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Dedicated to Professor A. A. Kirillov on the occasion of his 70th birthday

ABSTRACT. We prove a version of the Poincaré–Birkhoff–Witt theorem for the twisted quantized enveloping algebra $U'_q(sp_{2n})$. This is a subalgebra of $U_q(g_{2n})$ and a deformation of the universal enveloping algebra $U(sp_{2n})$ of the symplectic Lie algebra. We classify finite-dimensional irreducible representations of $U'_q(sp_{2n})$ in terms of their highest weights and show that these representations are deformations of finite-dimensional irreducible representations of $sp_{2n}$.


Key words and phrases. Quantized enveloping algebra, symplectic Lie algebra, representation.

1. Introduction

Let $g$ denote the orthogonal or symplectic Lie algebra $o_N$ or $sp_{2n}$ over the field of complex numbers. There are at least two different $q$-analogues of the universal enveloping algebra $U(g)$. These are the quantized enveloping algebra $U_q(g)$ introduced by Drinfeld [3] and Jimbo [11] and the twisted (or nonstandard) quantized enveloping algebra $U'_q(g)$ introduced by Gavriliuk and Klimyk [6] for $g = o_N$ and by Noumi [19] for $g = sp_{2n}$. If $q$ is a complex number that is neither zero nor a root of unity, then the representation theory of $U_q(g)$ is very much similar to that of the Lie algebra $g$; e.g., see Chari and Pressley [2, Chap. 10]. The description of the finite-dimensional irreducible representations of $U'_q(o_N)$ given by Iorgov and Klimyk [10] both exhibits similarity with the classical theory and reveals some new phenomena specific to the quantum case.

In this paper, we are concerned with the description of finite-dimensional irreducible representations of the twisted quantized enveloping algebra $U'_q(sp_{2n})$. We introduce a class of highest weight representations for this algebra and show that any finite-dimensional irreducible representation of $U'_q(sp_{2n})$ is highest weight. Then we give necessary and sufficient conditions on a highest weight representation to

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be finite-dimensional. By the main theorem (Theorem 6.3), the finite-dimensional irreducible representations of $U_q'(\mathfrak{sp}_{2n})$ are naturally parameterized by the $n$-tuples

$$\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n}),$$

where $m_1 \leq m_2 \leq \cdots \leq m_n$ are positive integers and each $\sigma_i$ is equal to 1 or $-1$.

Regarding this parametrization, the algebra $U_q'(\mathfrak{sp}_{2n})$ appears to be closer to the quantized enveloping algebra $U_q(\mathfrak{sp}_{2n})$ than to its orthogonal counterpart $U_q(\mathfrak{o}_N)$.

We work with the presentation of $U_q'(\mathfrak{sp}_{2n})$ introduced in [18], which is a slight modification of Noumi’s definition [19]. The defining relations are written in matrix form as a reflection equation for the matrix of generators. A version of the Poincaré–Birkhoff–Witt theorem for this algebra over the field $\mathbb{C}(q)$ was proved in [18]. Here we consider $q$ to be a nonzero complex number such that $q^2 \neq 1$ and prove the corresponding theorem for $U_q'(\mathfrak{sp}_{2n})$ (Theorem 3.6), relying on the Poincaré–Birkhoff–Witt theorem for the quantized enveloping algebra $U_q(\mathfrak{gl}_N)$.

Note that a similar reflection equation presentation exists for the algebra $U_q'(\mathfrak{o}_N)$. These presentations were derived in [19] by regarding $U_q'(\mathfrak{o}_N)$ and $U_q'(\mathfrak{sp}_{2n})$ as subalgebras of $U_q(\mathfrak{gl}_N)$ for appropriate $N$; see also Gavrilik, Iorgov, and Klimyk [5].

These subalgebras are coideals with respect to the coproduct on $U_q(\mathfrak{gl}_N)$. A more general description of coideal subalgebras of quantized enveloping algebras was given by Letzter [15, 16]. We outline a new proof of the Poincaré–Birkhoff–Witt theorem for $U_q'(\mathfrak{o}_N)$, similar to the symplectic case (see Remark 3.8 below); cf. Iorgov and Klimyk [9].

The algebra $U_q'(\mathfrak{o}_N)$ and its representations were studied by many authors. In particular, it plays the role of the symmetry algebra for the $q$-oscillator representation of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2n})$; see Noumi, Umeda, and Wakayama [20]. A quantum analog of the Brauer algebra associated with $U_q'(\mathfrak{o}_N)$ was constructed in [17]. Some families of Casimir elements were produced by Noumi, Umeda, and Wakayama [20], Havlíček, Klimyk, and Pošta [7], and also by Gavrilik and Iorgov [4] for the algebra $U_q'(\mathfrak{o}_N)$ and by Molev, Ragoucy, and Sorba [18] for both $U_q'(\mathfrak{o}_N)$ and $U_q'(\mathfrak{sp}_{2n})$. The paper [18] also provides a construction of some Yangian-type algebras associated with twisted quantized enveloping algebras. Their applications to spin chain models were discussed in Arnaudon et al. [1]. The algebra $U_q'(\mathfrak{sp}_{2n})$ has apparently received much less attention in the literature than its orthogonal counterpart, which we hope to remedy by this paper.

2. Definitions and Preliminaries

Fix a nonzero complex parameter $q$ such that $q^2 \neq 1$. Following Jimbo [13], we introduce the $q$-analog $U_q(\mathfrak{gl}_N)$ of the universal enveloping algebra $U(\mathfrak{gl}_N)$ as the associative algebra generated by the elements $t_1, \ldots, t_N, t_1^{-1}, \ldots, t_N^{-1}, e_1, \ldots, e_{N-1}, f_{N-1}$ with the defining relations

$$t_it_j = t_jt_i, \quad t_it_i^{-1} = t_i^{-1}t_i = 1,$$

$$t_ie_jf_i^{-1} = e_jq^{\delta_{ij}-\delta_{i,j+1}}, \quad t_if_jf_i^{-1} = f_jq^{-\delta_{ij}+\delta_{i,j+1}}.$$
The $q$-analogs of the root vectors are defined inductively by
\[ e_{i+1} = e_i, \quad e_{i+1,i} = f_i, \]
\[ e_{ij} = e_{ik}e_{kj} - q e_{kj}e_{ik} \quad \text{for} \quad i < k < j, \]
\[ e_{ij} = e_{ik}e_{kj} - q^{-1} e_{kj}e_{ik} \quad \text{for} \quad i > k > j, \]
and the elements $e_{ij}$ are independent of the choice of values of the index $k$.

Following [13], [21], consider the $R$-matrix presentation of the algebra $U_q(\mathfrak{g}_N)$. The $R$-matrix is given by
\[ R = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ij} \otimes E_{ji}, \tag{2.1} \]
which is an element of $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$, where the $E_{ij}$ denote the standard matrix units and the subscripts run over the set \{1, \ldots, N\}. The $R$-matrix satisfies the Yang–Baxter equation
\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.2} \]
where both sides range in $\text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$ and the subscripts indicate the copies of $\text{End } \mathbb{C}^N$; e.g., $R_{12} = R \otimes 1$ etc.

The quantized enveloping algebra $U_q(\mathfrak{g}_N)$ is generated by the elements $t_{ij}$ and $\bar{t}_{ij}, 1 \leq i, j \leq N$, subject to the relations
\[ t_{ij} = \bar{t}_{ji} = 0, \quad 1 \leq i < j \leq N, \]
\[ t_{ii}t_{ii} = t_{ii}t_{ii}, \quad 1 \leq i \leq N, \tag{2.3} \]
\[ RT_1T_2 = T_2T_1R, \quad R\bar{T}_1\bar{T}_2 = \bar{T}_2\bar{T}_1R, \quad R\bar{T}_1T_2 = T_2\bar{T}_1R. \]
Here $T$ and $\bar{T}$ are the matrices
\[ T = \sum_{i,j} t_{ij} \otimes E_{ij}, \quad \bar{T} = \sum_{i,j} \bar{t}_{ij} \otimes E_{ij}, \tag{2.4} \]
which are treated as elements of the algebra $U_q(\mathfrak{g}_N) \otimes \text{End } \mathbb{C}^N$. Both sides of each of the $R$-matrix relations in (2.3) are elements of $U_q(\mathfrak{g}_N) \otimes \text{End } \mathbb{C}^N \otimes \text{End } \mathbb{C}^N$, and the subscripts on $T$ and $\bar{T}$ indicate the copies of $\text{End } \mathbb{C}^N$ where $T$ or $\bar{T}$ acts; e.g., $T_1 = T \otimes 1$. In terms of the generators, the defining relations between the $t_{ij}$ can be written as
\[ q^{t_{ij}}t_{ia}t_{jb} - q^{\delta_{ab}}t_{ib}t_{ia} = (q - q^{-1})(\delta_{b < a} - \delta_{i < j})t_{ia}t_{ib}, \tag{2.5} \]
where $\delta_{b < a}$ is equal to 1 if $i < j$ and 0 otherwise. The relations between the $\bar{t}_{ij}$ are obtained by replacing $t_{ij}$ by $\bar{t}_{ij}$ everywhere in (2.5), and the relations involving both $t_{ij}$ and $\bar{t}_{ij}$ have the form
\[ q^{t_{ij}}\bar{t}_{ia}t_{jb} - q^{\delta_{ab}}t_{ib}\bar{t}_{ia} = (q - q^{-1})(\delta_{b < a}t_{ia}\bar{t}_{ib} - \delta_{i < j}t_{ia}\bar{t}_{ib}). \tag{2.6} \]
An isomorphism between the two presentations is given by the formulas
\[ t_i \mapsto t_{ii}, \quad t_i^{-1} \mapsto t_{ii}, \quad e_{ij} \mapsto \frac{t_{ij} t_{ii}}{q - q^{-1}}, \quad e_{ji} \mapsto \frac{t_{ii} t_{ji}}{q - q^{-1}}, \quad i < j; \quad (2.7) \]
e.g., see [14, Sec. 8.5.5]. We shall identify the corresponding elements of \( U_q(\mathfrak{gl}_N) \) via this isomorphism.

