A PROJECTIVE INVARIANT FOR SWALLOWTAILS AND GODRONS, AND GLOBAL THEOREMS ON THE FLECNODAL CURVE

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Abstract. We show some generic (robust) properties of smooth surfaces immersed in the real 3-space (Euclidean, affine or projective), in the neighbourhood of a godron (called also cusp of Gauss): an isolated parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. With the help of these properties and a projective invariant that we associate to each godron we present all possible local configurations of the tangent plane, the self-intersection line, the cuspidal edge and the flecnodal curve at a generic swallowtail in \( \mathbb{R}^3 \). We present some global results, for instance: In a hyperbolic disc of a generic smooth surface, the flecnodal curve has an odd number of transverse self-intersections (hence at least one self-intersection).


Key words and phrases. Geometry of surfaces, tangential singularities, swallowtail, parabolic curve, flecnodal curve, cusp of Gauss, godron, wave front, Legendrian singularities.

1. Introduction

A generic smooth surface in \( \mathbb{R}^3 \) has three (possibly empty) parts: an open hyperbolic domain at which the Gaussian curvature \( K \) is negative, an open elliptic domain at which \( K \) is positive and a parabolic curve at which \( K \) vanishes. A godron is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve. We present various robust geometric properties of generic surfaces, associated to the godrons. For example (Theorem 3):

Any smooth curve of a surface of \( \mathbb{R}^3 \) tangent to the parabolic curve at a godron \( g \) has at least 4-point contact with the tangent plane of the surface at \( g \).

The line formed by the inflection points of the asymptotic curves in the hyperbolic domain is called flecnodal curve. The next theorem is well known.
Theorem 1 [22], [14], [21], [16], [7]. At a godron of a generic smooth surface the flecnodal curve is (simply) tangent to the parabolic curve.

For any generic smooth surface we have the following global result (Proposition 5 and Theorem 10):

A closed parabolic curve bounding a hyperbolic disc has a positive even number of godrons, and the flecnodal curve lying in that disc has an odd number of transverse self-intersections (thus at least one self-intersection point).

The conodal curve of a surface $S$ is the closure of the locus of points of contact of $S$ with its bitangent planes (planes which are tangent to $S$ at least at two distinct points). It is well known (see [22], [14]) that:

At a godron of a generic smooth surface the cononal curve is (simply) tangent to the parabolic curve.

So the parabolic, flecnodal and conodal curves of a surface are mutually tangent at the godrons. At each godron, these three tangent curves determine a projective invariant $\rho$, as a cross-ratio (see the cr-invariant below). We show all possible configurations of these curves at a godron, according to the value of $\rho$ (Theorem 5). There are six generic configurations, see Fig. 2.

The invariant $\rho$ and the geometric properties of the godrons presented here are useful for the study of the local affine (projective) differential properties of swallowtails. So, for example, we present all generic configurations formed by the tangent plane, the self-intersection line, the cuspidal edge and the flecnodal curve in the neighbourhood of a swallowtail point of a surface of $\mathbb{R}^3$ in general position (see Fig. 7 for Theorem 13 and Fig. 8 for Theorem 14).

Our results are related to several mathematical theories as, for instance, implicit differential equations (Davidov [11]), contact geometry and Legendrian singularities (Arnold [4], [5]), and more closely to differential geometry, singularities of projections and tangential singularities (Bruce, Giblin, Tari [9], [10], Goryunov [12], Landis [16], Platonova [21], Banchoff, Thom [8]).

The paper is organised as follows. In Section 2, we recall the classification of points of a generic smooth surface in terms of the order of contact of the surface with its tangent lines. In Section 3, we give some definitions and present our results, proving some of them directly. Finally, in Section 4, we give the proofs of the theorems.

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2. Projective Properties of Smooth Surfaces

The points of a generic smooth surface in the real 3-space (projective, affine or Euclidean) are classified in terms of the contact of the surface with its tangent lines. In this section, we recall this classification and some terminology.

