MOSCOW MATHEMATICAL JOURNAL
Volume 6, Number 2, April–June 2006, Pages 359–388

AN INTRODUCTION TO CONWAY'S GAMES AND NUMBERS

DIERK SCHLEICHER AND MICHAEL STOLL

Abstract. This note attempts to furnish John H. Conway's combinatorial game theory with an introduction that is easily accessible and yet mathematically precise and self-contained and which provides complete statements and proofs for some of the folklore in the subject.

Conway's theory is a fascinating and rich theory based on a simple and intuitive recursive definition of games, which yields a very rich algebraic structure. Games form an abelian GROUP in a very natural way. A certain subgroup of games, called numbers, is a FIELD that contains both the real numbers and the ordinal numbers. Conway's theory is deeply satisfying from a theoretical point of view, and at the same time it has useful applications to specific games such as Go.

2000 Math. Subj. Class. 91-02, 91A05, 91A46, 91A70.

Key words and phrases. Conway game, surreal number, combinatorial game theory.

1. Combinatorial Game Theory

Combinatorial game theory is a fascinating and rich theory based on a simple and intuitive recursive definition of games, which yields a very rich algebraic structure: games can be added and subtracted in a very natural way, forming an abelian GROUP (Section 2). There is a distinguished sub-GROUP of games called *numbers*, which can also be multiplied and which form a FIELD (Section 3): this field contains both the real numbers (Section 3.2) and the ordinal numbers (Section 4). (In fact, Conway's definition generalizes both Dedekind sections and von Neumann's definition of ordinal numbers.) All Conway numbers can be interpreted as games that can actually be played in a natural way; in some sense, if a game is identified as a number, then it is understood well enough so that it would be boring to actually play it (Section 5). Conway's theory is deeply satisfying from a theoretical point of view, and at the same time it has useful applications to specific games such as Go [Go]. There is a beautiful microcosmos of numbers and games which are infinitesimally close to zero (Section 6), and the theory contains the classical and complete Sprague–Grundy theory of impartial games (Section 7).

Received November 14, 2004.

The theory was founded by John H. Conway in the 1970's. Classical references are the wonderful books *On Numbers and Games* [ONAG] by Conway and *Winning Ways* by Berlekamp, Conway and Guy [WW]; they have recently appeared in their second editions. The book [WW] is a most beautiful book bursting with examples and results but with less emphasis on mathematical rigor and exactness of some statements. The book [ONAG] is still the definitive source of the theory but is rather difficult to read for novices; even the second edition shows that it was originally written in one week, and we feel that the order of presentation (first numbers, then games) makes it harder to read and adds unnecessary complexity to the exposition. The book [SN] is an entertaining story about discovering surreal numbers on an island.

This note attempts to furnish combinatorial game theory with an introduction that is easily accessible and yet mathematically precise and self-contained and which provides complete statements and proofs for some of the folklore in the subject. We have written this note having in mind readers who have enjoyed looking at books like [WW] and are now eager to come to terms with the underlying mathematics before embarking on a deeper study in [ONAG], [GONC] or elsewhere. While this note should be complete enough for readers without previous experience with combinatorial game theory, we recommend looking at [WW], [GONC], or [AGBB] to pick up the playful spirit of the theory. We felt no need for duplicating many motivating examples from these sources, and we have no claims concerning the originality of any of the results.

2. The Group of Games

2.1. What is a game? (See [ONAG, Secs. 7, 0], [WW, Secs. 1, 2].) Our notion of a game tries to formalize the abstract structure underlying games such as Chess or Go: these are two-person games with complete information; there is no chance or shuffling. The two players are usually called *Left* (L) and *Right* (R). Every game has some number of *positions*, each of which is described by the set of allowed *moves* for each player. Each move (of Left or Right) leads to a new position, which is called a (left or right) *option* of the previous position. Each of these options can be thought of as another game in its own right: it is described by the sets of allowed moves for both players.

From a mathematical point of view, all that matters is the sets of left and right options that can be reached from any given position—we can imagine the game represented by a rooted tree with vertices representing positions and with oriented edges labeled L or R according to the player whose moves they reflect. The root represents the initial position, and the edges from any position lead to another rooted (sub-)tree, the root of which represents the position just reached.

Being identified with the initial position, a game is completely described by the sets of left and right options, each of which is another game. This leads to the recursive Definition 2.1 (1). Note that the sets L and R of options may well be infinite or empty. The Descending Game Condition (2) simply says that every game must eventually come to an end no matter how it is played; the number of moves until the end can usually not be bounded uniformly in terms of the game alone.

Definition 2.1 (Game)**.**

(1) Let L and R be two sets of games. Then the ordered pair $G := (L, R)$ is a *game.*

(2) (Descending Game Condition (DGC)). There is no infinite sequence of games $G^i = (L^i, R^i)$ with $G^{i+1} \in L^i \cup R^i$ for all $i \in \mathbb{N}$.

Logically speaking, this recursive definition does not tell you what games *are*, and it does not need to: it only needs to specify the axiomatic properties of games. A major purpose of this paper is of course to explain the meaning of the theory; see, for example, the creation of games below.

Definition 2.2 (Options and Positions)**.**

(1) (Options). The elements of L and R are called *left* (respectively, *right*) *options* of G.

(2) (Positions). The *positions* of G are G and all positions of any option of G.

In the recursive definition of games, a game consists of two sets of games. Before any game is "created," the only set of games we have is the empty set: the simplest game is the "zero game" $0 = (\{\ \}, \{\ \})$ with $L = R = \{\ \}$; in this game, no player has a move. Now that we have a nonempty set of games, the next simpler games are $1 = (\{0\}, \{\ \})$ (whose name indicates that it represents one free move for Left), $-1 = (\{\ \}, \{0\})$ (a free move for Right), and $* = (\{0\}, \{0\})$ (a free move for whoever gets to take it first).

Notation. We simplify (or abuse?) notation as follows: let $L = \{G^{L_1}, G^{L_2}, \ldots\}$ and $R = \{G^{R_1}, G^{R_2}, \ldots\}$ be two arbitrary sets of games (we do *not* mean to indicate that L or R are countable or nonempty); then for

$$G = (L, R) = \big(\{G^{L_1}, G^{L_2}, \ldots\}, \{G^{R_1}, G^{R_2}, \ldots\}\big)$$

we write $G = \{G^{L_1}, G^{L_2}, \ldots \mid G^{R_1}, G^{R_2}, \ldots\}$. Hence a game is actually a set with two distinguished kinds of elements, the left and right options.[1] With this notation, the four simplest games introduced so far can be written more easily as

$$0 = \{\ \mid\ \} \qquad 1 = \{0 \mid\ \} \quad -1 = \{\ \mid 0\} \qquad * = \{0 \mid 0\}.$$

Eventually, we will want the two players to move alternately (this will be formalized in Section 2.2), but the Descending Game Condition will be needed to hold even when players do not move alternately; see Section 2.3.

The simple recursive (and at first mind-boggling) definition of games has its counterpart in the following equally simple induction principle that is used in almost every proof in the theory.

Theorem 2.3 (Conway Induction)**.** *Let P be a property which games might have such that any game G has property P whenever all left and right options of G have this property. Then every game has property P.*

More generally, for $n \geq 1$, let $P(G_1, \ldots, G_n)$ be a property that any n-tuple of games might have (i. e., an n-place relation). Suppose that $P(G_1, \ldots, G_i, \ldots, G_n)$

---

[1] It is customary to abuse notation and write $\{L \mid R\}$ for the ordered pair (L, R). We shall try to avoid that in this paper.

holds whenever all $P(G_1, \ldots, G_i', \ldots, G_n)$ *hold (for all* $i \in \{1, \ldots, n\}$ *and all left and right options* $G_i' \in L_i \cup R_i$*, where* $G_i = (L_i, R_i)$*). Then* $P(G_1, \ldots, G_n)$ *holds for every* n*-tuple of games.*

Proof. Suppose there is a game G which does not satisfy P. If all left and right options of G satisfy P, then so does G by hypothesis, and so there is an option G' of G which does not satisfy P. Continuing this argument inductively, we obtain a sequence $G, G', G'', \ldots$ of games, each an option of its predecessor, which violates the Descending Game Condition. Note that formalizing this argument needs the axiom of choice.

The general statement follows similarly: if $P(G_1, \ldots, G_i, \ldots, G_n)$ is false, then so is some $P(G_1, \ldots, G_i', \ldots, G_n)$ (for some i and some $G_i' \in L_i \cup R_i$); thus either some $P(G_1, \ldots, G_i'', \ldots, G_n)$ or some $P(G_1, \ldots, G_i', \ldots, G_j', \ldots, G_n)$ is false, and it is easy to extract an infinite sequence $G_i, G_i', G_i'', \ldots$ of games that are options of their respective predecessors, which is again a contradiction. □

Note that Conway Induction does not need an explicit induction base (as opposed to ordinary induction for natural numbers, which must be based at 0): the empty game $0 = \{\,|\,\}$ satisfies property P automatically, because all its options do—there is no option which might fail to have property P.

As a typical illustration of how Conway Induction works, we show that its first form implies the Descending Game Condition.

Proposition 2.4. *Conway Induction implies the Descending Game Condition.*

Proof. Consider the property $P(G)$: *there is no infinite chain of games* $G, G', G'', \ldots$ *starting with* G *so that every game is followed by one of its options.* This property clearly is of the kind described by Conway Induction, and so it holds for every game. □

Conway's definition of a game [ONAG, Secs. 0, 7] consists of part (1) in Definition 2.1, together with the statement "*all games are constructed in this way.*" One way of making this precise is by Conway Induction: a game is "constructed in this way" if all its options are, and so Conway's axiom becomes a property which all games enjoy. We have chosen to use the equivalent Descending Game Condition in the definition so as to treat induction for one or several games on an equal footing.

Another easy consequence of the Conway Induction principle is that the positions of a game form a set (and not a proper CLASS).

2.2. Winning a game. (See [ONAG, Secs. 7, 0], [WW, Sec. 2].) From now on, suppose that the two players must move alternately. When we play a game, the most important aspect is usually whether we can win it or will lose it. In fact, most of the theory is about deciding which player can force a win in certain kinds of games. So we need some formal definition of who wins and who loses; there are no ties or draws in this theory.[2] The basic decision we make here is that we consider a player to have lost a game when it is his turn to move but he is unable to do so

[2] This is one reason why Chess does not fit well into our theory; another one is that addition is not natural for Chess. The game of Go, however, fits quite well.

(because his set of options is empty): the idea is that we cannot win if we do not have a good move, let alone no move at all. This *Normal Play Convention*, as we shall see, leads to a very rich and appealing theory.

There is also a *Misère Play Convention* that the loser is the one who makes the last move; with that convention, most of our theory would fail, and there is no comparably rich theory known: our fundamental equality $G = G$ for every game G rests very much on the normal play convention (Theorem 2.10); see also the end of Section 7 for the special case of impartial games. Another possible winning convention is by score; while scores are not built into our theory, they can often be simulated: see the remark after Definition 5.4 and [WW, Part 3].