For any \( N \)-tuple \( \varsigma = (\varsigma_1, \ldots, \varsigma_N) \) with \( \varsigma_i \in \{-1, 1\} \), the mapping
\[ t_{ij} \mapsto \varsigma_i t_{ij}, \quad t_{ij} \mapsto \varsigma_j t_{ij}, \quad (2.8) \]
defines an automorphism of the algebra \( U_q(\mathfrak{gl}_N) \).

By the Poincaré–Birkhoff–Witt theorem for the \( A_n \) type quantized enveloping algebra, the monomials
\[
\prod_{i,j} k_{ij} e_{ij} \quad \text{where} \quad \begin{cases} 
\varsigma_{ij} = 1, & \varsigma_{ij} \in \{-1, 1\}, \\
\varsigma_{ij} t_{ij} & \text{otherwise},
\end{cases}
\]
(run over integers, also form a basis of the algebra of \( U_q(\mathfrak{gl}_N) \); see [22] and [2, Proposition 9.2.2]. (This basis corresponds to the reduced decomposition
\[ w_0 = s_{N-1}s_{N-2} \cdots s_3 s_1 s_2 \cdots s_{N-1} \]
of the longest element of the Weyl group.) Using the isomorphism (2.7), we can conclude that the monomials
\[
\prod_{i,j} k_{ij} e_{ij} \quad \text{where} \quad \begin{cases} 
\varsigma_{ij} = 1, & \varsigma_{ij} \in \{-1, 1\}, \\
\varsigma_{ij} t_{ij} & \text{otherwise},
\end{cases}
\]
(run over nonnegative integers and the \( l_{ij} \) run over integers, also form a basis of \( U_q(\mathfrak{gl}_N) \). This follows from the relations
\[ t_{ii} t_{jj} = q^{\delta_{ij} - \delta_{ii}} t_{jj} t_{ii}, \quad t_{ii} t_{jj} = q^{\delta_{ij} - \delta_{ii}} t_{jj} t_{ii}. \quad (2.11) \]

Let \( U^- \) denote the subalgebra of \( U_q(\mathfrak{gl}_N) \) generated by the elements \( t_{ii} \) with \( i = 1, \ldots, N \) and \( t_{ij} \) with \( 1 \leq j < i \leq N \). Fix a permutation \( \pi \) of the indices 2, 3, \ldots, \( N \) and consider a linear ordering \( < \) on the set of elements \( t_{ij} \) with \( 1 \leq j < i \leq N \) such that \( \pi(i) < \pi(k) \) implies \( t_{ij} < t_{kl} \) for all possible \( j \) and \( l \).

**Proposition 2.1.** For the linear ordering defined above, the ordered monomials
\[
\prod_{i,j} k_{ij}^{m_{ij}} \prod_{i} t_{ii}^{m_{ii}}, \quad (2.12)
\]
where \( k_{ij} \) and \( m_i \) are nonnegative integers, form a basis of \( U^- \).
Proof. For any nonnegative integer \( l \), consider the subspace \( U^-_l \) of elements of degree at most \( l \) in the generators. By the Poincaré–Birkhoff–Witt theorem, a basis of \( U^-_l \) is formed by the monomials
\[
q^{k_N} N_{N-1} q^{k_{N-2}} N_{N-3} \cdots N_{N-2} k_N q^{k_{N-1}} N_{N-2} \cdots N_{N-3} \cdots N_{N-l} N_{N-1} \cdots N_{N-2} k_{N-l} N_{N-2} \cdots N_{N-3} \cdots N_{N-1} \cdots N_{N-l}
\]
with the sum of all exponents not exceeding \( l \). Hence it suffices to show that the ordered monomials (2.12) span \( U^- \). The statement will then follow by counting the number of ordered monomials of degree not exceeding \( l \).

By the defining relations (2.5), we have
\[
t_{ia} t_{jb} = t_{jb} t_{ia} + (q - q^{-1}) t_{ja} t_{ib},
\]
for \( i > j > a > b \), while
\[
t_{ia} t_{ab} = t_{ab} t_{ia} + (q - q^{-1}) t_{ia} t_{ab},
\]
for \( i > a > b \). Given a monomial \( t_{i_1 a_1} \cdots t_{i_r a_r} \), with \( i_r > a_r \) for all \( r \), one easily shows by induction on the degree \( p \) that, modulo elements of degree \( < p \), this monomial can be written as a linear combination of monomials of the form \( t_{j_1 b_1} \cdots t_{j_m b_m} \), where \( \pi(j_1) \leq \cdots \leq \pi(j_m) \). By the second relation in (2.13), this monomial coincides with the ordered monomial up to a factor that is a power of \( q \). The proof is completed by taking into account the first relation in (2.11).

We shall also use the extended quantized enveloping algebra \( \hat{U}_q(\mathfrak{gl}_N) \). This is an associative algebra generated by elements \( t_{ij} \) and \( \bar{t}_{ij} \) with \( 1 \leq i \leq j \leq N \) and elements \( t_{ii}^{-1} \) and \( \bar{t}_{ii}^{-1} \) with \( 1 \leq i \leq N \) subject to the relations
\[
t_{ij} = \bar{t}_{ij} = 0, \quad 1 \leq i < j \leq N,
\]
\[
t_{ii} \bar{t}_{ii} = \bar{t}_{ii} \bar{t}_{ii}, \quad t_{ii} t_{ii}^{-1} = t_{ii}^{-1} t_{ii} = 1, \quad \bar{t}_{ii} \bar{t}_{ii}^{-1} = \bar{t}_{ii}^{-1} \bar{t}_{ii} = 1, \quad 1 \leq i \leq N,
\]
\[
R T_1 T_2 = T_2 T_1 R, \quad R T_1 T_2 = T_2 T_1 R, \quad R T_2 T_1 = T_2 T_1 R,
\]
where we use the notation of (2.3). Obviously, we have the natural epimorphism \( \hat{U}_q(\mathfrak{gl}_N) \twoheadrightarrow U_q(\mathfrak{gl}_N) \) that takes the generators \( t_{ij} \) and \( \bar{t}_{ij} \), respectively, of \( \hat{U}_q(\mathfrak{gl}_N) \) to the elements of \( U_q(\mathfrak{gl}_N) \) with the same notation. More generally, for any nonzero complex numbers \( \rho_i, i = 1, \ldots, N \), the mapping
\[
\varrho : t_{ij} \mapsto \rho_i t_{ij}, \quad \bar{t}_{ij} \mapsto \rho_i \bar{t}_{ij}
\]
defines an epimorphism \( \hat{U}_q(\mathfrak{gl}_N) \twoheadrightarrow U_q(\mathfrak{gl}_N) \). For any \( i \in \{1, \ldots, N\} \), the element \( t_{ii} \bar{t}_{ii} \) belongs to the center of \( \hat{U}_q(\mathfrak{gl}_N) \), while \( t_{ii} \bar{t}_{ii} - \rho_i^2 \) is contained in the kernel of \( \varrho \).

Let \( \hat{U}^0 \) denote the (commutative) subalgebra of \( \hat{U}_q(\mathfrak{gl}_N) \) generated by the elements \( t_{ii}, t_{ii}^{-1}, \bar{t}_{ii}, \bar{t}_{ii}^{-1}, i = 1, \ldots, N \), and let \( \hat{U}^- \) denote the subalgebra of \( \hat{U}_q(\mathfrak{gl}_N) \) generated by \( \hat{U}^0 \) and all elements \( t_{ij} \) with \( i > j \).

Fix a permutation \( \pi \) of the indices \( 2, 3, \ldots, N \) and consider a linear ordering \( \prec \) on the set of elements \( t_{ij} \), \( 1 \leq j < i \leq N \), such that \( \pi(i) \prec \pi(k) \) implies \( t_{ij} \prec t_{kl} \) for all possible \( j \) and \( l \).

**Corollary 2.2.** The subalgebra \( \hat{U}^0 \) is isomorphic to the algebra of Laurent polynomials in the variables \( t_{ii} \) and \( \bar{t}_{ii} \). Moreover, for the linear ordering defined as
above, the ordered monomials in the elements \( t_{ij} \), \( 1 \leq j < i \leq N \), form a basis of the right \( \hat{U}^0 \)-module \( \hat{U}^- \).