A generic smooth surface $S$ is divided in three (possibly empty) parts:
(E) An open domain of elliptic points: there is no real tangent line exceeding 2-point contact with $S$;

(H) An open domain of hyperbolic points: there are two such lines, called asymptotic lines (their directions at the point of tangency are called asymptotic directions); and

(P) A smooth curve of parabolic points: a unique, but double, asymptotic line. The parabolic curve, divides $S$ into the elliptic and hyperbolic domains.

In the closure of the hyperbolic domain there is:

(F) A smooth immersed flecnodal curve: it is formed by the points at which an asymptotic tangent line exceeds 3-point contact with $S$.

One may also encounter isolated points of the following four types:

(g) A godron is a parabolic point at which the (unique) asymptotic direction is tangent to the parabolic curve;

(hn) A hyperbonode is a point of the simplest self-intersection of the flecnodal curve;

(b) A biflecnode is a point of the flecnodal curve at which one asymptotic tangent exceeds 4-point contact with $S$ (it is also called biinflection);

(en) An ellipnode is a real point in the elliptic domain of the simplest self-intersection of the complex conjugate flecnodal curves associated to the complex conjugate asymptotic lines.

In Fig. 1 the hyperbolic domain is represented in gray colour and the elliptic one in white. The flecnodal curve has a left branch $F_l$ (white) and a right branch $F_r$ (black). These branches will be defined in the next section.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The 8 tangential singularities of a generic smooth surface.}
\end{figure}

The term “godron” is due to R. Thom [13]. In other papers one can find the terms “special parabolic point” or “cusp of the Gauss map”. We keep Thom’s terminology since it is shorter. Here we will study the local projective differential properties of the godrons.

The above 8 classes of tangential singularities, Theorem 1 and all the theorems presented in this paper are projectively invariant and are robust features of a smooth surface, that is, they are stable in the sense that under a sufficiently small perturbation (taking derivatives into account) they do not vanish but only deform.
slightly. Seven of these classes were known at the end of the 19th century in the context of the enumerative geometry of complex algebraic surfaces, with prominent works of Cayley, Zeuthen and Salmon, see [22]. For these seven classes, the normal forms of surfaces at such points up to the 5-jet, under the group of projective transformations, were independently found by E. E. Landis [16] and O. A. Platonova [21]. The ellipnodes were found by D. Panov [20] who called them special elliptic points.

For surfaces in $\mathbb{R}^3$, these tangential singularities depend only on the affine structure of $\mathbb{R}^3$ (because they depend only on the contact with lines), that is, they are independent of any Euclidean structure defined on $\mathbb{R}^3$ and of the Gaussian curvature of the surface which could be induced by such a Euclidean structure.

Another definition of godron (cusp of Gauss) is in terms of the contact with the tangent plane. In this setting, a useful tool for analysis is the so called ‘height function’ (cf. [9], [10]), which can be defined once some Euclidean structure is fixed. Anyway, the singularities of the contact with the tangent plane ‘are expressed geometrically’, independently of any Euclidean structure, by the singularities of the dual surface: An ordinary godron corresponds to a swallowtail point of the dual surface, that is, to an $A_3$ (Legendre) singularity (cf. [6], [5]). There are also double (unstable) godrons, corresponding to an $A_4$ bifurcation (two swallowtails born or dying), that is, to an $A_4$ (Legendre) singularity (see Section 3.5). Of course, more degenerated godrons are also possible.

We will say that a godron is simple if it corresponds to a swallowtail point of the dual surface. All godrons of a surface in general position are simple.

Both types of tangential singularities (contact with lines and contact with planes) were extensively used by Cayley, Zeuthen and Salmon, see [22]. Besides the smooth surfaces, we also consider surfaces admitting wave front singularities (Section 3.7) and we study the behaviour of the flecnodal curve near the swallowtail points.

3. Statement of Results

Consider the pair of fields of asymptotic directions in the hyperbolic domain. An asymptotic curve is an integral curve of a field of asymptotic directions.

**Left and right asymptotic and flecnodal curves.** Fix an orientation in the 3-space $\mathbb{RP}^3$ (or in $\mathbb{R}^3$). The two asymptotic curves passing through a point of the hyperbolic domain of a generic smooth surface can be distinguished in a natural geometric way: One twists like a left screw and the other like a right screw. More precisely, a regularly parametrised smooth curve is said to be a left (right) curve if its first three derivatives at each point form a negative (resp. a positive) frame.