Every game G will be of one of the following four outcome classes: (1) Left can enforce a win, no matter who starts; (2) Right can enforce a win, no matter who starts; (3) the first player can enforce a win, no matter who it is; (4) the second player can enforce a win, no matter who. We shall abbreviate these four possibilities by $G > 0$, $G < 0$, $G \parallel 0$, and $G = 0$, respectively: here, $G \parallel 0$ is usually read "G is fuzzy to zero"; the justification for the notation $G = 0$ will become clear in Section 2.3. We can contract these as usual: $G \geq 0$ means $G > 0$ or $G = 0$, i. e. Left can enforce a win (at least) if he is the second player; $G \leq 0$ means that Right can win as second player; similarly, $G \rhd 0$ means $G > 0$ or $G \parallel 0$, i. e. Left can win as first player ("G is greater than or fuzzy to zero"), and $G \lhd 0$ means that Right can win as first player.

It turns out that only $G \geq 0$ and $G \leq 0$ are fundamental: if $G \geq 0$, then Left wins as second player, and so Right has no good opening move. A good opening move for Right would be an option G^R in which Right could win; since Left must start in G^R, this would mean $G^R \leq 0$. This leads to the following formal definition:

Definition 2.5 (Order of Games)**.** We define:

- $G \geq 0$ unless there is a right option $G^R \leq 0$;
- $G \leq 0$ unless there is a left option $G^L \geq 0$;

The interpretation of winning needs to be based on games where Left or Right wins immediately: this is the *Normal Play Convention* that a player loses when it is her turn but she has no move available. Formally, if Left has no move at all in G, then clearly $G \leq 0$ by definition, and so Right wins when Left must start but cannot move. Note that the convention "both players move alternately" enters the formal theory in Definition 2.5.

As is often the case, Definition 2.5 is recursive: to decide whether $G \geq 0$ or not, we must know whether $G^R \leq 0$, etc. The Descending Game Condition makes this well defined: if there were a game G for which $G \geq 0$ or $G \leq 0$ is not well defined, then this could only be so because there was an option G^L or G^R for which these relations are not well defined, etc., and this would eventually violate the DGC.

It is convenient to introduce the following conventions.

Definition 2.6 (Order of Games)**.** We define:

- $G = 0$ if $G \geq 0$ and $G \leq 0$, i. e., if there are no options $G^R \leq 0$ or $G^L \geq 0$;
- $G > 0$ if $G \geq 0$ but not $G \leq 0$, i. e., if there is an option $G^L \geq 0$ but no $G^R \leq 0$;

- $G < 0$ if $G \leq 0$ but not $G \geq 0$, i. e., if there is an option $G^R \leq 0$ but no $G^L \geq 0$;
- $G \parallel 0$ if neither $G \geq 0$ nor $G \leq 0$, i. e., if there are options $G^L \geq 0$ and $G^R \leq 0$.
- $G \rhd 0$ if $G \leq 0$ is false, i. e., if there is a left option $G^L \geq 0$;
- $G \lhd 0$ if $G \geq 0$ is false, i. e., if there is a right option $G^R \leq 0$;

A game G such that $G = 0$ is often called a "zero game" (not to be confused with *the* zero game $0 = \{\,|\,\}$!).

All these cases can be interpreted in terms of winning games; for example, $G \rhd 0$ means that Left can win when moving first: indeed, the condition ensures the existence of a good opening move for Left to $G^L \geq 0$, in which Left plays second.

Note that these definitions immediately imply the claim, made above, that for every game G exactly one of the following statements is true: $G = 0$, $G > 0$, $G < 0$, or $G \parallel 0$. They are the four cases depending on the two independent possibilities $\exists\, G^L \geq 0$ and $\exists\, G^R \leq 0$; see also Figure 1.

		if Right starts, then	
		Left wins	Right wins
if Left starts, then	Left wins	$G > 0$	$G \parallel 0$
	Right wins	$G = 0$	$G < 0$

FIGURE 1. The four outcome classes.

When we say "Left wins" etc., we mean that Left can enforce a win by optimal play; this does *not* mean that we assume a winning strategy to be actually known, or that Right might not win if Left plays badly. For example, for the beautiful game of Hex [Hex1, pp. 73–83], [Hex2], there is a simple proof that the first player can enforce a win, though no winning strategy is known unless the board size is very small—and there are serious Hex tournaments.[3]

The existence of a strategy for exactly one player (supposing that it is fixed who starts) is built into the definitions: to fix ideas, suppose that Right starts in a game $G \geq 0$. Then there is no right option $G^R \leq 0$, so that either G has no right option at all (and Left wins effortlessly) or all $G^R \rhd 0$, so that every G^R has a left option $G^{RL} \geq 0$: whatever Right's move, Left has an answer leading to another game $G^{RL} \geq 0$, and Left will never be the one who runs out of moves when it is his turn. By the Descending Game Condition, the game eventually stops at a position where there are no options left for the player whose turn it is, and this must be Right. Therefore, Left wins. Note that this argument does not assume that a strategy for Left is known, nor does it provide an explicit strategy.

Below we shall define equality of games as an equivalence relation. What we have so far is equality of games in the set-theoretic sense; to distinguish notation,

[3]The rules are usually modified to eliminate the first player's advantage. With the modified rules, one can prove that the second player can enforce a win (if he only knew how!), and the situation is then similar.

we use a different word for this and say that two games G and H are *identical* and write $G \equiv H$ if they have the same sets of (identical) left (respectively, right) options.

For the four simplest games, we have the following outcome classes. We obviously have $0 = 0$, since no player has a move; then it is easy to see that $1 > 0$, $-1 < 0$, and $* \parallel 0$.

A note on set theory. Definition 2.1 might look simple and innocent, but the CLASS of games thus defined is a proper CLASS (as opposed to a set): one way to see this is to observe that every ordinal number is a game (Section 4.1). We have adopted the convention (introduced in [ONAG]) of writing GROUP, FIELD, etc. for algebraic structures that are proper classes (as opposed to sets).

The set-theoretic foundations of our theory are the Zermelo–Fraenkel axioms including the axiom of choice (ZFC) and expanded by proper classes. It is a little cumbersome to express our theory in terms of ZFC: a game is a set with two kinds of elements, and it might be more convenient to treat combinatorial game theory as an appropriately modified analog of ZFC. See the discussion in [ONAG, Appendix to Part 0], where Conway argues that "the complicated nature of these constructions [expressing our theory in terms of ZFC] tells us more about the nature of formalizations within ZF than about our system of numbers ... [formalization within ZFC] destroys a lot of its symmetry." In this note, we shall not go into details concerning such issues; we only note that our Descending Game Condition in Definition 2.1 corresponds to the Axiom of Foundation in ZFC. In the special case of *impartial games*, however, the Descending Game Condition is exactly the Axiom of Foundation; see Section 7.

2.3. Adding and comparing games. (See [ONAG, Secs. 7, 1], [WW, Sec. 2].) Let us now introduce one of the most important concepts of the theory: the *sum* of two games. Intuitively, we put two games next to each other and allow each player to move in one of the two according to his choice, leaving the other game unchanged; the next player can then decide independently whether to move in the same game as her predecessor. The negative of a game is the same game in which the allowed moves for both players are interchanged (in games like chess, they simply switch colors). The formal definitions are given below. Note that at this point it is really necessary to require the DGC in its general form (rather than only for alternating moves) so as to guarantee that the sum of two games is again a game (which ends after a finite number of moves).

Definition 2.7 (Sum and Negative of Games). Let $G = \{G^L, \dots \mid G^R, \dots\}$ and $H = \{H^L, \dots \mid H^R, \dots\}$ be two games. Then we define

$$G + H :\equiv \{G^L + H,\, G + H^L, \dots \mid G^R + H,\, G + H^R, \dots\},$$
$$-G :\equiv \{-G^R, \dots \mid -G^L, \dots\}, \qquad \text{and}$$
$$G - H :\equiv G + (-H).$$

These are again recursive definitions. The definition of $G + H$ requires knowing several sums of the form $G^L + H$ etc. which must be defined first. However, all these additions are easier than $G + H$: recursive definitions work by induction without base, similarly to Conway Induction (this time, for binary relations): the sum $G+H$ is well defined as soon as all options G^L+H etc. are well defined.[4] To see how things get off the ground, note that the set of left options of $G+H$ is

$$\bigcup_{G^L}\{G^L+H\} \cup \bigcup_{H^L}\{G+H^L\} \tag{2.1}$$

where G^L and H^L run through the left options of G and H. If G and/or H have no left options, then the corresponding unions are empty, and there might be no left options of $G+H$ at all, or they might be all of the form G^L+H (or $G+H^L$). Therefore, $G+H$ and $-G$ are games.

As an example, $-1 \equiv \{\,|\,0\}$ is indeed the negative of $1 \equiv \{0\,|\,\}$, justifying our notation. Also, $*+* \equiv \{*\,|\,*\}$, and the latter is easily seen to be a zero game (whoever begins, loses), so that $*+*=0$. The following properties justify the name "addition" for the operation just defined.

Theorem 2.8. *Addition is associative and commutative with* $0 \equiv \{\,|\,\}$ *as zero element. Moreover, all games* G *and* H *satisfy* $-(-G) \equiv G$ *and* $-(G+H) \equiv (-G)+(-H)$.

Proof. By (2.1), the left (right) options of $G+\{\,|\,\}$ are $G^L+\{\,|\,\}$ ($G^R+\{\,|\,\}$) only, and so the claim "$G+\{\,|\,\}\equiv G$" follows by Conway Induction.

Commutativity uses induction too (in the second equality):

$$\begin{aligned} G+H &\equiv \{G^L+H,\, G+H^L,\, \dots \mid G^R+H,\, G+H^R,\, \dots\} \\ &\equiv \{H+G^L,\, H^L+G,\, \dots \mid H+G^R,\, H^R+G,\, \dots\} \equiv H+G. \end{aligned}$$

Associativity works similarly (we write only left options):

$$\begin{aligned} (G+H)+K &\equiv \{(G+H)^L+K,\, (G+H)+K^L,\, \dots \mid \dots\} \\ &\equiv \{(G^L+H)+K,\, (G+H^L)+K,\, (G+H)+K^L,\, \dots \mid \dots\} \\ &\equiv \{G^L+(H+K),\, G+(H^L+K),\, G+(H+K^L),\, \dots \mid \dots\} \\ &\equiv \{G^L+(H+K),\, G+(H+K)^L,\, \dots \mid \dots\} \equiv G+(H+K). \end{aligned}$$

Moreover, omitting dots from now on,

$$-(-G) \equiv -\{-G^R \mid -G^L\} \equiv \{-(-G^L) \mid -(-G^R)\} \equiv \{G^L \mid G^R\} \equiv G,$$

where again induction was used in the third equality. Finally,

$$\begin{aligned} -(G+H) &\equiv -\{G^L+H,\, G+H^L \mid G^R+H,\, G+H^R\} \\ &\equiv \{-(G^R+H),\, -(G+H^R) \mid -(G^L+H),\, -(G+H^L)\} \\ &\equiv \{(-G^R)+(-H),\, (-G)+(-H^R) \mid (-G^L)+(-H),\, (-G)+(-H^L)\} \\ &\equiv \{(-G)^L+(-H),\, (-G)+(-H)^L \mid (-G)^R+(-H),\, (-G)+(-H)^R\} \\ &\equiv (-G)+(-H), \end{aligned}$$

where $-G^R$ means $-(G^R)$, etc. The third line uses induction again. □

[4] More formally, one could treat $G+H$ as a formal pair of games and then prove by Conway Induction that every such formal pair is in fact a game: if all formal pairs G^L+H, $G+H^L$, G^R+H and $G+H^R$ are games, then clearly so is $G+H$. Similar remarks apply to the definition of multiplication and elsewhere.

