MCKAY CORRESPONDENCE AND EQUIVARIANT SHEAVES ON $\mathbb{P}^1$

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Dedicated to my father on the occasion of his 70th birthday

Abstract. Let $G$ be a finite subgroup in $\text{SU}(2)$, and $Q$ the corresponding affine Dynkin diagram. In this paper, we review the relation between the categories of $G$-equivariant sheaves on $\mathbb{P}^1$ and $\text{Rep} Q_h$, where $h$ is an orientation of $Q$, constructing an explicit equivalence of corresponding derived categories.


Key words and phrases. McKay correspondence, equivariant sheaves, quiver representations.

Introduction

Let $G \subset \text{SU}(2)$ be a finite subgroup. According to McKay correspondence, such a subgroup gives rise to a graph $Q$ which turns out to be an affine Dynkin diagram of ADE type. Let $\mathfrak{g}$ be the corresponding affine Lie algebra.

There are several approaches allowing one to construct $\mathfrak{g}$ from $G$:

1. There is a geometric construction, due to Nakajima and Lusztig, of $\mathfrak{g}$ and its representations in terms of the quiver varieties associated to graph $Q$. These varieties are closely related to the moduli spaces of instantons on the resolution of singularities $\mathbb{C}^2/G$.

2. There is an algebraic construction, due to Ringel [R2], which allows one to get the universal enveloping algebra $U_{\mathfrak{n}_+}$ of the positive part of $\mathfrak{g}$ as the Hall algebra of the category $\text{Rep} Q_h$, where $h$ is an orientation of $Q$ and $Q_h$ is the corresponding quiver. It was shown by Peng and Xiao [PX] that replacing $\text{Rep} Q_h$ by the quotient $R = D^b(\text{Rep} Q_h)/T^2$ of the corresponding derived category, we can get a description of all of $\mathfrak{g}$. Different choices of orientation give rise to equivalent derived categories $R$, with equivalences given by Bernstein–Gelfand–Ponomarev reflection functors; in terms of $\mathfrak{g}$, these functors correspond to the action of the braid group of the corresponding Weyl group.

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Recently, a third approach was suggested by Ocneanu [O] (unpublished), in a closely related setting of a subgroup in quantum SU(2), for \( q \) being a root of unity. His approach is based on studying essential paths in \( \hat{Q} = Q \times \mathbb{Z}/h\mathbb{Z} \), where \( h \) is the order of the root of unity. This approach is purely combinatorial: all constructions are done using this finite graph and vector spaces of essential paths between points in this graph, without involving any categories at all.

This paper grew out of the author’s attempt to understand Ocneanu’s construction and in particular, find the appropriate categorical interpretation of his combinatorial constructions; however, for simplicity we do it for subgroups in classical SU(2), leaving the analysis of the subgroups in quantum SU(2) for future papers.

We show that Ocneanu’s essential paths in \( \hat{Q} \) have a natural interpretation in terms of the category \( \text{Coh}_G(\mathbb{P}^1) \) of \( G \)-equivariant \( \mathcal{O} \)-modules on \( \mathbb{P}^1 \) or, rather, its “even part” \( \mathcal{C} = \text{Coh}_G(\mathbb{P}^1)^0 \), for certain natural \( \mathbb{Z}/2 \) grading on \( \text{Coh}_G(\mathbb{P}^1) \). This also provides a relation with Ringel–Lusztig construction: for (almost) any choice of orientation \( h \) of \( Q \), we construct an equivalence of triangulated categories

\[
R\Phi_h : D^b(\mathcal{C}) \simeq D^b(\text{Rep} \ Q_h)
\]

These equivalences agree with the equivalences of \( D^b(\text{Rep} \ Q_h) \) for different choices of \( h \) given by reflection functors. As a corollary, we see that the Grothendieck group \( L = K(\mathcal{C}) \) is an affine root lattice, and the set \( \Delta \) of classes of indecomposable objects in \( D^b(\mathcal{C}) \) is an affine root system.

This construction of the affine root system via equivariant sheaves has a number of remarkable properties, namely:

1. This does not require a choice of orientation of \( Q \) (unlike the category \( D^b(\text{Rep} \ Q_h) \), where we first choose an orientation and then prove that the resulting derived category is independent of orientation).
2. The indecomposable objects in the category \( \text{Coh}_G(\mathbb{P}^1) \) can be explicitly described. Namely, they are the sheaves \( \mathcal{O}(n) \otimes X_i \), where \( X_i \) are irreducible representations of \( G \) (these sheaves and their translations correspond to real roots of \( g \) and torsion sheaves, whose support is a \( G \)-orbit in \( \mathbb{P}^1 \) (in particular, torsion sheaves whose support is an orbit of a generic point correspond to imaginary roots of \( g \)).
3. This construction of the affine root system does not give a set of simple positive roots in \( \Delta \). Instead, it gives a canonical Coxeter element in the corresponding affine Weyl group, which is given by \( C[\mathcal{F}] = [\mathcal{F} \otimes \mathcal{O}(-2)] \) (this also corresponds to the Auslander–Reiten functor \( \tau \)).
4. This construction gives a bijection of the vertices of the affine Dynkin diagram \( Q \) and (some of) the \( C \)-orbits in \( \Delta \) (as opposed to the quiver construction, where vertices of \( Q \) are in bijection with the simple positive roots).

For \( G = \{1\} \) (which is essentially equivalent to \( G = \{\pm 1\} \)), these results were first obtained in the paper [BK], where it was shown that the corresponding Hall algebra contains the subalgebra isomorphic to \( U(\widehat{\mathfrak{sl}_2}) \). This paper in turn was inspired by an earlier paper of Kapranov [Kap].

It should be noted that many of the results obtained here have already been proved in other ways. Most importantly, it had been proved by Lenzing [GL], [Len]
that the derived category of equivariant sheaves on \( \mathbb{P}^1 \) is equivalent to the derived category of representation of the corresponding quiver; this result has been used by Schiffmann [Sch1], [Sch2] for construction of a subalgebra in the corresponding affine Lie algebra via Hall algebra of the category of equivariant sheaves. However, the construction of equivalence in [Len] is different than the one suggested here. The primary difference is that in Lenzing’s construction, the Dynkin diagram \( Q \) is constructed as the star diagram (in fact, he uses one-point completion of the star diagram), with lengths of branches determined by the branching points of the cover \( \mathbb{P}^1 \to X = \mathbb{P}^1/G \), and he uses a standard orientation of this diagram. In the construction presented here, the diagram is \( Q \) is defined in a more standard way, using the set \( I \) of irreducible representations of \( G \); more importantly, we construct not a single equivalence but a collection of equivalences, one for each admissible orientation. In Lenzing’s construction, torsion sheaves naturally play a major role; in our construction, we concentrate on the study of locally free sheaves.

Lenzing’s results apply not only to \( \mathbb{P}^1/G \) but to a much larger class of “non-commutative projective curves”. However, the downside of this is that the language he uses is rather technical, making his papers somewhat hard for non-experts. For this reason, we had chosen to give independent proofs of some of the results, thus saving the reader the necessity of learning the language of non-commutative curves. Of course, we tried to clearly mark the results which had already been known.

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1. Basic Setup

Throughout the paper, we work over the base field \( \mathbb{C} \) of complex numbers. \( G \) is a finite subgroup in \( SU(2) \); for simplicity, we assume that \( \pm I \in G \) and denote \( \overline{G} = G/\{\pm I\} \) (this excludes \( G = \mathbb{Z}_n \), \( n \) odd; this case can also be included, but some of the constructions of this paper will require minor changes). We denote by \( V \) the standard 2-dimensional space, considered as a tautological representation of \( SU(2) \) (and thus of \( G \)) and

\[
V_k = S^k V
\]

is the \((k+1)\)-dimensional irreducible representation of \( SU(2) \) (and thus a representation, not necessarily irreducible, of \( G \)).

We denote by \( \text{Rep} G \) the category of finite-dimensional representations of \( G \) and by \( I = \text{Irr} G \) the set of isomorphism classes of simple representations; for \( i \in I \), we denote by \( X_i \) the corresponding representation of \( G \). Since \( G \supset \{\pm I\} \), the category \( \text{Rep} G \) is naturally \( \mathbb{Z}_2 \)-graded:

\[
\text{Rep} G = \text{Rep}_0 G \oplus \text{Rep}_1 G, \quad \text{Rep}_p G = \{V \in \text{Rep} G : (-I)|_V = (-1)^p \text{id}\}. \quad (1.1)
\]

For homogeneous object \( X \), we define its “parity” \( p(X) \in \mathbb{Z}_2 \) by

\[
p(X) = p \quad \text{if} \ X \in \text{Rep}_p G;
\]

\[
p(X) = 0 \quad \text{if} \ X \in \text{Rep}_0 G.
\]

(1.2)
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in particular, \( p(V) = 1 \). We will also define, for \( i \in I \), its parity \( p(i) = p(X_i) \). This gives a decomposition

\[
I = I_0 \sqcup I_1. \tag{1.3}
\]

We define the graph \( Q \) with the set of vertices \( V(Q) = I \) and for every two vertices \( i, j \), the number of edges connecting them is defined by

\[
n(i, j) = \dim \text{Hom}_G(X_i, X_j \otimes V).
\]

Note that decomposition (1.3) shows that this graph is bipartite: \( V(Q) = V_0(Q) \sqcup V_1(Q) \), and all edges connect vertices of different parities.