Proof. Obviously, the subalgebra \( \hat{U}^0 \) is spanned by Laurent monomials in the elements \( t_{ii} \) and \( t_{ii}^- \). We need to verify that the monomials are linearly independent. Suppose that

\[
\sum_{m,l} c_{m,l} t_{i1}^{m_1} t_{i1}^{l_1} \cdots t_{jN}^{m_N} t_{jN}^{l_N} = 0,
\]

where the sum is over \( N \)-tuples \( m = (m_1, \ldots, m_N) \) and \( l = (l_1, \ldots, l_N) \) of integers and only finitely many coefficients \( c_{m,l} \in \mathbb{C} \) are nonzero. Apply an epimorphism of the form (2.14) to both sides of this relation. This gives a relation in \( \hat{U}^0 \),

\[
\sum_{m,l} c_{m,l} \rho_1^{m_{i1} + l_{i1}} \cdots \rho_N^{m_{jN} + l_{jN}} t_{i1}^{m_1} \cdots t_{jN}^{m_N} = 0.
\]

Since the monomials \( t_{i1}^{m_1} \cdots t_{jN}^{m_N} \), \( r_i \in \mathbb{Z} \), are linearly independent, we obtain

\[
\sum_{m,l} c_{m,l} \rho_1^{m_{i1} + l_{i1}} \cdots \rho_N^{m_{jN} + l_{jN}} = 0
\]

for any given integer differences \( m_i - l_i \), \( i = 1, \ldots, N \). Varying the values of the parameters \( \rho_i \), we conclude that \( c_{m,l} = 0 \) for all \( m \) and \( l \). This proves the first part of the corollary.

Arguing as in the proof of Proposition 2.1, we find that the ordered monomials in the elements \( t_{ij} \) with \( i > j \) span \( \hat{U}^- \) as a right \( \hat{U}^0 \)-module. It remains to show that the ordered monomials are linearly independent over \( \hat{U}^0 \). Suppose that a linear combination of the ordered monomials is zero,

\[
\sum_i t_{i1}^{k_{1i} a_1} \cdots t_{im}^{k_{mi} a_m} d_k = 0, \quad d_k \in \hat{U}^0,
\]

where \( \{(i_1 a_1), \ldots, (i_m a_m)\} = \{(2, 1), (3, 1), (3, 2), \ldots, (N, N - 1)\} \) and \( k \) runs over a finite set of tuples of nonnegative integers \( k_{i,a} \). Apply an epimorphism of the form (2.14) to both sides of this relation. By Proposition 2.1, we obtain \( \varphi(d_k) = 0 \) for all \( k \). As in the first part of the proof, varying the parameters \( \rho_i \), we conclude that \( d_k = 0 \) for all \( k \). \( \square \)

Note that a similar argument can be used to demonstrate that the ordered monomials of the form

\[
t_{i1}^{k_{1i} a_1} \cdots t_{im}^{k_{mi} a_m} a_{im}^{l_{im}} \cdots a_{i1}^{l_{i1}}
\]

form a basis of the left or right \( \hat{U}^0 \)-module \( \hat{U}_q(\mathfrak{gl}_N) \).

Now we reproduce some results in [18], [19] concerning the twisted quantized enveloping algebra \( U'_q(\mathfrak{sp}_{2n}) \). This is an associative algebra generated by elements \( s_{ij}, i, j \in \{1, \ldots, 2n\} \) and \( s_{i,i+1}^{-1}, i = 1, 3, \ldots, 2n - 1 \). The generators \( s_{ij} \) are zero for \( j = i + 1 \) with even \( i \) and for \( j \geq i + 2 \) and all \( i \). We combine the \( s_{ij} \) into a matrix \( S \) as in (2.4),

\[
S = \sum_{i,j} s_{ij} \otimes E_{ij}, \quad (2.15)
\]
so that $S$ has a block triangular form with $n$ diagonal $2 \times 2$-blocks,

$$ S = \begin{pmatrix}
  s_{11} & s_{12} & 0 & 0 & \ldots & 0 & 0 \\
  s_{21} & s_{22} & 0 & 0 & \ldots & 0 & 0 \\
  s_{31} & s_{32} & s_{33} & s_{34} & \ldots & 0 & 0 \\
  s_{41} & s_{42} & s_{43} & s_{44} & \ldots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  s_{2n-1,1} & s_{2n-1,2} & s_{2n-1,3} & s_{2n-1,4} & \ldots & s_{2n-1,2n-1} & s_{2n-1,2n} \\
  s_{2n,1} & s_{2n,2} & s_{2n,3} & s_{2n,4} & \ldots & s_{2n,2n-1} & s_{2n,2n}
\end{pmatrix}. $$

Consider the transpose of the $R$-matrix (2.1), given by

$$ R' = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ji} \otimes E_{jj}. \quad (2.16) $$

The defining relations of $U'_q(\mathfrak{sp}_{2n})$ have the form of the reflection equation

$$ RS_1 R' S_2 = S_2 R' S_1 R \quad (2.17) $$

together with

$$ s_{i,i+1}^{-1} s_{i,i+1} = s_{i,i+1}^{-1} s_{i,i+1} = 1 \quad (2.18) $$

and

$$ s_{i+1,i+1} s_{i,i+1} - q^2 s_{i+1,i+1} s_{i,i+1} = q^3 \quad (2.19) $$

for $i = 1, 3, \ldots, 2n - 1$. In terms of the generators $s_{ij}$, relation (2.17) reads

$$ q^{\delta_{i,j} + \delta_{j,i}} s_{ia} s_{jb} - q^{\delta_{i,j} + \delta_{j,i}} s_{ia} s_{jb} = (q - q^{-1}) q^{\delta_{i,j} - \delta_{i,j}} s_{ja} s_{ib} 
+ (q - q^{-1}) (q^{\delta_{i,j} - \delta_{i,j}} s_{ja} s_{ib} - q^{\delta_{i,j} - \delta_{i,j}} s_{ja} s_{ib}) 
+ (q - q^{-1})^2 (\delta_{i,j} - \delta_{i,j}) s_{ji} s_{ab}, \quad (2.20) $$

where $\delta_{i,j} = 1$ if the inequality in the subscript is satisfied and 0 otherwise.

For any $2n$-tuple $\zeta = (\zeta_1, \ldots, \zeta_{2n})$ with $\zeta_1 \in \{-1, 1\}$, the mapping

$$ s_{ij} \mapsto \zeta_i \zeta_j s_{ij}, \quad (2.21) $$

defines an automorphism of the algebra $U'_q(\mathfrak{sp}_{2n})$. This can be verified directly from the defining relations of the algebra.

We introduce a $2n \times 2n$ matrix $G$ by setting

$$ G = q \sum_{k=1}^n E_{2k-1,2k} - \sum_{k=1}^n E_{2k,2k-1}, \quad (2.22) $$

so that

$$ G = \begin{pmatrix}
  0 & q & \ldots & 0 & 0 \\
  -1 & 0 & \ldots & 0 & 0 \\
  0 & 0 & \ldots & 0 & q \\
  0 & 0 & \ldots & -1 & 0
\end{pmatrix}. $$

The mapping

$$ S \mapsto TG^T \quad (2.23) $$

defines a homomorphism $U'_q(\mathfrak{sp}_{2n}) \to U_q(\mathfrak{gl}_{2n})$. Explicitly, in terms of generators, it is written as

$$s_{ij} \mapsto q \sum_{k=1}^{n} t_{i,2k-1} t_{j,2k} - \sum_{k=1}^{n} t_{i,2k} t_{j,2k-1}. \quad (2.24)$$

If $q$ is regarded as a variable, then the mapping (2.23) is an embedding of algebras over the field $\mathbb{C}(q)$; see [18, Theorem 2.8]. This follows from the fact that the algebra $U'_q(\mathfrak{sp}_{2n})$ specializes to $U(\mathfrak{sp}_{2n})$ as $q \to 1$. More precisely, set $A = \mathbb{C}[q, q^{-1}]$ and consider the $A$-subalgebra $U'_A$ of $U'_q(\mathfrak{sp}_{2n})$ generated by the elements

$$\sigma_{i,i+1} = s_{i,i+1} - q, \quad \sigma_{i+1,i} = s_{i+1,i} + 1, \quad i = 1, 3, \ldots, 2n - 1,$$

and

$$\sigma_{ij} = \frac{s_{ij}}{q-q^{-1}}, \quad i \geq j, \text{ excluding } i = j + 1 \text{ for } j \text{ odd}.$$ 

Then we have an isomorphism

$$U'_A \otimes_A \mathbb{C} \cong U(\mathfrak{sp}_{2n}), \quad (2.25)$$

where the action of $A$ on $\mathbb{C}$ is defined via the evaluation $q = 1$; see [18]. The symplectic Lie algebra $\mathfrak{sp}_{2n}$ is defined as the subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements

$$F_{ij} = \sum_{k=1}^{2n} (E_{ik}g_{kj} + E_{jk}g_{ki}),$$

where the $g_{ij}$ are the matrix elements of the matrix $G^2 = [g_{ij}]$ obtained by evaluating $G$ at $q = 1$,

$$G^2 = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \end{pmatrix}.$$

The images of the elements $\sigma_{ij}$ under the isomorphism (2.25) are the respective elements $F_{ij}$ of $\mathfrak{sp}_{2n}$.

In the next section, we prove that if $q$ is specialized to a nonzero complex number such that $q^2 \neq 1$, then the mapping (2.23) defines an embedding of the respective algebras over $\mathbb{C}$.

### 3. Poincaré–Birkhoff–Witt Theorem

Define the extended twisted quantized enveloping algebra $\tilde{U}'_q(\mathfrak{sp}_{2n})$ as follows. This is an associative algebra generated by the elements $s_{ij}, i, j \in \{1, \ldots, 2n\}$ and $s_{i,i+1}^{-1}, i = 1, 3, \ldots, 2n - 1$, where $s_{ij} = 0$ for $j = i + 1$ with even $i$ as well as for $j \geq i + 2$ and all $i$. The defining relations are given by (2.18) and (2.20). As with the algebra $U'_q(\mathfrak{sp}_{2n})$, we combine the $s_{ij}$ into a matrix $S$ that has a block triangular form with $n$ diagonal $2 \times 2$-blocks.