Conway calls inductive proofs like the preceding ones "one-line proofs" (even if they do not fit on a single line): resolve the definitions, apply induction, and plug in the definitions again.

Note that

$$G - H :\equiv G + (-H) \equiv \{G^L - H, \ldots, G - H^R, \ldots \mid G^R - H, \ldots, G - H^L, \ldots\}.$$

From now on, we omit the dots in games like this (as was already done in the previous proof).

As examples, consider the games $2 :\equiv 1 + 1 \equiv \{0 + 1, 1 + 0 \mid \} \equiv \{1 \mid \}$, $3 :\equiv 2 + 1 \equiv \{1 + 1, 2 + 0 \mid \} \equiv \{2 \mid \}$, $4 \equiv \{3 \mid \}$, etc., as well as $-2 \equiv \{ \mid -1\}$ etc.

Definition 2.9. We write $G = H$ if $G - H = 0$, $G > H$ if $G - H > 0$, $G \parallel H$ if $G - H \parallel 0$, etc.

It is obvious from the definition and the preceding result that these binary relations extend the unary relations $G = 0$ etc. defined earlier.

Theorem 2.10. *Every game G satisfies $G = G$, or, equivalently, $G - G = 0$. Moreover, $G^L \lhd G$ for all left options G^L, and $G \lhd G^R$ for all right options G^R of G.*

Proof. By induction, we may suppose that $G^L - G^L \geq 0$ and $G^R - G^R \leq 0$ for all left and right options of G. By definition, we have $G - G^R \geq 0$ unless there is a right option $(G - G^R)^R \leq 0$, and indeed such an option is $G^R - G^R \leq 0$. Therefore, $G - G^R \lhd 0$, or $G \lhd G^R$. Similarly, $G^L \lhd G$.

Now $G - G \equiv \{G^L - G, G - G^R \mid G^R - G, G - G^L\} \geq 0$ unless there is any right option $(G - G)^R \leq 0$; but we have just shown that the right options are $G^R - G \rhd 0$ and $G - G^L \rhd 0$, and so indeed $G - G \geq 0$ and similarly $G - G \leq 0$; hence $G - G = 0$ and $G = G$. □

The equality $G - G = 0$ means that in the sum of any game with its negative, the second player has a winning strategy: indeed, if the first player makes any move in G, then the second player has the same move in $-G$ available and can copy the first move; the same holds if the first player moves in $-G$ because $-(-G) \equiv G$. Therefore, the second player can never run out of moves before the first does, and so the Normal Play Convention awards the win to the second player. This is sometimes paraphrased like this: when playing against a Grand Master simultaneously two games of chess, one with white and one with black, you can force at least one win (if draws are not permitted, as in our theory)! In Misère Play, we would not have the fundamental equality $G = G$.

The following results show that the ordering of games is compatible with addition.

Lemma 2.11. (1) *If $G \geq 0$ and $H \geq 0$, then $G + H \geq 0$.*
(2) *If $G \geq 0$ and $H \rhd 0$, then $G + H \rhd 0$.*

Note that $G \rhd 0$ and $H \rhd 0$ implies nothing about $G + H$: the sum of two fuzzy games can be in any outcome class. (Find examples!)

Proof. We prove both statements simultaneously using Conway Induction (with the binary relation $P(G, H)$: "for the pair of games G and H, the statement of the Lemma holds"). The following proof can easily be rephrased in the spirit of "Left has a winning move unless. . . "

(1) $G \geq 0$ and $H \geq 0$ mean there are no $G^R \leq 0$ and no $H^R \leq 0$, so all $G^R \triangleright 0$ and all $H^R \triangleright 0$. By the inductive hypothesis, all $H + G^R \triangleright 0$ and $G + H^R \triangleright 0$, so that $G + H$ has no right options $(G + H)^R \leq 0$ and thus $G + H \geq 0$.

(2) Similarly, $H \triangleright 0$ means there is an $H^L \geq 0$. By the inductive hypothesis, $G + H^L \geq 0$, so that $G + H$ has a left option $G + H^L \geq 0$ and thus $G + H \triangleright 0$. □

Theorem 2.12. *The addition of a zero game never changes the outcome: if $G = 0$, then $H > 0$, $H < 0$, $H = 0$, or $H \parallel 0$ if and only if $G + H > 0$, $G + H < 0$, $G + H = 0$, or $G + H \parallel 0$, respectively.*

Proof. If $H \geq 0$ or $H \leq 0$, then $G + H \geq 0$ or $G + H \leq 0$ by Lemma 2.11, and similarly if $H \triangleright 0$ or $H \triangleleft 0$, then $G + H \triangleright 0$ or $G + H \triangleleft 0$. Since $H = 0$ is equivalent to $H \geq 0$ and $H \leq 0$, $H > 0$ is equivalent to $H \geq 0$ and $H \triangleright 0$, etc., the "only if" part follows. The "if" part then follows from the fact that every game is in exactly one outcome class. □

Corollary 2.13. *Equal games are in the same outcome classes: if $G = H$, then $G > 0$ if and only if $H > 0$, etc.*

Proof. Consider $G + (H - H) \equiv H + (G - H)$, which by Theorem 2.12 has the same outcome class as G and H. □

Corollary 2.14. *Addition respects the order: for any triple of games, $G > H$ is equivalent to $G + K > H + K$, etc.*

Proof. $G+K > H+K \iff (G-H)+(K-K) > 0 \iff G-H > 0 \iff G > H$. □

Theorem 2.15. *The relation $\geq$ is reflexive, antisymmetric and transitive, and equality $=$ is an equivalence relation.*

Proof. The reflexivity of $\geq$ and $=$ is Theorem 2.10, and the antisymmetry of $\geq$ and the symmetry of $=$ hold by definition. The transitivity of $\geq$ and thus of $=$ follows like this: $G \geq H$ and $H \geq K$ implies $G - H \geq 0$ and $H - K \geq 0$, and hence $G - K + (H - H) \geq 0$ by Lemma 2.11. By Theorem 2.12, this implies $G - K \geq 0$ and $G \geq K$. □

Theorem 2.16. *The equivalence classes formed by equal games form an additive abelian GROUP in which the zero element is represented by any game $G = 0$.*

Proof. First, we have to observe that addition and negation are compatible with the equivalence relation: if $G = G'$ and $H = H'$, then $G-G' = 0$ and $H-H' = 0$; hence $(G+H)-(G'+H') \equiv (G-G')+(H-H') = 0$ by Lemma 2.11 and $G+H = G'+H'$ as needed. Easier yet, $G = G'$ implies $0 = G - G' \equiv -(-G) + (-G') \equiv (-G') - (-G)$, and hence $-G' = -G$.

For every game G, the game $-G$ represents the inverse equivalence class by Theorem 2.10. Finally, addition is associative and commutative by Theorem 2.8. □

It is all well to define equivalence classes of games, but their significance sits in the fact that replacing a game by an equivalent one never changes the outcome, even when this happens for games that are themselves parts of other games.

Theorem 2.17 (Equal Games)**.** *If $H = H'$, then $G+H = G+H'$ for all games G. If $G = \{G^{L_1}, G^{L_2}, \dots \mid G^{R_1}, G^{R_2}, \dots\}$ and $H = \{H^{L_1}, H^{L_2}, \dots \mid H^{R_1}, H^{R_2}, \dots\}$ are two games such that $G^{L_i} = H^{L_i}$ and $G^{R_i} = H^{R_i}$ for all left and right options, then $G = H$: replacing any option by an equivalent one (or any set of options by equivalent options) yields an equivalent game.*

Proof. The first part is self-proving: $(G+H)-(G+H') = (G-G)+(H-H') = 0$. The second part is similar but easier to write in words: in $G - H$, Left might move in G to some $G^L - H$ or in H to some $G - H^R$, and Right's answer will be either in H to a $G^L - H^{L'} = 0$ (with $H^{L'}$ chosen so that $H^{L'} = G^L$) or in G to a $G^{R'} - H^R = 0$. The situation is similar if Right starts. □

2.4. Simplifying games. (See [WW, Sec. 3], [ONAG, Sec. 10].) Since equality of games is defined as an equivalence relation, there are many ways of writing down a game that has a certain value (i. e., lies in a certain equivalence class). Some of these will be simpler than others, and there may even be a simplest or canonical form of a game. In this section, we show how one can simplify games and prove that simplest forms exist for an interesting class of games.

Definition 2.18 (Gift Horse)**.** Let G and H be games. If $H \lhd G$, then H is a *left gift horse* for G; if $H \rhd G$, then H is a *right gift horse* for G.

Lemma 2.19 (Gift Horse Principle)**.** *If $H_L, \dots$ are left gift horses and $H_R, \dots$ are right gift horses for $G = \{G^L, \dots \mid G^R, \dots\}$, then*

$$\{H_L, \dots, G^L, \dots \mid H_R, \dots, G^R, \dots\} = G.$$

(Here $\{H_L, \dots\}$ and $\{H_R, \dots\}$ can be arbitrary sets of games.)

Proof. Let $G' \equiv \{H_L, \dots, G^L, \dots \mid H_R, \dots, G^R, \dots\}$. Then $G' - G \geq 0$, since the right options are $G^R - G \rhd 0$ (by Theorem 2.10), $H_R - G \rhd 0$ (by assumption), and $G' - G^L$, which has the left option $G^L - G^L = 0$, so that $G' - G^L \rhd 0$. In the same way, we see that $G' - G \leq 0$, and it follows that $G' = G$. □

This "Gift Horse Principle" tells us how to offer extra options to a player without changing the value of a game (since no player really wants to move to these options), so that we know how to make games more complicated. Now we wish to see how we can *remove* options and thereby make a game simpler. Intuitively, an option that is no better than another option can as well be left out, since a reasonable player will never use it. This is formalized in the following definition and lemma.

Definition 2.20 (Dominated Option)**.** Let G be a game. A left option G^L is *dominated* by another left option $G^{L'}$ if $G^L \leq G^{L'}$. Similarly, a right option G^R is *dominated* by another right option $G^{R'}$ if $G^R \geq G^{R'}$.

Lemma 2.21 (Deleting Dominated Options)**.** *Let G be a game with fixed left and right options G^L and G^R. Then the value of G remains unchanged if some or all* *left options which are dominated by G^L are removed and similarly if some or all right options which are dominated by G^R are removed.*

Proof. Let G' be the game obtained from G by removing all or some left options that are dominated by G^L (but keeping G^L itself). Then all deleted options are left gift horses for G', since for such an option H we have $H \le G^L \lhd G'$. We can therefore add all these options to G', thereby obtaining G, without changing the value. The same argument works for dominated right options. □

As simple examples, we have $2 \equiv \{1 \mid \} = \{0, 1 \mid \}$, $3 \equiv \{2 \mid \} = \{0, 1, 2 \mid \}$ etc., and so we recover von Neumann's definition of natural numbers. Another example is $\{0, 1 \mid 2, 3\} = \{1 \mid 2\}$. Note that it is possible that all options are dominated but this does not mean that all options can be removed: as an example, consider $\omega :\equiv \{0, 1, 2, \ldots \mid \}$.

There is another way of simplifying a game, which works by introducing shortcuts rather than removing options. The idea is as follows. Suppose Left has a move G^L that Right can counter to some fixed $G^{LR} \le G$, a position at least as good for Right as G was. The claim is that replacing the single option G^L by all left options of G^{LR} does not change the value of G: this does not hurt Left (if Left wants to move to G^L, then he must expect the answer G^{LR} and then has all left options of G^{LR} available); on the other hand, it does not help Left if G is replaced by $G^{LR} \le G$. Precise statements are like this.