It is well-known that one can construct an isomorphism

\[
\{ \text{Paths of length } l \text{ in } Q \text{ connecting } i, j \} = \text{Hom}_G(X_i, V \otimes^l \otimes X_j)
\]

and that \( \text{Hom}_G(X_i, V \otimes^l \otimes X_j) \) can be described as the space of “essential paths” in \( Q \), which is naturally a direct summand in the space of all paths (see [CG]). The algebra of essential paths is also known as the preprojective algebra of \( Q \).

By McKay correspondence, \( Q \) must be an affine Dynkin diagram. We denote by \( \Delta(Q) \) the corresponding affine root system; it has a basis of simple roots \( \alpha_i, i \in I \).

\[
L(Q) = \bigoplus_{i \in I} \mathbb{Z} \alpha_i = \mathbb{Z} I
\]

the corresponding root lattice. It has a natural bilinear form given by \( \langle \alpha_i, \alpha_j \rangle = 2 \) and \( \langle \alpha_i, \alpha_j \rangle = -n(i, j) \); as is well-known, this form is positive semidefinite. The kernel of this form is \( \mathbb{Z} \delta \), where \( \delta \) is the imaginary root of \( \Delta(Q) \).

We denote by \( s_i : L(Q) \to L(Q) \) the reflection around root \( \alpha_i \).

Finally, let \( K(G) \) be the Grothendieck group of the category \( \text{Rep}_G \). It is freely generated by classes \( [X_i], i \in I \); thus, we have a natural isomorphism

\[
K(G) \cong L(Q), \quad [X_i] \mapsto \alpha_i. \tag{1.4}
\]

2. Quiver \( Q_h \)

We will consider a special class of orientations of \( Q \).

**Definition 2.1.** A height function \( h \) is a map \( I \to \mathbb{Z} \) such that \( h(i) - h(j) = \pm 1 \) if \( i, j \) are connected by an edge in \( Q \), and \( h(i) \equiv p(i) \) (mod 2).

Every height function gives rise to orientation of edges of \( Q \): if \( h(j) = h(i) - 1 \) then all edges connecting \( i \) and \( j \) are directed towards \( j \):

\[
i \to j \quad \text{if} \quad h(j) \equiv h(i) - 1.
\]

We will denote by \( Q_h \) the quiver given by this orientation. We will write \( i \to j \) if there exists an edge whose tail is \( i \) and head is \( j \). The notation \( \sum_{j : i \to j} \) will mean the sum over all vertices \( j \) connected with \( i \) by an edge \( j \to i \); if there are multiple edges, the corresponding vertex \( j \) will be taken more than once.

Orientations obtained in this way will be called admissible (note: our use of this word is slightly different from the use in other sources). It is easy to check that if \( Q \) has no loops, then any orientation of \( Q \) is admissible. For type \( A \), an orientation is admissible if the total number of clockwise arrows is equal to the number of
counterclockwise ones (which again rules out type $\tilde{A}_n$, $n$ even, corresponding to $G = \mathbb{Z}_{n+1}$). It is also obvious that $Q_h$ has no oriented loops, and that adding a constant to $h$ gives the same orientation.

Given a height function $h$, we will draw $Q$ in the plane so that $h$ is the $y$-coordinate; then all edges of $Q_h$ are directed down. Figure 1 shows an example of a height function for quiver of type $D$.

$$\text{Figure 1. Example of a height function for Dynkin diagram } \tilde{D}_7$$

**Definition 2.2.** Let $h$ be a height function on $Q$, and $i \in I$ be a sink in $Q_h$: there are no edges of the form $i \rightarrow j$ (in terms of $h$, it is equivalent to saying that $h$ has a local minimum at $i$). We define new height function

$$s_i^+ h(j) = \begin{cases} h(j) + 2, & j = i, \\ h(j), & j \neq i. \end{cases}$$

Similarly, if $i$ is a source, i.e., there are no vertices of the form $j \rightarrow i$ (equivalently, $h$ has a local maximum at $i$), then we define

$$s_i^- h(j) = \begin{cases} h(j) - 2, & j = i, \\ h(j), & j \neq i. \end{cases}$$

One easily sees that $s_i^+ h$, $s_i^- h$ are again height functions; the quiver $Q_{s_i^\pm h}$ is obtained from $Q_h$ by reversing orientation of all edges adjacent to $i$. We will refer to $s_i^\pm$ as (elementary) orientation reversal operations.

The following lemma is known; however, for the benefit of the reader we included the proof.

**Lemma 2.3.** Any two height functions $h$, $h'$ can be obtained one from another by a sequence of orientation reversal operations $s_i^\pm$.

**Proof.** Define the “distance” between two height functions by

$$d(h, h') = \sum_{i} |h(i) - h'(i)| \in \mathbb{Z}_+.$$

We will show that if $d(h, h') > 0$, then one can find $s_i^\pm$ such that $s_i^\pm$ can be applied to $h$, and $d(s_i^\pm h, h') < d(h, h')$; from this the theorem clearly follows.
Let $I_+ = \{i: h(i) > h'(i)\}$. One easily sees that if $i \in I_+$, and $j \rightarrow i$ in $Q_h$, then $j \in I_+$. Thus, if $I_+$ is non-empty, it must contain at least one source $i$ for $Q_h$. But then $d(s_i^- h, h') = d(h, h') - 2$.

Similarly, let $I_- = \{i: h(i) < h'(i)\}$. If $I_-$ is non-empty, similar argument shows it must contain at least one sink $i$ for $Q_h$. But then $d(s_i^+ h, h') = d(h, h') - 2$. □

3. Representations of Quivers

For readers convenience, we recall here the known results about the relation between representations of quiver $Q_h$ and root system $\Delta(Q)$ (see, e.g., [DR], [KR]).

Let $h$ be a height function on $Q$ and $Q_h$ the corresponding quiver. Consider the category $\text{Rep} Q_h$ of finite-dimensional representations of $Q_h$; similarly, let $\mathcal{D}^b(Q_h)$ be the bounded derived category of $\text{Rep} Q_h$. We denote by $K(Q_h)$ the Grothendieck group of $\text{Rep} Q_h$ which can also be described as the Grothendieck group of $\mathcal{D}^b(Q_h)$. It is well-known that the category $\text{Rep} Q_h$ is a hereditary abelian category:

$$\text{Ext}^i(X, Y) = 0 \quad \text{for all } i > 1.$$ 

For a representation $M = (M_i)_{i \in I}$ of $Q_h$, we define its dimension $\dim X \in L(Q)$ by $\dim M = \sum (\dim M_i) \alpha_i$ (recall that $L(Q) = \mathbb{Z}[I]$ is the root lattice of the root system $\Delta(Q)$, see Section 1).

The following theorem summarizes some of the known results about representations of quivers and root system $\Delta(Q)$.

**Theorem 3.1.** (1) The map $[X] \mapsto \dim X$ gives an isomorphism $K(Q_h) \cong L(Q)$. Under this isomorphism, the bilinear form on $L(Q)$ is identified with the following bilinear form on $K(Q_h)$:

$$\langle x, y \rangle = \langle x, y \rangle + \langle y, x \rangle$$

where by definition

$$\langle [X], [Y] \rangle = \dim \text{RHom}(X, Y) = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y).$$

(2) The set of dimensions of indecomposable modules is exactly the set $\Delta_+(Q)$ of positive roots in $\Delta(Q)$. For real roots $\alpha$, there is exactly one up to isomorphism indecomposable module $M_{\alpha}$ of dimension $\alpha$; for imaginary root $\alpha$, there are infinitely many pairwise non-isomorphic modules of dimension $\alpha$.

There is also an explicit description of indecomposables in $\mathcal{D}^b(Q_h)$ (see [Hap, Lemma I.5.2]).

**Theorem 3.2.** Indecomposable objects in $\mathcal{D}^b(Q_h)$ are of the form $M[n]$, where $M$ is an indecomposable object in $\text{Rep} Q_h$, $n \in \mathbb{Z}$.

We will also need reflection functors of Bernstein–Gelfand–Ponomarev. Recall that if $i$ is a sink in $Q_h$, then one has a natural functor

$$S_i^+: \text{Rep}(Q_h) \rightarrow \text{Rep}(Q_{s_i^- h}); \quad (3.1)$$

similarly, if $i$ is a source, one has a natural functor

$$S_i^-: \text{Rep}(Q_h) \rightarrow \text{Rep}(Q_{s_i^+ h}) \quad (3.2)$$

(see definition in [DR], [KR]).
The following result is known and easy to prove, so we skip the proofs.

**Theorem 3.3.** (1) Functors $S^+_i$ is left exact and $S^-_i$ is right exact.

We will denote by $RS^+_i, LS^-_i : Db(Q_h) \to Db(Q_{i^+_h})$ the corresponding derived functors.

(2) The functors $RS^+_i, LS^-_i$ are equivalences of categories $Db(Q_h) \to Db(Q_{i^+_h})$, which induce the usual reflections $s_i$ on the Grothendieck group:

$$\dim RS^+_i(X) = s_i(\dim X), \quad \dim LS^-_i(X) = s_i(\dim X).$$

(3) If $i, j$ are not neighbors in $Q$, then $RS^+_i, RS^+_j$ commute (i.e., compositions in different orders are isomorphic) and similarly for $LS^-_i$.

In particular, for a given height function $h$ let $s^+_i \ldots s^+_r$ be a sequence of elementary orientation reversal operations such that

$$s^+_i \ldots s^+_r(h) = h + 2.$$

One easily sees that this condition is equivalent to requiring that every index $i \in I$ appears in the sequence \{ $i_1, \ldots, i_r$ \} exactly once. The same argument as in the proof of Lemma 2.3 shows that for every height function $h$, such sequences of orientation reversal operations exist. For such a sequence, the corresponding element of the Weyl group

$$c^+_h = s_{i_1} \ldots s_{i_r},$$

is called the **Coxeter element**, and the corresponding composition of reflection functors

$$RC^+_h = RS^+_i \ldots RS^+_r : Db(Q_h) \to Db(Q_{h+2})$$

will be called the **Coxeter functor**. Note that since $Q_h \simeq Q_{h+2}$ as a quiver, we can consider $RC^+_h$ as an autoequivalence of $Db(Q_h)$.