Consider the algebra $\tilde{U}_q(\mathfrak{gl}_{2n})$ introduced in the previous section, and denote by $\tilde{U}_q(\mathfrak{gl}_{2n})$ its quotient by the ideal generated by the central elements $t_{ii} - 1$ for all
even \( i \). We keep the same notation for the images of the generators of \( \tilde{U}_q'(\mathfrak{gl}_{2n}) \) in \( \tilde{U}_q'(\mathfrak{gl}_{2n}) \). The mapping given by

\[
\psi: S \mapsto \tau G T^i
\]  

defines a homomorphism \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \to \tilde{U}_q'(\mathfrak{gl}_{2n}) \). This can be verified by the same calculation as with the homomorphism (2.23); see [19] and [18]. In particular,

\[
s_{i,i+1} \mapsto q t_i t_{i+1}, \quad s_{i,i+1}^{-1} \mapsto q^{-1} t_i^{-1} t_{i+1}
\]

for \( i = 1, 3, \ldots, 2n - 1 \).

**Theorem 3.1.** The mapping (3.1) defines an embedding \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \hookrightarrow \tilde{U}_q'(\mathfrak{gl}_{2n}) \).

**Proof.** We only need to show that the kernel of the homomorphism (3.1) is zero. We shall use a weak form of the Poincaré–Birkhoff–Witt theorem for the algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \), provided by the following lemma.

**Lemma 3.2.** The algebra \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \) is spanned by the monomials

\[
k_{2,1,1} \ldots k_{2n,2n} s_{2n,1} s_{2n,2n} s_{2n-2,2n-2} \ldots s_{2,2} k_{21, k_{22}} \times s_{1,2,1} \ldots s_{1,2n-1} s_{1,2n-3,1} \ldots s_{1,2} s_{1,3,1} s_{1,3,2} k_{1,12} \ldots k_{1,2n,12}
\]

where \( k_{1,2}, k_{1,3}, \ldots, k_{2n-1,2n} \) run over all integers and the remaining \( k_{ij} \) run over nonnegative integers.

**Proof.** We follow the argument in [18], where a similar statement was proved (see Lemma 2.1 there). We shall prove that any monomial in the generators can be written as a linear combination of monomials of the form (3.2). The defining relations (2.20) imply that

\[
q^{-\delta_{i,i+1} k_{s_{i,i+1}} s_{kl}} = q^{\delta_{i,i+1} k_{s_{i,i+1}} s_{kl}}
\]

for any \( i = 1, 3, \ldots, 2n - 1 \). Therefore, it suffices to consider monomials where the generators \( s_{12}, \ldots, s_{2n-1,2n} \) occur in nonnegative powers. For any monomial

\[
s_{i_1 a_1} \ldots s_{i_p a_p},
\]

we introduce its weight \( w = i_1 + \cdots + i_p \). We shall use induction on \( w \). The defining relations (2.20) for \( \tilde{U}_q'(\mathfrak{sp}_{2n}) \) imply that, modulo products of weight less than \( i + j \),

\[
q^{\delta_{i,i+1} s_{ia} s_{jb}} \equiv q^{\delta_{i,i+1} s_{ia} s_{jb}} + (q - q^{-1}) q^{\delta_{i,a} (\delta_{b,c} - \delta_{c,j}) s_{ia} s_{ib}}.
\]

Let \( \pi \) denote the permutation of the indices 1, 2, \ldots, 2n such that \( \pi^{-1} \) is given by (2n, 2n−2, \ldots, 2, 2n−1, 2n−3, \ldots, 1). Relation (3.5) allows us to represent (3.4), modulo monomials of weight less than \( w \), as a linear combination of monomials \( s_{j_1 b_1} \ldots s_{j_p b_p} \) of weight \( w \) such that \( \pi(j_1) \leq \cdots \leq \pi(j_p) \). Consider a submonomial

\[
s_{i_{c_1}} \ldots s_{i_{c_d}}
\]

containing generators with the same first subscript. By (3.5),

\[
s_{ia} s_{ib} \equiv q^{\delta_{ia} - \delta_{ia} + 1} s_{ib} s_{ia}
\]

for \( a > b \). Using this relation repeatedly, we bring the submonomial to the desired form.
Lemma 3.3. The monomials

\[ k_{2n,1}^{22}, \ldots, k_{2n,2n-2}^{22}, k_{2n,2n}, \ldots, s_{21}^{41}, s_{42}, s_{44}, s_{22}, s_{21}, k_{2n-1,1}, \ldots, s_{2n-1,2n-2}, s_{31}, s_{32}^{22} \times s_{2n,2n-1} s_{2n-1,2n}, \ldots, s_{21}^{11} s_{12}, s_{2n-1,2n-1}, \ldots, s_{11}^{11}, \]  \hspace{1cm} (3.7)

where \( k_{12}, k_{34}, \ldots, k_{2n-1,2n} \) run over all integers and the remaining \( k_{ij} \) run over nonnegative integers, are linearly independent in \( U_q(\mathfrak{sp}_{2n}) \).

Proof. Let \( \mu_i \) and \( \bar{\mu}_i, i = 1, \ldots, 2n \), be arbitrary nonzero complex numbers such that \( \mu_i = \bar{\mu}_i = 1 \) for all even \( i \). Consider the corresponding Verma module \( M(\mu, \bar{\mu}) \) over the algebra \( U_q(\mathfrak{g}_{2n}) \), which is defined as the quotient of \( U_q(\mathfrak{g}_{2n}) \) by the left ideal generated by the elements \( t_{ij} \) with \( i < j \) and \( t_{ii} - \mu_i, t_{ii} - \bar{\mu}_i \) with \( i = 1, \ldots, 2n \). Corollary 2.2 implies that the elements

\[ \bar{p}_{i_1}^{k_{i_1}}, \ldots, \bar{p}_{i_m}^{k_{i_m}}, \xi \]

form a basis of \( M(\mu, \bar{\mu}) \), where \( \xi \) denotes its highest vector and the generators \( t_{i_r a_r} \) with \( i_r > a_r \) are written in accordance with a certain linear ordering determined by the permutation \( \pi \) defined in the proof of Lemma 3.2.

Now suppose that a nontrivial linear combination of the monomials (3.7) is zero. For each odd \( i \), the image of the power \( s_{ii}^k \) under the homomorphism (3.1) is given by

\[ \psi: s_{ii}^k \mapsto q^{\frac{3n-k^2}{2}} t_{i,ii}^{k_{i_1},i_{1+1}}. \]  \hspace{1cm} (3.8)

In the set of monomials that occur in the linear combination, take a monomial such that each exponent \( k_{ii} \) takes the minimal possible value \( \kappa_i \) for each odd \( i \). Now take the image of the linear combination under (3.1) and apply this image to the vector

\[ t_{21}^{\kappa_1} \cdots t_{2n,2n-1}^{\kappa_{2n-1}} \xi \in M(\mu, \bar{\mu}). \]

By the choice of the parameters \( \kappa_i \), the nonzero contribution to the resulting expression can only come from the monomials (3.7) with \( k_{ii} = \kappa_i \) for all odd \( i \). However, by (3.8),

\[ \psi(s_{ii}^{\kappa_i}) t_{i,ii+i}^{\kappa_i} \xi = c_i \xi, \]

where \( c_i \) is a constant depending on \( \mu_i \) and \( \bar{\mu}_j \), which can easily be calculated. Choosing the parameters \( \mu_j \) and \( \bar{\mu}_j \) in such a way that \( c_i \neq 0 \) for each odd \( i \), we conclude that the image under \( \psi \) of a certain nontrivial linear combination of monomials of the form

\[ k_{2n,1}^{22}, \ldots, k_{2n,2n-2}^{22}, k_{2n,2n}, \ldots, s_{41}^{41}, s_{42}, s_{44}, s_{22}, s_{21}, k_{2n-1,1}, \ldots, s_{2n-1,2n-2}, s_{31}, s_{32}^{22} \times s_{2n,2n-1} s_{2n-1,2n}, \ldots, s_{21}^{11} s_{12}, s_{2n-1,2n-1}, \ldots, s_{11}^{11}, \]  \hspace{1cm} (3.9)

acts as zero when applied to the highest vector \( \xi \) of the Verma module \( M(\mu, \bar{\mu}) \).