**Proposition 1.** At a hyperbolic point of a surface one asymptotic curve is left and the other one is right.

A proof is given (for generic surfaces) in Euclidean Remark below.

The hyperbolic domain is therefore foliated by a family of left asymptotic curves and by a family of right asymptotic curves. The corresponding asymptotic tangent lines are called respectively left and right asymptotic lines.
Definition. The left (right) flecnodal curve $F_l$ (resp. $F_r$) of a surface $S$ consists of the points of the flecnodal curve of $S$ whose asymptotic line, having higher order of contact with $S$, is a left (resp. right) asymptotic line.

The following statement (complement to Theorem 1) is used and implicitly proved (almost explicitly) in [24, 27]. A proof is given in Section 4, see Fig. 1:

**Theorem 2.** A simple godron separates locally the flecnodal curve into its right and left branches.

**Definition.** A flattening of a generic curve is a point at which the first three derivatives are linearly dependent. Equivalently, a flattening is a point at which the curve has at least 4-point contact with its osculating plane.

The flattenings of a generic curve are isolated points separating the right and left intervals of that curve.

**Euclidean Remark.** If we fix an arbitrary Euclidean structure in the affine oriented space $\mathbb{R}^3$, then the lengths of the vectors and the angles between vectors are defined. Therefore, for such Euclidean structure, the torsion $\tau$ of a curve and the Gaussian curvature $K$ of a surface are defined. In this case a point of a curve is right, left or flattening if the torsion at that point satisfies $\tau > 0$, $\tau < 0$ or $\tau = 0$, respectively. The Gaussian curvature $K$ on the hyperbolic domain of a smooth surface is negative. The Beltrami–Enneper Theorem states that the values of the torsion of the two asymptotic curves passing through a hyperbolic point with Gaussian curvature $K$ are given by $\tau = \pm \sqrt{-K}$. This proves Proposition 1.

**Definition.** An inflection of a (regularly parametrised) smooth curve is a point at which the first two derivatives are linearly dependent. Equivalently, an inflection is a point at which the curve has at least 3-point contact with its tangent line.

A generic curve in the affine space $\mathbb{R}^3$ has no inflection. However, a generic 1-parameter family of curves can have isolated parameter values for which the corresponding curve has one isolated inflection. The inflections of curves in $\mathbb{R}^3$ (usually avoided) have a very interesting geometry (see [26]).

**Theorem 3.** Let $S$ be a smooth surface. All smooth curves of $S$ which are tangent to the parabolic curve at a godron $g$ have either a flattening or an inflection at $g$, and their osculating plane is the tangent plane of $S$ at $g$.

**Corollary 1.** If a point is a godron of a generic smooth surface, then it is a flattening of both the parabolic curve and the flecnodal curve.

**Remark.** The converse is not true: A flattening of the parabolic curve or of the flecnodal curve is not necessarily a godron.

### 3.1. The cr-invariant and classification of godrons

**The conodal curve.** Let $S$ be a smooth surface. A bitangent plane of $S$ is a plane which is tangent to $S$ at least at two distinct points (which form a pair of conodal points). The conodal curve $D$ of a surface $S$ is the closure of the locus of points of contact of $S$ with its bitangent planes.
At a godron of $S$, the curve $D$ is simply tangent to the curves $P$ (parabolic) and $F$ (flecnodeal). This fact will be clear from our calculation of $D$ for Platonoova’s normal form of godrons.

**The projective invariant.** At any simple godron $g$, there are three tangent smooth curves $F$, $P$ and $D$, to which we will associate a projective invariant:

Consider the Legendrian curves $L_F$, $L_P$, $L_D$ and $L_g$ (of the 3-manifold of contact elements of $S$, $PT^*S$) consisting of the contact elements of $S$ tangent to $F$, $P$, $D$ and to the point $g$, respectively (the contact elements of $S$ tangent to a point are just the contact elements of $S$ at that point, that is, $L_g$ is the fibre over $g$ of the natural projection $PT^*S \to S$). These four Legendrian curves are tangent to the same contact plane $\Pi$ of $PT^*S$. The tangent directions of these curves determine four lines $\ell_F$, $\ell_P$, $\ell_D$ and $\ell_g$, through the origin of $\Pi$.

