STATISTICS OF YOUNG DIAGRAMS OF CYCLES OF DYNAMICAL SYSTEMS FOR FINITE TORI AUTOMORPHISMS

V. ARNOLD

ABSTRACT. A permutation of a set of $N$ elements is decomposing this set into $y$ cycles of lengths $x_i$, defining a partition $N = x_1 + \cdots + x_y$. The length $X_1$, the height $y$ and the fullness $\lambda = N/xy$ of the Young diagram $x_1 \geq x_2 \geq \cdots \geq x_y$ behave for the large random permutation like $x \sim aN$, $y \sim b\ln N$, $\lambda \sim c/\ln N$.

The finite 2-torus $M$ is the product $\mathbb{Z}_m \times \mathbb{Z}_m$, and its Fibonacci automorphism sends $(u, v)$ to $(2u + v, u + v) \pmod{m}$. This permutation of $N = m^2$ points of the finite torus $M$ defines a peculiar Young diagram, whose behavior (for large $m$) is very different from that of a random permutation of $N$ points.

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1. Parameters of Young Diagrams of Random Permutations and of Automorphisms of Finite Tori

An automorphism of the continuous torus manifold $T^m = \mathbb{R}^m/\mathbb{Z}^m$ is defined by a matrix $A \in \text{GL}(m, \mathbb{Z})$. The periodical orbits of this dynamical system form the discrete finite tori $M = \mathbb{Z}_n^m$, consisting of the points with rational coordinates,

$z = (z_1, \ldots, z_m)$, $z_j = u_j/n$, $u_j \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

The number of such periodic points (for a fixed value of denominator $n$) is finite:

$|\mathbb{Z}_n^m| = n^m$.

The resulting permutation $A: M \to M$ of finite set $M$ consists of $y$ cycles of lengths $x_1 \geq x_2 \geq \cdots \geq x_y$, where $x_1 + \cdots + x_y = n^m$. This partition of number $n^m$ is described by its Young diagram (consisting of $y$ lines, containing $x_1$ unite squares in the first line, $x_2$ in the second line and so on). The area of this diagram equals $n^m$.

The present article discusses the shapes of these diagrams. For instance, we shall discuss the question, whether it is similar to the typical shape of the diagram of a

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random permutation \( a: M \to M \) of a finite set, consisting of \( N = |M| \) elements. Namely, we shall compare the parameters of the permutation \( A \), defined by the automorphism of the torus, with the mean values of these parameters, averaged along all the \( N! \) permutations \( a \in S(M) \) of finite set \( M \), having \( N \) elements.

For instance, we shall study the following parameters of a Young diagram:

- \( x \) = the length (the length \( x_1 \) of the longest cycle);
- \( y \) = the height (the number of the lines);
- \( \lambda \) = the fullness \( (\lambda = N/(xy)) \) shows which part of the circumscribed rectangle of the diagram is covered by it);
- \( \mu = y/x \) (the diagram’s asymmetry).

The question on the behavior of the mean values of these parameters of random permutations of \( N \) objects for \( N \to \infty \) leads to interesting conjectures (like \( x \sim c_1 N, \ y \sim c_2 \ln N, \ \lambda \sim c_3/\ln N, \ \mu \sim c_4(\ln N)/N \), at least for the Cesaro averages in \( N \).

The main result of the present article is the observation, that parameters of the Young diagrams of the cycles of the Fibonacci automorphisms \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) of the torus \( T^2 \), (computed in the article for \( n \leq 20 \)) are very different from the mean values, described above (the values of \( \lambda \) and of \( \mu \) being much larger for the case of the automorphisms, than for the typical random permutations of \( N = n^2 \) elements).

2. Young Diagrams of Cycles of the Golden Ratio
   Fibonacci Automorphism, \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \)

The results of the calculations of these cycles lengths diagrams for the automorphisms \( A: M \to M \), where \( M = \mathbb{Z}^2_n \) are presented below. We denote the partition \( N = x_1 + x_2 + \cdots + x_y, \ x_1 \geq x_2 \geq \cdots \geq x_y \)

by the monomial symbol

\[
D = X_1^{Y_1} \cdot X_2^{Y_2} \cdots X_s^{Y_s},
\]

where \( X_1 > X_2 > \cdots > X_s \) are the different lengths of the lines of the diagram, the numbers \( Y_1, \ldots, Y_s \) being their multiplicities (for instance \( x_1 = \cdots = x_{Y_1} > x_{Y_1+1} \)).

The values \( Y_i = 1 \) shall be omitted in the monomial symbol notation \( D \) of the diagram (thus, \( D = 21^2 \) denotes the diagram \( \square_1 \)).