Since the image of \( s_{ij} \) under the homomorphism \( \psi \) is given by (2.24), we find that

\[ \psi(s_{i,i+1}) \xi = q \mu_i \xi, \quad \psi(s_{i+1,i}) \xi = -\bar{\mu}_i \xi \]

for each odd \( i \). Moreover, using (2.24) again, we arrive at the formulas

\[ \psi(s_{ij}) \xi = \begin{cases} qt_{i,j-1} \xi & \text{if } j \text{ is even and } i \geq j, \\ -\bar{\mu}_j t_{i,j+1} \xi & \text{if } j \text{ is odd and } i \geq j + 2. \end{cases} \]
Varying the parameters $\mu_i$ and $\tilde{\mu}_i$, we conclude that the elements
\begin{equation}
\ell_{2n,1}^{k_{2n,1}} \cdots \ell_{2n,2n-2}^{k_{2n,2n-2}} \ell_{2n,2n}^{k_{2n,2n}} \ell_{2n,2n-3}^{k_{2n,2n-3}} \cdots \ell_{2n-1,2n}^{k_{2n-1,2n-2}} \ell_{2n-1,2n-1}^{k_{2n-1,2n-1}} \ell_{2n-1,2n-2}^{k_{2n-1,2n-2}} \cdots
\end{equation}
of the algebra $\mathcal{U}_q(\mathfrak{g}_{2n})$ are linearly dependent. This contradicts Corollary 2.2, thus completing the proof of the lemma. 
\end{proof}

Now let us denote by $\mathcal{U}^{++}$ the subalgebra of $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$ generated by the elements $s_{i,i+1}$ for all odd $i$ and $s_{ij}$ for all $i \geq j$. For any nonnegative integer $m$, consider the subspace $\mathcal{U}^{++}_m$ of $\mathcal{U}^{++}$ of elements of degree at most $m$ in the generators. Lemma 3.2 implies that $\mathcal{U}^{++}_m$ is spanned by the monomials (3.2) of total degree $\leq m$ with nonnegative powers of the generators. On the other hand, by Lemma 3.3, the monomials (3.7) of total degree $\leq m$ with nonnegative powers of the generators are linearly independent. Since the numbers of both types of monomials of total degree $\leq m$ coincide, we conclude that each of these families of monomials is a basis in $\mathcal{U}^{++}_m$. It follows that each family of monomials (3.2) and (3.7) is a basis of $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$. In particular, any element $u$ of $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$ can be written as linear combination of the monomials (3.7). However, the proof of Lemma 3.3 shows that $\psi(u) = 0$ implies $u = 0$, thus proving that the kernel of $\psi$ is zero. 
\end{proof}

The following version of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$ was already noted in the proof of Theorem 3.1. 

Corollary 3.4. Each of the families (3.2) and (3.7) of monomials is a basis of the algebra $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$.
\end{corollary}

For any $i = 1, 3, \ldots, 2n - 1$, set
\begin{equation}
\tilde{\vartheta}_i = s_{i+1,i+1}s_{ii} - q^2 s_{i+1,i}s_{i,i+1}.
\end{equation}

As was observed in [18, Sec. 2.2], the elements $\tilde{\vartheta}_i$ belong to the center of $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$.

\section*{Proposition 3.5. The elements}
\begin{equation}
k_{2n,1} \cdots k_{2n,2n-2} k_{2n,2n} \ell_{2n,2n-1}^{k_{2n,2n-1}} \ell_{2n-1,2n}^{k_{2n-1,2n-1}} \cdots \ell_{2n-1,2n-2}^{k_{2n-1,2n-2}} \ell_{2n-1,2n-1}^{k_{2n-1,2n-1}} \ell_{2n-1,2n-2}^{k_{2n-1,2n-2}} \cdots
\end{equation}
run over all integers for odd $i$ and $k_i$, as well as the remaining $k_{r,j}$, run over nonnegative integers, form a basis of the algebra $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$.

\begin{proof}
First, we prove that by the monomials
\begin{equation}
k_{2n,1} \cdots k_{2n,2n-2} k_{2n,2n} k_{2n,2n-1} k_{2n-1,2n} \cdots k_{2n-1,2n-2} \cdots k_{2n-1,2n-2} \cdots
\end{equation}
where the $k_{i,i+1}$ are integers for odd $i$ and the remaining $k_{r,j}$ are nonnegative integers, form a basis in $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$. Indeed, the argument used in the proof of Lemma 3.2, together with (3.3), implies that the monomials (3.12) span the algebra $\mathcal{U}_q^{'}(\mathfrak{sp}_{2n})$; the only additional observation required is that
\begin{equation}
s_{ii}s_{jb} = s_{jb}s_{ii}
\end{equation}
(3.13)
for all odd \( i \) such that \( i \geq j \geq b \); see (2.20). Then, as in the proof of Theorem 3.1, we conclude that the monomials (3.12) form a basis of \( U'_q(\mathfrak{sp}_{2n}) \).

Now we show that the elements (3.11) span the algebra \( U'_q(\mathfrak{sp}_{2n}) \). Since the defining relations between the generators \( s_{ii}, s_{i,i+1}, s_{i+1,i}, \) and \( s_{i+1,i+1} \) do not involve any other generators, it suffices to consider the special case \( n = 1 \). We have

\[
 s_{22}^k s_{21}^l s_{12}^m s_{11}^r = -q^{-2} s_{22}^k s_{12}^{l-1} s_{11}^{-1} s_{12}^r + \sum_{a=1}^l c_a s_{22}^{k+1} a^{-1} s_{21}^{m+a-1} s_{12}^{r+1}
\]

for some complex coefficients \( c_a \). Arguing by induction on \( l \), we find that

\[
 s_{22}^k s_{21}^l s_{12}^m s_{11}^r = (-1)^l q^{-2l} s_{22}^k s_{12}^{m-l} s_{11}^r
\]

+ a linear combination of \( s_{22}^{'k'} s_{12}^{m'} s_{11}^{r'} \) with \( l' < l \). (3.14)

Thus the elements (3.11) span \( U'_q(\mathfrak{sp}_{2n}) \). Relations (3.14) can obviously be inverted to obtain similar expressions of the elements \( s_{22}^k s_{12}^{m} s_{11}^r \) via the monomials \( s_{22}^k s_{21}^l s_{12}^m s_{11}^r \). It follows that the elements (3.11) are linearly independent. \( \square \)

By the definition of the algebra \( U'_q(\mathfrak{sp}_{2n}) \), we have a surjective homomorphism \( U'_q(\mathfrak{sp}_{2n}) \to U'_q(\mathfrak{sp}_{2n}) \), which takes the generators \( s_{ij} \) to the elements of \( U'_q(\mathfrak{sp}_{2n}) \) with the same name. In other words, we have an isomorphism

\[
 U'_q(\mathfrak{sp}_{2n}) \cong U'_q(\mathfrak{sp}_{2n}) / I,
\]

where \( I \) is the ideal of \( U'_q(\mathfrak{sp}_{2n}) \) generated by the central elements \( \vartheta_i = q^3 \) for all odd \( i = 1, 3, \ldots, 2n - 1 \). Hence the following Poincaré–Birkhoff–Witt theorem for the algebra \( U'_q(\mathfrak{sp}_{2n}) \) follows from Proposition 3.5 and relations (3.3) and (3.13).

**Theorem 3.6.** The elements

\[
 s_{22}^{k_{2,2}} s_{21}^{k_{2,1}} \ldots s_{2n,2n-2}^{k_{2n,2}} s_{2n,2n}^{k_{2n,1}} \ldots s_{41}^{k_{4,4}} s_{42}^{k_{4,3}} s_{43}^{k_{4,2}} s_{44}^{k_{4,1}} \times s_{11}^{k_{1,1}} s_{12}^{k_{1,2}} s_{31}^{k_{3,1}} s_{32}^{k_{3,2}} s_{33}^{k_{3,3}} s_{34}^{k_{3,4}} \ldots s_{2n-1,2n-1}^{k_{2n-1,2n-1}} s_{2n-1,2n}^{k_{2n-1,2n}},
\]

where the \( k_{i,i+1} \) run over all integers for odd \( i \) and the remaining \( k_{r,s} \) run over nonnegative integers, form a basis of the algebra \( U'_q(\mathfrak{sp}_{2n}) \). \( \square \)

**Corollary 3.7.** The mapping (3.1) defines an embedding \( U'_q(\mathfrak{sp}_{2n}) \to U_q(\mathfrak{gl}_{2n}) \).

**Proof.** By Theorem 3.1, the algebra \( U'_q(\mathfrak{sp}_{2n}) \) can be identified with a subalgebra of \( U_q(\mathfrak{gl}_{2n}) \). Then for each \( i = 1, 3, \ldots, 2n - 1 \) the relation \( \vartheta_i = q^3 \) is equivalent to \( t_{ii}t_{ii} = 1 \). However, the quotient of \( U'_q(\mathfrak{sp}_{2n}) \) by the relations \( t_{ii}t_{ii} = 1 \) for all odd \( i \) is isomorphic to \( U_q(\mathfrak{gl}_{2n}) \). Hence, the claim follows from Theorem 3.6. \( \square \)

Using Corollary 3.7, we shall treat \( U'_q(\mathfrak{sp}_{2n}) \) as a subalgebra of \( U_q(\mathfrak{gl}_{2n}) \).

**Remark 3.8.** An analog of the Poincaré–Birkhoff–Witt theorem for the algebra \( U'_q(\mathfrak{o}_N) \) was proved in [9] with the use of the diamond lemma. If \( U'_q(\mathfrak{o}_N) \) is regarded as an algebra over \( \mathbb{C}(q) \), then the theorem can also be proved by a specialization argument; see [18, Corollary 2.3]. If \( q \) is a nonzero complex number such that \( q^2 \neq 1 \), then a proof can be given in a way similar to the above argument with
some simplifications. Indeed, recall that $U'_q(\mathfrak{o}_N)$ is generated by elements $s_{ij}$ with the defining relations
\begin{align*}
s_{ij} &= 0, \quad 1 \leq i < j \leq N, \\
s_{ii} &= 1, \quad 1 \leq i \leq N,
\end{align*}
using the same notation as for the symplectic case. The monomials
\begin{equation}
\begin{array}{c}
s_{21} s_{31} s_{32} \cdots s_{N1} s_{N2} \cdots s_{NN-1} \\
\end{array}
\end{equation}
span the algebra $U'_q(\mathfrak{o}_N)$; see [18, Lemma 2.1]. To prove the linear independence of these monomials, consider their images under the homomorphism $\phi: U'_q(\mathfrak{o}_N) \to U_q(\mathfrak{gl}_N)$ defined by $S \mapsto T\bar{T}$. Consider the Verma module $M(\mu)$ over $U_q(\mathfrak{gl}_N)$ with $\mu = (1, \ldots, 1)$ and the highest vector $\xi$ so that
\begin{equation}
\bar{t}_{ij} \xi = 0, \quad i < j, \quad t_{ii} \xi = 1, \quad i = 1, \ldots, N.
\end{equation}

Applying the image of the monomial (3.16) to the highest vector $\xi$, we obtain
\begin{align*}
\bar{t}_{21}^{k_{21}} \bar{t}_{31}^{k_{31}} \bar{t}_{32}^{k_{32}} \cdots \bar{t}_{N1}^{k_{N1}} \bar{t}_{N2}^{k_{N2}} \cdots \bar{t}_{NN-1}^{k_{NN-1}} \xi, \\
\end{align*}
We complete the proof by using Proposition 2.1. This argument also shows that the homomorphism $\phi$ is an embedding.