It is easy to show (see [DR], [Shi]) that the Coxeter element $c^+_h$ only depends on $h$ and not on the choice of the sequence $i_1, \ldots, i_r$; moreover, the proof of this only uses the fact that $s_i, s_j$ commute if $i, j$ are not connected in $Q$ and does not use the braid relations. Thus, by Theorem 3.3, this implies that up to an isomorphism, $RC^+_h$ also depends only on $h$; this justifies the notation $RC^+_h$.

Similarly, we can define functors

$$LC^-_h : Db(Q_h) \to Db(Q_{h-2}) \simeq Db(Q_h)$$

using sequences of orientation reversals $s^-_{i_1} \ldots s^-_{i_r} h = h - 2$; the corresponding element of the Weyl group will be denoted by $c^-_h$. As before, it can be shown that $LC^-_h, c^-_h$ only depend on $h$.

For readers familiar with the theory of Auslander–Reiten sequences (see [ARS], [Hap]), we add that the category $\text{Rep} Q_h$ has Auslander–Reiten sequences, and the Auslander–Reiten functor $\tau$ is given by $\tau = C^-_h$. 

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4. Equivariant Sheaves

In this section, we introduce the main object of this paper, the category of equivariant sheaves on \( \mathbb{P}^1 \). Most of the results of this section are well-known and given here only for the convenience of references. Lemma 4.4 does not seem to be easily available in the literature, but is very easy to prove.

Let \( V^* \) be the dual of the tautological representation \( V \) of \( \text{SU}(2) \). Since \( G \) is a finite subgroup in \( \text{SU}(2) \), it naturally acts on \( \mathbb{P}^1 = \mathbb{P}(V^*) \), and the structure sheaf \( \mathcal{O} \) has a standard \( \text{SU}(2) \)- (and thus \( G \))-equivariant structure. Moreover, all twisted sheaves \( \mathcal{O}(n) \) also have a standard equivariant structure, so that the space of global sections \( \Gamma(\mathcal{O}(n)) \) is a representation of \( G \):

\[
\Gamma(\mathcal{O}(n)) = \begin{cases} 
S^n V = V_n, & n \geq 0, \\
0, & n < 0.
\end{cases} \tag{4.1}
\]

Similarly, the higher homology spaces are naturally representations of \( G \); it is well-known that \( H^i(\mathbb{P}^1, \mathcal{O}(n)) = 0 \) for \( i > 1 \), and

\[
H^1(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} 
S^{-n-2} V^* = V^*_{-n-2}, & n \leq -2, \\
0, & n \geq -1.
\end{cases} \tag{4.2}
\]

Let \( \text{Qcoh}_G(\mathbb{P}^1) \), \( \text{Coh}_G(\mathbb{P}^1) \) be the categories of \( G \)-equivariant quasi-coherent (respectively, coherent) \( \mathcal{O} \)-modules on \( \mathbb{P}^1 \) (see, e.g., [BKR, Section 4] for definitions). Note that we are considering isomorphisms \( \lambda_g : \mathcal{F} \to g^* \mathcal{F} \) as part of the structure of the \( G \)-equivariant sheaf. For brevity, we will denote morphisms and Ext groups in \( \text{Qcoh}_G(\mathbb{P}^1) \) by \( \text{Hom}_G(X, Y) \), \( \text{Ext}_G(X, Y) \). For an equivariant sheaf \( \mathcal{F} \) we will denote

\[
\mathcal{F}(n) = \mathcal{O}(n) \otimes_\mathcal{O} \mathcal{F}
\]

with the obvious equivariant structure. Similarly, for a finite-dimensional representation \( X \) of \( G \), we denote

\[
X(n) = \mathcal{O}(n) \otimes_\mathbb{C} X.
\]

We list here some of the basic properties of equivariant sheaves; proofs can be found in [BKR, Section 4].

**Theorem 4.1.** (1) \( \text{Qcoh}_G(\mathbb{P}^1) \) is an abelian category, and a sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) is exact in \( \text{Qcoh}_G(\mathbb{P}^1) \) if and only if it is exact in \( \text{Qcoh}(\mathbb{P}^1) \).

(2) For any \( \mathcal{F}, \mathcal{G} \in \text{Qcoh}_G(\mathbb{P}^1) \), the space \( \text{Hom}_G(\mathcal{F}, \mathcal{G}) \) is a representation of \( G \), and \( \text{Hom}_G(\mathcal{F}, \mathcal{G}) = (\text{Hom}_G(\mathcal{F}, \mathcal{G}))^G \). Similarly, \( \text{Ext}_G^i(\mathcal{F}, \mathcal{G}) = (\text{Ext}_G^i(\mathcal{F}, \mathcal{G}))^G \); in particular, \( \text{Ext}_G^i(\mathcal{F}, \mathcal{G}) = 0 \) for \( i > 1 \).

(3) For any \( \mathcal{F}, \mathcal{G} \in \text{Coh}_G(\mathbb{P}^1) \), the spaces \( \text{Hom}_G(\mathcal{F}, \mathcal{G}), \text{Ext}_G^i(\mathcal{F}, \mathcal{G}) \) are finite-dimensional.

(4) For any \( \mathcal{F}, \mathcal{G} \in \text{Coh}_G(\mathbb{P}^1) \), one has

\[
\text{Hom}_G(\mathcal{F}, \mathcal{G}(n)) = 0 = \text{Hom}_G(\mathcal{F}(-n), \mathcal{G}) \quad \text{for } n \ll 0,
\]

\[
\text{Ext}_G^i(\mathcal{F}, \mathcal{G}(n)) = 0 = \text{Hom}_G(\mathcal{F}(-n), \mathcal{G}) \quad \text{for } n \gg 0.
\]

As an immediate corollary of this, we see that the category \( \text{Coh}_G(\mathbb{P}^1) \) has the Krull–Schmidt property: every object \( \mathcal{F} \in \text{Coh}_G(\mathbb{P}^1) \) can be written as a direct
sum of indecomposable modules, and the multiplicities do not depend on the choice of such decomposition. It is also a hereditary category (recall that a category is called hereditary if $\text{Ext}^2(A, B) = 0$ for any objects $A, B$; in particular, this implies that a quotient of an injective object is injective, as can be easily seen from the long exact sequence of Ext groups).

We say that $F \in \text{Coh}_G(\mathbb{P}^1)$ is locally free if it is locally free as an $O$-module.

**Theorem 4.2.**

(1) Every locally free $F \in \text{Coh}_G(\mathbb{P}^1)$ is isomorphic to direct sum of sheaves of the form

$$X_i(n) = O(n) \otimes X_i,$$

where $X_i$ is an irreducible representation of $G$. (4.3)

(2) If $F \in \text{Coh}_G(\mathbb{P}^1)$ is locally free, then the functor $\otimes F: \text{Coh}_G(\mathbb{P}^1) \rightarrow \text{Coh}_G(\mathbb{P}^1)$ is exact.

(3) Every coherent $G$-equivariant sheaf has a resolution consisting of locally free sheaves.

(4) (Serre Duality.) For any two locally free sheaves $F, G$ we have isomorphisms

$$\text{Ext}^1_G(F, G(-2)) = \text{Ext}^1(F(2), G) = \text{Hom}_G(G, F)^*.$$

It immediately follows from computation of homology of $O(n)$ as coherent sheaves that

$$\text{Hom}_G(X_i(n), X_j(k)) = \begin{cases} \text{Hom}_G(X_i, V_{k-n} \otimes X_j), & k \geq n, \\ 0, & k < n; \end{cases}$$

$$\text{Ext}^1_G(X_i(n), X_j(k)) = \begin{cases} \text{Hom}_G(X_i, V_{n-k-2}^* \otimes X_j), & k \leq n-2, \\ 0, & k \geq n-1. \end{cases}$$

Finally, we note that the action of $(-I) \in \text{SU}(2)$ gives a decomposition of $\text{Coh}_G(\mathbb{P}^1)$ into even and odd part. Namely, since $-I$ acts trivially on $\mathbb{P}^1$, structure of $G$-equivariant sheaf gives an isomorphism $\lambda_{-I}: F \rightarrow (-I)^* F = F$. We define

$$\text{Coh}_G(\mathbb{P}^1)_p = \{ F \in \text{Coh}_G(\mathbb{P}^1): \lambda_{-I} = (-1)^p \}, \quad p \in \mathbb{Z}_2.$$

From now on, the main object of our study will be the category

$$\mathcal{C} = \text{Coh}_G(\mathbb{P}^1)_0.$$ (4.6)

In particular, a locally free sheaf $X_i(n) \in \mathcal{C}$ iff $n + p(X_i) \equiv 0 \mod 2$, where $p(X_i)$ is the parity defined by (1.2).

One easily sees that $\mathcal{C}$ is a full subcategory in $\text{Coh}_G(\mathbb{P}^1)$ closed under extensions. Equivalently, it can be described as follows.

**Lemma 4.3.** Let $\bar{G} = G/\{ \pm I \}$. Then $\mathcal{C}$ is naturally equivalent to the category $\text{Coh}_{\bar{G}}(\mathbb{P}^1)$ of $\bar{G}$-equivariant coherent sheaves on $\mathbb{P}^1$.