**Definition.** The cr-invariant $\rho(g)$ of a godron $g$ is defined as the cross-ratio of the lines $\ell_F$, $\ell_P$, $\ell_D$ and $\ell_g$ of $\Pi$:

$$\rho(g) = (\ell_F, \ell_P, \ell_D, \ell_g).$$

**Platonoova’s normal form.** According to Platonoova’s Theorem [21], the 4-jet of a surface at a godron can be sent by projective transformations to the normal form

$$z = \frac{y^2}{2} - x^2y + \lambda x^4 \quad \text{(for some } \lambda \neq 0, \frac{1}{4}). \quad \text{(G1)}$$

**Theorem 4.** Let $g$ be a godron, with cr-invariant value $\rho$, of a generic smooth surface $S$. Put the 4-jet of $S$ (after projective transformations) in Platonoova’s normal form (G1). Then the coefficient $\lambda$ equals $\rho/2$.

Of course, among the 6 possible choices of the above cross-ratio, we have chosen that whose relation with Platonoova’s normal form is simpler.

It turns out that among the 2-jets of the curves in $S$, tangent to $P$ at a godron, there is a special 2-jet at which “something happens”. We introduce it in the following lemma.

**Tangential Map and Separating 2-jet.** Let $g$ be a godron of a generic smooth surface $S$. The tangential map of $S$, $\tau_S : S \to (\mathbb{R}P^3)^\vee$, associates to each point of $S$ its tangent plane at that point. The image $S^\vee$ of $\tau_S$ is called the dual surface of $S$.

Write $J^2(g)$ for the set of all 2-jets of curves of $S$ tangent to $P$ at $g$. By the image of a 2-jet $\gamma$ in $J^2(g)$ under the tangential map $\tau_S$ we mean the image, under $\tau_S$, of any curve of $S$ whose 2-jet is $\gamma$. By Theorem 3, all the 2-jets of $J^2(g)$ (and also the 3-jets of curves on $S$ tangent to $P$ at $g$) are curves lying in the tangent plane of $S$ at $g$. In suitable affine coordinates, the elements of $J^2(g)$ can be identified with the curves $t \mapsto (t, ct^2, 0)$, $c \in \mathbb{R}$.

**Separating 2-jet Lemma.** There exists a unique 2-jet $\sigma$ in $J^2(g)$ (that we call separating 2-jet at $g$) satisfying the following properties:

(a) The images, under $\tau_S$, of all elements of $J^2(g)$ different from $\sigma$ are cusps of $S^\vee$ sharing the same tangent line $\ell_g^\vee$, at $\tau_S(g)$.

(b) The image of $\sigma$ under $\tau_S$ is a singular curve of $S^\vee$ whose tangent line at $\tau_S(g)$ is different from $\ell_g^\vee$. 
(c) (separating property): The images under $\tau_S$ of any two elements of $J^2(g)$, separated by $\sigma$, are cusps pointing in opposite directions.

**Remark.** Once the 4-jet of a smooth surface, at a godron with cr-invariant $\rho$, is sent by projective transformations to the normal form $z = y^2/2 - x^2y + px^4/2$, the separating 2-jet is independent of $\rho$: It is given by the mapping $t \mapsto (t, t^2, 0)$.

For almost all values of $\rho$ the curves $F$, $P$ and $D$ are simply tangent one to the others. However, for isolated values of $\rho$ two of these curves may have higher order of tangency and then some bifurcation occurs. We will look for the values of $\rho$ at which "something happens".

