \( p \) to a two-dimensional component of \( P \) or a singular point of the restriction of \( p \) to a triple line of \( P \), or a vertex of \( P \). In the first case, the difference between \( K_- \) and \( K_+ \) is nothing but the Morse transformation of the level set \( Q = t - t_0 \) of a real quadratic form \( Q(x, y) = \pm x^2 \pm y^2 \), see Fig. 12.

![Figure 13. Transformation of \( K \) induced by a minimum point on a triple line.](image)

To explore the second case, suppose that the critical point of the restriction of \( p \) to a triple line of \( P \) is a minimum point of \( p \); denote this point by \( O \). By \( v_i \), \( i = 1, 2, 3 \), denote the unit vector tangent to the \( i \)th two-dimensional component \( \sigma_i \) at \( O \) and orthogonal to the triple line. Since \( O \) is not a critical point of \( p|_{\sigma_i} \), we have \( p_* v_i \neq 0 \). Here the \( \sigma_i \) are the components of \( (P \setminus SP) \cap U(O) \) of the two-dimensional strata of \( P \) in the neighborhood \( U(O) \) of \( O \); it does not matter whether the \( \sigma_i \) actually belong to different components of \( P \setminus SP \). Let us say that the component \( \sigma_i \) is ascending at \( O \) if the vector \( p_* v_i \in T_{t_0} S^1 \) induces a positive orientation of \( S^1 \) and descending if this orientation is negative. There may be 0, 1, 2 or 3 descending components. Their number defines the transformation between \( K_- \) and \( K_+ \) up to isotopy. The graphs \( K_- \) and \( K_+ \) differ only inside of the dotted circle, see Fig. 13. If \( O \) is a maximum point of \( p \) restricted to a triple line, one has to revert all arrows and signs in the lower indices in Fig. 13.

The last case (where the fiber \( F_0 \) contains a vertex \( V \in P \)) is more complicated. We will return to it later.

**Remark.** The double of a disk is \( S^2 \). Draw the fragment \( K_- \cap U(O) \) of the graph \( K_- \) (that is, the picture inside of the left dotted circle, see Figs. 12, 13) on the lower hemisphere of \( S^2 \) and the fragment \( K_+ \cap U(O) \) of the graph \( K_+ \) (that is, the picture inside of the right dotted circle) on the upper hemisphere. In all four cases, we get an embedding of the link of \( O \) in \( P \) to the sphere with the equator drawn on it by a dotted line. The number of intersection points of the equator and the link of \( O \) is twice the number of descending components at \( O \).

This is not a mere coincidence but a general recipe for describing the difference between \( K_- \) and \( K_+ \). Suppose that there is exactly one singular point \( O \) between
Consider an \(\varepsilon\)-neighborhood \(U(O)\) of \(O\), where \(\varepsilon < t_0 - t_-\) and \(\varepsilon < t_+ - t_0\). The intersection \(P \cap U(O)\) is the cone over the graph \(\text{lk}\, O\), which is the circle with three radii, see the definition of a simple polyhedron (Definition 1 in Section 2). By \(D_-, D_0\), and \(D_+\) denote the intersections of \(U(O)\) with \(F_-, F_0\), and \(F_+\), respectively. Obviously, \(K_+ \setminus D_+\) is isotopic to \(K_- \setminus D_-\). Thus, the difference between \(K_+\) and \(K_-\) is “hidden inside of \(U(O)\)”.

The singular fiber \(F_0\) cuts the sphere \(S^2 = \partial U(O)\) into two hemispheres \(S^2_+\) and \(S^2_-\). Put \(K'_+ = (K_0 \setminus U(O)) \cup (P \cap S^2_+)\) and \(K'_- = (K_0 \setminus U(O)) \cup (P \cap S^2_-)\). These graphs are embedded into tori \((F_0 \setminus U(O)) \cap S^2_+\) and \((F_0 \setminus U(O)) \cap S^2_-\).

**Lemma 5.** The graph \(K_-,\) respectively, \(K_+\), is isotopic to the graph \(K'_-,\) respectively, \(K'_+\).

**Proof.** Let us define a surface \(G_s\) as the union of \(F_{t_0+s(t_0-t_-)} \setminus U(O)\) and \(\{ x \in S^2 \mid p(x) < t_0 + s(t_0 - t_-) \}\), where \(-1 \leq s \leq 0\), and put \(K'_s = P \cap G_s\). In \(U(O) \setminus O\), the edges and 2-components of \(P\) are transversal to all fibers and to the spheres centered at \(O\). So all \(G_s\), \(-1 \leq s \leq 0\), are transversal to \(P\), and the family \(K'_s\) provides an isotopy between \(K'_{-1} \subset G_{-1}\) and \(K'_0 \subset G_0\). It remains to note that \(G_{-1} = F_-\), \(K'_{-1} = K_-\), and \(K'_0 = K'_+\). An isotopy between \(K_+\) and \(K'_+\) can be constructed in a similar way.

![Figure 14. Mutual position of \(\text{lk}_{F_0}(O)\) (dotted line) and \(\text{lk}_P(O)\) in \(S^2\)](image)

So to describe a “surgery” in the graph \(K\) near a singular point \(O\), it is sufficient to draw the links of \(O\) in \(F_0\) and in \(P\) on the sphere \(S^2 = \partial U(O)\) and cut \(S^2\) into two disks \(S^2_+\) and \(S^2_-\) along the former link (the equator). The mutual position of \(\text{lk}_{F_0}\, O\) and \(\text{lk}_P\, O\) may be considered as a real analogue for the Milnor fiber of an isolated singularity on a complex hypersurface, see [22]. Figure 14 represents the mutual position of these links on \(S^2\) for minimax and saddle points on a two-dimensional component of \(P\) and for the four cases of Fig. 13, where \(O\) is a minimax point of the restriction of \(p\) to an edge of \(P\). Note that there are, up to isotopy and symmetries, exactly four different embeddings of a circle with a diameter into a sphere with an equator, provided that any edge of the graph intersects the equator at most twice and both vertices lie in the same hemisphere.

The same argument works if a singular point is a vertex \(V\) of a simple polyhedron \(P\). In this case, \(\text{lk}_P(V)\) is a circle with three radii; denote this graph by \(\Delta\). Any edge of \(\Delta\) is the link of \(V\) in one of the six two-dimensional components of \((P \setminus SP) \cap U(V)\), and we assumed that \(V\) is not a critical point of the restriction
of $p$ to the closures of these components. This implies that any edge of $\Delta$ intersects the equator at most twice, because it is close to an arc of a great circle obtained as the intersection of $S^2 = \partial U(V)$ and the tangent plane to a component $\sigma$ at $V$. Since the differential $dp|_e$ does not vanish at $V$ for any edge $e$ of $P$, the vertices of $\Delta$ lie in either $S^2_+$ or $S^2_-$ but not at the equator $\text{lk}_{F_0}(V)$.

Without loss of generality, we can assume that $P \cap U(V)$ consists of plane pieces. Now it follows that any edge of $\Delta$ connecting vertices in different hemispheres intersects the equator exactly once. An edge with both endpoints in the same hemisphere can have either none or two intersection points with the equator; in the latter case, these two points are opposite. An edge is said to be long if it intersects the equator twice. We have to consider three cases.

![Figure 15. Transformation of $K$ near a vertex $V$](image)

Case 1. All four vertices of $\Delta$ lie in one hemisphere. Then any long edge contains a diameter of the other hemisphere. Since any two diameters intersect one another, there is at most one long edge. Thus in Case 1 there are, up to isotopy, only two possibilities to draw $\Delta$ on a sphere with the equator, see Fig. 15(a,b).

Figure 15 also shows the difference between $K_-$ and $K_+$.