Theorem 1. The parameters of the cycles’ diagram \( D \) of the Fibonacci permutations

\[
A(z_1, z_2) = (2z_1 + z_2, z_1 + z_2)
\]
of the finite tori \( \mathbb{Z}_n^2 \), have, for \( n \leq 20 \), the following values:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( D )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3:1</td>
<td>3</td>
<td>2</td>
<td>( 2/3 \approx 0.67 )</td>
<td>( 2/3 \approx 0.67 )</td>
</tr>
<tr>
<td>3</td>
<td>4:2:1</td>
<td>4</td>
<td>3</td>
<td>( 3/4 = 0.75 )</td>
<td>( 3/4 = 0.75 )</td>
</tr>
<tr>
<td>4</td>
<td>3:5:1</td>
<td>3</td>
<td>6</td>
<td>( 8/9 \approx 0.89 )</td>
<td>( 2 = 2.00 )</td>
</tr>
<tr>
<td>5</td>
<td>( 10^2 \cdot 2^2:1 )</td>
<td>10</td>
<td>5</td>
<td>( 1/2 = 0.50 )</td>
<td>( 1/2 = 0.50 )</td>
</tr>
<tr>
<td>6</td>
<td>( 12^2 \cdot 4^2:3:1 )</td>
<td>12</td>
<td>6</td>
<td>( 1/2 = 0.50 )</td>
<td>( 1/2 = 0.50 )</td>
</tr>
<tr>
<td>7</td>
<td>( 8^6:1 )</td>
<td>8</td>
<td>7</td>
<td>( 7/8 \approx 0.88 )</td>
<td>( 7/8 \approx 0.88 )</td>
</tr>
<tr>
<td>8</td>
<td>( 6^8:3^5:1 )</td>
<td>6</td>
<td>14</td>
<td>( 16/21 \approx 0.76 )</td>
<td>( 7/3 \approx 2.33 )</td>
</tr>
<tr>
<td>9</td>
<td>( 12^6:4^2:1 )</td>
<td>12</td>
<td>9</td>
<td>( 3/4 = 0.75 )</td>
<td>( 3/4 = 0.75 )</td>
</tr>
<tr>
<td>10</td>
<td>( 30^2 \cdot 10^2:6^3:2^2:1 )</td>
<td>30</td>
<td>10</td>
<td>( 1/3 \approx 0.33 )</td>
<td>( 1/3 = 0.33 )</td>
</tr>
<tr>
<td>11</td>
<td>( 5^{34}:1 )</td>
<td>5</td>
<td>25</td>
<td>( 121/125 \approx 0.97 )</td>
<td>( 5 = 5.00 )</td>
</tr>
<tr>
<td>12</td>
<td>( 12^{10} \cdot 4^2:3^5:1 )</td>
<td>12</td>
<td>18</td>
<td>( 2/3 \approx 0.67 )</td>
<td>( 3/2 = 1.5 )</td>
</tr>
<tr>
<td>13</td>
<td>( 14^{12}:1 )</td>
<td>14</td>
<td>13</td>
<td>( 13/14 \approx 0.93 )</td>
<td>( 13/14 \approx 0.93 )</td>
</tr>
<tr>
<td>14</td>
<td>( 24^8 \cdot 3^3:1 )</td>
<td>24</td>
<td>14</td>
<td>( 7/12 \approx 0.58 )</td>
<td>( 7/12 \approx 0.58 )</td>
</tr>
<tr>
<td>15</td>
<td>( 20^8 \cdot 10^2 \cdot 4^{10} \cdot 2^2:1 )</td>
<td>20</td>
<td>23</td>
<td>( 45/92 \approx 0.49 )</td>
<td>( 23/20 \approx 1.15 )</td>
</tr>
<tr>
<td>16</td>
<td>( 12^{10} \cdot 6^8:3^5:1 )</td>
<td>12</td>
<td>30</td>
<td>( 32/45 \approx 0.71 )</td>
<td>( 5/2 = 2.50 )</td>
</tr>
<tr>
<td>17</td>
<td>( 18^{16}:1 )</td>
<td>18</td>
<td>17</td>
<td>( 17/18 \approx 0.94 )</td>
<td>( 17/18 \approx 0.94 )</td>
</tr>
<tr>
<td>18</td>
<td>( 12^{20} \cdot 4^2:3:1 )</td>
<td>12</td>
<td>30</td>
<td>( 9/10 = 0.90 )</td>
<td>( 5/2 = 2.50 )</td>
</tr>
<tr>
<td>19</td>
<td>( 9^{40}:1 )</td>
<td>9</td>
<td>41</td>
<td>( 361/369 \approx 0.98 )</td>
<td>( 41/9 \approx 4.56 )</td>
</tr>
<tr>
<td>20</td>
<td>( 30^{10} \cdot 10^2 \cdot 6^{10} \cdot 3^5:2^2:1 )</td>
<td>30</td>
<td>30</td>
<td>( 4/9 \approx 0.44 )</td>
<td>( 1 = 1.00 )</td>
</tr>
</tbody>
</table>

**Remark 1.** The table suggests several conjectures. For the prime values \( n = p > 5 \) the symbol of the diagram is of the form \( D = X^Y_1 \). One sees suggesting relations of \( D(pq) \) to \( D(p) \) and \( D(q) \). We shall leave to the reader the pleasure to formulate such conjectures.

**Remark 2.** Some examples of the diagrams of cycles of some special automorphisms of finite tori had been proved by Persival and Vivaldi in [1]. But their special examples form a small part of the total number of cases and one is unable to conclude from these examples what is the general shape of the typical diagram of cycles lengths (even for the cases of the golden ratio automorphism \( \frac{2}{1} \frac{1}{1} \)) and for the typical values of \( n \).

**Remark 3.** Together with the finite torus \( M = \mathbb{Z}_p^2 \), one might consider the finite projective line,

\[
P = P^1(\mathbb{Z}_p) = (\mathbb{Z}_p^2 \setminus 0) / (\mathbb{Z}_p \setminus 0),
\]

consisting of \( p + 1 \) points.