Definition 2.22 (Reversible Option)**.** Let G be a game. A left option G^L is said to be *reversible* (*through* G^{LR}) if G^L has a right option $G^{LR} \le G$. Similarly, a right option G^R is said to be *reversible* (*through* G^{RL}) if G^R has a left option $G^{RL} \ge G$.

Lemma 2.23 (Bypassing Reversible Options)**.** *If G has a left option H that is reversible through $K = H^R$, then*

$$G = \{H, G^L, \ldots \mid G^R, \ldots\} = \{K^L, \ldots, G^L, \ldots \mid G^R, \ldots\}$$

(*here G^L runs through all left options of G other than H*)*. In words*: *the value of G is unchanged when we replace the reversible left option H by all left options of K. A similar statement holds for right options.*

Proof. Let $G' = \{K^L, \ldots, G^L, \ldots \mid G^R, \ldots\}$ and $G'' = \{H, K^L, \ldots, G^L, \ldots \mid G^R, \ldots\}$. We claim that H is a left gift horse for G'. This can be seen as follows. First, for all K^L we have $K^L \lhd G'$, since K^L is a left option of G'. Also, $K \le G \lhd G^R$, and so $K \lhd G^R$ for all G^R. These statements together imply that $K \le G'$. Since K is a right option of H, this in turn says that $H \lhd G'$, as was to be shown. By the Gift Horse Principle, we now have $G' = G''$. On the other hand, $K^L \lhd K \le G$, and so all K^L are left gift horses for G, whence $G = G'' = G'$. □

One aspect of reversible options might be surprising: if G^L is reversible through G^{LR}, this means that Left may bypass the move to G^L and Right's answer to G^{LR} and move directly to some left option of G^{LR}; but what if there was another right option $G^{LR'}$ which Right might prefer over G^{LR}: is Right deprived of her better move? For the answer, note that $\{ \mid 1\} = \{ \mid 100\} = 0$: although Right might prefer that her only move was 1 rather than 100, the first player to move will always lose, which is all that counts. Similarly, depriving Right of her better answer $G^{LR'}$ would make a difference only if there was a game S such that $G+S \leq 0$ but $G^{LR}+S \vartriangleright 0$ (our interest is in the case that Left starts: these conditions mean that Left cannot win in $G+S$, but he can when jumping directly to G^{LR}); however, the first condition and the hypothesis imply $G^{LR}+S \leq G+S \leq 0$, contradicting the second condition.

Given the simplifications of games described above, the question arises whether there is a simplest form of a game, a form that cannot be further simplified by removing dominated options and bypassing reversible options. The example $\omega = \{0, 1, 2, \dots \mid \}$ shows that this is not the case in general. But such a simplest form exists if we impose a natural finiteness condition, which is satisfied by most real-life games.

Definition 2.24 (Short)**.** A game G is said to be *short* if it has only finitely many positions.

Theorem 2.25 (Normal Form)**.** *In each equivalence class of short games, there is a unique game that has no dominated or reversible positions.*

Proof. Since both ways of simplifying games reduce the number of positions, we eventually reach a game that cannot be simplified further. This proves existence.

To prove uniqueness, we assume that G and H are two equal (short) games, both without dominated and reversible positions. We have to show that $G \equiv H$. Let G^L be some left option of G. Since $G^L \vartriangleleft G = H$, there must be a right option $G^{LR} \leq H$ or a left option H^L such that $G^L \leq H^L$. The first is impossible, since G^L is not reversible. Similarly, there is some $G^{L'}$ such that $H^L \leq G^{L'}$, and so $G^L \leq G^{L'}$. But there are no dominated options either, and so $G^L = H^L = G^{L'}$. By induction, $G^L \equiv H^L$. In that way, we see that G and H have the same set of (identical) left options, and the same is true for the right options. □

3. The FIELD of Numbers

3.1. What is a number? (See [ONAG, Secs. 0, 1], [WW, Sec. 2].) We have already encountered games like 0, 1, −1, 2 that we have denoted by numbers and that behave like numbers. In particular, they measure which player has got how many free moves left and are therefore easy to compare. We now wish to extend this to a class of games that is as large as possible (and whose elements are to be called *numbers*).

The guiding idea is that numbers should be totally ordered, i. e., no two numbers should ever be fuzzy to each other. Recall that by Theorem 2.10 we always have $G^L \vartriangleleft G$ and $G \vartriangleleft G^R$. If G, G^L and G^R are to be numbers, this forces $G^L < G < G^R$, so that we must at least require that $G^L < G^R$. For numbers to be preserved under playing, we need to require that all options of numbers are numbers. This leads to the following definition.

Definition 3.1 (Number)**.** A game $x = \{x^L, \dots \mid x^R, \dots\}$ is called a *number* if all left and right options x^L and x^R are numbers and satisfy $x^L < x^R$.

As it turns out, this simple definition leads not only to a totally ordered additive subGROUP of games but even to a real algebraically closed FIELD that simultaneously contains the real and ordinal numbers!

We shall use lowercase letters x, y, z, ... to denote numbers. The simplest numbers are 0, 1 and -1. A slightly more interesting number is $\frac{1}{2} :\equiv \{0 \mid 1\}$ (one checks easily that $\frac{1}{2}+\frac{1}{2} = 1$, justifying the name). There are also "infinite numbers" like $\omega = \{0, 1, 2, \ldots \mid \}$.

Lemma 3.2. *Every number $x = \{x^L, \ldots \mid x^R, \ldots\}$ satisfies $x^L < x < x^R$.*

Proof. The left options of $x^L - x$ are of the form $x^L - x^R$ or $x^{LL} - x$. Since x is a number, we have $x^L - x^R < 0$. We use the inductive hypothesis $x^{LL} < x^L$ and $x^L - x \lhd 0$ in Theorem 2.10. Therefore, Lemma 2.11 implies $x^{LL} - x = (x^{LL} - x^L) + (x^L - x) \lhd 0$.

If $x^L - x \rhd 0$ were true, we would need some $(x^L - x)^L \geq 0$, which we have just excluded. Hence $x^L \leq x$ for all left options x^L of x, and similarly $x \leq x^R$ for all right options x^R. The claim now follows, because $x^L \lhd x \lhd x^R$ by Theorem 2.10. □

Theorem 3.3. *If x and y are numbers, then $x + y$ and $-x$ are numbers, so that (equivalence classes of) numbers form an abelian subGROUP of games.*

Proof. Since $-x = \{-x^R, \ldots \mid -x^L, \ldots\}$, we have $(-x)^L = -x^R < -x^L = (-x)^R$, so the options of $-x$ are ordered as required. Conway Induction now shows that $-x$ is a number.

In $x + y = \{x^L + y, x + y^L, \ldots \mid x^R + y, x + y^R, \ldots\}$, we have the inequalities $x^L + y < x^R + y$ and $x + y^L < x + y^R$ by Corollary 2.14. By Lemma 3.2, we also have $x^L + y < x + y < x + y^R$ and $x + y^L < x + y < x^R + y$, so that $x + y$ is a number as soon as all its options are, and Conway Induction applies. □

Theorem 3.4. *Numbers are totally ordered: every pair of numbers x and y satisfies exactly one of the relations $x < y$, $x > y$, or $x = y$.*

Proof. Suppose there is a number $z \parallel 0$. This would imply the existence of options $z^L \geq 0 \geq z^R$, which is excluded by definition: numbers are never fuzzy.

Now if there were two numbers $x \parallel y$, then $x - y$ would be a number by Theorem 3.3 and $x - y \parallel 0$, but this is impossible, as we have just shown. □

3.2. Short numbers and real numbers. (See [ONAG, Sec. 2], [WW, Sec. 2].) A short number is simply a number that is a short game, i. e., a game with only finitely many positions. In particular, it then has only finitely many options, and since numbers are totally ordered, we can eliminate dominated options so as to leave at most one left and at most one right option.

By the definition of negation, addition, and multiplication (see below in Section 3.3), it is easily seen that the set (!) of (equivalence classes of) short numbers is a unitary ring.

Theorem 3.5. *The ring of short numbers is (isomorphic to) the ring $\mathbb{Z}[\frac{1}{2}]$ of dyadic fractions.*

Proof. We have already seen that $\{0 \mid 1\} = \frac{1}{2}$; therefore, $\mathbb{Z}[\frac{1}{2}]$ is contained in the ring of short numbers. For the converse, see [ONAG, Theorem 12]. The main step in proving the converse is to show that

$$\left\{ \frac{m}{2^n} \,\middle|\, \frac{m+1}{2^n} \right\} = \frac{2m+1}{2^{n+1}}$$

for integers m and natural numbers n. □

Let S denote the ring of short numbers (or dyadic fractions). We can represent every element x of S in the form

$$x = \{y \in S\colon y < x \mid y \in S\colon y > x\},$$

where both sets of options are nonempty. In fact, the set of all numbers satisfying this property is exactly the field $\mathbb{R}$ of real numbers: we are taking Dedekind sections in the ring S. (More precisely, this is the most natural model of real numbers within our Conway numbers: it is the only one where all real numbers have all their options in S. Other embeddings are obtained by choosing a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ and then changing the images of this basis by some infinitesimal amounts.) For some more discussion, see [ONAG, Chap. 2].

3.3. Multiplication of numbers. To turn numbers into a FIELD, we need a multiplication.

Definition 3.6 (Multiplication)**.** Given two numbers $x = \{x^L, \dots \mid x^R, \dots\}$ and $y = \{y^L, \dots \mid y^R, \dots\}$, we define the product

$$x \cdot y := \{x^L \cdot y + x \cdot y^L - x^L \cdot y^L,\ x^R \cdot y + x \cdot y^R - x^R \cdot y^R, \dots \mid \\ x^L \cdot y + x \cdot y^R - x^L \cdot y^R,\ x^R \cdot y + x \cdot y^L - x^R \cdot y^L, \dots\}.$$

As with addition in (2.1), the left and right options are all terms as in the definition that can be formed with left and right options of x and y. More precisely, the left options are indexed by pairs (x^L, y^L) and pairs (x^R, y^R), and similarly for the right options. In particular, this means that if x has no left options (say), then $x \cdot y$ will not have left and right options of the first type shown. We shall usually omit the dot and write xy for $x \cdot y$.

While this definition might look complicated, it is actually not. It is motivated by $x^L < x < x^R$ and $y^L < y < y^R$, and so we want multiplication to satisfy $(x - x^L)(y - y^L) > 0$, and hence $xy > x^L y + x y^L - x^L y^L$, which motivates the first type of left options. The other three types are obtained in a similar way.

One might try the simpler definition $xy = \{x^L y, xy^L \mid x^R y, xy^R\}$ for multiplication, motivated by $x^L < x < x^R$ and $y^L < y < y^R$. But the inequalities would be wrong for negative numbers. In fact, this would be just a different notation for addition!

Recall the two special numbers $0 \equiv \{\mid\}$ and $1 \equiv \{0 \mid\}$.