Case 2. Three vertices of $\Delta$ lie in one hemisphere and the last one in the other. Again, there is at most one long edge and thus only two nonisotopic embeddings of $\Delta$, see Fig. 15(c,d).

Case 3. There are two vertices in upper hemisphere and two other in the lower one. The maximal number of long edges is two: at most one in each hemisphere. If there are no long edges, we have the situation of Fig. 15(e); if there is one long edge, the picture is like Fig. 15(f); finally, if there are two long edges, we obtain the situation of Fig. 15(g).
Note that all results of this section apply to the level sets of Morse functions on arbitrary simple polyhedra, not only to simple spines of $M(A)$.

8. \( \theta \)-Curves in the Fibers

Recall that a spine $P$ of a closed 3-manifold $M$ intersects any loop that is nontrivial in $\pi_1(M)$.

**Lemma 6.** For any $t$, the graph $K_t = P \cap F_t$ intersects any loop representing a nonzero element of $\pi_1(F_t) = \mathbb{Z}^2$.

**Proof.** Consider the exact sequence

$$\cdots \to \pi_2(S^1) \to \pi_1(T^2) \xrightarrow{i_*} \pi_1(M^3) \to \cdots$$

of the fibering $p: M^3 \xrightarrow{\pi} S^1$. Since $\pi_2(S^1) = 0$, it follows that $i_*$ is a monomorphism, that is, any nontrivial loop in $T^2$ is nontrivial in $M^3$, too. Hence any spine of $M$ intersects this loop, and the Lemma follows. \( \square \)

We assumed that $p$ is an $S^1$-valued Morse function on $P$. In this case, all critical points of $p$ are isolated, and all but finite number of values of $p$ are regular. Let $F$ be a fiber corresponding to a regular value of $p$. Then the intersection $F \cap P$ is a trivalent graph $K \subset F$.

**Lemma 7.** Suppose that a trivalent graph $K \subset T^2$ intersects any nontrivial loop in $T^2$. Then $K$ contains a subgraph that is a $\theta$-curve.

**Proof.** Let $K_1, \ldots, K_n$ be connected components of $K$. If a component $K_i$ contains no cycles nontrivial in $T^2$, then there is a disk $D_i^2 \subset T^2$ such that $K \cap D_i^2 = K_i$. If there are several components $K_i$ without nontrivial cycles, then there exists a disjoint union of disks $U$ such that $K \cap U$ coincides with the set of all connected components of $K$ containing no nontrivial cycles. Put $K' = K \setminus U$. All connected components of $K'$ (if any) contain nontrivial cycles.

Any cycle in $\pi_1(T^2)$ is homotopic to a cycle contained in $T^2 \setminus U$. This means that any nontrivial cycle intersects $K'$, and thus $K' \neq \emptyset$. Since any component $K'_i$ of $K'$ contains nontrivial cycles, we can choose some cycles $\gamma_i \subset K'_i$ represented by simple closed curves. If there are several connected components of $K'$, the cycles $\gamma_i$ do not intersect one another and thus are homotopic. In this case, among connected components of $T^2 \setminus K'_i$ there is an annulus containing other components of $K'$. It contains a cycle $\gamma$ (homotopic to all the $\gamma_i$) going along a boundary circle of the annulus. The cycle $\gamma$ is nontrivial and does not intersect $K$. Since this is impossible, the graph $K'$ is connected.

Let $U_1, \ldots, U_k$ be the connected components of $T^2 \setminus K'$. Every $U_i$ is an orientable surface with boundary. It contains no closed curves nontrivial in $U_i$. Otherwise, such a curve would be either trivial or nontrivial in $T^2$. In the former case, it would split the torus into two disjoint parts that contain different connected components of $K'$, which is impossible since the graph $K'$ is connected. The latter case is impossible, too, because any cycle nontrivial in $T^2$ intersects $K'$. A surface with boundary containing no nontrivial cycles is a disk. Thus $T^2 \setminus K'$ is a collection of $s$ two-dimensional cells, and $K'$ defines a cell decomposition of $T^2$. 

Let \( v, \) respectively, \( e, \) be the number of vertices, respectively, edges of \( K'. \) Note that \( e = \frac{3}{2}v, \) because \( K' \) is a trivalent graph. Also, we have \( v - e + s = \chi(T^2) = 0, \) which implies that \( v = 2s \) and \( e = 3s. \) If the number \( s \) is greater than 1, it can be decreased by deleting an edge separating two different cells. When \( s = 1, \) the graph \( K' \) is a \( \theta \)-curve. □

**Theorem 9.** Let \( F \) be a nonsingular fiber of the fibering \( p: M^3 \to T^2 \to S^1, \) and \( K = P \cap F. \) Then \( K \) contains a subgraph \( L \) that is a \( \theta \)-curve. The number of pairwise nonisotopic \( \theta \)-curves contained in \( K \) is finite.

**Proof.** Combine Lemmas 6 and 7 and note that the number of all subgraphs of \( K \) is finite. □

Let \( F_- \) and \( F_+ \) be two close nonsingular fibers and \( L_- \subset K_- \) be a \( \theta \)-curve in \( F_. \) If the interval \((t_-, t_+)\) contains no singular values of \( p|_P, \) then the graphs \( K_- \) and \( K_+ \) are isotopic, and \( K_+ \) contains a \( \theta \)-curve \( L_+ \) isotopic to \( L_-. \) If there is a singular value \( t_0 \in (t_-, t_+), \) then \( K_+ \) may not contain a \( \theta \)-curve isotopic to \( L_-, \) see, for example, Fig. 16, where a saddle Morse transformation occurs in a 2-component of \( P \) as \( t = t_0. \) By Theorem 9, \( K_+ \) still contains \( \theta \)-curves. It is important that some of them are not too distant from \( L_- \) in the graph \( \Gamma, \) that is, can be obtained from \( L_- \) by a small number of flips.

![Figure 16. Transformation of \( L \) induced by a saddle point](image)

**Theorem 10.** Let both trivalent graphs \( K_-, K_+ \subset T^2 \) contain \( \theta \)-curves and differ by one of the transformations shown in Figs. 12, 13, and 15. Then for any \( \theta \)-curve \( L_- \subset K_- \) there exists a \( \theta \)-curve \( L_+ \subset K_+ \) such that \( d(\Delta_+, \Delta_-) \leq 5, \) where \( \Delta_+ \) is the Farey triangle corresponding to \( L_+ \) and \( d \) is the distance in \( \Gamma \) (see Section 3).

For a graph \( K, \) we can consider the set \( w(K) \subset \Gamma \) of Farey triangles in \( \mathbb{Z}^2 \) that correspond to all \( \theta \)-curves contained in \( K. \) Theorem 10 proclaims that \( d(w(K_-)), w(K_+)) \leq 5, \) where \( d(X, Y) \) is the distance from a subspace \( X \) to a subspace \( Y \) of a metric space \( \Gamma \) defined by the formula \( d(X, Y) = \max_{x \in X} \min_{y \in Y} d(x, y); \) note that \( d(X, Y) \) need not be symmetric.

**Proof.** First, suppose that the whole graph \( \text{lk}_P V \) lies in one hemisphere, where \( V \in F_0 \) is a singular point of \( p|_P \) or a vertex. This happens in the situations of Fig. 15 (a) and of the leftmost pictures of Figs. 12 and 13. Then \( U(V) \) contains only one isolated connected component of \( K_+ \) (or \( K_- \)), which has nothing to do with \( \theta \)-curves in \( K_\pm. \) So the transformation of \( K \) arising from a singularity at \( V \)
does not affect $L_-$, and the graph $K_+$ contains a $\theta$-curve $L_+$ isotopic to $L_-$. Of course, in this case $d(\Delta_+, \Delta_-) = 0$.