Operator \( A : M \to M \) acts on \( P \) as some special (projective) permutation,

\[
A_p \in \text{GP}(\mathbb{Z}_p) \subset S(p+1)
\]

of \( p + 1 \) points. This permutation is the quotient of \( A : M \to M \) (modulo the scalars subgroup of the linear group).

The cycles of these permutations define new Young diagrams (of area \( p + 1 \)).
Theorem 2. The parameters of the Young diagrams of cycles of the projective permutations $A_p$ of the set $P$ of $p+1$ elements, defined by the Fibonacci automorphism

$$A(z_1, z_2) = (2z_1 + z_2, z_1 + z_2) \pmod{p \leq 20}$$

have the following values:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$D$</th>
<th>$x$</th>
<th>$y$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>$1 = 1,00$</td>
<td>$1/3 \approx 0.33$</td>
</tr>
<tr>
<td>3</td>
<td>$2^2$</td>
<td>2</td>
<td>2</td>
<td>$1 = 1,00$</td>
<td>$1 = 1,00$</td>
</tr>
<tr>
<td>5</td>
<td>$5^1$</td>
<td>5</td>
<td>2</td>
<td>$3/5 = 0,60$</td>
<td>$2/5 = 0,40$</td>
</tr>
<tr>
<td>7</td>
<td>$4^2$</td>
<td>4</td>
<td>2</td>
<td>$1 = 1,00$</td>
<td>$1/2 = 0,50$</td>
</tr>
<tr>
<td>11</td>
<td>$5^1 \cdot 4^1$</td>
<td>5</td>
<td>4</td>
<td>$3/5 = 0,60$</td>
<td>$4/5 = 0,80$</td>
</tr>
<tr>
<td>13</td>
<td>$7^2$</td>
<td>7</td>
<td>2</td>
<td>$1 = 1,00$</td>
<td>$2/7 \approx 0,29$</td>
</tr>
<tr>
<td>17</td>
<td>$9^2$</td>
<td>9</td>
<td>2</td>
<td>$1 = 1,00$</td>
<td>$2/9 \approx 0,22$</td>
</tr>
<tr>
<td>19</td>
<td>$9^2 \cdot 1^2$</td>
<td>9</td>
<td>4</td>
<td>$19/36 \approx 0,53$</td>
<td>$4/9 \approx 0,44$</td>
</tr>
</tbody>
</table>

Proof of Theorem 1. One simply calculates the orbits of permutation $A: M \to M$. To abbreviate the notations we shall write the relation

$$Az = w \quad (\text{where } z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix})$$

in the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad (\text{where } z_j \in \mathbb{Z}_n, w_k \in \mathbb{Z}_n).$$

Calculating next $A^2z = Aw$, and so on, we obtain the sequence of vectors in $M$, providing the cyclical orbit.

For $n = 5$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ we get the following orbits:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{cycle 0 of length 1};$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \text{cycle I of length 10};$$

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \text{cycle II of length 10};$$

$$\begin{pmatrix} 0 \\ 3 \end{pmatrix} \in \Pi, \begin{pmatrix} 0 \\ 4 \end{pmatrix} \in \Pi;$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Pi, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \Pi, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \Pi, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \in \Pi;$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \text{cycle III of length 2};$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \in \Pi, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \Pi, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in \Pi, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \Pi;$$
\[
\begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \ldots \quad \implies \text{cycle IV of length 2.}
\]

The 5 cycles, described above, provide
\[|O| + |I| + |II| + |III| + |IV| = 1 + 10 + 10 + 2 + 2 = 25 = |M|\]
points, therefore there are no other cycles.

We have thus proved the statement of Theorem 1 for \( n = 5 \). The other cases are proved similarly, the calculations being sometimes longer (due to the longer cycles and to the higher numbers of cycles). \( \square \)

The same calculations provide also Theorem 2. For example, consider the case \( n = p = 5 \). The scalars \( c \) act on the cycles: \( cO = O \), while the cycle I moves the following way:
\[2I = II, \quad 3I = II, \quad 4I = I.\]

These relations show also, that
\[2II = I, \quad 3II = I, \quad 4II = II.\]

Similarly, we calculate, that
\[2III = IV, \quad 3III = IV, \quad 4III = III,\]
and hence the cycle IV moves similarly:
\[2IV = III, \quad 3IV = III, \quad 4IV = IV.\]

It follows, that the projective permutation \( A_p \) (\( p = 5 \)) acts the following way on the 6 points
\[\xi = 0, \ 1, \ 2, \ 3, \ 4, \ \infty,\]
(\( \text{representing the 6 lines} \ z_2 = \xi z_1 \)):
\[A_p: 0 \mapsto 3 \mapsto \infty \mapsto 1 \mapsto 4 \mapsto 0, \quad 2 \mapsto 2.\]

We have thus calculated the Young diagram \( D(A_5) = 5 \cdot 1 \), proving Theorem 2 for \( p = 5 \).

The calculation of the projective permutation \( A_p: P \to P \) is simpler, than the direct calculation of the orbits of \( A: M \to M \) in Theorem 1. One might first prove Theorem 2 and then use it to obtain the table of Theorem 1. This simple reasoning, using the projective geometry, is, unfortunately, hidden in the paper [1], where it is replaced by complicated algebraic notations of fields extensions theory.