We will denote the bounded derived category of $\mathcal{C}$ by $D^b(\mathcal{C})$.

For future use, we will also need an analogue of induction functor. Recall that for any finite subgroup $H \subset G$, we have an induction functor $\text{Ind}^G_H: \text{Rep} H \rightarrow \text{Rep} G$. 

In particular, for \( H = \{1\} \) and the trivial representation \( C \) it gives the regular representation of \( G \):

\[
R = \text{Ind}_{\{1\}}^G C \simeq \bigoplus_{i \in I} d_i X_i, \quad d_i = \dim X_i.
\] (4.7)

As any representation of \( G \), \( R \) can be decomposed into even and odd part (cf. (1.1)): \( R = R_0 \oplus R_1 \), where

\[
R_p = \bigoplus_{i \in I_p} d_i X_i.
\] (4.8)

It is easy to check that \( R_0 \) is exactly the regular representation of \( \bar{G} \).

Similarly, we have an induction functor from the category of \( H \)-equivariant sheaves to \( G \)-equivariant sheaves; in particular, for \( H = \{1\} \), we get a functor \( \text{Coh}(\mathbb{P}^1) \to \text{Coh}_G(\mathbb{P}^1) \). However, it will be more convenient to consider the functor

\[
\text{Ind}: \text{Coh}(\mathbb{P}^1) \to \mathcal{C} = \text{Coh}_\bar{G}(\mathbb{P}^1), \quad F \mapsto \bigoplus_{g \in \bar{G}} g^* F.
\] (4.9)

The following lemma lists some of the properties of this functor; the proof is straightforward and left to the reader.

**Lemma 4.4.** Let \( \text{Ind}: \text{Coh}(\mathbb{P}^1) \to \mathcal{C} \) be defined by (4.9). Then for even \( n \), \( \text{Ind} \mathcal{O}(n) \) is naturally isomorphic as an equivariant sheaf to \( R_0 \otimes \mathcal{O}(n) \); for odd \( n \), \( \text{Ind} \mathcal{O}(n) \) is naturally isomorphic as an equivariant sheaf to \( R_1 \otimes \mathcal{O}(n) \).

**5. Auslander–Reiten Relations**

The following result will play a key role in the study of the category \( \text{Coh}_G(\mathbb{P}^1) \).

**Theorem 5.1.** For any \( n \in \mathbb{Z} \), \( i \in I \), there is short exact sequence in \( \text{Coh}_G(\mathbb{P}^1) \):

\[
0 \to X_i(n) \to \sum_j X_j(n+1) \to X_i(n+2) \to 0
\] (5.1)

where the sum is over all neighbors \( j \) of \( i \) in \( Q \).

We will call sequences of the form (5.1) Auslander–Reiten sequences.

**Proof.** Notice first that we have a short exact sequence of \( \mathcal{O} \)-modules:

\[
0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to \mathcal{O}(2) \to 0.
\]

Let us tensor it with \( X_i(n) \). Since tensoring with a locally free sheaf is an exact functor (Theorem 4.2), this gives short exact sequence

\[
0 \to X_i(n) \to (V \otimes X_i)(n+1) \to X_i(n+2) \to 0.
\] (5.2)

Since \( V \otimes X_i = \sum_j X_j \), we get the statement of the theorem. \( \square \)

**Remark 5.2.** For readers familiar with the general theory of Auslander–Reiten (AR) sequences (see [ARS] for the overview of the theory), we point out that our use of this name is justified: it can be shown, using Serre duality that these sequences do satisfy the usual definition of AR sequences (see [RVdB]). This, in particular,
shows that the Auslander–Reiten functor $\tau$ for locally free equivariant sheaves is given by

$$\tau(\mathcal{F}) = \mathcal{F}(-2).$$

However, we are not going to use the general theory of AR sequences in this paper.

**Corollary 5.3.** For any $n \in \mathbb{Z}$, $i \in I$ we have the following relations in the Grothendieck group $K(\text{Coh}_G(\mathbb{P}^1))$

$$[X_i(n)] - \sum_j [X_j(n + 1)] + [X_i(n + 2)] = 0 \quad (5.3)$$

where the sum is over all neighbors $j$ of $i$ in $Q$.

Later we will study the Grothendieck group $K(\text{Coh}_G(\mathbb{P}^1))$ in more detail, in particular showing that relations (5.3) is the full set of relations among classes $[X_i(n)]$ (see Corollary 9.2).

6. **Indecomposable Objects in $\text{Coh}_G(\mathbb{P}^1)$**

In this section, we describe the indecomposable sheaves in $\text{Coh}_G(\mathbb{P}^1)$. Again, these results are not new; they follow from results of [Len], [Sch1] who considers more general setting of “non-commutative projective curves”. So this section is included just for the reader’s convenience.

We start by describing torsion sheaves.

For every point $x \in \mathbb{P}^1$, $n \in \mathbb{Z}_+$, denote $\mathcal{O}_n[x] = \mathcal{O}/m_n^x$, where $m_n$ is the ideal of functions vanishing at $x$. This sheaf is supported at point $x$; choosing a local coordinate $z$ at $x$, we can identify the stalk of this sheaf at $x$ with $\mathbb{C}[z]/z^n\mathbb{C}[z]$.

It is well known that sheaves $\mathcal{O}_n[x]$ are indecomposable, and that every coherent $\mathcal{O}$-module is isomorphic to a direct sum of a locally free sheaf and sheaves of the form $\mathcal{O}_n[x]$.

Assume now that $x \in \mathbb{P}^1$ is such that $\text{Stab}_G(x) = \{ \pm I \}$ and thus $\text{Stab}_\bar{G} x = \{ 1 \}$ (recall that $\bar{G} = G/\{ \pm I \}$). Such points will be called generic. Define

$$\mathcal{O}_{n[Gx]} = \text{Ind} \mathcal{O}_n[x] = \bigoplus_{g \in \bar{G}} \mathcal{O}_{n[gx]} \quad (6.1)$$

where the functor Ind is defined by (4.9). This sheaf is supported on the orbit $Gx$ and has a canonical $G$-equivariant structure which can be lifted to a $\bar{G}$-equivariant structure. It is easy to see that the space of global sections $\Gamma(\mathcal{O}_{n[Gx]})$ is isomorphic to $n$ copies of the regular representation of $\bar{G}$.

This construction can be generalized to non-generic points.

**Theorem 6.1.** Let $x \in \mathbb{P}^1$ be non-generic: $H = \text{Stab}_G x \neq \{ 1 \}$. Then

1. $H$ is a cyclic group.
2. For suitable choice of the generator $\sigma$ of $H$, its action in the one-dimensional vector space $\mathcal{O}_n[x] = \mathcal{O}/m_n^x$ is given by $\sigma \mapsto e^{2\pi i/n}|H|$.
3. Consider the completed local ring $\hat{\mathcal{O}}_{\mathbb{P}^1,x}$; choosing a local coordinate $z$ at $x$ identifies this algebra with $\mathbb{C}[[z]]$. It has a natural action of $H$. Then we
have a natural equivalence of categories
\[
\{ \text{\(\bar{G}\)-equivariant coherent sheaves supported on \(Gx\)} \}
\simeq \{ \text{\(H\)-equivariant finite-dimensional modules over \(\mathcal{O}_{P^1,x}\)} \}
\]
given by
\[
\mathcal{F} \mapsto \text{stalk of } \mathcal{F} \text{ at } x.
\]

The proof of this theorem is straightforward and left to the reader.

Definition 6.2. Let \(H = \text{Stab}_{\bar{G}} x\), and let \(Y\) be a finite-dimensional representation of \(H\). Then we denote by \(\mathcal{O}_{n[Gx],Y}\) the \(\bar{G}\)-equivariant sheaf supported on the orbit \(Gx\) and whose stalk at \(x\) is isomorphic as an \(H\)-module to \(Y \otimes (\mathcal{O}_{P^1,x}/m^n_x)\).

It is easy to show that the space of global sections of \(\mathcal{O}_{n[Gx],Y}\) is isomorphic to \(\text{Ind}^G_H(Y \otimes (\mathcal{O}/m^n_x))\).

Theorem 6.3. (1) The following is the full list of indecomposable objects in \(C\):

- Torsion sheaves \(\mathcal{O}_{n[Gx]}\), where \(x \in P^1\), \(\text{Stab}_{\bar{G}} x = \{1\}\), \(n \in \mathbb{Z}_+\).
- Torsion sheaves \(\mathcal{O}_{n[Gx],Y}\), where \(x \in P^1\), \(H = \text{Stab}_{\bar{G}} x \neq \{1\}\), \(Y\) is an irreducible representation of \(H\).
- Free sheaves \(X_i(n)\), where \(n \in \mathbb{Z}\), \(i \in I\), \(n + p(i) \equiv 0 \mod 2\).

(2) Indecomposable objects in \(D^b(C)\) are of the form \(M[n]\), where \(M\) is an indecomposable object in \(C\), \(n \in \mathbb{Z}\).

(3) There are no injective and no projective indecomposable objects in \(C\).

Remark 6.4. Of course, sheaves \(\mathcal{O}_{n[Gx]}, H = \text{Stab}_{\bar{G}} x = \{1\}\), can be considered as special case of sheaves \(\mathcal{O}_{n[Gx],Y}\). However, we choose to list cases \(H = \{1\}\) and \(H \neq \{1\}\) separately.