Theorem 3.7. *For all numbers x, y, z, we have the identities*

$$0 \cdot x \equiv 0, \quad 1 \cdot x \equiv x, \quad xy \equiv yx, \quad (-x)y \equiv y(-x) \equiv -xy$$

and the equalities

$$(x+y)z = xz + yz, \qquad (xy)z = x(yz).$$

Proof. The proofs are routine "one-line-proofs"; the last two are a bit lengthy and can be found in [ONAG, Theorem 7]. □

The reason why multiplication is defined only for numbers rather than for arbitrary games is that there are games G, G', and H with $G = G'$ but $GH \neq G'H$. For example, we have $\{1 \mid \} = \{0, 1 \mid \}$, but $\{1 \mid \} \cdot * = \{* \mid *\} = 0$, whereas $\{0, 1 \mid \} \cdot * = \{0, * \mid 0, *\} \parallel 0$. (Note that we have $0 \cdot G = 0$ and $1 \cdot G = G$ for any game G. Furthermore, since games form an abelian GROUP, we always have integral multiples of arbitrary games.)

The following theorem shows that our multiplication behaves as expected. Its proof is the most complicated inductive proof in this paper. It required quite some work to produce a concise version of the argument in the proof, even with Conway's [ONAG, Theorem 8] at hand. The main difficulty is to organize simultaneous induction for three different statements with different arguments.

Theorem 3.8. (1) *If x and y are numbers, then so is xy.*

(2) *If x_1, x_2, and y are numbers such that $x_1 = x_2$, then $x_1 y = x_2 y$.*

(3) *If x_1, x_2, y_1, and y_2 are numbers such that $x_1 < x_2$ and $y_1 < y_2$, then $x_1 y_2 + x_2 y_1 < x_1 y_1 + x_2 y_2$. In particular, if $y > 0$, then $x_1 < x_2$ implies $x_1 y < x_2 y$.*

Proof. We shall prove most of the statements simultaneously using Conway Induction. More precisely, let $P_1(x, y)$, $P_2(x_1, x_2, y)$, and $P_3(x_1, x_2, y_1, y_2)$ stand for the statements above. For technical reasons, we also introduce the statement $P_4(x, y_1, y_2)$:

(4) *If x, y_1, and y_2 are numbers with $y_1 < y_2$, then $P_3(x^L, x, y_1, y_2)$ and $P_3(x, x^R, y_1, y_2)$ hold for all options x^L and x^R of x.*

We begin by proving P_1, P_2 and P_4 simultaneously. For this part, we assume that all numbers involved are positions in a fixed number z (we can take $z = \{x, x_1, x_2, y, y_1, y_2 \mid \}$). Then for a position z' in z, we define $n(z')$ to be the distance between z' and the root z in the rooted tree representing the game z, that is, the number of moves needed to reach z' from the starting position z (a nonnegative integer). Formally, we set $n(z) = 0$ and, if z' is a position in z, $n({z'}^L) = n(z') + 1$ and $n({z'}^R) = n(z') + 1$.

For each of the statements P_1, P_2, P_4, we measure its "depth" by a pair of natural numbers (r, s) (where $s = \infty$ is allowed) as follows.

- The depth of $P_1(x, y)$ is $(n(x) + n(y), \infty)$.
- The depth of $P_2(x_1, x_2, y)$ is $(\min\{n(x_1), n(x_2)\} + n(y), \max\{n(x_1), n(x_2)\})$.
- The depth of $P_4(x, y_1, y_2)$ is $(n(x) + \min\{n(y_1), n(y_2)\}, \max\{n(y_1), n(y_2)\})$.

The inductive argument consists in showing that each statement follows from statements that have greater depths (in the lexicographic ordering) and only involve positions of the games occurring in the statement under consideration. If the statement were false, it would follow that there exists an infinite chain of positions

z' in z of unbounded depth $n(z')$, each a position of its predecessor; this contradicts the Descending Game Condition.

Properties of addition will be used without explicit mention.

(1) We begin with $P_1(x, y)$. We may assume that all terms $x^L y$, xy^L, $x^L y^L$, etc. are numbers, using $P_1(x^L, y)$ etc.; therefore all options of xy are numbers. It remains to show that the left options are smaller than the right options. There are four inequalities to show; we treat one of them in detail (the others being analogous). We show that

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_2}y + xy^R - x^{L_2}y^R.$$

There are three cases. First, suppose that $x^{L_1} = x^{L_2}$. Then, using $P_2(x^{L_1}, x^{L_2}, y)$ and $P_2(x^{L_1}, x^{L_2}, y^R)$, we see that the statement is equivalent to $P_3(x^{L_1}, x, y^L, y^R)$, which is in turn a special case of $P_4(x, y^L, y^R)$.

Now suppose that $x^{L_1} < x^{L_2}$. Then we use $P_3(y^L, y, x^{L_1}, x^{L_2})$ (which follows from $P_4(y, x^{L_1}, x^{L_2})$) and $P_3(x^{L_2}, x, y^L, y^R)$ (which follows from $P_4(x, y^L, y^R)$) to get

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_2}y + xy^L - x^{L_2}y^L < x^{L_2}y + xy^R - x^{L_2}y^R.$$

Similarly, if $x^{L_1} > x^{L_2}$, we use $P_4(x, y^L, y^R)$ and $P_4(y, x^{L_2}, x^{L_1})$ to get

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_1}y + xy^R - x^{L_1}y^R < x^{L_2}y + xy^R - x^{L_2}y^R.$$

(2) For $P_2(x_1, x_2, y)$, note that $z_1 = z_2$ if $z_1^L < z_2 < z_1^R$ and $z_2^L < z_1 < z_2^R$ for all relevant options. So we have to show a number of statements of the type $(x_1y)^L < x_2y$ or $x_2y < (x_1y)^R$. We carry this out for the left option $(x_1y)^L = x_1^L y + x_1 y^L - x_1^L y^L$; the other possible cases are done in the same way. The statement $P_2(x_1, x_2, y^L)$ gives $x_1y^L = x_2y^L$, and $P_4(y, x_1^L, x_2)$ gives $x_1^L y + x_2 y^L < x_1^L y^L + x_2 y$, which together imply $(x_1y)^L < x_2y$.

(3) Now we consider $P_4(x, y_1, y_2)$. Since $y_1 < y_2$, there is some y_1^R such that $y_1 < y_1^R \le y_2$, or there is some y_2^L such that $y_1 \le y_2^L < y_2$. We consider the first case; the second one is analogous. First, note that from $P_1(x, y_1)$ we get $(xy_1)^L < xy_1 < (xy_1)^R$ for all left and right options of xy_1. We therefore obtain the inequalities

$$xy_1 + x^L y_1^R < x^L y_1 + xy_1^R \quad \text{and} \quad x^R y_1 + xy_1^R < xy_1 + x^R y_1^R. \tag{3.1}$$

Now if $y_1^R = y_2$, then, by $P_2(y_1^R, y_2, x)$, $P_2(y_1^R, y_2, x^L)$, and $P_2(y_1^R, y_2, x^R)$, we are done. Otherwise, $y_1^R < y_2$, and $P_4(x, y_1^R, y_2)$ says that

$$x^L y_2 + xy_1^R < x^L y_1^R + xy_2 \quad \text{and} \quad xy_2 + x^R y_1^R < xy_1^R + x^R y_2. \tag{3.2}$$

Adding the left (respectively, right) inequalities in (3.1) and (3.1) and canceling like terms proves the claim.

This shows that every statement $P_1(x, y)$, $P_2(x_1, x_2, y)$, and $P_4(x, y_1, y_2)$ follows from similar statements which use the same arguments or some of their options, and it is easily verified that all statements used have greater depths. This proves P_1, P_2, and P_4.

(4) It remains to show $P_3(x_1, x_2, y_1, y_2)$. This is done by Conway Induction in the normal way, using the statements we have already shown.

Since $x_1 < x_2$, there is some $x_1^R \le x_2$ or some $x_2^L \ge x_1$. Assume the first possibility (the other one is treated in the same way). If $x_1^R = x_2$, then we apply P_2 to get $x_1^R y_1 = x_2 y_1$ and $x_1^R y_2 = x_2 y_2$. Using $P_4(x_1, y_1, y_2)$, we also have $x_1 y_2 + x_1^R y_1 < x_1 y_1 + x_1^R y_2$; together, these imply the desired conclusion. Finally, if $x_1^R < x_2$, then, by induction ($P_3(x_1^R, x_2, y_1, y_2)$), we get $x_1^R y_2 + x_2 y_1 < x_1^R y_1 + x_2 y_2$ and, using $P_4(x_1, y_1, y_2)$ again, also $x_1 y_2 + x_1^R y_1 < x_1 y_1 + x_1^R y_2$. Adding them together and canceling like terms proves our claim. □

3.4. Division of numbers. The definition of division is more complicated than that of addition or multiplication—necessarily so, since, for example, $3 = \{2 \mid \}$ is a very simple game with only finitely many positions, while $\frac{1}{3} = \{\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \dots \mid \dots, \frac{11}{32}, \frac{3}{8}, \frac{1}{2}\}$ has infinitely many positions which must all be "generated" somehow from the positions of 3.

It suffices to find a multiplicative inverse for every $x > 0$. It is convenient to rewrite positive numbers as follows.

Lemma 3.9. *For every number $x > 0$, there is a number y without negative options such that $y = x$.*

Proof. To achieve this, we simply add the left Gift Horse 0 and then delete all negative left options, which are now dominated by 0. □

Theorem 3.10. *For a number $x > 0$ without negative options, define*

$$y = \left\{ 0,\ \frac{1 + (x^R - x)y^L}{x^R},\ \frac{1 + (x^L - x)y^R}{x^L} \;\middle|\; \frac{1 + (x^L - x)y^L}{x^L},\ \frac{1 + (x^R - x)y^R}{x^R} \right\}$$

where all options $x^L \neq 0$ and x^R of x are used. Then y is a number with $xy = 1$.

Note that this definition is recursive as always: to find $y = 1/x$, we need to know $1/x^L$ and $1/x^R$ first. However, this time we also need to know left and right options of y. We actually view this as an algorithmic definition: initially, make 0 a left option of y. Then, for every left option y^L generated so far, produce new left and right options of y with all x^L and all x^R; similarly, for every right option y^R already generated, do the same. This step is then iterated (countably) infinitely many times. More precisely, we define a sequence of pairs Y_n^L, Y_n^R of sets of numbers recursively as follows.

$$Y_0^L = \{0\}, \quad Y_0^R = \varnothing;$$

$$Y_{n+1}^L = Y_n^L \cup \bigcup_{x^R} \left\{ \frac{1+(x^R-x)y^L}{x^R} : y^L \in Y_n^L \right\} \cup \bigcup_{x^L} \left\{ \frac{1+(x^L-x)y^R}{x^L} : y^R \in Y_n^R \right\},$$

$$Y_{n+1}^R = Y_n^R \cup \bigcup_{x^L} \left\{ \frac{1+(x^L-x)y^L}{x^L} : y^L \in Y_n^L \right\} \cup \bigcup_{x^R} \left\{ \frac{1+(x^R-x)y^R}{x^R} : y^R \in Y_n^R \right\}.$$

Then

$$y = \left\{ \bigcup_{n \in \mathbb{N}} Y_n^L \;\middle|\; \bigcup_{n \in \mathbb{N}} Y_n^R \right\}.$$

If x is a real number (which always can be written with at most countably many options), then at each step we generate new sets of left and right options, the suprema and infima of which converge to $1/x$. In this case, we have a convergent (infinite) algorithm specifying a Cauchy sequence of numbers. General numbers x might need arbitrarily big *sets* of options, so that the option sets of y can become arbitrarily big too. However, the necessary number of iteration steps in the construction of y is still at most countable.