In the cases represented by Fig. 15 (b) and by the second picture of Fig. 13, it is easy to see that the part of any $\theta$-curve lying inside of any dotted circle is a part of an edge of the $\theta$-curve (if nonempty), and the part of the $\theta$-curve $L_-$ lying outside of the dotted circle can be augmented with a path in $K_+$ lying inside of the dotted circle and homotopic to the path in $K_-$ involved in the $\theta$-curve $L_-$. Similar argument works for the case of Fig. 15 (c), where the part of $L_-$ lying inside of the dotted circle may be a tripod (a neighborhood of a trivalent vertex) or a part of an edge of $L_-$ or the empty set. So, $d(\Delta_+ , \Delta_-) = 0$ in all the cases where $\text{lk}_P V$ intersects the equator in less than four points.

The following Lemma is necessary to deal with the seven remaining cases (Fig. 15 (d–g), two last pictures of Fig. 13, and the saddle transformation shown in Fig. 12, right).

Lemma 8. Let $K_-, K_+ \subset T^2$ be trivalent graphs and $L_- \subset K_-$ be a $\theta$-curve. Suppose that $K_+$ differs from $K_-$ by adding one edge only and $K_+$ contains a $\theta$-curve, too.

1. If $K_+$ is obtained from $K_-$ by adding one extra edge, then it contains a $\theta$-curve $L_+$ such that $d(\Delta_+, \Delta_-) = 0$.

2. If $K_+$ is obtained from $K_-$ by deleting an edge $e$, then it contains a $\theta$-curve $L_+$ such that $d(\Delta_+, \Delta_-) \leq 1$.

Proof. The first statement is obvious, since we can put $L_+ = L_-$. In the second case, we also can put $L_+ = L_-$ unless $e \subset L_-$. Suppose that the edge $e$ is a part of an edge $l$ of $L_-$. Cut the torus $T^2$ along $L_-$ into a hexagonal 2-cell $H$. The boundary $\partial H$ contains two arcs $e_1 \subset l_1, e_2 \subset l_2$ arising from $e$, and two arcs $f_1 = FAB, f_2 = CDE$ complementary to the arcs $l_1 = FE, l_2 = BC$ that arise from $l$, see Fig. 17.

There is the following alternative: either there exists a path $\gamma_1$ in $H \setminus K_+$ from the midpoint of $e_1$ to the midpoint of $e_2$, or there exists a path $\gamma_2$ in $K_+$ from a point in $f_1$ to a point in $f_2$; this path $\gamma_2$ may include edges of $K_+$ belonging to $l \setminus e$. The first case is impossible, because the path $\gamma_1$ yields a nontrivial cycle in $T^2 \setminus K_+$; so we have the second case.

Put $L_+ = (L_- \setminus l) \cup \gamma_2$. Obviously, $L_+$ is a trivalent subgraph of $K_+$ with two vertices at the endpoints of $\gamma_2$. It contains two edges of $L_-$ different from $l$, which form a nontrivial cycle $\sigma$ homotopic to $\gamma_1$. Thus $L_+$ intersects any cycle $m\sigma + n\mu \in \ldots$
$\pi_1(T^2)$ with $n \neq 0$, where $\mu$ is any cycle such that $\sigma$ and $\mu$ generate $\pi_1(T^2)$. By construction, $L_+$ intersects any cycle homotopic to $\sigma$, too. Thus, $L_+$ is a $\theta$-curve.

The path $\gamma_2$ considered above divides $\partial H$ in two arcs. If the vertices $A, D$ of $H$, which correspond to two different vertices of $L_-$ (see Fig. 17), belong to the same arc, then $L_+$ is isotopic to $L_- \cup \Delta_+$ and $\Delta_+ = \Delta_-$. If they belong to different arcs, then $d(\Delta_+, \Delta_-) = 1$. □

Let us return to the proof of Theorem 10. Consider the case of Fig. 15 (e). Realize the graph $K_-$ as the union of solid and dotted lines on the middle picture of Fig. 18. First, add the dashed edge to $K_-$. Then delete the dotted edge from the resulting graph. This results in $K_+$. By Lemma 8, the first step does not affect any $\theta$-curve in $K_-$, and the graph obtained after the second step contains a $\theta$-curve $L_+$ such that $d(\Delta_+, \Delta_-) \leq 1$.

![Figure 18. Step by step transformation of $K_-$ to $K_+$](image)

The same reasoning applies in all other cases. For instance, for the saddle transformation (see Fig. 12, right) we need first to add two extra edges that will be included in $K_+$, and then delete two edges of $K_+ \setminus K_-$ from the graph obtained. Lemma 8 is applicable at each step, because any auxiliary graph contains either $K_-$ (whenever an edge is added at the previous step) or $K_+$ (whenever an edge is deleted at the next step) and thus contains $\theta$-curves. So for the saddle transformation we can guarantee that $d(\Delta_+, \Delta_-) \leq 2$ for some $\theta$-curve $L_+ \subset K_+$. Figure 16 shows that this estimate is exact.

It is easy to see that in all remaining cases the graphs $K_-$ and $K_+$ can be related by a sequence of $m$ operations of adding an edge followed by $n$ operations of deleting an edge, where $m, n \leq 5$. In fact, there are sequences giving the pairs $(m, n)$ equal to $(2, 1)$ and $(4, 3)$ for the last two pictures of Fig. 13, $(2, 1)$ for the case of Fig. 15 (d), $(3, 3)$ and $(5, 5)$ for the cases of Fig. 15 (f) and Fig. 15 (g). By Lemma 8, there exists a $\theta$-curve $L_+ \subset K_+$ such that $d(\Delta_+, \Delta_-)$ does not exceed the number of deletion steps, which is $n$ for “left to right” (or “bottom to top”) transformations shown in Figs. 12, 13, and 15, and $m$ for reverse transformations. In any case, this number is at most 5, which proves the theorem. □

Remarks. Theorem 10 gives an upper bound for $d(w(K_+), w(K_-))$. As we already saw (Fig. 16), this estimate is exact in the case of the saddle transformation. In fact, there are examples showing that the upper bound obtained in Theorem 10 is attainable for the third picture of Fig. 13, for the cases shown in Fig. 15 (d,e), and for the “top to bottom” transformation of Fig. 15 (f). In particular, the distance $d(w(K_+), w(K_-))$ is not symmetric for some graphs $K_+$ and $K_-$ related by a transformation shown in Fig. 15 (d) or on the third picture of Fig. 13.
The estimate for \(d(w(K_+), w(K_-))\) given in Theorem 11 exceeds 3 only in the case of Fig. 15(g) and for the “right to left” transformation from the right-most picture of Fig. 13. Even in these cases, no examples are known where \(d(w(K_+), w(K_-)) > 3\). We believe that such examples do not exist. This would imply that the upper bound 5 in Theorem 10 can be replaced by 3; the existing examples show that it cannot be replaced by a number smaller than 3.

9. Proof of Theorem 8

Let \(t \in \mathbb{R}/\mathbb{Z}\) be a parameter on the circle \(S^1 = \mathbb{R}/\mathbb{Z}\). Without loss of generality, it can be assumed that the fiber \(F_0 = p^{-1}(0)\) contains no singular points of \(p\).

The transversality hypothesis in Theorem 8 means that \(p\) has singular points neither inside of 2-components nor inside of edges of \(P\). Thus the only singularities of \(p|_P\) are the vertices of \(P\); we can assume that the vertices lie in pairwise different fibers. Let \(t_1 < t_2 < \cdots < t_n\) be their projections on \(S^1\) (that is, the singular values of \(p\)); note that \(n = c(P)\). For any \(i = 1, \ldots, n - 1\) consider a nonsingular fiber \(F_i = F_i(t_i+t_{i+1})/2\). Also put \(F_n = F_0\).