**Canonical coefficients.** Consider a godron $g$ with cr-invariant $\rho$ and suppose that the 4-jet of the surface was sent (by projective transformations) to the normal form $z = y^2/2 - x^2y + px^4/2$. The plane curves $F$, $P$ and $D$, which are the projections of $F$, $P$ and $D$ to the $(x, y)$-plane along the $z$-axis, have the same 2-jet as $F$, $P$ and $D$, respectively (since, according to Theorem 3, $F$, $P$ and $D$ have at least 4-point contact with the $(x, y)$-plane). These 2-jets correspond to three parabolas $y = c_Fx^2$, $y = c_Px^2$ and $y = c_Dx^2$, whose coefficients $c_F$, $c_P$ and $c_D$ we call the canonical coefficients of the curves $F$, $P$ and $D$, respectively.

The configuration of the curves $F$, $P$ and $D$ with respect to the asymptotic line and the separating 2-jet at $g$ is equivalent to the configuration at the origin of the curves $F$, $P$ and $D$ with respect to the parabolas $y = 0 \cdot x^2 = 0$ and $y = 1 \cdot x^2$ on the $(x, y)$-plane (see the above Remark). This configuration is determined by the relative positions of the canonical coefficients $c_F$, $c_P$, $c_D$, with respect to the numbers $c_\sigma = 1$ and $c_{ad} = 0$, in the real line:

**Theorem 5.** Given a simple godron $g$ of a smooth surface, there are six possible generic configurations of the curves $F$, $P$ and $D$ with respect to the separating 2-jet and to the asymptotic line at $g$ (they are represented in Fig. 2). The actual configuration at $g$ depends on which of the following six open intervals the cr-invariant $\rho(g)$ belongs to, respectively:

\[
\begin{align*}
\rho &\in (1, \infty) \quad \iff \quad 1 < c_D < c_P < c_F; \\
\rho &\in \left( \frac{3}{2}, 1 \right) \quad \iff \quad 0 < c_P < c_F < c_D < 1; \\
\rho &\in \left( \frac{2}{3}, \frac{3}{2} \right) \quad \iff \quad c_P < 0 < c_F < c_D < 1; \\
\rho &\in (0, \frac{2}{3}) \quad \iff \quad c_P < c_F < 0 < c_D < 1; \\
\rho &\in (-\frac{1}{2}, 0) \quad \iff \quad c_P < c_D < 0 < c_D < 1; \\
\rho &\in (-\infty, -\frac{1}{2}) \quad \iff \quad c_P < 0 < c_D < 1 < c_F.
\end{align*}
\]

Besides the 5 exceptional values of $\rho$, given in Theorem 5, we will present separately (Sections 3.9 and 3.10) other important exceptional values of $\rho$.

**3.2. The index of a godron**

**Definition.** A godron is said to be positive or of index $+1$ (resp. negative or of index $-1$) if at the neighbouring parabolic points the half-asymptotic lines, directed to the hyperbolic domain, point towards (resp. away from) the godron (Fig. 3). Some authors use the term hyperbolic (resp. elliptic)).
The asymptotic double of the hyperbolic domain. A godron $g$ can be positive or negative, depending on the index of the direction field, which is naturally associated to $g$, on the asymptotic double $A$ of $S$: The asymptotic double of $S$ is the surface $A$ in the manifold of contact elements of $S$, $PT^*S$, consisting of the field of asymptotic directions. It doubly covers the hyperbolic domain, and its projection to $S$ has a fold singularity over the parabolic curve. There is an asymptotic lifted field of directions on the surface $A$, constructed in the following way. At each point of the contact manifold $PT^*S$ a contact plane is applied, in particular at each point of $A$. Consider a point of the smooth surface $A$ and assume that the tangent plane of $A$ at this point does not coincide with the contact plane. Then these two planes intersect along a straight line tangent to $A$. The same holds at all nearby points in $A$. This defines a smooth direction field on $A$ which vanishes only at the points where those planes coincide: over the godrons.

If $g$ is a positive godron, then the index of this direction field at its singular point equals $+1$, the point being a node or a focus; if $g$ is negative, the index equals $-1$ and the point is a saddle. See Fig. 4.

**Proposition 2.** A godron $g$ is positive (negative) if and only if the value of its cr-invariant $\rho$ satisfies: $\rho(g) > 1$ (resp. $\rho(g) < 1$).