\section{4. Highest Weight Representations}

Here we introduce the highest weight representations of $U'_q(\mathfrak{sp}_{2n})$ and prove that every finite-dimensional irreducible representation of this algebra is highest weight. The arguments is quite standard; cf. [8] and [2, Chap. 10]. From now on, $q$ is a nonzero complex number that is not a root of unity.

We use the following notation. For any two $n$-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, we denote by $\alpha \cdot \beta$ the $n$-tuple $(\alpha_1 \beta_1, \ldots, \alpha_n \beta_n)$. Next, if $\alpha$ is an $n$-tuple of integers, then $q^{\alpha}$ stands for the $n$-tuple $(q^{\alpha_1}, \ldots, q^{\alpha_n})$.

A representation $V$ of $U'_q(\mathfrak{sp}_{2n})$ is called a \textit{highest weight representation} if $V$ is generated by a nonzero vector $v$ such that
\begin{align*}
s_{ij}v &= 0, \quad i = 1, 3, \ldots, 2n - 1, \quad j = 1, 2, \ldots, i, \\
s_{i,i+1}v &= \lambda_i v, \quad i = 1, 3, \ldots, 2n - 1,
\end{align*}
for some complex numbers $\lambda_i$. These numbers are necessarily nonzero in view of relation (2.18). The $n$-tuple $\lambda = (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1})$ will be called the \textit{highest weight} of $V$. Note that the elements $s_{i,i+1}$ with odd $i$ pairwise commute owing to relations (2.20). The commutative subalgebra of $U'_q(\mathfrak{sp}_{2n})$ generated by these elements will play the role of a Cartan subalgebra.

Consider the root system $\Delta$ of type $C_n$, which is the subset of $\mathbb{R}^n$ consisting of the vectors
\begin{equation}
\pm 2 \varepsilon_i, \quad 1 \leq i \leq n, \quad \pm \varepsilon_i \pm \varepsilon_j, \quad 1 \leq i < j \leq n,
\end{equation}
where $\varepsilon_i$ is the $n$-tuple with 1 in the $i$-th position and zeros elsewhere. This set splits into positive and negative roots, $\Delta = \Delta^+ \cup (-\Delta^+)$, where the set $\Delta^+$ of
positive roots consists of the vectors
\[ 2\varepsilon_i, \quad 1 \leq i \leq n, \quad \varepsilon_i + \varepsilon_j, \quad -\varepsilon_i + \varepsilon_j, \quad 1 \leq i < j \leq n. \]

For any \( n \)-tuple \( \mu = (\mu_1, \mu_3, \ldots, \mu_{2n-1}) \) of nonzero complex numbers, we define the corresponding \textit{weight subspace} of \( V \) by setting
\[ V_\mu = \{ w \in V : s_{i,i+1}w = \mu_iw, \quad i = 1, 3, \ldots, 2n-1 \}. \]

Any nonzero vector \( w \in V_\mu \) is called a \textit{weight vector} of weight \( \mu \).

Given an \( n \)-tuple \( \lambda = (\lambda_1, \lambda_3, \ldots, \lambda_{2n-1}) \) of nonzero complex numbers, the corresponding \textit{Verma module} \( M(\lambda) \) over \( U'_q(\mathfrak{sp}_{2n}) \) is defined as the quotient of \( U'_q(\mathfrak{sp}_{2n}) \) by the left ideal generated by the elements \( s_{ij}, \quad i = 1, 3, \ldots, 2n-1, \quad j = 1, 2, \ldots, i, \quad (4.1) \)

and
\[ s_{i,i+1} - \lambda_i, \quad i = 1, 3, \ldots, 2n-1. \]

The Verma module is obviously a highest weight representation. The image \( \xi \) of the element \( 1 \in U'_q(\mathfrak{sp}_{2n}) \) in \( M(\lambda) \) is the highest vector, and \( \lambda \) is the highest weight. By the Poincaré–Birkhoff–Witt theorem for the algebra \( U'_q(\mathfrak{sp}_{2n}) \) (see Theorem 3.6), \( M(\lambda) \) has the basis consisting of the elements
\[ \prod_{i=2,4,\ldots,2n} s_{i,1}^{k_{i,1}} s_{i,2}^{k_{i,2}} \cdots s_{i,i-2}^{k_{i,i-2}} s_{i,i}^{k_{i,i}} \xi, \]

where the \( k_{ij} \) run over nonnegative integers. By (3.3), we have the weight space decomposition
\[ M(\lambda) = \bigoplus_{\mu} M(\lambda)_\mu. \]

The weight subspace \( M(\lambda)_\mu \) is nonzero if and only if \( \mu \) has the form \( \mu = q^{-\omega} \cdot \lambda \), where \( \omega \) is a linear combination of elements of \( \Delta^+ \) with nonnegative integer coefficients. The dimension of \( M(\lambda)_\mu \) is given by the same formula as in the classical case; e.g., see [8]. In particular, the weight space \( M(\lambda)_\lambda \) is one-dimensional and is spanned by \( \xi \).

Every highest weight module \( V \) with highest weight \( \lambda \) is a homomorphic image of \( M(\lambda) \). Thus \( V \) is the direct sum of its weight subspaces, \( V = \bigoplus V_\mu \). By a standard argument, \( M(\lambda) \) has a unique maximal submodule that does not contain the vector \( \xi \). The quotient of \( M(\lambda) \) by this submodule is, up to an isomorphism, the unique irreducible highest weight module with highest weight \( \lambda \). We denote this quotient by \( L(\lambda) \).

**Proposition 4.1.** Each finite-dimensional irreducible representation \( V \) of \( U'_q(\mathfrak{sp}_{2n}) \) is isomorphic to \( L(\lambda) \) for some highest weight \( \lambda \).

**Proof.** This can be verified by a standard argument. We need to show that \( V \) contains a weight vector annihilated by all operators (4.1). Since the operators \( s_{i,i+1}, \quad i = 1, 3, \ldots, 2n-1 \), pairwise commute on \( V \), it follows that \( V \) necessarily contains a weight vector \( w \) of some weight \( \mu \). If \( w \) is not annihilated by the operators (4.1), then, by applying these operators to \( w \), we obtain other weight vectors with weights of the form \( q^{-\omega} \cdot \mu \), where \( \omega \) is a linear combination of elements of \( \Delta^+ \) with
nonnegative integer coefficients. Since $\dim V < \infty$, we can complete the proof by the classical argument; e. g., see [8]. □

By Proposition 4.1, to describe finite-dimensional irreducible representations of $U'_q(sp_{2n})$, it suffices to find necessary and sufficient conditions on $\lambda$ for the representation $L(\lambda)$ to be finite-dimensional. As in the classical theory, the case $n = 1$ plays an important role.

5. Representations of $U'_q(sp_2)$

Using (2.19), we can treat $U'_q(sp_2)$ as an algebra with generators $s_{11}, s_{22}, s_{12},$ and $s_{-12}^{-1}$. The defining relations acquire the form (2.18) with $i = 1$ together with

\begin{equation}
    s_{11}s_{22} = q^{-2}s_{22}s_{11} - (q - q^{-1})(s_{12}^2 - q^2) \tag{5.1}
\end{equation}

and

\begin{equation}
    s_{12}s_{11} = q^2s_{11}s_{12}, \quad s_{12}s_{22} = q^{-2}s_{22}s_{12}. \tag{5.2}
\end{equation}

For a nonzero complex number $\lambda$, the corresponding Verma module $M(\lambda)$ has the basis $s^k_{22}\xi$, $k \geq 0$. Using (5.1) and (5.2), we obtain

\begin{equation*}
    s_{11}s_{22}^k\xi = q^3(1 - \lambda^2q^{-2k})(1 - q^{-2k})s_{22}^{k-1}\xi.
\end{equation*}

Hence the module $M(\lambda)$ is reducible if and only if $\lambda = \sigma q^m$ for some positive integer $m$ and $\sigma \in \{-1, 1\}$. Thus we have the following.

**Proposition 5.1.** The irreducible highest weight module $L(\lambda)$ over $U'_q(sp_2)$ is finite-dimensional if and only if $\lambda = \sigma q^m$ for some positive integer $m$ and $\sigma \in \{-1, 1\}$.

In this case, $L(\lambda)$ has a basis $\{v_k\}$, $k = 0, 1, \ldots, m - 1$, with the action of $U'_q(sp_2)$ given by

\begin{align*}
    s_{12}v_k &= \sigma q^{m-2k}v_k, \quad s_{22}v_k = v_{k+1}, \quad s_{11}v_k = q^3(1 - q^{2m-2k})(1 - q^{-2k})v_{k-1},
\end{align*}

where $v_{-1} = v_m = 0$.