By virtue of Theorem 9, the graph \(K_0 = P \cap F_0\) contains a finite number \(N > 0\) of \(\theta\)-curves. Let \(L_0\) be one of them. By Theorem 10, we can construct step by step a sequence \(L_i \subset K_i = P \cap F_i\), \(i = 1, \ldots, n\), of \(\theta\)-curves such that \(d(\Delta_i, \Delta_{i+1}) \leq 5\).

This implies the inequality \(d(\Delta_n, \Delta_0) \leq 5n\).

Recall that \(L_n\) is one of the \(N\) \(\theta\)-curves contained in \(K_0\). First suppose that \(L_n\) coincides with \(L_0\). For the corresponding Farey triangles \(\Delta_0\) and \(\Delta_n\) this means that \(\Delta_n = A(\Delta_0)\); here \(A\) is the matrix of the monodromy operator \(A\). Consequently, \(d(\Delta_0, \Delta_n) = c(A)\). Combining this with the inequality \(d(\Delta_n, \Delta_0) \leq 5n\), we get \(c(A) \leq 5n\) or, equivalently, \(c(P) \geq c(A)/5\).

Note that there are at least two loose vertices of \(P\), that is, vertices where the graph \(K\) undergoes the transformation shown in Fig. 15(c). One of them corresponds to the minimum point of \(\tilde{p}\) and another to the maximum point, where \(\tilde{p}: M(A) \to \mathbb{R}\) is the function that covers \(p\) under the universal covering \(\tilde{M}(A) \to M(A)\) and \(D\) is one of the preimages of the 3-cell \(M(A)\) under this covering. Since loose vertices do not affect \(\theta\)-curves and corresponding hexagons, we have \(d(\Delta_{i_0}, \Delta_0) \leq 5(n-k)\), where \(k \geq 2\) is the number of the loose vertices of \(P\). This proves the inequality \(c(P) \geq \frac{1}{2}c(A) + k \geq \frac{1}{2}c(A) + 2\) in the case \(L_n = L_0\).

Now suppose that \(L_n\) does not coincide with \(L_0\). Consider a sequence \(L_{n+1}, L_{n+2}, \ldots, L_{Nn}\), where \(L_i \subset K_{i'} = P \cap F_{i'}\) with \(i' \equiv i \mod n\), and \(d(\Delta_i, \Delta_{i+1}) \leq 5\) for all \(i = 1, \ldots, Nn\); we also assume that \(\Delta_i = \Delta_{i+1}\) whenever the vertex lying between \(F_{i'}\) and \(F_{i'+1}\) is a loose vertex. Since the graph \(K_0\) contains only \(N\) different \(\theta\)-curves, we have \(L_{sn} = L_{tn}\) for some \(s, t\) such that \(0 \leq s < t \leq N\). Then, as above, we have \(c(A^{t-s}) \leq d(\Delta_{sn}, \Delta_{tn}) \leq 5(t-s)(n-k)\), because \(\Delta_{sn} = A^{t-s}\Delta_{sn}\) and there are \(k \geq 2\) loose vertices. It follows from the results of Section 3 that \(c(A^{t-s}) = (t-s)c(A)\) provided that the monodromy operator \(A\) is parabolic or hyperbolic. Thus \(5(t-s)(c(P) - k) \geq (t-s)c(A)\), i.e., \(c(P) \geq \frac{1}{2}c(A) + k \geq \frac{1}{2}c(A) + 2\).

The proof of Theorem 8 is complete. \(\square\)

Remark. It was shown in [26] that a spine \(P\) can be deformed so that the only transformations of the level sets are two Morse transformations (see Fig. 12), the
tions shown on the second picture in Fig. 13, and flips, see Fig. 15 (d). (In fact, this statement is proved in [26] for functions with values in the interval [0, 1], but the proof applies to $S^1$-valued functions, too.) Note that a flip in $K$ induces at most one flip in $L$ and a saddle transformation induces at most two flips, while two other transformations do not require flips at all. This yields the estimate $c(P) \geq c(\sigma) + 2 - 2s$ (where $s$ is the number of saddle points of $p$), which holds for arbitrary spine of $M(A)$, implying $c(M(A)) \geq c(\sigma) + 2 - 2s$. However, it is unlikely that there exists any lower bound for $s$.

10. Conjectures

Let us recall the idea of the proof of Theorem 8. Given a spine $P$ of $M^3 = M^3(A)$, suppose that we can assign some object $\sigma_t$ to each fiber $F_t$ containing no vertices of $P$ in such a way that

1. $\sigma_t$ is an element of some metric space $(\Sigma, d)$;
2. there is an action of the group $\text{SL}(2, \mathbb{Z})$ on the set $\Sigma$, and monodromy $A$ takes $\sigma_t$ to $\sigma_{t+2\pi} = A\sigma_t$;
3. $d(\sigma_{t}, A\sigma_t) \geq c(A)$;
4. as $t$ varies, $\sigma_t$ remains unchanged until $F_t$ encounters a vertex $V$ in $P$;
5. if there is exactly one vertex of $P$ between $F_{t+}$ and $F_{t-}$ (we can assume that vertices of $P$ lie in pairwise different fibers), then $d(\sigma_{t+}, \sigma_{t-}) \leq 1$.

Then the number of vertices of $P$ is at least $c(A)$. This can be followed by an argument that shows that this number is in fact at least $c(A) + k$, where $k$ is some positive integer smaller than 6.

In the proof of Theorem 8, $\sigma_t$ is an isotopy class of $\theta$-curves in $F_t$, $\Sigma$ is the vertex set of the trivalent graph $\Gamma$ described in Section 3, $d$ is the distance in $\Gamma$, and $k = 2$. To preserve property 4, we had to restrict ourselves to spines transversal to the fibers. Still, we got $d(\sigma_{t+}, \sigma_{t-}) \leq 5$ instead of property 5; this leads to an annoying factor of $1/5$ in Theorem 8.

In this section we present another construction for $\sigma_t$; namely, it will be some class of 2-chains representing the fundamental class $[F_1]$. This approach applies to all special spines $P$ of $M(A)$ (recall that, by Theorem 2, any minimal spine of $M(A)$ is a special one), not only to those transversal to the fibers, and properties 1, 2, 4, and 5 hold, while property 3 remains unproved.

Given a special spine $P$ of $M^3$, by $P'$ denote a triangulation of $M^3$ dual to $P$. Its tetrahedra correspond to the vertices of $P$ (see Fig. 1 (f)), triangles correspond to the edges of the singular graph $SP$ (that is, to the triple lines) of the spine $P$, edges of $P'$ are dual to the 2-cells of $P$, and the only vertex of $P'$ is located somewhere inside of the 3-cell $M \setminus P$. Here and below we abuse the word “triangulation”: the intersection of two simplices (if nonempty) is a subset of the sets of their faces of smaller dimensions, but the cardinality of this subset may exceed 1.