3. Statistics of Young Diagrams of Random Permutations of \( N \) Points

For \( N \leq 7 \) we describe below all the Young diagrams \( D \) of the \( N! \) permutations of \( N \) points. For each diagram \( D \) of area \( N \) we show in the following table the number \( r(D) \) of its realizations by different permutations of the \( N \) elements (providing the whole symmetric group of order \( \sum r(D) = N! \)):

\[
\begin{array}{c|c|c|c}
\hline
\text{N} & \text{r} & \text{N = 2} & \text{N = 3} & \text{N = 4} \\
\hline
\text{D} & \text{r} & 2 & 1.1 & 3 & 2.1 & 1^3 & 4 & 3.1 & 2^2 & 2.1^2 & 1^4 \\
\text{r} & \text{N} & 1 & 1 & 2 & 3 & 1 & 6 & 8 & 3 & 6 & 1 \\
\hline
\end{array}
\]
For each Young diagram \( D \) it is easy to calculate its length, \( x(D) \), its height \( y(D) \), its fullness \( \lambda(D) = N(D)/\langle x(D)y(D) \rangle \) and its asymmetry \( \mu(D) = y(D)/x(D) \).

Then we calculate the mean values

\[
\hat{f}(N) = \frac{\sum (r(D)f(D))/N!}{N!}
\]

(for each parameter \( f = x, y, \lambda \) or \( \mu \)).

In the asymmetry case we introduce also the logarithmical measure of the asymmetry, \( l(D) = \log_{10} \mu(D) \). Next we calculate its quadratic mean value \( \sigma^2(N) \), defined by the relation

\[
\sigma^2(N) = \frac{\sum (r(D)l^2(D))/N!}{N!}.
\]

**Theorem 3.** For \( N \leq 7 \) the parameters of the \( N! \) permutations of \( N \) elements have the following mean values:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \hat{x} )</th>
<th>( \hat{y} )</th>
<th>( \hat{\lambda} )</th>
<th>( \hat{\mu} )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 3/2 = 1.50 )</td>
<td>( 3/2 = 1.50 )</td>
<td>1.00</td>
<td>5/4 = 1.25</td>
<td>0.30</td>
</tr>
<tr>
<td>3</td>
<td>( 2\frac{1}{5} \approx 2.17 )</td>
<td>( 1\frac{2}{5} \approx 1.83 )</td>
<td>7/8 ( \approx 0.88 )</td>
<td>10/9 ( \approx 1.11 )</td>
<td>0.33</td>
</tr>
<tr>
<td>4</td>
<td>( 2\frac{19}{23} \approx 2.79 )</td>
<td>( 2\frac{7}{12} \approx 2.08 )</td>
<td>( 2\frac{5}{36} \approx 0.80 )</td>
<td>( 1\frac{17}{14} \approx 0.95 )</td>
<td>0.36</td>
</tr>
<tr>
<td>5</td>
<td>( 3\frac{17}{20} \approx 3.42 )</td>
<td>( 2\frac{7}{16} \approx 2.28 )</td>
<td>( 2\frac{25}{35} \approx 0.75 )</td>
<td>( 1\frac{8}{29} \approx 0.82 )</td>
<td>0.26</td>
</tr>
<tr>
<td>6</td>
<td>( 4\frac{31}{75} \approx 4.04 )</td>
<td>( 2\frac{49}{75} \approx 2.57 )</td>
<td>0.72</td>
<td>0.76</td>
<td>0.40</td>
</tr>
<tr>
<td>7</td>
<td>4.68</td>
<td>2.71</td>
<td>0.69</td>
<td>0.70</td>
<td>0.42</td>
</tr>
</tbody>
</table>

I have proved no asymptotical formulae for the description of the behaviors of these mean values for larger \( N \), but the above table leads to the following conjectures.

1. The mean length of the area \( N \) diagrams of random permutations of \( N \) elements grows like

\[
\hat{x} \sim c_1 N
\]

(for some coefficient \( c_1 \) of order of 2/3).
2. The mean height of the diagrams of random permutations of $N$ elements grows with $N$ much slower, than the mean length, namely like

$$\hat{y} \sim c_2 \ln N$$

(growing approximately 2 times, while $N$ is replaced by $N^2$).

3. The mean fullness of the diagrams of random permutations of $N$ elements declines slowly, while $N$ is growing, in the average like

$$\hat{\lambda} \sim c_3 / \ln N$$

(declining approximately by a factor of 2, while $N$ is replaced by $N^2$).

4. The mean asymmetry $\hat{\mu}$ of the diagrams of random permutations of $N$ elements declines slowly, while $N$ is growing, but it remains rather large for a considerable part of the permutations (as shows the growth of the quadratic mean $\sigma(N)$ with $N$ in the preceding table).

Of course, all these empirical observations need a serious theoretical study. I shall describe here only some unrigorous ("physical") reasons for the above conjectures 1 and 2.

Consider a series of random independent choices, one after the other, of the elements of a finite set, consisting of $N$ elements.

It is natural to guess, that the repeated choice of an element would (in the average) occur after approximately $cN$ choices, for some constant $c < 1$.

This guess leads to the conclusion $x_1 \sim c_1 N$ for the length of the first cycle. The later choices might be considered as a new series of independent choices (of the elements, of the remaining set of $N_1 = (1 - c_1)N$ elements). The next repetition is expected after $x_2 = c_1 N_1$ choices, leaving for the rest the set of $N_2 = (1 - c_1)N_1 = (1 - c_1)^2 N$ remaining unchosen elements. Continuing this way, we shall leave after $y$ cycles the remaining set of $N_y = (1 - c_1)^y N$ unchosen elements. Therefore, the cycles lengths would form an approximately geometrical progression.