Note also that all finite-dimensional irreducible $U'_q(sp_2)$-modules can be obtained by restriction from $U'_q(gl_2)$-modules. Indeed, consider the irreducible highest weight module over $U'_q(gl_2)$ generated by a vector $w$ satisfying

\begin{align*}
    \hat{t}_{12}w &= 0, \quad t_{11}w = \varsigma_1\mu_1^2w, \quad t_{12}w = \varsigma_2\mu_2^2w,
\end{align*}

where $\varsigma_1, \varsigma_2 \in \{-1, 1\}$ and $\mu_1, \mu_2 \in \mathbb{Z}$. If $\mu_1 - \mu_2 \geq 0$, then this module has dimension $m = \mu_1 - \mu_2 + 1$ and its restriction to the subalgebra $U'_q(sp_2)$ is isomorphic to $L(\lambda)$ with $\lambda = \sigma q^m$, where $\sigma = \varsigma_1/\varsigma_2$. This can readily be derived with the use of (2.24).
6. Classification Theorem

Consider an arbitrary irreducible highest weight representation \( L(\lambda) \) of \( U'_q(\mathfrak{sp}_{2n}) \). Our first aim is to find necessary conditions on \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}) \) for \( L(\lambda) \) to be finite-dimensional. Thus suppose that \( \dim L(\lambda) < \infty \).

For any index \( k = 1, 2, \ldots, n \), the subalgebra of \( U'_q(\mathfrak{sp}_{2n}) \) generated by the elements \( s_{2k-1,2k-1}, s_{2k,2k}, s_{2k-1,2k} \) and \( s_{2k-1,2k}^{-1} \) is isomorphic to \( U'_q(\mathfrak{sp}_2) \). The cyclic span of the highest vector \( \xi \) of \( L(\lambda) \) with respect to this subalgebra is a highest weight module with highest weight \( \lambda_{2k-1} \). The irreducible quotient of this module is finite-dimensional, and so, by Proposition 5.1, \( \lambda_{2k-1} = \sigma_k q^{m_k} \) for some positive integer \( m_k \) and \( \sigma_k \in \{-1, 1\} \). Thus the highest weight \( \lambda \) of \( L(\lambda) \) must have the form

\[
\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n}).
\]

Consider the composition of the action of \( U'_q(\mathfrak{sp}_{2n}) \) on \( L(\lambda) \) with the automorphism (2.21), where

\[
\varsigma_i = \sigma_i, \quad i = 1, 3, \ldots, 2n - 1, \quad \varsigma_i = 1, \quad i = 2, 4, \ldots, 2n.
\]

This composition is isomorphic to an irreducible finite-dimensional highest weight module with highest weight \( (q^{m_1}, \ldots, q^{m_n}) \). Thus, without loss of generality, it suffices to consider the modules \( L(\lambda) \) with highest weight \( \lambda = (q^{m_1}, \ldots, q^{m_n}) \) for some positive integers \( m_i \).

Now note that, for each \( k = 1, 2, \ldots, n - 1 \), if we restrict the range of subscripts of the generators of \( U'_q(\mathfrak{sp}_{2n}) \) to the subset \( \{2k - 1, 2k, 2k + 1, 2k + 2\} \), then the corresponding elements generate a subalgebra of \( U'_q(\mathfrak{sp}_{2n}) \) isomorphic to \( U'_q(\mathfrak{sp}_4) \). The cyclic span of the highest vector \( \xi \) of \( L(\lambda) \) with respect to this subalgebra is a highest weight module with highest weight \( (q^{m_k}, q^{m_{k+1}}) \). The irreducible quotient of this module is finite-dimensional. Hence, considering irreducible highest modules over \( U'_q(\mathfrak{sp}_4) \), we can obtain necessary conditions on the \( m_k \).

The generators of \( U'_q(\mathfrak{sp}_4) \) are the nonzero entries of the matrix

\[
S = \begin{pmatrix}
   s_{11} & s_{12} & 0 & 0 \\
   s_{21} & s_{22} & 0 & 0 \\
   s_{31} & s_{32} & s_{33} & s_{34} \\
   s_{41} & s_{42} & s_{43} & s_{44}
\end{pmatrix}
\]

together with the elements \( s_{12}^{-1} \) and \( s_{34}^{-1} \). The highest vector \( \xi \) of \( L(\lambda) \) with highest weight \( \lambda = (q^{m_1}, q^{m_2}) \) is annihilated by \( s_{11}, s_{31}, s_{32}, s_{33} \), and we have

\[
s_{12} \xi = q^{m_1} \xi, \quad s_{34} \xi = q^{m_2} \xi,
\]

where \( m_1 \) and \( m_2 \) are positive integers. Consider the subspace \( L^0 \) of \( L(\lambda) \) defined by

\[
L^0 = \{ v \in L(\lambda): s_{11} v = s_{31} v = s_{33} v = 0 \}.
\]

Note that \( \xi \in L^0 \), and so \( L^0 \neq 0 \).
Lemma 6.1. The subspace $L^0$ is invariant under the action of each of the operators $s_{32}, s_{41}, s_{12}, s_{34}, s_{12}^{-1}, s_{34}^{-1}$. Moreover, these operators satisfy the relation

$$s_{32}s_{41} - s_{41}s_{32} = (q^2 - 1)(s_{12}^{-1}s_{34} - s_{12}s_{34}^{-1})$$

on $L^0$.

Proof. The first statement is immediate from (3.3) for the elements $s_{12}, s_{34}$ and their inverses. It follows from the defining relations (2.20) that

$$s_{33}s_{32} = s_{32}s_{33}, \quad s_{11}s_{32} = s_{32}s_{11} + (q^{-2} - 1)s_{12}s_{31},$$

and

$$s_{41}s_{32} = q^{-1}s_{32}s_{41} + (q - q^{-1})(q^{-1}s_{21}s_{33} - s_{12}s_{33}),$$

which implies the statement for the operator $s_{32}$. Furthermore, we have

$$s_{11}s_{41} = s_{41}s_{11}, \quad s_{33}s_{41} = s_{41}s_{33} + (q^{-1} - q)s_{34}s_{31},$$

and

$$s_{31}s_{41} = q^{-1}s_{41}s_{31} + (q - q^{-1})(q^{-1}s_{43}s_{11} - s_{34}s_{11}),$$

completing the proof of the first statement.

Now, by (2.20),

$$s_{32}s_{41} = s_{41}s_{32} + (q - q^{-1})(s_{12}s_{43} - s_{34}s_{21}).$$

However, (2.19) gives

$$s_{21} = q^{-2}s_{12}^{-1}s_{22}s_{11} - q^{-1}s_{12},$$

so that $s_{21}$ coincides with $-qs_{12}^{-1}$ as an operator on $L^0$. Similarly, $s_{43}$ coincides with the operator $-qs_{34}^{-1}$, thus yielding the desired relation. \(\square\)

Lemma 6.1 implies that $L^0$ is a representation of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$. Indeed, the action is defined by

$e \mapsto \frac{s_{32}}{q - q^{-1}}, \quad f \mapsto \frac{s_{41}}{q(q - q^{-1})}, \quad k \mapsto s_{12}^{-1}s_{34},$

where $e, f, k, k^{-1}$ are the standard generators of $U_q(\mathfrak{sl}_2)$ satisfying

$$ke = q^2ek, \quad kf = q^{-2}fk, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Since

$$e\xi = 0 \quad \text{and} \quad k\xi = q^{-m_1+m_2}\xi,$$

it follows that the cyclic span of $\xi$ with respect to $U_q(\mathfrak{sl}_2)$ is a highest weight module. Since this module is finite-dimensional, we must have $m_2 - m_1 \geq 0$ by the classification theorem for the finite-dimensional irreducible representations of the algebra $U_q(\mathfrak{sl}_2)$; e.g., see [2, Chap. 10]. Thus we have proved the following.

Proposition 6.2. Suppose that $\lambda = (q^{m_1}, \ldots, q^{m_n})$, where $m_1, \ldots, m_n$ are positive integers. If the representation $L(\lambda)$ of the algebra $U_q'(\mathfrak{sp}_{2n})$ is finite-dimensional, then $m_1 \leq m_2 \leq \cdots \leq m_n$. \(\square\)
Now our aim is to show that these conditions are also sufficient for the representation $L(\lambda)$ to be finite-dimensional. We treat $U'_q(\mathfrak{sp}_{2n})$ as a subalgebra of $U_q(\mathfrak{gl}_{2n})$ and use a version of the Gelfand–Tsetlin basis for representations of $U_q(\mathfrak{gl}_N)$; see [12]. Let $\nu = (\nu_1, \ldots, \nu_N)$ be an $N$-tuple of integers such that $\nu_1 \geq \cdots \geq \nu_N$. The corresponding finite-dimensional irreducible representation $V(\nu)$ of $U_q(\mathfrak{gl}_N)$ is generated by a nonzero vector $\zeta$ such that

$$\bar{t}_{ij} \zeta = 0, \quad 1 \leq i < j \leq N, \quad t_{ii} \zeta = q^{\nu_i} \zeta, \quad 1 \leq i \leq N.$$  

For any integer $m$, we set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$  