Proof. (Compare [ONAG, Theorem 10].) By induction, all options of y are numbers. (There are of course two inductive processes involved, one with respect to x and the other with respect to n above.) We now prove that $xy^L < 1 < xy^R$ for all left and right options of y. This is done by induction on n. The statement is obvious for $n = 0$. We give one of the four cases for the inductive step in detail: suppose $y^L = (1 + (x^R - x)y^{L'})/x^R$ with $y^{L'} \in Y_n^L$; then by induction we have $xy^{L'} < 1$, and so (by Theorem 3.8) $(x^R - x)xy^{L'} < x^R - x$, which is equivalent to the claim.

To prove that y is a number, we must show that the left options are smaller than the right options. It is easy to see that the right options are positive: for $(1 + (x^L - x)y^L)/x^L$, this follows from $1 - xy^L > 0$; for $(1 + (x^R - x)y^R)/x^R$, this follows by induction ($y^R > 0$). There are then four more cases involving the two different kinds of generated left (respectively, right) options. We look at two of them; the other two are dealt with analogously. So suppose we wish to show that $(1 + (x^R - x)y^{L_1})/x^R < (1 + (x^L - x)y^{L_2})/x^L$. This is equivalent to

$$x^R(1 + (x^L - x)y^{L_2}) - x^L(1 + (x^R - x)y^{L_1}) > 0.$$

The left-hand side of this equation can be written in each of the two ways

$$\begin{aligned}&(x^R - x^L)(1 - xy^{L_1}) + (y^{L_1} - y^{L_2})x^R(x - x^L)\\ &\quad = (x^R - x^L)(1 - xy^{L_2}) + (y^{L_2} - y^{L_1})x^L(x^R - x),\end{aligned}$$

showing that it is always positive. Now suppose we wish to show that $(1 + (x^{R_1} - x)y^L)/x^{R_1} < (1 + (x^{R_2} - x)y^R)/x^{R_2}$. This is equivalent to

$$x^{R_1}(1 + (x^{R_2} - x)y^R) - x^{R_2}(1 + (x^{R_1} - x)y^L) > 0.$$

Again, the left-hand side can be written as

$$\begin{aligned}&(y^R - y^L)x^{R_1}(x^{R_2} - x) + (x^{R_1} - x^{R_2})(1 - xy^L)\\ &\quad = (y^R - y^L)x^{R_2}(x^{R_1} - x) + (x^{R_2} - x^{R_1})(xy^R - 1),\end{aligned}$$

showing that it is always positive. Note that we use the inductive result that the "earlier" options y^R and y^L satisfy $y^R > y^L$.

Finally, to prove that $xy = 1$, we have to show that $(xy)^L < 1 < (xy)^R$ (since $0 = 1^L < xy$ trivially). For example, take

$$\begin{aligned}(xy)^R &= x^R y + xy^L - x^R y^L\\ &= 1 + x^R\left(y - \frac{1 + (x^R - x)y^L}{x^R}\right)\\ &= 1 + x^R(y - y^{L'}) > 1.\end{aligned}$$

□

Corollary 3.11. *The (equivalence classes of) numbers form a totally ordered FIELD.*

In fact, this field is real algebraically closed. This is shown in [ONAG, Chapter 4].

4. Ordinal Numbers

4.1. Ordinal numbers. (See [ONAG, Sec. 2].)

Definition 4.1 (Ordinal Number)**.** A game G is an *ordinal number* if it has no right options and all of its left options are ordinal numbers.

We shall use small Greek letters like α, β, γ, ... to denote ordinal numbers.

An ordinal number is really a number in our sense, as the following lemma shows.

Lemma 4.2. (1) *Every ordinal number is a number.*

(2) *If α is an ordinal number, then the class of all ordinal numbers $\beta < \alpha$ is a set.*

(3) *If α is an ordinal number, then $\alpha = \{\beta\colon \beta < \alpha \mid \}$, where β runs through the ordinal numbers.*

(4) *If α is an ordinal number, then $\alpha + 1 = \{\alpha \mid \}$.*

Proof. Note that $\beta < \alpha$ implies that there is an α^L such that $\beta \le \alpha^L$. This follows from $\alpha - \beta = \{\alpha^L - \beta \mid \dots\}$ and the definition of $G \rhd 0$.

(1) We proceed by induction. By hypothesis, all α^L are numbers. Since there are no α^R, the condition $\alpha^L < \alpha^R$ for all pairs (α^L, α^R) is trivially satisfied; hence α is itself a number.

(2) We claim that $\{\beta\colon \beta < \alpha\} = \{\alpha^L\} \cup \bigcup_{\alpha^L} \{\beta\colon \beta < \alpha^L\}$. The statement follows from this by induction and the fact that the union of a family of sets indexed by a set is again a set.

The right-hand side is certainly contained in the left-hand side, since all $\alpha^L < \alpha$. Now let $\beta < \alpha$. Then $\beta \le \alpha^L$ for some α^L; hence $\beta < \alpha^L$ or $\beta = \alpha^L$, showing that β is an element of the right-hand side.

(3) Let $\gamma = \{\beta\colon \beta < \alpha \mid \}$. (This is a game by what we have just shown.) Then $\alpha - \gamma = \{\alpha^L - \gamma \mid \alpha - \gamma^L\}$ (there are no right options of α or γ). Since all α^L are ordinal numbers and $\alpha^L < \alpha$, it follows that every α^L is a left option of γ, and hence $\alpha^L < \gamma$ and all $(\alpha - \gamma)^L < 0$. By the definition of γ, all $\gamma^L < \alpha$, and so all $(\alpha - \gamma)^R > 0$. Therefore $\alpha - \gamma = 0$.

(4) We have $\alpha + 1 = \{\alpha, \alpha^L + 1 \mid \}$, and so we have to show that $\alpha^L + 1 \le \alpha$ for all α^L. Let $\beta = \alpha^L$. By induction, $\beta + 1 = \{\beta \mid \} \le \{\beta, \dots \mid \} = \alpha$. ☐

Simple examples of ordinal numbers are the natural numbers: $0 = \{\mid\}$, $1 = \{0 \mid \}$, $2 = \{1 \mid \} = \{0, 1 \mid \}$, ... The next ordinal number after all natural numbers is quite important; it is $\omega = \{0, 1, 2, \dots \mid \}$, the smallest infinite ordinal.

Recall that an ordered set or class is said to be *well ordered* if every nonempty subset or subclass has a smallest element. This is equivalent to the requirement that there be no infinite descending chain of elements $x_0 > x_1 > x_2 > \dots$.

Proposition 4.3. *The class of ordinal numbers is well ordered.*

Proof. Let $\mathcal{C}$ be some nonempty class of ordinal numbers. Then there is some $\alpha \in \mathcal{C}$. Replace $\mathcal{C}$ by the *set* $\mathcal{S} = \{\beta \in \mathcal{C} \colon \beta \leq \alpha\}$; then it suffices to show that the set $\{\beta \colon \beta \leq \alpha\}$ is well ordered. But every descending chain in this set is a chain of options of $\{\alpha \mid \}$ and therefore must be finite by the DGC. □

The principle of Conway Induction applied to ordinal numbers results in the Theorem of Ordinal Induction (sometimes called "transfinite induction").

Theorem 4.4 (Ordinal Induction)**.** *Let P be a property which ordinal numbers might have. If "β satisfies P for all $\beta < \alpha$" implies "α satisfies P", then all ordinal numbers satisfy P.*

Proof. Apply Conway Induction to the property "if G is an ordinal number, then G satisfies P" and recall that $\alpha = \{\beta < \alpha \mid \}$. □

On the other hand, one could use the concept of birthdays (see below) to prove the principle of Conway Induction from the Theorem of Ordinal Induction.

Of course, we then also have a principle of Ordinal Recursion. For example, we can recursively define the following numbers.

Definition 4.5. $2^{-\alpha} := \{0 \mid 2^{-\beta} \colon \beta < \alpha\}$ (where α and β are ordinal numbers).

These numbers are all positive and approach zero, in a similar way as the ordinal numbers approach infinity—for every positive number $z > 0$, there is some ordinal number α such that $2^{-\alpha} < z$. As an example, we have $2^{-\omega} = \omega^{-1}$. The notation is justified, since one shows easily that $2 \cdot 2^{-(\alpha+1)} = 2^{-\alpha}$.

Finally, there is another important property of ordinal numbers, which is some kind of dual to the well-ordering property.

Proposition 4.6. *Every* set *of ordinal numbers has a least upper bound within the ordinal numbers.*

Proof. Let $\mathcal{S}$ be such a set. Then $\alpha = \{\mathcal{S} \mid \}$ is an ordinal number[5] and an upper bound for $\mathcal{S}$. Hence the class of ordinal upper bounds is nonempty, therefore (because of well-ordering) there is a least upper bound. □

4.2. Birthdays. The concept of birthday of a game is a way of making the history of creation of numbers and games precise. It assigns to every game an ordinal number, which can be understood as the "number of steps" that are necessary to create this game "out of nothing" (i. e., starting from the empty set).

Definition 4.7 (Birthday)**.** Let G be a game. The *birthday* of G, $b(G)$, is defined recursively by $b(G) = \{b(G^L), b(G^R) \mid \}$.

For example, $b(0) = 0$ and $b(1) = b(-1) = b(*) = 1$; more generally, for ordinal numbers α, one has $b(\alpha) = \alpha$. A game is short if and only if its birthday is finite (see below). All nonshort real numbers have birthday ω. The successive "creation" of numbers with the first few birthdays is illustrated by the "Australian Number Tree" in [WW, Sec. 2, Fig. 2] and [ONAG, Fig. 0].

[5] This is the customary abuse of notation: we mean the ordered pair $\alpha = (\mathcal{S}, \{\ \}) = \{s \colon s \in \mathcal{S} \mid \}$.

By definition, the birthday is an ordinal number. It has the following simple properties.

Lemma 4.8. *Let G be a game. Then $b(G^L) < b(G)$ for all G^L and $b(G^R) < b(G)$ for all G^R. Furthermore, $b(-G) = b(G)$.*

Proof. Immediate from the definition. □

Note that two games that are equal can have different birthdays. For example, $\{-1 \mid 1\} = 0$, but the first has birthday 2, whereas $b(0) = 0$. But there is a well-defined minimal birthday among the games in an equivalence class.

Proposition 4.9. *A game G is short if and only if it has birthday $b(G) < \omega$ (i. e., $b(G) = n$ for some $n \in \mathbb{N} = \{0, 1, \dots\}$).*

Proof. If G is short, then by induction all of its finitely many options have finite birthdays. Let b be the maximum of these. Then $b(G) = \{b \mid \} = b + 1 < \omega$. The converse implication follows from the fact that there are only finitely many games with any given finite birthday (this is easily seen by ordinary induction). □

The birthday is sometimes useful if one needs a bound on a game.

Proposition 4.10. *If G is a game, then $-b(G) \le G \le b(G)$.*

Proof. It suffices to prove the upper bound; the lower bound follows by replacing G with $-G$, since both have the same birthday.

Let $\alpha = b(G)$. $G \le \alpha$ means that for all G^L, we have $G^L \lhd \alpha$, and for all α^R, we have $G \lhd \alpha^R$. But there are no α^R, so that we can forget about the second condition. Now by induction, $G^L \le b(G^L) < b(G) = \alpha$, giving the first part. □

5. Games and Numbers

In some sense, numbers are the simplest games — since they are totally ordered, we know exactly what happens (i. e., who wins) when we add games that are numbers. It is much more difficult to deal with general games. To make life easier, we try to get as much mileage as we can out of comparing games with numbers. The references for this chapter are [ONAG, Secs. 8, 9] and [WW, Secs. 2, 6].