Let $f : \mathbb{R}^1 \to S^1$ be the standard covering defined by the rule $f(t) = t \mod 2\pi$. By $\tilde{f} : \tilde{M}^3 \to M^3$ denote the covering such that $\tilde{f}_\ast(\pi_1(\tilde{M}^3)) = i_\ast(\pi_1(F))$, where $i : F \to M^3$ is the embedding map of some fiber $F$ into $M^3$. Then a projection $\tilde{p} : \tilde{M} \to \mathbb{R}^1$ can be defined by the relation $f \circ \tilde{p} = p \circ \tilde{f}$. By $F_t \subset \tilde{M}$ denote the fiber $\tilde{p}^{-1}(t)$, where $t \in \mathbb{R}^1$. Put $\tilde{P} = \tilde{f}^{-1}(P)$ and $\tilde{P}' = \tilde{f}^{-1}(P')$. Note that $\tilde{P}$ is
a spine of \( \tilde{M} \) punctured infinitely many times (at all preimages of the vertex \( V' \) of \( P' \)) and \( P' \) is a triangulation of \( M \). Furthermore, \( \tilde{M} \) is nothing but \( T^2 \times \mathbb{R} \) equipped with the deck transformation \( (x, t) \mapsto (Ax, t + 2\pi) \), where \( x \in T^2 \) and \( t \in \mathbb{R} \), and both \( \tilde{P} \) and \( P' \) are invariant under the deck transformation. Fix a cartesian product structure on \( \tilde{M} \) and define the forgetting projection \( j: \tilde{M} \to T^2 \) by \( j(x, t) = x \).

By \( M_0 \), respectively, \( \tilde{M}_0 \), denote the complement to the vertices of \( P \) in \( M \), respectively, to the vertices of \( \tilde{P} \) in \( \tilde{M} \). It is easy to see that there is a strict deformation retraction \( r: M_0 \to \text{sk}_2 P' \), which is covered by a strict deformation retraction \( \tilde{r}: \tilde{M}_0 \to \text{sk}_2 \tilde{P}' \). If a fiber \( F_t \) contains no vertices of \( \tilde{P} \), the image \( \tilde{r}(F_t) \subset \text{sk}_2 \tilde{P}' \) defines a 2-cycle \( \tau^2_t \in Z_2(\tilde{P}') \). The group \( Z_2(\tilde{P}') \) is generated by the 2-cells of \( \tilde{P}' \), so we have \( \tau^2_t = \sum \alpha_k(t) \tau^2_k \), where \( \{ \tau^2_k \} \) is the set of the 2-cells of \( \tilde{P}' \).

The coefficients \( \alpha_k(t) \) are equal to the intersection indices of \( F_t \) with the oriented edges \( e_k \) of \( \tilde{P} \) corresponding to the (oriented) 2-cells \( \tau^2_k \). Let \( A_k \) and \( B_k \) be the endpoints of an edge \( e_k \) oriented from \( A_k \) to \( B_k \). If \( A_k \) and \( B_k \) lie both below or both above \( F_t \) (that is, if \( \bar{p}(A_k) - t)(\bar{p}(B_k) - t) > 0 \), then \( \alpha_k(t) = 0 \); otherwise, \( \alpha_k(t) = \text{sign}(\bar{p}(B_k) - \bar{p}(A_k)) = \pm 1 \). Obviously, only finite number of the coefficients can be nonzero, and property 4 holds for all \( \alpha_k(t) \), whence for \( \tau^2_t \) as well.

Let us consider a 2-chain \( j_*(\tau^2_t) \). By construction, it is a cycle and its homology class is \([T^2]\) (because \( j_*(\tau^2_t) = j_*(F_t) \)). All triangles that constitute \( j_*(\tau^2_t) \) have all their vertices mapped to \( j(V') \) (where \( V' \) is the vertex of \( P' \)). Let \( \sigma_t \) be the linearization of \( j_*(\tau^2_t) \), that is, the 2-chain obtained from \( j_*(\tau^2_t) \) by replacing the characteristic mappings of all triangles by homotopic (with fixed vertices) linear mappings; a mapping of a triangle to the torus is linear if it is a linear mapping to the plane followed by the projection \( \mathbb{R}^2 \to T^2 \). Roughly speaking, we define \( \Sigma \) as the set of all linear 2-cycles in \( T^2 \) that are homological to \([T^2]\) and have the only vertex, which is placed at \( j(V') \). Accurately speaking, any element of \( \Sigma \) is a (homological to \([T^2]\)) cycle \( \sum \beta_k \Delta_k \), where the coefficients \( \beta_k \) are integers, all but finite number of them are zero, and \( \{ \Delta_k \} \) is the set of projections (from \( \mathbb{R}^2 \) to \( T^2 \)) of all different oriented triangles with vertices in \( \mathbb{Z}^2 \); the triangles are equal if they come from the same edge \( e_k \) of \( \tilde{P}' \), so even geometrically coinciding triangles may be different in this sense. This difference will be exploited later, but unless otherwise stated, we identify all coinciding triangles. Degenerated triangles (those with three vertices along a line) are allowed, and their orientation is, as usually, a cyclic ordering of their vertices. The metric on \( \Sigma \) will be constructed later.

Obviously, we have \( \sigma_t \in \Sigma \). Further, the group \( \text{SL}(2, \mathbb{Z}) \) acts on \( \Sigma \) in a natural way, and \( \sigma_{t+2\pi} = A \sigma_t \), where \( A \) is the monodromy of the fibration \( p: M^3 \to S^1 \). Thus properties 2 and 4, see above, are satisfied; further, property 1 is satisfied for any choice of metric \( d \).

To find the difference between \( \sigma_{t+} \) and \( \sigma_{t-} \) (where the interval \((t-, t+)\) contains the projection \( \bar{p}(V) \) of exactly one vertex \( V \) of \( \tilde{P} \)), consider the difference between \( F_{t+} \) and \( F_{t-} \). It is easy to see that \( F_{t+} \) can be obtained as a connected sum of \( F_{t-} \) and a small two-dimensional sphere \( S^2(V) \) centered at \( V \). Thus \( \sigma_{t+} \) is obtained from \( \sigma_{t-} \) by adding \( j_*(\bar{p}(S^2(V))) \), that is, by one of the two-dimensional Pachner moves, see [25] and Section 4 above. Indeed, there are four edges of \( \tilde{P} \).
emanating from $V$. Suppose that they are all different, that is, they are not loops. Then $\tilde{r}(S^2(V))$ is the boundary of a tetrahedron triangulated into four triangles $\tau_k^2$, from which 0 to 4 can annihilate with triangles that constitute $\tilde{r}(F_k)$. So $\sigma_{t_+}$ is obtained from $\sigma_{t_-}$ in one of the following five ways:

(0) by adding the projection of the boundary of some tetrahedron, that is, by adding triangles $BCD$, $CAD$, $ABD$, and $BAC$ (mind the orientations!), or
(1) by replacing a triangle $ABC$ of $\sigma_{t_-}$ by three triangles $ABD$, $BCD$, and $CAD$, or
(2) by replacing two triangles $ABC$ and $ACD$ of $\sigma_{t_-}$ (the segment $AC$ contributes to both their boundaries, but with opposite signs) by the triangles $ABD$ and $BDC$, or
(3) by replacing three triangles $ABD$, $BCD$, and $CAD$ by a single triangle $ABC$, or
(4) by erasing four triangles $BCD$, $CAD$, $ABD$, and $BAC$.

Of course, moves 3 and 4 may not be applicable to arbitrary $\sigma_t \in \Sigma$.

If one of the edges of $P$ incident to $V$ is a loop, then two of the triangles that form $\tilde{r}(S^2(V))$ annihilate with one another, and it is easy to see that $\sigma_{t_+} = \sigma_{t_-}$.

Consider a graph with vertices at the elements of $\Sigma$ and edges corresponding to the moves described above. Let $d$ be the distance function on this graph. This completes the construction of the metric space $(\Sigma, d)$. Property 5 now holds by construction. Recall that properties 1, 2, and 4 hold as well. Property 3 is a conjecture. If it holds, then the inequality $c(M(A)) \geq c(A)$ follows immediately.