We define the Gelfand–Tsetlin pattern $\Omega$ (associated with $\nu$) as an array of integer row vectors, where $\nu_{N,i} = \nu_i$ for $i = 1, \ldots, N$, so that the top row coincides with $\nu$, and

$$\nu_{k+1,i} \geq \nu_{ki} \geq \nu_{k+1,i+1}, \quad 1 \leq i \leq k \leq N - 1.$$  

There exists a basis $\{\zeta_\Omega\}$ of $V(\nu)$ parameterized by the patterns $\Omega$ such that the action of the generators of $U_q(\mathfrak{gl}_N)$ is given by

$$t_k \zeta_\Omega = q^{w_k} \zeta_\Omega, \quad w_k = \sum_{i=1}^{k-1} \nu_{ki} - \sum_{i=1}^{k} \nu_{k-1,i},$$  

$$e_k \zeta_\Omega = -\sum_{i=1}^{k} \frac{[l_{k+1,i} - l_{ki}] \cdots [l_{k+1,k+1} - l_{k+1,i}] \zeta_{\Omega+\delta_{ki}}}{[l_{k+1,i} - l_{ki}] \cdots [l_{k+1,k+1} - l_{k+1,i}]} \zeta_{\Omega+\delta_{ki}},$$  

$$f_k \zeta_\Omega = \sum_{i=1}^{k} \frac{[l_{k+1,i} - l_{ki}] \cdots [l_{k+1,k+1} - l_{k+1,i}] \zeta_{\Omega-\delta_{ki}}}{[l_{k+1,i} - l_{ki}] \cdots [l_{k+1,k+1} - l_{k+1,i}]} \zeta_{\Omega-\delta_{ki}},$$  

where $l_{ki} = \nu_{ki} - i + 1$ and the symbol $\wedge$ indicates that the zero factor in the denominator is skipped. The array $\Omega \pm \delta_{ki}$ is obtained from $\Omega$ by replacing $\nu_{ki}$ with $\nu_{ki} \pm \delta_{ki}$. The vector $\zeta_\Omega$ is considered to be zero if the array $\Omega$ is not a pattern.

Now let $m_1, \ldots, m_n$ be positive integers satisfying $m_1 \leq m_2 \leq \cdots \leq m_n$. Consider the representation $V(\nu)$ of $U_q(\mathfrak{gl}_{2n})$, where the highest weight $\nu$ is defined by

$$\nu = (r_n, \ldots, r_1, 0, \ldots, 0), \quad r_i = m_i - 1.$$
We introduce the pattern $\Omega^\nu$ associated with $\nu$ as follows:

\[
\begin{array}{cccccccc}
r_n & r_{n-1} & r_{n-2} & \cdots & r_1 & 0 & \cdots & 0 & 0 \\
r_n & r_{n-1} & r_{n-2} & \cdots & r_1 & 0 & \cdots & 0 & 0 \\
r_{n-1} & r_{n-2} & \cdots & r_1 & 0 & \cdots & 0 & 0 \\
r_{n-1} & r_{n-2} & \cdots & r_1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
r_2 & r_1 & 0 & 0 \\
r_2 & r_1 & 0 \\
r_1 & 0 \\
r_1 & 0 
\end{array}
\]

For each $k = 1, 2, \ldots, n$, row $2k-1$ from the bottom is $(r_k, r_{k-1}, \ldots, r_1, 0, \ldots, 0)$ with $k-1$ zeros, while row $2k$ from the bottom is $(r_k, r_{k-1}, \ldots, r_1, 0, \ldots, 0)$ with $k$ zeros. By the above formulas for the action of the generators of $U_q(\mathfrak{gl}_{2n})$ in the Gelfand–Tsetlin basis, we have

\[
\tilde{t}_{i,i+1}\zeta_{\Omega^\nu} = 0, \quad i = 1, 3, \ldots, 2n-1, \quad \tilde{t}_{i,i-1}\zeta_{\Omega^\nu} = 0, \quad i = 3, \ldots, 2n-1.
\]

The defining relations of $U_q(\mathfrak{gl}_{2n})$ (see Section 2) imply that the vector $\zeta_{\Omega^\nu}$ is also annihilated by all generators $t_{jk}$ with $j-k \geq 2$ and odd $j$ as well as by $t_{kj}$ with $j-k \geq 2$ and even $j$.

Consider the restriction of the $U_q(\mathfrak{gl}_{2n})$-module $V(\nu)$ to the subalgebra $U_q'(\mathfrak{sp}_{2n})$. Using (2.6) and (2.24), we obtain

\[
s_{ij}\zeta_{\Omega^\nu} = 0, \quad i = 1, 3, \ldots, 2n-1, \quad j = 1, 2, \ldots, i, \\
s_{2k-1,2k}\zeta_{\Omega^\nu} = q^{r_k+1}\zeta_{\Omega^\nu} = q^{m_k}\zeta_{\Omega^\nu}, \quad k = 1, 2, \ldots, n.
\]

Hence the cyclic span of the vector $\zeta_{\Omega^\nu}$ with respect to the subalgebra $U_q'(\mathfrak{sp}_{2n})$ is a highest weight module with highest weight $\lambda = (q^{m_1}, \ldots, q^{m_n})$. Since the representation $V(\nu)$ is finite-dimensional, we conclude that so is the representation $L(\lambda)$ of $U_q'(\mathfrak{sp}_{2n})$.

Thus, combining this argument with Propositions 4.1 and 6.2, we obtain the following theorem.

**Theorem 6.3.** Every finite-dimensional irreducible representation of the algebra $U_q'(\mathfrak{sp}_{2n})$ is isomorphic to a highest weight representation $L(\lambda)$, where the highest weight $\lambda$ is an $n$-tuple of the form

\[
\lambda = (\sigma_1 q^{m_1}, \ldots, \sigma_n q^{m_n})
\]

with positive integers $m_1 \leq m_2 \leq \cdots \leq m_n$ and each $\sigma_i$ is 1 or $-1$. In particular, the isomorphism classes of finite-dimensional irreducible representations of $U_q'(\mathfrak{sp}_{2n})$ are parameterized by such $n$-tuples. \hfill \Box

It is likely that the structure of $L(\lambda)$ is very much similar to that of the representation of the Lie algebra $\mathfrak{sp}_{2n}$ with highest weight $(r_n, \ldots, r_1)$, where $r_i = m_i - 1$. In particular, these representations should have the same dimensions and characters. This can be proved for the case in which $U_q'(\mathfrak{sp}_{2n})$ is treated as an algebra over
\( \mathbb{C}(q) \) by following the argument in [2, Sec. 10.1]. Indeed, recall the \( \mathcal{A} \)-subalgebra \( U'_\mathcal{A} \) of \( U'_q(\mathfrak{sp}_{2n}) \) introduced in Section 2. Let \( \xi \) denote the highest vector of the \( U'_q(\mathfrak{sp}_{2n}) \)-module \( L(\lambda) \) with \( \lambda = (q^{m_1}, \ldots, q^{m_n}) \), where the positive integers \( m_i \) satisfy \( m_1 \leq m_2 \leq \cdots \leq m_n \). Set

\[
L(\lambda)_\mathcal{A} = U'_\mathcal{A}\xi.
\]

Then \( L(\lambda)_\mathcal{A} \) is a \( U'_\mathcal{A} \)-submodule of \( L(\lambda) \) such that

\[
L(\lambda)_\mathcal{A} \otimes \mathcal{A} \mathbb{C}(q) \cong L(\lambda)
\]

in an isomorphism of vector spaces over \( \mathbb{C}(q) \). Moreover, \( L(\lambda)_\mathcal{A} \) is the direct sum of its intersections with the weight spaces of \( L(\lambda) \), and each intersection is a free \( \mathcal{A} \)-module; cf. [2, Proposition 10.1.4]. Now set

\[
\overline{L}(\lambda) = L(\lambda)_\mathcal{A} \otimes \mathcal{A} \mathbb{C},
\]

where the \( \mathcal{A} \)-action on \( \mathbb{C} \) is defined by the evaluation at \( q = 1 \). By the specialization isomorphism (2.25), \( \overline{L}(\lambda) \) is a module over the Lie algebra \( \mathfrak{sp}_{2n} \). By the specialization formulas of Section 2, we find

\[
F_{2k-1, 2k}\xi = (m_k - 1)\xi, \quad k = 1, 2, \ldots, n,
\]

where \( \xi \) denotes the image of \( \xi \) in \( \overline{L}(\lambda) \). Moreover,

\[
F_{ij}\xi = 0, \quad i = 1, 3, \ldots, 2n - 1, \quad j = 1, 2, \ldots, i.
\]

Thus, taking the weights with respect to the basis \((F_{2n-1, 2n}, \ldots, F_{12})\) of the Cartan subalgebra of \( \mathfrak{sp}_{2n} \), we conclude that \( \overline{L}(\lambda) \) is a highest weight module over \( \mathfrak{sp}_{2n} \). Since \( \dim \overline{L}(\lambda) < \infty \), we see that the module \( \overline{L}(\lambda) \) must be irreducible. Thus we arrive at the following result.

**Theorem 6.4.** The \( \mathfrak{sp}_{2n} \)-module \( \overline{L}(\lambda) \) is isomorphic to the finite-dimensional irreducible module with highest weight \((r_n, \ldots, r_1)\), \( r_i = m_i - 1 \). In particular, the character of \( U'_q(\mathfrak{sp}_{2n}) \)-module \( L(\lambda) \) is given by the Weyl formula, and its dimension over \( \mathbb{C}(q) \) is the same as the dimension of \( \overline{L}(\lambda) \) over \( \mathbb{C} \).

\[\square\]

**References**


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