5.1. When is a game already a number? If we wish to compare games with numbers, the first question we have to answer is whether a given game is already (equal to) a number. The following result gives a general recipe for deciding that two games are equal; it can be used to provide a criterion for a game to be a number.

Proposition 5.1 (General Simplicity Theorem)**.** *Let G and H be games such that*

(1) $\forall\, G^L\colon G^L \lhd H$ *and* $\forall\, G^R\colon H \lhd G^R$;
(2) $\forall\, H^L \exists\, G^L\colon H^L \le G^L$ *and* $\forall\, H^R \exists\, G^R\colon H^R \ge G^R$.

Then $G = H$.

Proof. By the first assumption, all G^L are left gift horses for H, and all G^R are right gift horses for H. So $H = K$, where K is the game whose set of left (respectively, right) options is the union of the left (respectively, right) options of G and of H.

Then by the second assumption, all options in K that came from H are dominated by options that came from G, so we can eliminate all options coming from H and get $H = K = G$. □

Note that if H is a number, then the second condition means that the first condition does not hold with H replaced by an option of H. This gives the usual statement of the Simplicity Theorem for comparing games with numbers (the last claim in the following corollary).

Corollary 5.2. *Suppose G is a game and x is a number such that $\forall\, G^L : G^L \lhd x$ and $\forall\, G^R : x \lhd G^R$. Then G is equal to a number; in fact, G is equal to a position of x.*

If no option of x satisfies the assumption in place of x, then $G = x$.

Proof. (See also [ONAG, Theorem 11].) The second statement is a special case of Proposition 5.1. If there is an option x' of x such that $G^L \lhd x' \lhd G^R$, then replace x with x' and use induction. □

The notion of "simple" games is defined in terms of birthdays: the earlier a game is created (i. e., the smaller its birthday), the simpler it is. The simplest game is $0 = \{\,|\,\}$, which is created first, and subsequently more and more complicated games with later birthdays are created out of simpler (older) ones. The name "Simplicity Theorem" for the last statement in the corollary above comes from the fact that in this case, x is the "simplest" number satisfying the assumption (because none of its options do). If G is a number, the statement can then be interpreted as saying that G is equal to the simplest number that fits between G's left and right options.

For example, a game that has no right options must be equal to a number, since $G^L \leq b(G^L) < b(G)$. In fact, this number is a position of $b(G)$, and hence G is even an ordinal number.

Note that a game G such that $G^L < G^R$ for all pairs (G^L, G^R) is not necessarily a number. A simple counterexample is given by $G = \{0 \mid \uparrow\}$, where $\uparrow = \{0 \mid *\}$, which is a positive game smaller than all positive (Conway) numbers (an *all small* game, see below in Section 6), but we have $0 < \uparrow$.

5.2. How to play with numbers. It is clear how one has to play in a number: choose an option which is as large (or as small) as possible. It is always a disadvantage to move in a number because one has to move to a position worse than before. In any case, we can easily predict from the sign of the number who will win the game, and to achieve this win, it is only necessary to choose options of the correct sign. As far as playing is concerned, numbers are pretty boring! But what is good play in a sum $G + x$, where x is a number and G is not?

Theorem 5.3 (Weak Number Avoidance Theorem)**.** *If G is a game that is not equal to a number and x is a number, then*

$$x \lhd G \iff \exists\, G^L : x \leq G^L.$$

Proof. The implication "⇐" is trivial. Now assume $x \lhd G$. This means that either $\exists\, G^L \geq x$ (and we are done), or $\exists\, x^R \leq G$ and we can assume that $x \rhd G^L$ for all G^L. Since G is not equal to a number by assumption, Corollary 5.2 implies that there is some G^R with $x \geq G^R$. But then we get $G \geq x^R > x \geq G^R$, in contradiction to the basic fact $G \lhd G^R$. □

If we apply this to G and $-x$, it says $G + x \rhd 0 \iff G^L + x \geq 0$ for some G^L. In words, this means that if there is a winning move in the sum $G + x$, then there is already a winning move in the G component. In short:

> *To win a game, you do not have to move in a number, unless there is nothing else to do.*

This does *not* mean, however, that the other options $G + x^L$ are redundant (i. e., dominated or reversible). This is only the case in general when G is *short*. To prove this stronger version of the Number Avoidance Theorem, we need some preparations.

Definition 5.4 (Left and Right Stops)**.** Let G be a short game. We define (short) numbers $L(G)$ and $R(G)$, the *left* and *right stops* of G, as follows.

If G is (equal to) a number, we set $L(G) = R(G) = G$. Otherwise,

$$L(G) = \max_{G^L} R(G^L) \quad \text{and} \quad R(G) = \min_{G^R} L(G^R).$$

Since G has only finitely many options, the maxima and minima exist. Note that G must have both left and right options; otherwise G would be a number.

One can think of $L(G)$ as the best value Left can achieve as first player in G at the point where the game becomes a number. Similarly, $R(G)$ is the best value Right can achieve when moving first. (Since numbers are pretty uninteresting games, we can stop playing as soon as the game turns into a number; this number can then conveniently be interpreted as the score of the game.)

Proposition 5.5. *Let G be a* short *game.*

(1) *If y is a number, then the following implications hold.*

$$y > L(G) \Longrightarrow y > G, \qquad y < L(G) \Longrightarrow y \lhd G,$$
$$y < R(G) \Longrightarrow y < G, \qquad y > R(G) \Longrightarrow y \rhd G.$$

(2) *If $z > 0$ is a positive number and G is not equal to a number, then $G < G^L + z$ for some G^L.*

Proof. (1) If G is a number, then $G = L(G) = R(G)$, and the statements are trivially true. If G is not a number, we proceed by induction.

Assume $y > L(G)$. Then, by definition, we have $y > R(G^L)$ for all G^L. By the induction hypothesis, this implies $y \rhd G^L$ for all G^L. By Theorem 5.3, this means that $y \geq G$, and hence $y > G$, since y cannot be equal to G (y is a number, but G is not).

Now assume $y < L(G)$. Then, by definition, $y < R(G^L)$ for some G^L. By the induction hypothesis, $y < G^L$ for this G^L. But this implies $y \lhd G$.

The other two statements are proved analogously.

(2) Let $y = L(G) + z/2$. By the first part, we then have $y > G$ and $y - z \lhd G$. By Theorem 5.3, there is a G^L with $y - z \leq G^L$. Hence $G < y \leq G^L + z$. □

The first part of the preceding proposition can be interpreted as saying that the *confusion interval* of G (the set of numbers G is fuzzy to) extends from $R(G)$ to $L(G)$. (The endpoints may or may not be included; this depends on who has the move when the game reaches its stopping position; compare the Temperature Theory of games in [ONAG, Sec. 9] and [WW, Sec. 6].)

With these preparations, we can now state and prove the Number Avoidance Theorem in its strong form.

Theorem 5.6 (Strong Number Avoidance Theorem)**.** *If G is a* short *game that is not equal to a number and x is a number, then $G + x = \{G^L + x \mid G^R + x\}$.*

Proof (Cf. [ONAG, Thm. 90]). We have $G+x = \{G^L+x, G+x^L \mid G^R+x, G+x^R\}$. Consider an option $G+x^L$. By the second part of Prop. 5.5, applied to $z = x-x^L$, there is some G^L such that $G < G^L + x - x^L$. Therefore $G + x^L < G^L + x$ is dominated and can be removed. An analogous argument applies to $G + x^R$. □

Let us demonstrate that the assumption on G is really necessary. Consider the game

$$G = \{\mathbb{Z} \mid \mathbb{Z}\} = \{\dots, -2, -1, 0, 1, 2, \dots \mid \dots, -2, -1, 0, 1, 2, \dots\}.$$

Then $\{G^L + 1 \mid G^R + 1\} = \{\mathbb{Z}+1 \mid \mathbb{Z}+1\} = \{\mathbb{Z} \mid \mathbb{Z}\} = G$, but (of course) $G+1 \neq G$, since $1 \neq 0$.

This game G also provides a counterexample to the second part of Proposition 5.5 for nonshort games. It is easily seen that $G \parallel n$ for all integers n; hence $G < G^L + 1$ is impossible.

The deeper reason for this failure is that it is not possible to define left and right stops for general games. And this is because there is nothing like a supremum of an arbitrary set of numbers (for example, $\mathbb{Z}$ has no least upper bound). This should be contrasted with the situation we have with ordinal numbers, where every set of ordinal numbers has a least upper bound within the ordinal numbers.

6. Infinitesimal Games [WW, Sec. 8]

If a game is approximately the size of a positive (or negative) number, we know that Left (or Right) will win it. But there are games which are less than all positive numbers and greater than all negative numbers, and we do not get a hint as to who is favored by the game. Such a game is said to be *infinitesimal*.

Definition 6.1 (Infinitesimal Game)**.**

(1) A game G is said to be *infinitesimal* if $-2^{-n} < G < 2^{-n}$ for all natural numbers n.
(2) A game G is said to be *strongly infinitesimal* if $-z < G < z$ for all positive numbers z.

An example of an infinitesimal but not strongly infinitesimal game is given by $2^{-\omega} = \{0 \mid 2^{-n} \colon n \in \mathbb{N}\}$. The standard example of a positive strongly infinitesimal game is $\uparrow = \{0 \mid *\}$ (pronounced "up"). More examples are provided by the following class of games.

Definition 6.2 (All Small)**.** A game G is said to be *all small* if every position of G that has left options also has right options and vice versa.

Since a game without left (or right) options is equal to a number (see the example after Corollary 5.2), this definition is equivalent to "no number other than 0 occurs as a position of G." The simplest all small games are 0, $*$, $\uparrow = \{0 \mid *\}$ and $\downarrow = \{* \mid 0\}$ (note that $\{* \mid *\} = 0$). They show that an all small game can fall into any of the four outcome classes.

Proposition 6.3. *If G is an all small game, then G is strongly infinitesimal.*

Proof. If G is a number, then $G = 0$, and the claim holds trivially. Otherwise, G has left and right options, all of which are strongly infinitesimal by induction. Let $z > 0$ be a number. Let z^R be some right option of z. Then for all and hence for some G^R we have $G^R < z < z^R$ and so $G \lhd z^R$. On the other hand, we have $G^L < z$ for all G^L. Together, these two facts imply that $G \le z$. The inequality $G \ge -z$ is shown in the same way. □

Not all strongly infinitesimal games are all small. Examples are provided by "tinies" and "minies" like $\{0 \mid \{0 \mid -1\}\}$ [WW, Sec. 5].

We shall see in a moment that for short games, "infinitesimal" and "strongly infinitesimal" are the same. More precisely, the following theorem tells us that a short infinitesimal game is already bounded by some integral multiple of $\uparrow$. (This is mentioned in [WW, Sec. 20: "The Paradox"], but we do not know of a published proof.) For example, we have $* < 2\uparrow$, and so also $2\uparrow + * > 0$.

Theorem 6.4. *Let G be a* short *game.*

(1) *If $G \lhd 2^{-n}$ for all positive integers n, there is some positive integer m such that $G \lhd m\uparrow$.*
(2) *If $G \le 2^{-n}$ for all positive integers n, there is some positive integer m such that $G \le m\uparrow$.*

Proof. We can assume that G is not a number, because otherwise the assumption implies in both cases that $G \le 0$. We use induction.