This conjecture is supported by the following observation. For any triangle of $\sigma_t$, draw three segments connecting its baricenter with the midpoints of its sides. All the tripods obtained in this way form a graph $K$. In some simple cases (for example, for spines constructed in Section 4), $K$ is a trivalent graph embedded into $T^2$. The Pachner moves 0 and 4 affect $K$ according to Fig. 15(a) ("left to right" transformation of Fig. 15(a) for move 0 and "right to left" for move 4); moves 1 and 3 affect $K$ according to Fig. 15(c); finally, moves 2 correspond to flips, see Fig. 15(e). It is possible now to choose a $\theta$-curve in $K$ and apply the argument of Section 9, taking into account that moves of Figs. 15(a) and 15(c) do not affect the isotopy classes of $\theta$-curves and any flip in the graph $K$ implies at most one flip of any $\theta$-curve in it. However, in the general case the graph $K$ neither is embedded into the torus nor is trivalent, because there may be 4, 6 etc. triangles having the same edge in common, even for spines transversal to the fibers.

There is a more geometrical reformulation of this conjecture. For a chain $c = \sum r_1 \tau_i$, where the sum is finite, $r_i \in \mathbb{Z}$, and the $\tau_i$ are singular simplices, put $||c|| = \sum |r_i|$, and consider the minimum value $l^1(A)$ of $||c||$ over all 3-chains in $T^2 \times I$, $I = [0, 2\pi]$, representing the fundamental class $[T^2 \times I]$ and satisfying the condition $\sigma_{2\pi} = A\sigma_0$, where 2-chains $\sigma_{2\pi}$ and $\sigma_0$ are the intersections of the chain $c$ with $T^2 \times \{2\pi\}$ and $T^2 \times \{0\}$, and $A(x, 0) = (Ax, 2\pi)$ for $(x, 0) \in T^2 \times \{0\}$. It is easy to see that $l^1(A) \leq c(A)$. The conjecture can be formulated as follows: $l^1(A) = c(A)$. The definition of $l^1(A)$ is similar to Gromov’s definition of the simplicial $l^1$-norm (with some boundary conditions imposed), see [11]. However,
Let us say that a vertex Conjecture
An edge
For all \(d\)hibit moves 0 and 4; only moves 1, 2, and 3 are allowed. Obviously, the inequality
\[d_m, n\] is defined by replacing a mark (\(n\) this triangle and
T 2-cycles homological to \(\sigma\) prohibit cancellation of the triangles in these cases. Then we get the set \(\Sigma\) (which states that property 3 holds).

\[\sigma\in\Sigma,\] which means that \(\sigma\) contains \(m\) positively oriented copies of this triangle and \(n\) negatively oriented copies of it. A natural mapping \(s:\Sigma\rightarrow\Sigma\) is defined by replacing a mark \((m, n)\) by the coefficient \(m - n\). To define a distance \(d'\) on \(\Sigma\), we follow the construction of the distance function \(d\); however, we prohibit moves 0 and 4; only moves 1, 2, and 3 are allowed. Obviously, the inequality \(d'(\sigma_1, \sigma_2) \geq d(s(\sigma_1), s(\sigma_2))\) holds. Note also that \(s(A\sigma) = A\sigma,\) where \(\sigma\in\Sigma\) and \(A\in SL(2, \mathbb{Z}).\) So the following statement is weaker than the conjecture above (which states that property 3 holds).

**Conjecture 3.** For all \(\sigma\in\Sigma\) and \(A\in SL(2, \mathbb{Z})\), the inequality \(d'(A\sigma, \sigma) \geq c(A)\) holds.

However, this weaker conjecture still implies the inequality \(c(M(A)) \geq c(A)\).

**Theorem 11.** Conjecture 3 implies the estimate \(c(M(A)) \geq c(A)\).

**Proof.** Let us say that a vertex \(V\) of \(\tilde{P}\) is maximal, respectively, minimal, if the endpoints of all four edges going from \(V\) are not above, respectively, not below \(V\) (for \(A, B\in \tilde{M}\), we say that \(A\) lies above \(B\) if \(\tilde{p}(A) > \tilde{p}(B)\); further, we assume that \(\tilde{p}(A) \neq \tilde{p}(B)\) whenever vertices \(A\) and \(B\) are different). Maximal and minimal vertices are also called critical. If a vertex \(V\in \tilde{P}\) is maximal (respectively, minimal, critical), then its projection \(\tilde{f}(V)\in P\) is said to be a maximal (respectively, minimal, critical) vertex of \(P\). Obviously, maximal (minimal, critical) vertices of \(P\) are well-defined, because for any vertex \(V\in P\) all its preimages \(\tilde{f}^{-1}(V)\) are or are not critical (maximal, minimal) vertices of \(\tilde{P}\) simultaneously.

First, suppose that there are no critical vertices. Then any vertex of \(\tilde{P}\) induces a change of \(\sigma_1\in\Sigma'\) by one of the moves 1, 2, 3. (However, it can happen that the chain \(s(\sigma_1)\in\Sigma\) undergoes the move 0 or 4. That is why we had to introduce \(\Sigma'\) instead of \(\Sigma.\)) It follows that in this case \(d(s(\sigma_1), s(\sigma_{1+2\pi})) = d'(\sigma_1, \sigma_{1+2\pi})\), so Conjecture 3 implies property 3 (see the beginning of Section 10), and the spine \(P\) contains at least \(c(A)\) vertices.

Suppose that \(\tilde{P}\) contains critical vertices, but there is an isotopy of the embedding of \(P\) in \(M^3\) such that the deformed spine \(P_1\) yields the deformed covering spine \(\tilde{P}_1\) with neither minimal no maximal vertices. Then the number of the vertices of \(P_1\) is at least \(c(A)\) (provided that Conjecture 3 is true); obviously, \(P\) and \(P_1\) have the same number of vertices.

The notion of peripheric edge is necessary for the case of arbitrary spines.

**Definition 12.** An edge \(e\) of the singular graph \(SP\) of a spine \(P\) is said to be peripheric if there exists a vertex \(V\) of \(P\) with the following property: if a cycle

with the original definition by Gromov, we have \(\|M^3(A)\|_t = 0.\) Also see the concluding remarks in [31].
\( \gamma \subset SP \) contains \( e \) and \( p_*(\gamma) \neq 0 \) (here \( p \) is the projection \( M^3 \rightarrow S^3 \)), then every connected component of \( \gamma \setminus V \) containing \( e \) is a loop \( \gamma' \) (with endpoints at \( V \)) such that \( p_*(\gamma') = 0 \); in the other words, the map \( p_* \) restricted to the closure of the connected component of \( SP \setminus V \) containing \( e \) is trivial. Edges that are not peripheric are called regular.

For example, a loop \( e \) is peripheric if and only if \( p_*(e) = 0 \). Unlike the property of a vertex to be critical, the property of an edge to be peripheric is an isotopy invariant.