Proof. It is known that every coherent sheaf \(\mathcal{F}\) has a maximal torsion subsheaf \(T\) so that \(\mathcal{F}/T\) is locally free, and short exact sequence \(0 \to T \to \mathcal{F} \to \mathcal{F}/T \to 0\) splits. If we additionally assume that \(\mathcal{F}\) is a \(\bar{G}\)-equivariant sheaf, then \(T\) and thus \(\mathcal{F}/T\) must also be \(\bar{G}\)-equivariant. It easily follows from Theorem 4.1 that then \(0 \to T \to \mathcal{F} \to \mathcal{F}/T \to 0\) splits in the category of \(\bar{G}\)-equivariant sheaves; thus, every coherent \(\bar{G}\)-equivariant sheaf is a direct sum of a free sheaf and a torsion sheaf.

By Theorem 4.2, indecomposable free sheaves are of the form \(X_i(n)\). For torsion sheaves, it is easy to see that an indecomposable torsion sheaf must be supported on an orbit \(Gx\). Now the classification of indecomposable torsion sheaves follows from Theorem 6.1.

Classification of indecomposable objects in \(D^b\) follows from [Hap, Lemma I.5.2].

His argument was given for the category of representations of a hereditary algebra, but actually works in any hereditary category: it only uses that a quotient of an injective object is injective, which easily follows from long exact sequence of Ext spaces and the fact that \(\text{Ext}^2(\mathcal{F}, \mathcal{G}) = 0\) for any \(\mathcal{F}, \mathcal{G}\).\hfill \Box
7. Auslander–Reiten Quiver \( \hat{Q} \)

We can now define a combinatorial structure which will play a crucial role in our paper. Let \( \hat{Q} \) be the set of isomorphism classes of locally free indecomposable objects in \( \mathcal{C} \); by Theorem 6.3, we can write

\[
\hat{Q} = \{(i, n) : n \in \mathbb{Z}, i \in I, n + p(i) \equiv 0 \mod 2\}
\] (7.1)

We will frequently use the notation

\[
X_q = X_i(n) = \mathcal{O}(n) \otimes X_i, \quad q = (i, n) \in \hat{Q}.
\]

We turn \( \hat{Q} \) into a quiver by defining, for \( q = (i, n) \), \( q' = (j, n + 1) \),

\[
\text{(Number of edges } q \rightarrow q') = \dim \text{Hom}_\mathcal{C}(X_q, X_{q'}) = \dim \text{Hom}_G(X_i, X_j \otimes V) = n(i, j)
\]

(recall that \( n(i, j) \) is the number of edges between \( i \) and \( j \) in \( Q \) and \( V = \Gamma(P^1, \mathcal{O}(1)) \) is the tautological representation of \( SU(2) \)).

Note that the edges only connect vertices with \( n \) differing by one, and all edges are directed “up” (i.e., so that \( n \) increases). The figure below shows an example of the quiver \( \hat{Q} \) when \( G \) is of type \( \hat{D}_7 \).

\[\text{Figure 2. Auslander–Reiten quiver } \hat{Q} \text{ for } Q = \hat{D}_7\]

Remark 7.1. Again, for users familiar with the Auslander–Reiten theory, we note that \( \hat{Q} \) is exactly the stable AR quiver of the category \( \text{Coh}_G(P^1) \). As was mentioned in the introduction, it is the picture of this quiver in Ocneanu’s work (who defined it purely combinatorially, without reference to the category of sheaves) that inspired the current paper.

The following easy lemma is left as an exercise to the reader.

Lemma 7.2. \( \hat{Q} \) is a connected quiver.
This is actually the reason for considering the even part \( C = \text{Coh}_G(\mathbb{P}^1)_0 \) rather than all of \( \text{Coh}_G(\mathbb{P}^1) \): otherwise, the resulting quiver would not be connected.

Note that we have a natural pairing

\[
\text{Hom}_G(X_i, X_j \otimes V) \otimes \text{Hom}_G(X_j, X_i \otimes V) \to \mathbb{C}
\]
given by composition

\[
X_i \xrightarrow{\varphi} X_j \otimes V \xrightarrow{\psi \otimes 1} X_i \otimes V \xrightarrow{1 \otimes \text{ev}} X_i
\]

where \( \text{ev}: V \otimes 2 \to \mathbb{C} \) is a \( G \)-equivariant pairing (which is defined uniquely up to a constant). This gives rise to duality

\[
\text{Hom}_G(X_i(n), X_j(n+1)) = \text{Hom}_G(X_j(n+1), X_i(n+2))^*.
\]

(7.2)

As usual in quiver theory, we define path algebra of \( \hat{Q} \) by

\[
\text{Path}_{\hat{Q}}(q, q') = \text{Span}(\text{paths from } q \text{ to } q' \text{ in } \hat{Q}),
\]

\[
\text{Path}_{\hat{Q}} = \bigoplus_{q, q' \in \hat{Q}} \text{Path}_{\hat{Q}}(q, q').
\]

(7.3)

One easily sees that \( \text{Path}_{\hat{Q}}((i, k), (j, l)) \) coincides with the space of paths in \( Q \) of length \( l - k \) between \( j \) and \( j \) (if \( l - k > 0 \); otherwise, \( \text{Path}_{\hat{Q}}((i, k), (j, l)) = 0 \). We also define an analogue of preprojective algebra by

\[
\Pi_{\hat{Q}} = \text{Path}_{\hat{Q}} / (\mathcal{N}), \quad \mathcal{N} = \sum_{e: q \to q'} e_1 e^*: q \to q(2),
\]

(7.4)

where for \( q = (i, n) \), we denote \( q(2) = (i, n+2) \), the sum is over all edges originating at \( q \), and \( e_1, e^* \) are dual bases in \( \text{Hom}_C(X_{q}, X_{q'}) \) and \( \text{Hom}_C(X_{q'}, X_{q(2)}) \) with respect to the pairing (7.2). (This algebra is isomorphic to Ocneam’s algebra of essential paths, or, in the terminology of AR quivers, to the mesh algebra of \( \hat{Q} \) considered as a polarized translation quiver.)

**Theorem 7.3.** Let \( q, q' \in \hat{Q} \). Then

\[
\text{Hom}_C(X_q, X_{q'}) = \Pi_{\hat{Q}}(q, q').
\]

**Proof.** Let \( \Pi_{Q} \) be the preprojective algebra of the graph \( Q \), and \( \Pi_{Q}' \) subspace of paths of length \( l \). Then for \( q = (i, k), q' = (j, l) \) it follows from the definition that for \( l < k \) we have

\[
\Pi_{Q}(q, q') = 0 = \text{Hom}_C(X_i(k), X_j(l))
\]

and for \( l \geq k \),

\[
\Pi_{Q}(q, q') = \Pi_{Q}'(X_i, X_j) = \text{Hom}_C(X_i, V \otimes (l-k) \otimes X_j) = \text{Hom}_C(X_i(k), X_j(l)).
\]

(\( \square \))

This theorem shows that morphisms between free indecomposable modules in \( C \) are described by essential paths in \( \hat{Q} \), so the structure of the subcategory \( \text{Free}(C) \), consisting of locally free sheaves, can be recovered from \( \hat{Q} \). Note however that realization of \( \text{Free}(C) \) in terms of equivariant sheaves gives much more than Hom spaces: it embeds \( \text{Free}(C) \) into an abelian category with sufficiently many injectives, allowing one to define Ext functors and derived category \( D^b(C) \). Both of these would be difficult to define in terms of \( \hat{Q} \).
8. Equivalence of Categories

Let $Q$, $\hat{Q}$ be as defined above, and let $h$ be a height function on $Q$ (see Definition 2.1). Recall that such a height function defines a choice of orientation on $Q$; the corresponding quiver is denoted $Q_h$. Then this height function defines an embedding

$$i_h: Q_h^\text{opp} \rightarrow \hat{Q}, \quad i \mapsto (i, h_i)$$

where $Q_h^\text{opp}$ is the quiver obtained from $Q_h$ by reversing all arrows. The figure below shows an example of such an embedding.

![Example of embedding](image)

Remark 8.1. Note that once we have embedded $Q_h \subset \hat{Q}$, we can identify the quiver $\hat{Q}$ with $\mathbb{Z}Q_h$ (see, e.g., [R1] for the definition).

From now on, we will consider $Q_h^\text{opp}$ as a subset of $\hat{Q}$, omitting $i_h$ in our notation.

Definition 8.2. Let $q_1, q_2 \in \hat{Q}$. We say that $q_1$ is a predecessor of $q_2$ (notation: $q_1 \prec q_2$) if there exists a path $q_1 \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow q_2$ in $\hat{Q}$. In this case, we also say that $q_2$ is a successor of $q_1$ and write $q_2 \succ q_1$.

Lemma 8.3. Let $q = (i, n) \in \hat{Q}$. Then

1. $n > h_i$ if and only if $q \not\in Q_h$, $q \prec q'$ for some $q' \in Q_h$. In this case, we will say that $q$ is above $Q_h$ and write $q > Q_h$.
2. $n < h_i$ if and only if $q \not\in Q_h$, $q \prec q'$ for some $q' \in Q_h$. In this case, we will say that $q$ is below $Q_h$ and write $q < Q_h$.

Lemma 8.4. (1) If $q_1 \succeq Q_h$, $q_2 \prec Q_h$, then $\text{Hom}_C(X_{q_1}, X_{q_2}) = 0$.

(2) If $q_1 \preceq Q_h$, $q_2 \succeq Q_h$, then $\text{Ext}^1_C(X_{q_1}, X_{q_2}) = 0$. In particular, if $q_1, q_2 \in Q_h$, then $\text{Ext}^1_C(X_{q_1}, X_{q_2}) = 0$.