The process will end, when there would remain only 1 element, $N_y \sim 1$. It leads to the guess, that

$$y \sim \ln(1 - c_1)^{-1} N = \frac{\ln N}{-\ln(1 - c_1)},$$

explaining conjecture 2.

The explicit calculations of the statistics of the random permutations of $N$ elements for $N > 7$ become very long, and even the computers would not be able to compute the needed sums of the $N!$ terms, say, for $N = 100$ (which is the case that we need for the comparison with the situation of the automorphisms of Section 2, where $N = n^2$, $n = 2, 3, \ldots, 20$).

Therefore, I had tried a different approach, creating artificially some "random permutations" of a hundred of elements and calculating for the resulting Young diagrams the "typical" values of the parameters ($x, y, \lambda, \mu$).

I describe in the next section my ways to generate such "random" permutations. The resulting values of the parameters confirm, in general, the above conjectures 1–4, as we shall see.
4. Experimental Ways to Generate some “Random” Permutations

I had used two ways described in Section 4 and in Section 3. The first way starts from a table of random numbers. To generate a permutation of the $N = 100$ elements, interpreted as the two-digits decimal numbers (00, 01, . . . , 99), one starts from the table of random digits ($\alpha, \beta, \gamma, \delta, \ldots$).

The first element we choose will be ($\alpha, \beta$). If ($\gamma, \delta$) is different from it, we choose it to be the second one. If not, we move to the following element ($\varepsilon, \zeta$), and so on.

If the first $k$ different pairs are ready, we move till the first pair, different from all the chosen ones (in our random digits table).

We obtain this way an ordering of all the $N = 100$ pairs, defining the desired permutation of $N$ elements. We guess it to be a “random” permutation, provided that the initial table was sufficiently random.

**Example.** From the following table of random numbers, consisting of 100 two-digits decimal numbers, the preceding algorithm had chosen in a succession 64 different two-digits long numbers (the 36 rejected candidates, appearing repeatedly, are enclosed in brackets):

47 99 07 32 02 91 52 66 21 81 27 82 70 43 17 65 76 28 63 08
94 11 01 95 (52) (76) 87 (65) 29 16 20 80 10 25 37 (65) (32) 35 (21) 74
05 36 18 (24) 73 (48) 90 18 75 12 (02) 15 41 72 38 61 (73) (73) (63) (11)
24 83 56 (32) (74) 06 84 (56) (81) 67 14 03 (83) (56) 96 (48) (27) (37) 97 (08)
(37) 89 (02) (97) (38) (52) 44 19 (24) (28) (12) (01) 13 69 (20) (17) (84) 88 53 (61).

The mean waiting time to meet a new two-digits number grows from 1 in the first line to 1.4 in the second, 1.65 in the third, 1.75 in the forth line and 2.6 in the last, fifth, one. The average waiting time (for all the 100 candidates choices) is, therefore, 1.68.

The theory, described below, suggests for the choice of the ($k + 1$)-st candidate from $N$ objects the waiting time (number of the attempts) of order $N/(N - k)$. This theory provides for the 64 successful choices (in our model with $N = 100$ candidates) the mean waiting time

$$\frac{100}{64} \ln \frac{100}{100 - 65} \approx 1.56 \ln 2.86 \approx 1.62$$

(excellently similar to the above observed value 1.68).

To use the above technology, one needs a random numbers table. I had used the following 3 variants:

1) using the list of the members of the National Academy of Science of USA, write the two middle digits (say, the 4th and the 5th) of the telephone number of each member of the Academy (ordered alphabetically).

2) write the first and the third digits of the car’s number for the sequence of the cars, moving along the Vavilova street (in front of the Steklov Mathematical Institute at Moscow).

3) copy few lines from the book “Abime’s number” by V. Varapetian (Moscow, New Time, 1998), which book consists of many millions of random digits.

Some defect of these technologies is the delay, due to the many repetitions of the chosen numbers, especially at the end of the choices of $N$ different objects.
I guess, that the total number of the random elements, needed to get a permutation of $N$ elements, grows with $N$ like $N \ln N$.

Indeed, at the first remaining candidate choice among the $N$ candidates the probability to meet it in the random elements table is $1/N$, and the expected waiting time is approximately $N$.

At the preceding candidate choice the probability not to reject is $2/N$, the waiting time expectation being, therefore, $N/2$.

Similarly, at $j$ choices before the end the probability to meet a new element in the table of random elements is $j/N$. The expectation of the waiting time is $N/j$.

Thus, the total number of the random elements to check would be (in the average)

$$\sum_{j=1}^{N} (N/j) = N \sum_{j=1}^{N} (1/j) \sim N \ln N.$$ 

Therefore, to create a permutation of $N = 100$ two-digits long numbers one would need approximately 500 random telephone numbers. The National Academy of Sciences list is sufficiently long for this goal.

However, to avoid the delaying factor $\ln N$, I had invented faster ways to create random permutations.

One might use the above algorithm for the first $N/2$ choices (when there are still not so many delaying repetitions). The remaining set of $N/2$ candidates should be permuted along the $N/2$ remaining places by the same method, using the preceding algorithm to order $N/2$ elements (instead of $N$).