**Lemma 9.** If \( P \) contains no peripheric edges, then it is isotopic to a spine with no critical vertices.

**Proof.** Take any edge \( e_1 \in P \). Since \( e_1 \) is a regular edge, there exists a cycle \( \gamma_1 \subset SP \) such that \( e_1 \subset \gamma_1 \), \( p_*(\gamma_1) \neq 0 \), and \( \gamma_1 \) does not pass twice through any vertex. Then there exists a spine \( P_1 \) isotopic to \( P \) such that the projection \( \bar{p}: M \rightarrow \mathbb{R}^1 \) restricted to a connected component \( \bar{\gamma}_1' \) of \( \bar{f}^{-1}(\gamma_1') \), where \( \gamma_1' \subset P_1 \), takes the sequence of the vertices of \( \gamma_1' \) to a strictly monotone sequence \( \{a_i: i \in \mathbb{Z}\} \) of real numbers; here \( \gamma_1' \) is obtained from \( \gamma_1 \) under an isotopy that takes \( P \) to \( P_1 \). To construct \( P_1 \), fix a monotone sequence \( \{a_i\} \) satisfying the condition \( a_{i+m} = a_i + 2\pi k \), where \( m \) is the length of \( \gamma_1 \) and \( k \in \mathbb{Z} = \pi_1(S^3) \) is equal to \( p_*(\gamma_1) \). Let \( \{b_i: i \in \mathbb{Z}\} \) be the sequence of the projections \( \bar{p}(V_i) \), where the \( V_i \) are consecutive vertices of \( \gamma_1 \). The differences \( c_k = a_k - b_k \) form an \( m \)-periodic sequence. Let \( \{\delta_i: i = 1, \ldots, m\} \), be a family of arcs in \( M \) such that the endpoints of \( \delta_i \) are \( V_i \) and \( V'_i \), where \( \bar{p}(V'_i) = a_i \) and the \( \delta_i \) do not intersect \( SP \); suppose also that the projections \( \bar{f}(\delta_i) \) do not intersect one another. It is easy to see that such a family exists, and that it is possible to get a spine \( P_1 \) with the properties described above by an isotopy of the mapping \( \text{id}: M^3 \rightarrow M^3 \), fixed outside of a small collar neighborhood of \( \bigcup_{i=1}^{m} s(\delta_i) \).

Now any vertex \( V \in \gamma_1 \) is not critical. If there remain critical vertices, take an edge \( e_2 \in P_1 \) incident to one of them and a cycle \( \gamma_2 \subset SP_1 \) containing \( e_2 \) such that \( p_*(\gamma_2) \neq 0 \) and \( \gamma_2 \) does not pass twice through any vertex. If \( \gamma_2 \cap \gamma_1 = \emptyset \), repeat the construction above. Otherwise, take the connected component \( \gamma_2^s \) of \( \gamma_2 \setminus \gamma_1 \) that contains \( e_2 \), and move the inner vertices of this component as above, but do not move its endpoints. In the spine \( P_2 \) obtained after this step, any vertex \( V \in \gamma_1 \cup \gamma_2^s \) is not critical. Indeed, any vertex of \( \gamma_2^s \) is a noncritical vertex of \( P_2 \) by construction, and any vertex \( V \in \gamma_1 \) has neighboring (in \( \gamma_1 \)) vertices above and below \( V \) in \( P_2 \); \( P_2 \) inherits this property from \( P_1 \), because the isotopy converting \( P_1 \) into \( P_2 \) does not affect \( \gamma_1 \).

If there still remain critical vertices, repeat the construction described above. Then we get a spine \( P_3 \) isotopic to \( P_2 \). The set of the vertices of \( \gamma_1 \cup \gamma_2^s \cup \gamma_2^o \), which certainly are not critical in \( P_3 \), is larger than the similar set \( \gamma_1 \cup \gamma_2^s \) in \( P_2 \). Hence, in a finite number of steps we obtain a spine \( P_k \) isotopic to \( P \) and having no critical vertices.

According to Lemma 9, it follows from Conjecture 3 that any spine of \( M(A) \) without peripheric edges contains at least \( c(A) \) vertices.

The degree of any vertex \( V \in SP \) equals four. Suppose that there are \( k \) peripheric and \( 4-k \) regular half-edges incident to \( V \) (we consider half-edges, because the
Suppose that there is an edge $e$ in the triangulation $P$. Consider the connected component of $SP \setminus V$ containing the peripheric edge $e$ incident to $V$. Let it contain $m$ edges and $n$ vertices different from $V$. Then the number of half-edges in this component is equal to $2m$ and to $4n + 1$ simultaneously, which is impossible.

Let $P$ be an arbitrary spine of $M(A)$. Let us say that a vertex is regular if it is incident to regular edges only, semiregular if it is incident to two peripheric and two regular half-edges, and peripheric if it is incident to peripheric edges only.

Following the proof of Lemma 9, replace $P$ by an isotopic spine $P_1$ such that all regular and semiregular vertices of $P_1$ are critical. For any semiregular vertex $V$ of $P_1$, by $b(V)$ denote the connected component of $SP \setminus V$ that consists of peripheric edges only (i.e., lies “behind $V$”). Following the proof of Lemma 9, replace $P_1$ by a spine $P_2$ such that $P_2$ is isotopic to $P_1$, the isotopy between $P_1$ and $P_2$ does not affect regular edges, $p(V_i) = p(V)$ for all semiregular vertices $V_i \in P_2$ and all peripheric vertices $V_i \in b(V)$, and $p(e)$ is a zero-homotopic loop in $S^1$ for any peripheric edge $e$.

Consider 2-cycles $\sigma_i \in \Sigma'$ constructed from the spine $P_2$. If there is exactly one regular vertex between $F_{t_+}$ and $F_{t_-}$, then $\sigma_{t_+}$ is obtained from $\sigma_{t_-}$ by one of the moves 1, 2, 3, so $d'(\sigma_{t_+}, \sigma_{t_-}) = 1$.

Suppose that there is exactly one semiregular vertex $V \in \tilde{P}_2$ (together with all peripheric vertices of $b(V)$) between $F_{t_+}$ and $F_{t_-}$. Compare $\sigma_{t_-}$ and $\sigma_{t_+}$. The former cycle consists of the triangle corresponding to the regular edge $e_-$ of $\tilde{P}_2$ incident to $V$ such that the other endpoint of $e_-$ lies below $V$, and of some other triangles. The latter cycle includes the triangle that corresponds to the other regular edge $e_+$ of $\tilde{P}_2$ incident to $V$; other triangles that constitute $\sigma_{t_+}$ are the same as in $\sigma_{t_-}$. Note that all triangles but one of a 2-cycle $\sigma \in \Sigma'$ determine the remaining triangle $\Delta$ uniquely: directions of its sides are those of the boundary of the 2-chain formed by the remaining triangles and thus denote $\Delta$ up to dilatation with the coefficient $\pm 1$, and this coefficient is uniquely determined by the condition that $\sigma$ is homological to $[T^2]$; informally, $\Delta$ is the difference between $[T^2]$ and the 2-chain formed by the other triangles of $\sigma$. So in this case we have $\sigma_{t_+} = \sigma_{t_-}$.

We have shown that regular vertices of $P_2$ shift $\sigma_t$ by distance 1, while semiregular and peripheric vertices do not affect $\sigma_t$ at all. Recall that regularity, semiregularity, and periphericity are preserved under isotopy. Consequently, Conjecture 3 implies that the number of regular vertices of an arbitrary spine $P$ of $M(A)$ is at least $c(A)$, so the number of all vertices of $P$ is greater than or equal to $c(A)$, too. This completes the proof of Theorem 11.

Finally, let us provide a sketchy explanation (or, maybe, rather motivation) for the “+5” summand in Conjecture 2. Suppose that there is an edge $e \in P'$ of the triangulation $P'$ dual to a given spine $P$ such that $p_2(e) = \pm 1$ in the group $\pi_1(S^1) = \mathbb{Z}$. Note that the cartesian product structure on $\tilde{M}$ and the corresponding projection $j : \tilde{M} \to T^2$ are not uniquely defined; in our case we can choose any
element of $\mathbb{Z}^2 = \pi_1(T^2)$ to be the image $j_*(e)$. Choose $j$ so that $j_*(e) = 0$. Cycles $\sigma_t$ can include degenerated triangles with an edge $j_*(e)$, which are projections of the triangles of $\tilde{P}^d$ incident to $e$. However, if vertices $A$ and $B$ of a triangle $ABC$ coincide, the triangle $ABC$ cannot be distinguished from $BAC$, which is equal, on the other hand, to $ABC$ with the opposite orientation, so such triangles can be ignored in $\sigma_t$. It can be easily shown that Pachner moves 1, 2, and 3 involving triangles of this type do not affect $\sigma_t$ and thus require $k$ “additional” vertices of $P$, where $k$ is the number of triangles of $\tilde{P}^d$ incident to $e$, that is, the number of sides of the 2-cell of $\tilde{P}$ dual to $e \subset \tilde{P}$. We can assume $P$ to be a minimal spine of $M(A)$. In this case $P$ is a special spine, and, according to Section 2, it contains $n + 1$ 2-cells, where $n = c(M^3)$ is the number of its vertices. The total number of the vertices of 2-cells equals $6n$, so the average number of sides for a 2-cell of $P$ is $\frac{6n}{n+1} > 5$ (because $n \geq 6$). Thus it is reasonable to expect at least 5 “spare” vertices, in addition to $c(A)$ vertices that do change $\sigma_t$.