Proof. It follows from $\text{Hom}_C(X_{q_1}, X_{q_2}) = \Pi\hat{Q}(X_{q_1}, X_{q_2})$ that $\text{Hom}_C(X_{q_1}, X_{q_2})$ can be nonzero only if $q_1 \preceq q_2$. But then $q_2 \succeq q_1 \succeq Q_h$, which proves part (a).

Part (b) follows from part (a) and Serre duality (Theorem 4.2).
Lemma 8.5. The sheaves $X_q$, $q \in Q_h$, generate the category $D^b(C)$ as a triangulated category: the smallest full triangulated subcategory in $D^b(C)$ containing all $X_q$, $q \in Q_h$, is $D^b(C)$.

Proof. Let $D$ be the smallest triangulated subcategory in $D^b(C)$ containing all $X_q$, $q \in Q_h$. If $i$ is a sink for $Q_h$, then it follows from AR exact sequence (5.1) that $X_i(h_i + 2) \in D$. Thus, all $X_q$, $q \in Q_h$, are in $D$. Similarly, if $i$ is a source for $Q_h$, then all $X_q$, $q \in Q_{-h_i}$, are in $D$. By Lemma 2.3, this implies that all $X_q$, $q \in \hat{Q}$, are in $D$, so $D$ contains all locally free equivariant sheaves. Since every coherent sheaf admits a locally free resolution, this completes the proof. 

Define now a functor
\[
\Phi_h : C \to \text{Rep}(Q_h) \quad (8.2)
\]
by
\[
\Phi_h(F)(i) = \text{Hom}_C(X_i(h_i), F)
\]
and for an edge $e : j \to j$ in $Q_h$, we define the corresponding map by
\[
\text{Hom}_C(X_j(h_j), F) \to \text{Hom}_C(X_i(h_i), F), \quad \varphi \mapsto \varphi \circ h(e)
\]
where $h(e) : (i, h_i) \to (j, h_j)$ is the edge in $\hat{Q}$ corresponding to $e$ under embedding (8.1).

Lemma 8.6. (1) $\Phi_h$ is left exact.
(2) $R^i\Phi_h = 0$ for $i > 1$, and
\[
(R^1\Phi_h(F))(i) = \text{Ext}_{C}^{1}(X_i(h_i), F).
\]

Proof. Follows from left exactness of functor $\text{Hom}_C(X_i(h_i), -)$ and definitions. 

Example 8.7. Let $h$ be the “standard” height function: $h(i) = 0$ for $i \in I_0$, $h(i) = 1$ for $i \in I_1$. Then the map $\Phi_h$ can also be described as follows. Consider the functor $\text{Rep} Q_h \to \text{Rep} G$ defined by \[V_i \mapsto \bigoplus V_i \otimes X_i.\] Then the composition
\[
\mathcal{C} \overset{\Phi_h}{\longrightarrow} \text{Rep} Q_h \to \text{Rep} G
\]
is given by
\[
\mathcal{F} \mapsto \Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{F}(-1)).
\]

Indeed, for each representation $M$ of $G$ we have
\[
M \simeq \bigoplus_{i \in I} \text{Hom}_G(X_i, M) \otimes X_i.
\]
Applying it to $\Gamma(\mathcal{F}) = \text{Hom}_C(O, \mathcal{F})$ we get
\[
\Gamma(\mathcal{F}) = \bigoplus_{i \in I} \text{Hom}_C(X_i \otimes O, \mathcal{F}) \otimes X_i = \bigoplus_{i \in I} \text{Hom}_C(X_i \otimes O, \mathcal{F}) \otimes X_i.
\]
Since $\mathcal{F} \in \mathcal{C} = \text{Coh}_G(\mathbb{P}^1)_h$, we have $\text{Hom}_G(X_i \otimes O, \mathcal{F}) = 0$ for $i \in I_1$; thus,
\[
\Gamma(\mathcal{F}) = \bigoplus_{i \in I_0} \text{Hom}_G(X_i \otimes O, \mathcal{F}) \otimes X_i.
\]

Similar argument shows that
\[
\Gamma(\mathcal{F}(-1)) = \bigoplus_{i \in I_1} \text{Hom}_G(X_i \otimes O(1), \mathcal{F}) \otimes X_i.
\]
Thus,
\[ \Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{F}(-1)) = \left( \bigoplus_{i \in I_0} \text{Hom}_C(X_i, \mathcal{F}) \otimes X_i \right) \oplus \left( \bigoplus_{i \in I_1} \text{Hom}_C(X_i(1), \mathcal{F}) \otimes X_i \right) \]
as desired.

Since \( \Phi_h \) is left exact, we can define the associated derived functor \( R\Phi_h : \mathcal{D}^b(C) \to \mathcal{D}^b(\text{Rep} \, Q_h) \). The following two theorems are the main results of this paper.

**Theorem 8.8.** For any height function \( h \), the functor \( R\Phi_h : \mathcal{D}^b(C) \to \mathcal{D}^b(Q_h) \) defined by (8.2) is an equivalence of triangulated categories.

**Proof.** It follows from Lemma 8.4 that the object \( T = \bigoplus_{q \in Q_h} X_q \) satisfies \( \text{Ext}^1(T, T) = 0 \). By Lemma 8.5, direct summands of \( T \) generate \( \mathcal{D}^b(C) \) as a triangulated category. Therefore, \( T \) is the tilting object in \( \mathcal{D}^b(C) \) as defined in [Hap], [GL]. By general result of [GL], functor \( \Phi : \mathcal{D} \mapsto \text{Hom}(T, \mathcal{F}) \) gives rise to equivalence of corresponding derived categories \( \mathcal{D}^b(C) \approx \mathcal{D}^b(\text{mod} \, -A) \), where \( A = \text{End}(T) \). But by Theorem 7.3, \( A \) is exactly the path algebra of \( Q_h^\text{opp} \), so the category of right \( A \)-modules is equivalent to the category of representations of quiver \( Q_h \), which completes the proof. \( \square \)

**Theorem 8.9.** Let \( i \) be a sink for \( h \), and \( S_i^+ \) the reflection functor defined by (3.1). Then the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{D}^b(C) & \xrightarrow{R\Phi_h} & \mathcal{D}^b(Q_h) \\
\downarrow & & \downarrow \\
\mathcal{D}^b(Q_{s_i^+h}) & \xrightarrow{RS_i^+} & \mathcal{D}^b(Q_h)
\end{array}
\]

and similarly for \( S_i^- \).

**Proof.** Let us first prove that if \( q \succ Q_h \), then
\[ R\Phi_{s_i^+h}(X_q) = RS_i^+ \circ R\Phi_h(X_q). \]
Indeed, in this case it follows from Lemma 8.6, Lemma 8.4 that \( R^i \Phi_h(X_q) = 0 \) for \( i > 0 \), so \( R\Phi_h(X_q) = \Phi_h(X_q) \); similarly, \( R\Phi_{s_i^+h}(X_q) = \Phi_{s_i^+h}(X_q) \).

Let \( i \) be a sink for \( Q_h \). Then we have the Auslander–Reiten exact sequence
\[ 0 \to X_i(h_i) \to \bigoplus_j X_{i+2}(h_j) \to X_i(h_i + 2) \to 0. \]
Applying to this sequence functor \( \text{Hom}_C(-, X_q) \) and using Lemma 8.4 which gives vanishing of \( \text{Ext}^3 \), we get a short exact sequence
\[ 0 \to \text{Hom}_C(X_i(h_i + 2), X_q) \to \bigoplus_j \text{Hom}_C(X_j(h_j), X_q) \to \text{Hom}_C(X_i(h_i), X_q) \to 0. \]
Comparing this with definition of $S^+_i$, we see that $\Phi^+_i h(X_q) = S^+_i h(X_q)$.

Thus, we have shown that for $q \succ Q_h$, we have $R\Phi^+_i h(X_q) = RS^+_i \circ R\Phi h(X_q)$.

To complete the proof, we now use the following easy result.

**Lemma 8.10.** Let $D, D'$ be triangulated categories, $\Phi_1, \Phi_2 : D \to D'$ triangle functors, and $\alpha : \Phi_1 \to \Phi_2$ a morphism of functors. Assume that there is a collection of objects $X_q \in D$ which generate $D$ as a triangulated category in the sense of Lemma 8.5 and such that for each $X_q$, $\alpha : \Phi_1(X_q) \to \Phi_2(X_q)$ is an isomorphism. Then $\alpha$ is an isomorphism of functors.

Applying this lemma to $D = D^b(C)$, $D' = D^b(Q_h^+, \iota)$, $\Phi_1 = R\Phi^+_i h$, $\Phi_2 = R\Phi h$ and the collection of objects $X_q, q \succ Q_h$ (which generate $D^b(C)$ by Lemma 8.5), we get the statement of the theorem. □

**Corollary 8.11.** We have the following commutative diagram

$\begin{array}{ccc}
D^b(C) & \xrightarrow{R\Phi_h} & D^b(Q_h) \\
\downarrow \rotate[0]{\circ \gamma(-2)} & & \downarrow R\Phi^+_h \\
D^b(C) & \xrightarrow{R\Phi_h} & D^b(Q_h) \\
\end{array}$

where $R\Phi_h$ is an equivalence of derived categories, $\Phi_h$ is an equivalence of abelian categories. For example, there are no injective or projective objects in $C$ while there are enough injectives and projectives in $\text{Rep} Q_h$. Similarly, the set of simple modules is very much different in these two categories.

9. The Root System and the Grothendieck Group

In this section, we list some important corollaries of the equivalence of categories constructed in the previous section. Again, many of these results can also be obtained form the equivalence of categories constructed by Lenzing; however, we choose to provide an independent exposition.