To choose this ordering, one might even use the ready random ordering of the first half of the candidates, identifying this first half with the second one by some bijection.

One might also use the old method, replacing the random digits table $(\alpha \beta)(\gamma \delta)\ldots$ be the table $(\beta \gamma)(\delta \varepsilon)\ldots$

An other reasonable suggestion is to order the remaining set of $N^1 = N/K$ elements, using as the new random elements table the sequence of the residues of the random elements of the set $\{1, \ldots, N\}$ of the initial table modulo $N'$.

**Example.** For $N = 16$ I had obtained this way the following ordering of the 16 residues $\{0, 1, \ldots, 15\}$:

0, 4, 3, 12, 9, 8, 7, 14, 5, 1, 2, 11, 6, 15, 10, 13.

To study the resulting permutation I rewrite this information in form of two tables

$$a = \begin{array}{ccc}
12 & 13 & 14 & 15 \\
8 & 9 & 10 & 11 \\
4 & 5 & 6 & 7 \\
0 & 1 & 2 & 3
\end{array}$$

$$b = \begin{array}{ccc}
6 & 15 & 10 & 13 \\
5 & 1 & 2 & 11 \\
9 & 8 & 7 & 14 \\
0 & 4 & 3 & 12
\end{array}$$

Namely, the permutation sends to the place of the element $j$ of table $a$ the element of the table $b$, occupying in table $b$ the set, occupied by $j$ in table $a$. For instance, we get

$0 \rightarrow 0, \quad 4 \rightarrow 9, \quad 8 \rightarrow 5, \quad \ldots, \quad 15 \rightarrow 13$. 
This method provides immediately the cycles of our permutation:

- $0 \ [0, \ldots] \ \text{of length 1}$;
- $1, 4, 9 \ [1, \ldots] \ \text{of length 3}$;
- $2, 3, 12, 6, 7, 14, 10 \ [2, \ldots] \ \text{of length 7}$;
- $5, 8 \ [5, \ldots] \ \text{of length 2}$;
- $11 \ [11, \ldots] \ \text{of length 1}$;
- $13, 15 \ [13, \ldots] \ \text{of length 2}$.

Therefore, the cycles Young diagram of our permutation of $N = 16$ elements has the symbol

$$D = 7\cdot3\cdot2^2\cdot1^2.$$  

We calculate the parameters values

$$x = 7, \quad y = 6, \quad \lambda = 16/42 \approx 0.38, \quad \mu = 6/7 \approx 0.86.$$  

The goal is to use these empirical “Monte Carlo” values instead of the mean values $(\hat{x}(16), \ldots, \hat{\mu}(16))$, whose calculation would require the summation of the parameters values for all the $16!$ permutations of our 16 elements.

I shall describe one more example of these “random” permutations calculations.

The ordering of some 64 numbers, provided by the random digits table at page 50 can be represented, similarly to the preceding case, by two tables of 64 numbers:

| $a = \begin{array}{cccccccc}
89 & 90 & 91 & 94 & 95 & 96 & 97 & 99 \\
76 & 80 & 81 & 82 & 83 & 84 & 87 & 88 \\
66 & 67 & 69 & 70 & 72 & 73 & 74 & 75 \\
47 & 48 & 52 & 53 & 56 & 61 & 63 & 65 \\
32 & 35 & 36 & 37 & 38 & 41 & 43 & 44 \\
19 & 20 & 21 & 24 & 25 & 27 & 28 & 29 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
01 & 02 & 03 & 05 & 06 & 07 & 08 & 10 \\
\end{array}$ | $b = \begin{array}{cccccccc}
97 & 89 & 44 & 19 & 73 & 69 & 88 & 53 \\
83 & 56 & 06 & 84 & 67 & 14 & 03 & 96 \\
75 & 12 & 15 & 41 & 72 & 38 & 61 & 24 \\
35 & 74 & 05 & 36 & 48 & 73 & 90 & 18 \\
87 & 29 & 16 & 20 & 80 & 10 & 25 & 37 \\
76 & 28 & 63 & 08 & 94 & 11 & 01 & 95 \\
21 & 81 & 27 & 82 & 70 & 43 & 17 & 65 \\
47 & 99 & 07 & 32 & 02 & 91 & 52 & 66 \\
\end{array}$ |

Both tables consist of the same 64 numbers, ordered in table $a$ as in the natural series, while table $b$ presents them in the order of page 50 (provided by the random numbers table).

The permutation sends the element $j$ of table $a$ to the element, written in table $b$ at the place, occupied by $j$ in table $a$. Thus, we easily find the orbits, like

$$01 \mapsto 47 \mapsto 35 \mapsto 29 \mapsto \ldots.$$  

These orbits explicit calculations provide the Young diagram of the cycles of our permutation,

$$D = 35\cdot15\cdot7\cdot3\cdot2\cdot1^2.$$  

Its 7 cycles are generated, for instance, by the following 7 starting elements,

$$01, \ 02, \ 38, \ 14, \ 18, \ 17, \ 72.$$
The values of the parameters of this Young diagram are
\[ x = 35, \quad y = 7, \quad \lambda = 64/245 \approx 0.26, \quad \mu = 7/35 \approx 0.20. \]