Of course, it can happen that there are no edges $e$ such that $p_*(e) = \pm 1$. However, there always exists an edge $e$ such that $p_*(e) = m > 0$. Then the argument above can be applied to an $m$-fold covering $M_m = M(A^m)$ of $M(A)$. A special spine $P$ of $M(A)$ (punctured once) is covered by a special spine $\tilde{P}_m$ of $\tilde{M}_m$ with $m$ punctures, and we can destroy $m - 1$ 2-cells of $\tilde{P}_m$ in order to get a special spine of $\tilde{M}_m$ punctured only once. Apart from this, we destroy an $m$th 2-cell of $\tilde{P}_m$ by the process described in the previous paragraph. Consequently, we obtain vertices of $m$ 2-cells of $\tilde{P}_m$ as additional, spare vertices, and $c(A^m) = mc(A)$ (as we can restrict ourselves to the cases of hyperbolic or parabolic monodromy) vertices of $\tilde{P}_m$ affecting $\sigma_t$; obviously, the number of vertices of $\tilde{P}_m$ is equal to $m$ times the number of the vertices of $P$. Unfortunately, the “averaging argument” provides no proof for the claim that, by destroying $m$ 2-cells during the above-described process, we can destroy at least $5m$ vertices (in fact, $4m + 1$ would already suffice), even though there is an essential ambiguity in the choice of 2-cells to be discarded. Because of pseudominimality of $P$, all boundary curves of 2-cells are not short, see [16] and the beginning of Section 4, but this does not readily yield $4m$ of $4m + 1$ spare vertices required, since a priori the boundary curve of a 2-cell can visit some vertices more than once. It is only possible to prove with this approach that $2m$ vertices are destroyed when $m$ 2-cells are discarded, and this results in 2 “additional” vertices of $P$, compare with Theorem 8.

11. Concluding Remarks

In this section we discuss possible applications of the results and methods of this paper to other 3-dimensional manifolds.

**Stallings manifolds.** Consider a fibration $p: M^3 \xrightarrow{F_g} S^1$, where $F_g$ is an orientable surface (of genus $g$) and $M^3$ is an orientable 3-manifold. It may happen that a manifold $M$ can be fibered over a circle in several different ways, and the genus of the fiber need not be defined uniquely by $M^3$. Choose any of the fibering structures and denote by $A$ the monodromy, which is an isotopy class of self-diffeomorphisms $F_g \to F_g$. 

The ideas of Sections 3–9 can be applied to this situation as well. Nothing changes in Section 7. Further, Lemma 6 remains true. Obviously, the set of isotopy classes of \( \theta \)-curves should be replaced by the set of isotopy classes of trivalent graphs \( L \subset F_g \) such that \( F_g \setminus L \) is a 2-cell. With this correction, analogues of Lemma 7 and Theorem 9 hold, while the proof of Lemma 7 requires slight modification in the second paragraph. However, the difference between 1-skeletons \( L_- \) and \( L_+ \) of \( F_g \) in an analogue of Lemma 8 is measured by at most one Dehn twist (rather than at most one flip), which may require up to \( 4g - 3 \) flips. So in Theorem 10 we only get the estimate \( d(\Delta_+, \Delta_-) \leq 5(4g - 3) \), where \( \Delta_+, \Delta_- \) denote isotopy classes of trivalent graphs embedded in \( F_g \) so that their complements are 2-cells, and \( d \) is the “flip-distance”. The remaining part of the reasoning of Sections 8, 9 goes smoothly, and we get an analogue of Theorem 8 with the factor \( \frac{1}{3(4g-3)} \) instead of \( \frac{1}{2} \). The approach discussed in Section 10 can be applied to this situation, too.

Another difference from the case of \( g = 1 \) is that we no longer know how to find \( c(A) \), that is, the minimal number of flips required to convert a trivalent graph \( L \) such that \( F_g \setminus L \) is a 2-cell into its monodromy image \( AL \). Recall that \( c(A) \) can be computed as explained in Section 3 whenever \( g = 1 \). The former approach is obstructed by the fact that the graph \( \Gamma \) is not a tree if \( g > 1 \). Probably, some estimates for \( c(A) \) are easier to obtain than its exact value. In order to do this using the latter approach, one should replace the Farey tesselation of \( H^2 \) by the cell decomposition of the Teichmüller space of \( F_g \) described in [5].

**Topological economy principle.** If Conjecture 2 holds, a minimal spine of a torus bundle space \( M(A) \) can be constructed as follows (compare with Section 4): fix a fiber \( F = p^{-1}(0) \subset M^3 \), then choose a \( \theta \)-curve \( L \subset F \) that requires \( c(A) \) flips only to be converted to \( AL \), construct a simple polyhedron \( P_0 \) from the evolution of \( L \), and, finally, add to \( P_0 \) an extra face that cuts any path connecting two boundary components of \( M^3 \setminus F \) in the complement of \( P_0 \).

According to Thurston’s geometrisation conjecture, any prime orientable compact 3-manifold \( M^3 \) can be cut into geometric pieces \( M_i \), \( i = 1, \ldots , m \), along incompressible tori \( T^2_j \), \( j = 1, \ldots , n \), see, for example, [33] or the last section of [27]. For nontrivial cycles in the \( T^2_j \) remain nontrivial in \( M^3 \) (by incompressibility), any spine \( P \) of \( M \) intersects all nontrivial cycles in any \( T^2_j \); assuming general position, the intersections \( P \cap T^2_j \) contain \( \theta \)-curves for all \( j \) by virtue of Lemma 7.

Given a family of \( \theta \)-curves \( L_j \subset T^2_j \), consider simple polyhedra \( P_t \subset M_i \) such that \( P_t \cap T^2_j = L_j \) for any boundary torus \( T^2_j \) of \( M_i \). Then add extra faces to \( P_t \) as necessary to get spines of the \( M_i \). The union \( P \) of obtained polyhedra is a spine of \( M^3 \) (maybe, with several punctures). Minimize the total number of vertices of \( P \) over all possible choices of the \( P_t \) and the extra faces and over all “boundary conditions”, that is, all families \( L_j \subset T^2_j \). Needless to say, it is not clear yet how to implement this program in the general case.

**Conjecture 4.** A minimal spine of any 3-manifold (that can be cut into geometric pieces) can be obtained by the procedure described in the paragraph above.

In other words, there exists a minimal spine of \( M^3 \) that intersects any torus \( T^2_j \) along one \( \theta \)-curve only. Similar conjecture can be formulated about graph-mani-
folds [19], [36], which can be cut along incompressible tori into several copies of $D^2 \times S^1$ and $N^2 \times S^1$, where $N^2$ stands for $D^2$ with two holes. Conjecture 5 is yet another manifestation of the “topological economy principle” expressed and illustrated in [4].

References


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