Throughout this section, we let $L = K(C)$ be the Grothendieck group of category $C$. We define the inner product on $L$ by

$$(x, y) = \langle x, y \rangle + \langle y, x \rangle$$

where by definition

$$(\langle X], [Y] \rangle = \dim \text{RHom}(X, Y) = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y)$$

(compare with Theorem 3.1). We define $\Delta \subset L$ by

$$\Delta = \{ [X], X \text{ is a non-zero indecomposable object in } D^b(C) \}.$$  \hspace{1cm} (9.2)

Finally, we define the map $C : \Delta \to \Delta$ by

$$C([X]) = [X(-2)].$$
Theorem 9.1. (1) The set $\Delta \subset L$ is an affine root system, and $C$ is a Coxeter element.

(2) Recall the lattice $L(Q)$ and root system $\Delta(Q)$ from Section 1. Then for any height function $h$ on $Q$, the map

$$R\Phi_h : L \to L(Q), \quad [\mathcal{F}] \mapsto \bigoplus (\dim \text{RHom}_C(X_i(h)), [\mathcal{F}]) \alpha_i$$

(9.3)

is an isomorphism of abelian groups which identifies $\Delta \subset L$ with $\Delta(Q) \subset L(Q)$ and $C$ with the Coxeter element $c_h^+$ defined by (3.3).

Proof. The first part follows from the second one; the second part follows from the fact that $R\Phi_h$ is an equivalence of categories (Theorem 8.8 and Corollary 8.11). □

Corollary 9.2. (1) For any height function $h$, the classes $[X_q], q \in Q_h$, are free generators of the Grothendieck group $K(C)$.

(2) $K(C)$ is generated by the classes $[X_i(n)], n \equiv p(i) \mod 2$, and $AR$ relations (5.3) is the full list of all relations among classes $[X_i(n)]$.

Proof. By Lemma 8.5, classes $[X_q], q \in Q_h$, generate $K(C)$. On the other hand, by Theorem 9.1, $K(C)$ is isomorphic to $\mathbb{Z}^I$ and thus has rank $|I|$, so these generators must be linearly independent.

To prove the second part, denote temporarily $L' = \text{Span}([X_i(n)])/J$, where $J$ is the subgroup generated by $AR$ relations (5.3). Since $[X_i(n)]$ generate $K(C)$ and $AR$ relations hold in $K(C)$, we see that $K(C)$ is a quotient of $L'$.

Now choose some height function $h$. It follows from Lemma 2.3 that $[X_q], q \in Q_h$, generate $L'$, so it has rank at most $|I|$. On the other hand, $K(C)$ has rank $|I|$. Thus, $L'$ has rank $|I|$ and $L' = K(C)$. □

Example 9.3. Let $h$ be the “standard” height function as defined in Example 8.7. Then the map $R\Phi_h : K(C) \to L(Q) \simeq K(G)$ is given by

$$[X_i] \mapsto \alpha_i, \quad i \in I_0; \quad [X_i(1)] \mapsto \alpha_i + \sum_{j=1} \alpha_j, \quad i \in I_1,$$

and the corresponding Coxeter element is $C = (\prod_{i \in I_1} s_i)(\prod_{i \in I_0} s_i)$.

This also implies that for this $h$, $[X_i(-1)] \mapsto -\alpha_i, i \in I_1$; in particular, we see that classes $\alpha_i, i \in I_0$, and $-\alpha_i, i \in I_1$, form a set of representatives of $C$-orbits on $Q$. An analogous statement for finite root system has been proved in [Kos2].

For completeness, we also describe here the indecomposable sheaves corresponding to imaginary roots of $\Delta$.

Theorem 9.4. Let $x \in \mathbb{P}^1$ be a generic point: $\text{Stab}_G x = \{1\}$, and let

$$\delta = [\mathcal{O}_{\mathbb{P}^1}] \in \Delta$$

(see (6.1)). Then:

(1) $\delta$ does not depend on the choice of point $x$. 

(2) \( \delta = \delta_0 - \delta_1 \), where
\[
\delta_0 = \sum_{i \in I_0} d_i [X_i] = [R_0],
\]
\[
\delta_1 = \sum_{i \in I_1} d_i [X_i(-1)] = [R_1(-1)],
\]
where \( d_i = \dim X_i \) and \( R_0, R_1 \) are even and odd parts of the regular representation defined by (4.8).

(3) \( C\delta = \delta \); \( C\delta_0 = \delta_0 - 2\delta \); \( C\delta_1 = \delta_1 - 2\delta \).

(4) For any \( \alpha \in L \), \( \langle \delta, \alpha \rangle = 0 \).

(5) \( \delta \) is a generator of the set of imaginary roots of \( \Delta \):
\[
\Delta^{im} = 2\delta.
\]

Proof. We start by proving (2), by explicitly constructing a resolution of \( O_{Gx} \).
Let \( \varphi_x \) be a holomorphic section of \( O(1) \) which has a simple zero at \( x \). Then we have a short exact sequence of sheaves (not equivariant):
\[
0 \rightarrow O \xrightarrow{\varphi_x} O(1) \rightarrow O_{Gx} \rightarrow 0.
\]
Tensoring it with \( O(-1) \), we get a sequence
\[
0 \rightarrow O(-1) \xrightarrow{\varphi_x} O \rightarrow O_{Gx} \rightarrow 0.
\]
Now let us apply to this sequence functor \( \text{Ind} \) defined by (4.9). Using Lemma 4.4, we see that it gives a \( \bar{G} \)-equivariant short exact sequence
\[
0 \rightarrow R_1 \otimes O(-1) \rightarrow R_0 \otimes O \rightarrow O_{Gx} \rightarrow 0
\]
which gives equality \( \delta = \delta_0 - \delta_1 \), thus proving part (2) of the theorem.

Part (1) follows from (2).
Since \( O_{Gx} \otimes O(-2) \simeq O_{Gx} \), we get \( C\delta = \delta \). To compute \( C\delta_0 \), note that the same argument as in the proof of part (2) also gives a short exact sequence
\[
0 \rightarrow R_0 \otimes O \rightarrow R_1 \otimes O(1) \rightarrow O_{Gx} \rightarrow 0
\]
thus giving \( C^{-1}\delta_1 = \delta + \delta_0 = 2\delta + \delta_1 \).

To prove part (4), recall notation \( \langle [X], [Y] \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y) \).
Then Serre duality immediately gives
\[
\langle x, y \rangle = -\langle y, Cx \rangle.
\]
Since \( C\delta = \delta \), we get
\[
\langle \delta, x \rangle = \langle x, \delta \rangle + \langle \delta, x \rangle = \langle x, \delta \rangle - \langle x, C\delta \rangle = 0.
\]

Part (4) implies that \( \delta \) is an imaginary root. Moreover, since by Corollary 9.2 classes \( [X_i], [X_i(-1)] \) are free generators of \( L \). Since some of \( d_i \) are equal to 1, it follows from part (2) that for any \( k > 1 \), \( \delta/k \notin L \); thus, \( \delta \) must be a generator of the set of imaginary roots.

Since every indecomposable object in \( D^b(C) \) is of the form \( F[n] \), where \( F \) is an indecomposable \( G \)-equivariant sheaf (Theorem 6.3), it follows that every root \( \alpha \in \Delta \) can be written as either \( \alpha = [F] \) or \( \alpha = -[F] \), thus giving some splitting of \( \Delta \) into positive and negative roots. This polarization can be described explicitly.
Recall from algebraic geometry (see, e.g., [Har, Exercises II.6.10, II.6.12]) that for any coherent sheaf, we can define two integer numbers, its rank and degree. In particular, for a locally free sheaf \( \mathcal{F} = X \otimes \mathcal{O}(n) \) (where \( X \) is a finite-dimensional vector space), we have
\[
\text{rk}(X \otimes \mathcal{O}(n)) = \dim X, \\
\text{deg}(X \otimes \mathcal{O}(n)) = n \dim X.
\]
Degree and rank give well-defined linear maps \( K \to \mathbb{Z} \), where \( K \) is the Grothendieck group of the category of coherent sheaves.

In particular, we can define rank and degree for a \( G \)-equivariant sheaf, ignoring the equivariant structure; this gives linear maps \( K(\mathcal{C}) \to \mathbb{Z} \), which we also denote by \( \text{rk}, \text{deg} \).

**Lemma 9.5.** (1) If \( \mathcal{F} \in \mathcal{C} \) is a non-zero free sheaf, then \( \text{rk} \mathcal{F} > 0 \).
(2) If \( \mathcal{F} \in \mathcal{C} \) is a non-zero torsion sheaf, then \( \text{rk} \mathcal{F} = 0, \text{deg} \mathcal{F} > 0 \).
(3) For any \( x \in K(\mathcal{C}) \), we have
\[
\text{rk}(x) = (x, \delta_0) = (x, \delta_1)
\]
where \( \delta_0, \delta_1 \) are defined in Theorem 9.4.

**Proof.** The first two parts are well-known.

To check \( \text{rk}(x) = (x, \delta_0) = (x, \delta_1) \), it suffices to check it for \( x = \{X_i\}, i \in I_0 \), and \( x = \{X_i(-1)\}, i \in I_1 \) (by Example 9.3, they generate \( L \)). If \( i \in I_0 \), then
\[
([X_i], \delta_0) = \dim \text{Hom}_G(X_i, \sum_{j \in I_0} d_j X_j) = d_i = \text{rk} X_i
\]
thus proving \( \text{rk}(x) = (x, \delta_0) \). Since \( \delta_0 - \delta_1 = \delta \) is in the kernel of \((\ , \ )\) (Theorem 9.4), this implies \( (x, \delta_0) = (x, \delta_1) \).