A similar random numbers table experiment with \( N = 100 \) had lead me to the table \( b \) of the form
\[
\begin{array}{cccccccc}
84 & 6 & 98 & 64 & 96 & 37 & 47 & 45 \\
87 & 9 & 19 & 76 & 75 & 4 & 77 & 31 \\
27 & 97 & 11 & 20 & 26 & 24 & 80 & 46 \\
23 & 89 & 66 & 3 & 33 & 74 & 44 & 39 \\
50 & 37 & 12 & 28 & 92 & 55 & 79 & 73 \\
48 & 63 & 0 & 52 & 30 & 93 & 16 & 35 \\
82 & 88 & 71 & 40 & 29 & 22 & 41 & 54 \\
83 & 38 & 18 & 10 & 59 & 17 & 8 & 42 \\
43 & 95 & 5 & 58 & 2 & 80 & 36 & 51 \\
25 & 72 & 90 & 62 & 32 & 56 & 78 & 70 \\
\end{array}
\]

the table \( a \) representing the standard ordering \( 0 < 1 < 2 < \cdots < 99 \) of the same 100 numbers.

Calculating the cycles decomposition from this table, we obtain
\[ D = 42 \cdot 36 \cdot 18 \cdot 2 \cdot 1^2 \]

(the corresponding 6 cycles starting, for instance, at the elements 10, 34, 0, 13, 50, 55). For example, the cycle of length 2 consists of the elements 13 and 58.

The parameters values for this permutation are
\[ x(D) = 42, \quad y(D) = 6, \quad \lambda(D) = 25/63 \approx 0.397, \quad \mu = 1/7 \approx 0.143. \]

For another “random” permutation of 100 elements I obtained \( D = 40 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \), leading to \( x = 90, \quad y = 5, \quad \lambda = 2/9 \approx 0.222, \quad \mu = 1/18 \approx 0.056. \)

These values of the parameters of “random” permutation of \( N = 100 \) elements confirm the 4 conjectures of page 48, once more. (Since \( \ln N \approx 4.6 \), we get \( c_3 \approx 1.12 \) in conjecture 3.)

5. Galois Fields Random Permutations

A different way to create “random” permutations is provided by the Galois fields tables (described in [2]). For \( N = p^2 \), where \( p \) is a prime, we can use directly the tables of article [2] (correcting the few misprints according to the tables symmetries).

We use the field’s table as table \( b \) of the algorithm of theory of Section 4, considering it as the table of a permutation of the \( p^2 \) residues (modulo \( p^2 \)), replacing the symbol “\( \infty \)” in the table of the field by the missing there residue 0.
For \( p = 3, 5, 7, 11 \) and 13 we obtain the following diagrams of the permutation’s cycles lengths (for the resulting permutations of \( N = p^2 \) elements):

<table>
<thead>
<tr>
<th>( p )</th>
<th>( N )</th>
<th>( D )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>6·2·1</td>
<td>6</td>
<td>3</td>
<td>( \frac{1}{2} \approx 0.50 )</td>
<td>( \frac{1}{2} \approx 0.50 )</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>14·5·4·1(^2)</td>
<td>14</td>
<td>5</td>
<td>( \frac{5}{14} \approx 0.36 )</td>
<td>( \frac{5}{14} \approx 0.36 )</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td>16·11·7·6·4·3·1(^2)</td>
<td>16</td>
<td>8</td>
<td>( \frac{49}{128} \approx 0.38 )</td>
<td>( \frac{1}{2} = 0.50 )</td>
</tr>
<tr>
<td>11</td>
<td>121</td>
<td>65·39·5·3(^3)·2·1</td>
<td>65</td>
<td>8</td>
<td>( \frac{121}{520} \approx 0.23 )</td>
<td>( \frac{5}{68} \approx 0.12 )</td>
</tr>
<tr>
<td>13</td>
<td>169</td>
<td>98·55·12·2·1(^2)</td>
<td>98</td>
<td>6</td>
<td>( \frac{169}{520} \approx 0.31 )</td>
<td>( \frac{1}{25} \approx 0.06 )</td>
</tr>
</tbody>
</table>

These values of the parameters are quite similar to those values for the “random” permutations, which are provided by the preceding section algorithm, applied to the “random” number from the telephones lists and from the “Abime’s number”. These empirical “random” permutations parameters of my experiments are:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( D )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \lambda )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>7·3·2(^2)·1(^2)</td>
<td>7</td>
<td>6</td>
<td>( \frac{3}{21} \approx 0.381 )</td>
<td>( \frac{9}{5} \approx 0.86 )</td>
</tr>
<tr>
<td>25</td>
<td>9·7·5·3·1</td>
<td>9</td>
<td>5</td>
<td>( \frac{5}{9} \approx 0.556 )</td>
<td>( \frac{9}{5} \approx 0.56 )</td>
</tr>
<tr>
<td>64</td>
<td>35·15·7·3·2·1(^2)</td>
<td>35</td>
<td>7</td>
<td>( \frac{64}{225} \approx 0.281 )</td>
<td>( \frac{1}{5} \approx 0.20 )</td>
</tr>
<tr>
<td>100</td>
<td>42·36·18·2·1(^2)</td>
<td>42</td>
<td>6</td>
<td>( \frac{25}{63} \approx 0.397 )</td>
<td>( \frac{1}{6} \approx 0.14 )</td>
</tr>
<tr>
<td>100</td>
<td>90·4·3·2·1</td>
<td>90</td>
<td>5</td>
<td>( \frac{7}{9} \approx 0.222 )</td>
<td>( \frac{9}{19} \approx 0.056 )</td>
</tr>
<tr>
<td>100</td>
<td>88·9·1(^3)</td>
<td>88</td>
<td>5</td>
<td>( \frac{88}{77} \approx 0.227 )</td>
<td>( \frac{5}{68} \approx 0.075 )</td>
</tr>
<tr>
<td>169</td>
<td>147·13·8·1</td>
<td>147</td>
<td>4</td>
<td>( \frac{169}{520} \approx 0.237 )</td>
<td>( \frac{1}{17} \approx 0.027 )</td>
</tr>
</tbody>
</table>