(1) $G \lhd 2^{-n}$ means by Theorem 5.3 that there is some G^R with $G^R \le 2^{-n}$. Since there are only finitely many G^R (G is short), there must be one G^R that works for infinitely many and hence for all n. By induction, we conclude that $G^R \le m\uparrow$ for this G^R and some m and therefore $G \lhd m\uparrow$ for this m.

(2) (a) If $G \le 2^{-n}$, then $G < 2^{-(n-1)}$. Hence G satisfies the assumption of the first part, and we conclude that there is some m_0 such that $G \lhd m\uparrow$ for all $m \ge m_0$.

(b) $G \le 2^{-n}$ implies that $G^L \lhd 2^{-n}$ for all G^L. By induction, for every G^L there is some m such that $G^L \lhd m\uparrow$. Since there are only finitely many G^L (G is short), there is some m_1 such that $G^L \lhd m_1\uparrow$ for all G^L.

(c) Now let $m = \max\{m_1, m_0 + 3\}$. By (a), we know that $G \lhd (m-3)\uparrow < (m-1)\uparrow + * = (m\uparrow)^R$. By (b), we know that $G^L \lhd m\uparrow$ for all G^L. Together, these imply $G \le m\uparrow$. □

The apparent asymmetry in this proof is due to the lack of something like an "Up Avoidance Theorem."

This result says that infinitesimal short games can be measured in (short) units of $\uparrow$. This is the justification behind the "atomic weight calculus" described in [WW, Secs. 7, 8].

Remark 6.5. It is perhaps tempting to think that a similar statement should be true for strongly infinitesimal general games. If one tries to mimic the above proof, one runs into two difficulties. In the first part, we have used the fact that any finite set of positive short numbers has a positive short lower bound. The corresponding conclusion would still be valid, since any set of positive numbers has a positive lower bound (which is a number). (For $z > 0$, it is easy to see that $z \geq 2^{-b(z)}$; the claim then follows from the statement on upper bounds for sets of ordinals.) In the second part, we have used the fact that any finite set of natural numbers (or of multiples $m\uparrow$) has an upper bound. This does not seem to generalize easily.

7. Impartial Games [WW, Sec. 3], [ONAG, Sec. 11]

7.1. What is an impartial game? An impartial game is one in which both players have the same possible moves in every position. Formally, this reads as follows.

Definition 7.1 (Impartial Game)**.** An *impartial game* is a game for which the sets of left and right options are equal and all options are impartial games themselves.

It follows that every impartial game G satisfies $G = -G$, and hence $G + G = 0$. Therefore, $G = 0$ or $G \parallel 0$. There is no need to distinguish the sets of left and right options, and so we simply write $G = \{G', G'', \dots\}$, where $\{G', G'', \dots\}$ is the set of options of G (again, this notation is not meant to suggest that the set of options should be countable or nonempty).

The standard examples include the game of Nim: it consists of a finite collection of heaps $H_1, \dots, H_n$, each of which is an ordinal number (some number of coins, matches, etc.; if this makes you feel more comfortable, think of natural numbers only). A move consists of reducing any single heap by an arbitrary amount (i. e., replacing one of the ordinal numbers by a strictly smaller ordinal number), leaving all other heaps unaffected. A move in the ordinal number 0 is of course impossible, and so the game ends when all heaps are reduced to 0. As usual, winner is the one who made the last move. Note that a single heap game is trivial: if the heap is nonzero, then the winning move consists in reducing the heap to zero, leaving no legal move to the opponent. A game with several heaps is the sum (in our usual sense) of its heaps: it is $H_1 + H_2 + \cdots + H_n$.

Conway coined the term *nimber* for a single Nim heap, and he writes $*n$ for a heap of size n (where n is of course an ordinal number). The rules of Nim can then simply be written as $*0 = 0$, $*n = \{*0, *1, \dots, *(n-1)\}$ (if n is finite) or $*n = \{*k \colon k < n\}$ (in general). We have $*n = *k$ if and only $n = k$: the equality $*n = *k$ means $0 = *n - *k = *n + *k$ (note that $*k = -*k$, since nimbers are impartial games), and in $*n + *k$ with $n \neq k$ the first player wins by reducing the larger heap so as to leave two heaps of equal size to the opponent). The nimbers inherit a total ordering from the ordinal numbers, so that every set of nimbers is well ordered (Proposition 4.3). But note that this is not the same as the ordering of general games restricted to impartial games: a Nim heap $*n$ of size $n > 0$ has $*n \parallel 0$, not $*n > 0$!

A second note on set theory. Since there is no need to distinguish the left and right options of impartial games and all these options are impartial games themselves, Definition 2.1 of Games simplifies to the following.

Definition 7.2 (Impartial Game)**.**

(1) Let G be a set of impartial games. Then G is an *impartial game.*
(2) (Descending Game Condition for Impartial Games.) There is no infinite sequence of games G^i with $G^{i+1} \in G^i$ for all $i \in \mathbb{N}$.

Therefore, impartial games are just sets in the sense of Zermelo and Fraenkel; the Descending Game Condition exactly reduces to the Axiom of Foundation.[6]

7.2. Classification of impartial games. There is a well-known classification of impartial games, due to Sprague and Grundy, which says that every impartial game is equal to a well-defined nimber $*n$ and hence equal to a Nim game with a single heap, which has a trivial winning strategy. Unfortunately, this does not make every impartial game easy to analyze: in practice, it might be hard to tell exactly which nimber an impartial game is equal to; as an example, we mention the game of *Sylver Coinage*[7] [WW, Sec. 18].

The theory of impartial games is based on the following definition.

Definition 7.3 (The mex: *m*inimal *ex*cluded number)**.** Let G be a set of nimbers. Then $\text{mex}(G)$ is the least nimber not contained in G.

Note that every *set* of nimbers has an upper bound by Proposition 4.6, and the well-ordering of nimbers implies that every *set* of nimbers has a well-defined minimal excluded nimber, so that the mex is well defined.

The first and fundamental step of the classification is the following observation.

Proposition 7.4 (Bogus Nim)**.** *Let G be any set of nimbers. Then $G = \text{mex}(G)$.*

Some remarks might be in order here. First, G is of course an impartial game itself, describing its set of options which are all nimbers. Now $\text{mex}(G)$ is a particular nimber, and the result asserts that the game G is equal as a game to the nimber $\text{mex}(G)$.

Proof. All we need to show is that $G - \text{mex}(G) = G + \text{mex}(G) = 0$. The main trick is to write this right: $\text{mex}(G) = *n = \{*k : k < n\}$ for an ordinal number n, while $G = *n \cup G'$, where G' is the set of options of G exceeding $*n$. (Note that by assumption $*n$ is not an option of G.) The first player has three kinds of possible moves: move in $\text{mex}(G) = *n$ to some $*k < *n$, move in G to an option $*k < *n$, or move in G to an option $*k > *n$. The first leads to $G + *k$ and is countered by

---

[6]The observation that every set can be viewed as an impartial game leads to amusing questions of the type "what is the Nim heap equivalent to $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, ...?" The answer of course depends on the exact way of representing these sets within set theory.

[7]Sylver Coinage is usually played in misère play; however, one can equivalently declare the number 1 illegal and use the normal winning convention.

the move in G to $*k + *k = 0$; the second leads to $*k + *n$ and is countered by the move in $*n$ to $*k + *k = 0$; finally, the third leads to $*k + *n$ (this time, with $k > n$) and is countered by a move in $*k$ to $*n + *n = 0$. In all three cases, the second player moves to 0 and wins. □

The name "Bogus Nim" refers to the following interpretation of this game: the game G is indeed a Nim heap $*n$ (offering moves to all $*k$ with $k < n$), but in addition it is allowed to increase the size of the heap. This increase is immediately reversed by the second player, bringing the heap back to $*n$ (in which no further increase is possible), so that all increasing moves are reversible moves.

A rather obvious corollary is the following: $G = *0$ if and only if no legal move in G leads to $*0$ (if $*0$ is no option of G, then clearly $\text{mex}(G) = *0$): if G has the option $*0$, then this is a winning move for the first player, and hence $G \parallel 0$; otherwise, all options of G (if any) lead to nimbers $*n \neq *0$ from which the second player wins.

Theorem 7.5 (The Classification of Impartial Games)**.** *Every impartial game is equal to a unique nimber.*

Proof. We use Conway induction: write $G = \{G', G'', \dots\}$, listing all options of G. By the inductive hypothesis, every option of G is equal to a nimber, and hence (using Theorem 2.17)

$$G = \{*k\colon \text{ there is an option of } G \text{ which is equal to } *k\}.$$

Therefore, all options of G are equal to nimbers; hence, by Proposition 7.4, G is equal to the mex of all its options, which is a nimber. Uniqueness is clear. □

We should mention that for the game Nim, there is a well-known explicit strategy, at least for heaps of finite size: in the game $G = *n_1 + *n_2 + \cdots + *n_s$, write each heap size n_i in binary form and form the EXCLUSIVE OR (XOR) of them. Then $G = 0$ if and only if the XOR is zero in every bit: in the latter case, it is easy to see that every option of G is nonzero, while if the XOR is nonzero, then every heap which contributes a `1` to the most significant bit of the XOR can be reduced in size so as to turn the game into a zero game.

Nim is often played in Misère play, and the binary strategy as just described works in this form almost without difference, except very near the end when there are at most two heaps of size exceeding one. It is sometimes wrongly concluded that there is a general theory of impartial games in Misère play, similar to Theorem 7.5. However, this is false: the essential difference is the use of Theorem 2.17, which allows us to replace an option by an equivalent one, and this is based on the usual winning convention. There is no analog in Misère play to the Sprague–Grundy theory: for any two impartial games G and H without reversible options which are different in form (i. e., $G \not\equiv H$), there is another impartial game K such that the winners of $G + K$ and $H + K$ are different; see [ONAG, Chap. 12].

Acknowledgements. This note grew out of a summer school on the subject which we taught for students of the *Studienstiftung des deutschen Volkes* in 2001 (in La Villa). We would like to thank the participants for the inspiring and interesting discussions and questions which eventually led to the writing of this note. We would also like to thank our Russian friends, in particular Alexei Belov, for encouragement and interest in Conway games.

References

[AGBB] J. H. Conway, *All games bright and beautiful*, Amer. Math. Monthly **84** (1977), no. 6, 417–434. MR 0446534

[Go] E. Berlekamp and D. Wolfe, *Mathematical Go*, A K Peters Ltd., Wellesley, MA, 1994. MR 1274921

[GONC] R. J. Nowakowski (ed.), *Games of no chance*, Mathematical Sciences Research Institute Publications, vol. 29, Cambridge University Press, Cambridge, 1996. MR 1427953

[Hex1] M. Gardner, *The Scientific American book of mathematical puzzles and diversions*, Simon&Schuster, New York, 1959.

[Hex2] C. Browne, *Hex strategy: making the right connections*, A K Peters Ltd., Natick, MA, 2000. MR 1810604

[ONAG] J. H. Conway, *On numbers and games*, Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976. MR 0450066

[SN] D. E. Knuth, *Surreal numbers*, Addison-Wesley Publishing Co., Reading, Mass.–London–Amsterdam, 1974. MR 0472278

[WW] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays,* Vols. 1, 2, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1982. MR 654501

School of Engineering and Science, International University Bremen, Postfach 750 561, D-28725 Bremen, Germany

E-mail address: dierk@iu-bremen.de

E-mail address: m.stoll@iu-bremen.de