If \( i \in I_1 \), then
\[
([X_i(-1)], \delta_1) = \dim \text{Hom}_G(X_i(-1), \sum_{j \in I_1} d_j X_j(-1)) = d_i = \text{rk} X_i
\]
thus proving \( \text{rk}(x) = (x, \delta_1) \). Since \( \delta_0 - \delta_1 = \delta \) is in the kernel of \((\ , \ )\) (Theorem 9.4), this implies \( (x, \delta_0) = (x, \delta_1) \). \( \square \)

Note that the functional \( \text{deg} x \) can not be written in terms of the form \((\ , \ )\): indeed, \( \text{deg} \delta = |\bar{G}| = |G|/2 \), but \((\delta, \cdot) = 0 \).

**Theorem 9.6.** Let \( \alpha \in \Delta \).

(1) \( \alpha = [\mathcal{F}] \) for some indecomposable free sheaf \( \mathcal{F} \in \mathcal{C} \) if and only if \( \text{rk} \alpha = (\alpha, \delta_0) > 0 \).
(2) \( \alpha = -[\mathcal{F}] \) for some indecomposable free sheaf \( \mathcal{F} \in \mathcal{C} \) if and only if \( \text{rk} \alpha = (\alpha, \delta_0) < 0 \).
(3) \( \alpha = [\mathcal{F}] \) for some indecomposable torsion sheaf \( \mathcal{F} \in \mathcal{C} \) if and only if \( \text{rk} \alpha = (\alpha, \delta_0) = 0, \text{deg} \alpha > 0 \).
(4) \( \alpha = -[\mathcal{F}] \) for some indecomposable torsion sheaf \( \mathcal{F} \in \mathcal{C} \) if and only if \( \text{rk} \alpha = (\alpha, \delta_0) = 0, \text{deg} \alpha < 0 \).
Thus, we see that we have a triangular decomposition of $\Delta$:

$$\Delta = \Delta_+ \cup \Delta_0 \cup \Delta_-,$$

$$\Delta_+ = \{ \alpha \in \Delta : \text{rk}(\alpha) > 0 \} = \{ [\mathcal{F}] , \mathcal{F} \text{ is a free indecomposable sheaf} \} ,$$

$$\Delta_- = \{ \alpha \in \Delta : \text{rk}(\alpha) < 0 \} = \{ -[\mathcal{F}] , \mathcal{F} \text{ is a free indecomposable sheaf} \} ,$$

$$\Delta_0 = \{ \alpha \in \Delta : \text{rk}(\alpha) = 0 \} = \{ \pm [\mathcal{F}] , \mathcal{F} \text{ is a free torsion sheaf} \} . \quad (9.4)$$

(note that $\Delta_+$ is exactly the set of vertices of $\hat{Q}$). The set $\Delta_+$ has been discussed by Schiffmann [Sch2], who denoted the corresponding subalgebra in the loop algebra by $\mathcal{L}n$. Note, however, that this notation is somewhat misleading: $\mathcal{L}n$ is not the loop algebra of a positive part of the finite dimensional algebra $\bar{g}$, which easily follows from the fact that there are real roots in $\Delta_0$ (see example in the next section).

**Theorem 9.7.** Let $g = |\hat{G}| = |\bar{G}|/2$. Then for any $x \in L$ we have

$$C^g(x) = x - 2(\text{rk } x)\delta. \quad (9.5)$$

**Proof.** Let $\varphi_x$ be a section of the sheaf $\mathcal{O}(1)$ which has a single zero at generic point $x$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\varphi_x} \mathcal{O}(1) \rightarrow \mathcal{O}_x \rightarrow 0$$

which, however, is not equivariant even under the action of $\{ \pm I \} \subset \text{SU}(2)$. To fix it we consider $\varphi_x^2$ which gives the following $\mathbb{Z}_2$-equivariant sequence

$$0 \rightarrow \mathcal{O}(-2) \xrightarrow{\varphi_x^2} \mathcal{O} \rightarrow \mathcal{O}_{2|x} \rightarrow 0.$$ 

Now let us take product of pullbacks of $\varphi_x^2$ under all $g \in \hat{G}$

$$0 \rightarrow \mathcal{O}(-2g) \xrightarrow{\prod_{s \in \hat{g}} g^r \varphi_x^2} \mathcal{O} \rightarrow \mathcal{O}_{2\hat{|G|x}} \rightarrow 0.$$ 

Tensoring it with any locally free sheaf $\mathcal{F}$, we get

$$0 \rightarrow \mathcal{F}(-2g) \xrightarrow{\prod_{s \in \hat{g}} g^r \varphi_x^2} \mathcal{F} \rightarrow \mathcal{O}_{2\hat{|G|x}} \otimes \mathcal{F}_x \rightarrow 0$$

which implies $C^g[\mathcal{F}] - [\mathcal{F}] + 2(\text{rk } \mathcal{F})\delta = 0. \quad \square$

This result—that $C^g$ is a translation—was known before and can be proved without the use of equivariant sheaves, see, e.g., Steinberg [Ste]. However, the approach via equivariant sheaves also provides a nice interpretation for the corresponding functional as the rank of the sheaf.

Comparing (9.5) with the description of the action of Coxeter element in the language of representations of the quiver, we see that rank is closely related to the notion of defect $\partial_x(x)$ as defined in [DR], namely

$$\text{rk}(x) = -\frac{1}{2} \partial_x(x).$$

Therefore, $\Delta_0$ is exactly the set of indecomposable objects of defect zero, which shows that torsion sheaves correspond to regular representations.

**Corollary 9.8.** (1) For any $\alpha \in \Delta_0$, $C^g x = x$; in particular, $C$-orbit of $\alpha$ is finite.

(2) $C$ acts freely on $\Delta_+$, and the set of orbits $\Delta_+/C$ is naturally in bijection with $Q$. 
10. Example: $\tilde{A}_n$

In this section, we consider the example of the cyclic group of even order: $G = \mathbb{Z}_n$, $n = 2k$.

The irreducible representations of this group are $X_i$, $i \in \mathbb{Z}_n$; all of them are one-dimensional. The corresponding Dynkin diagram $Q$ is the cycle of length $n$.

The root system $\Delta(Q)$ can be described as follows. Let $V$ be a real vector space of dimension $n + 1$, with basis $\delta, e_i, i \in \mathbb{Z}_n$. Define inner product in $V$ by $(e_i, e_j) = \delta_{ij}$, $(\nu, \delta) = 0$. Then

$$\Delta(Q) = \left\{ e_i - e_j + a\delta, i \neq j \in \mathbb{Z}_n, a \in \frac{j-i}{n} + \mathbb{Z} \right\}. \quad (10.1)$$

The simple roots are

$$\alpha_i = e_i - e_{i+1} + \frac{1}{n}\delta, \ i \in \mathbb{Z}_n$$

so that $\sum \alpha_i = \delta$. The simple reflections $s_i$ are given by

$$s_i(e_i) = e_{i+1} - \frac{1}{n}\delta, \quad s_i(e_{i+1}) = e_i + \frac{1}{n}\delta, \quad s_i(e_j) = e_j, \ j \neq i, i + 1.$$ 

It is easy to see that this description of $\Delta(Q)$, while unusual, is equivalent to the standard description of the affine root system $\tilde{A}_{n-1}$.

We choose standard height function $h$:

$$h(i) = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}. \end{cases} \quad (10.2)$$

The corresponding Coxeter element $C$ is

$$C = \left( \prod_{i \text{ odd}} s_i \right) \left( \prod_{i \text{ even}} s_i \right).$$

The action of $C$ on $\Delta(Q)$ is given by

$$C(e_i) = \begin{cases} e_{i+2} - \frac{2}{n}\delta, & i \text{ even}, \\ e_{i-2} + \frac{2}{n}\delta, & i \text{ odd}. \end{cases}$$

Thus, we have

$$C^{n/2}(e_i) = \begin{cases} e_i - \delta, & i \text{ even}, \\ e_i + \delta, & i \text{ odd}, \end{cases}$$

which implies

$$C^{n/2}(\alpha) = \alpha - (\alpha, \varepsilon)\delta, \quad \varepsilon = \sum_{i \text{ even}} e_i - \sum_{i \text{ odd}} e_i \equiv \sum_{i \text{ even}} \alpha_i \equiv - \sum_{i \text{ odd}} \alpha_i \mod \mathbb{Z}\delta$$

(compare with Theorem 9.7, Lemma 9.5).
Explicitly, we can write

\[ C_{n/2}(\alpha) = \begin{cases} 
\alpha, & i \equiv j \mod 2, \\
\alpha - 2\delta, & (i, j) \equiv (0, 1) \mod 2, \\
\alpha + 2\delta, & (i, j) \equiv (1, 0) \mod 2,
\end{cases} \]

Thus, in this case we have

\[ \Delta_0 = \{ e_i - e_j + a\delta \}, \quad i \equiv j \mod 2, \]
\[ \Delta_+ = \{ e_i - e_j + a\delta \}, \quad i \text{ even, } j \text{ odd}, \]
\[ \Delta_- = \{ e_i - e_j + a\delta \}, \quad i \text{ odd, } j \text{ even}. \]

Figure 3 shows a segment of the AR graph \( \hat{Q} \) for this root system.

**Figure 3.** Fragment of graph \( \hat{Q} \) for root system of type \( A \). Here \( i \) is an even number. The figure also shows, for each \( X_q \in \hat{Q} \), the image of \( R\Phi_h[X_q] \) for the standard choice of height function \( h \) as in Example 8.7.

**References**


MCKAY CORRESPONDENCE AND EQUIVARIANT SHEAVES ON $\mathbb{P}^1$


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