The similarity of the last two tables (generated by the Galois field on the first table and by the random numbers tables in the second one) confirms the high degree of the chaoticity of the Galois fields tables, conjectured in the article [2].

One might also see in this similarity one more confirmation of the 4 empirical conjectures of Section 3 (on the behavior of the mean values of the parameters of random permutations of \( N \) elements for growing \( N \)).

### 6. Comparison on the Cycles Numbers Statistics

Comparing the tables of the parameters values of the Young diagrams of the cycles length for the finite tori automorphisms (Section 2) with the values for the random permutations cycles diagrams (Sections 4–5), one sees a series of important differences.

For example, the fullness values, \( \lambda \), for the automorphisms periodic points cases are much larger, than for the random permutations cases. The fullness decline with the grows of the area of the diagram, observed for the random permutations, is rather missing for the automorphisms’ diagrams cases.

The asymmetries \( \mu \) of the Young diagrams of the automorphisms cases are much larger, than for the most of random diagrams of same area.
The automorphisms periodic points diagrams are generally higher ($\mu > 1$), while for the random permutations diagram they are rather low ($\mu < 1$), becoming even lower ($\mu \to 0$) for growing area $N$.

The large asymmetry of the observed Young diagrams of the automorphisms cycles lengths for the tables of Section 2 makes them different from the averaged behavior of the random permutations Young diagrams, both for the uniform averaging along the $N!$ equiprobable permutations of $N$ objects and for the averaging along the Plancherel–Vershik measure [5] (explained in [4]). Indeed, the mean value $\mu$ of the asymmetry parameter $\mu$ (averaged along the Plancherel–Vershik measure) shows, that the asymmetry vanishes asymptotically ($\mu \to 1$ for $N \to \infty$).

The heights $y$ of the periodic points' diagram for the automorphisms seem to grow with the area $N$ rather linearly, than logarithmically, (which is typical for the random permutations' diagrams).

Thus, we may conclude, that the Young diagrams of the periodic points of the automorphisms of the finite tori $M = \mathbb{Z}_n^2$, defined by the golden ratio Fibonacci matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, behave for the large area $N = n^2$ case very differently from the typical Young diagrams of the random permutations of $N$ elements. The permutations of the points of $M$, generated by the automorphisms, are not random and their Young diagrams have a different behavior for $N \to \infty$ (which is unknown and deserves to be studied asymptotically).

One might suppose, that this distinction would persist for other (non-Fibonacci) automorphisms $A: M \to M$ of finite tori. It would be interesting to study, at least empirically, the behavior for $n \to \infty$ of the mean values of the parameters $(x, y, \lambda, \mu)$ of the Young diagrams of the cycles of the automorphisms $A$ of an $n^2$-points torus $M$ taking the means along the group of the automorphisms (or at least the Cesaro asymptotical behavior of these means at $n \to \infty$).

It would also be interesting to compare the means values of the parameters of the diagrams of the permutations of $N$ elements along the whole symmetric group $S'(N)$ with their mean values along the subgroup of the projective permutations of the $N = p + 1$ points of finite projective line $P^1(\mathbb{Z}_p)$.

For the $m$-dimensional tori cases one should consider the permutations of the $N = p^{m-1} + \cdots + p + 1$ points of the finite projective $(m-1)$-space $P^{m-1}(\mathbb{Z}_p)$ (with a possible distinction between its finite Lobachevsky part $\Lambda$ of the Klein model and its relativistic finite De Sitter world part $P \setminus \Lambda$).

The behavior of all these objects for large dimensions, $m \to \infty$, might provide new interesting asymptotics of the large “Young diagrams”. For the automorphism of an $m$-torus

$$A: (\mathbb{Z}_n)^m \to (\mathbb{Z}_n)^m$$

these asymptotical formulae might depend on the higher-dimensional periodic continued fraction of the operator $A \in \text{GL}(n, \mathbb{Z})$.

It might even depend on the “triangulation” of the torus $(S^1)^{m-1}$ by the convex polyhedra (with their integral points), defined by the geometry of the “period” of this higher-dimensional continued fraction.

The periods’ Young diagram might be generalized in this case, to take into account the structure of the action of the commutative group $\mathbb{Z}^{m-1}$ of the symmetries.
of the periodic continued fraction (replacing the action of the single operator $A$, occurring for $m = 2$).

Even the mean values of the parameters of the Young diagrams of cycles of $A$ along the group of the automorphisms (which do not depend on the continued fraction geometry of operator $A$) deserve a detailed study (at least an empirical one, similar to the experiments, described above).

**References**


Steklov Mathematical Institute, 8, Gubkina street, 119991, Moscow, GSP-1, Russia