We say that the elliptic domain of a smooth surface $S$ is locally convex in the neighbourhood of a godron if, when projected to the tangent plane to $S$ (from any
Figure 4. The asymptotic double of the hyperbolic domain near a godron.

point, exterior to this plane), the image of the elliptic domain is locally convex: The tangent line of the parabolic curve being locally inside the image of the hyperbolic domain (the hyperbolic domain being locally convex if this line lies locally in inside the image of the elliptic domain).

Corollary 2 (of Theorem 5). At a godron \( g \) with \( cr \)-invariant \( \rho \) the hyperbolic (elliptic) domain is locally convex if and only if \( \rho > 2/3 \) (resp. \( \rho < 2/3 \)).

Proof. The hyperbolic (elliptic) domain is locally convex at \( g \) if and only if the canonical coefficient \( c_P \) of the parabolic curve is positive (resp. negative). \( \square \)

Theorem 6. (a) In the neighbourhood of any positive godron the hyperbolic domain is locally convex.

(b) There exist negative godrons for which the neighbouring hyperbolic domain is locally convex.

(c) At the negative godrons for which the neighbouring hyperbolic domain is locally convex, the flecnodal curve lies locally between \( P \) and \( D \) (see Fig. 2). Moreover, the \( cr \)-invariant satisfies: \( \frac{2}{3} < \rho < 1 \).

Proof. The theorem follows from Proposition 2 and Corollary 2. \( \square \)

Statements (a) and (b) of Theorem 6 were discovered by F. Aicardi [1].

Corollary 3. All godrons of a cubic surface in \( \mathbb{RP}^3 \) are negative.

Proof. By the definitions of asymptotic curve and of flecnodal curve, any straight line contained in a smooth surface is both an asymptotic curve and a connected component of the flecnodal curve of that surface.

Let \( S \) be an algebraic surface of degree 3. At a point of the flecnodal curve, an asymptotic line has at least 4-point contact with \( S \). Since \( S \) is a cubic surface, this line must lie completely in \( S \). So the flecnodal curve of \( S \) consists of straight lines.

At a godron \( g \) of \( S \), the tangent line to the parabolic curve (that is, the flecnodal curve) lies in the hyperbolic domain. Thus the neighbouring elliptic domain is locally convex. Therefore, by the above theorem, \( g \) is negative. \( \square \)

Factorisable polynomials. A set \( \{\ell_1, \ldots, \ell_n\} \) of real affine functions on the plane is said to be in general position if: (i) The lines \( \ell_i = 0, \ell_j = 0 \) are not parallel \((i \neq j, i, j \in \{1, \ldots, n\})\), and (ii) For any \( i \in \{1, \ldots, n\} \), the line \( \ell_i = 0 \) contains
no critical point of the function $\prod_{j \neq i} \ell_j$. The product $\prod \ell_i$ of $n$ real affine functions in general position is called a factorisable polynomial.

In [19], A. Ortiz-Rodríguez proved, among other things, that for any real factorisable polynomial of degree $n$, $f = \prod \ell_i$, the following holds: (i) The lines $\ell_i$ are the only components of the flecnodal curve of the graph of $f$, (ii) This graph has exactly $n(n - 2)$ godrons, and

**Proposition 3** (Theorem 1 and Lemma 13 of [19]). All godrons of the graph of a real factorisable polynomial are negative.

**Proof.** Theorem 6 provides an alternative and very simple proof of Proposition 3: Since the flecnodal curve consists of straight lines, at each godron $g$ the asymptotic tangent line and the asymptotic curve coincide with one of such straight lines. Thus, the elliptic domain is locally convex at $g$, and hence $g$ is a negative godron (by Theorem 6).

\[ \square \]

### 3.3. Locating the left and right branches of the flecnodal curve

**Remark on the co-orientation of the elliptic domain.** Each connected component of the elliptic domain is ‘naturally’ co-oriented: At each elliptic point the surface lies locally on one of the two half-spaces determined by its tangent plane at that point. This half-space, that we name positive half-space, determines a natural co-orientation on each connected component of the elliptic domain. By continuity, the natural co-orientation extends to the parabolic points (where the parabolic curve is smooth). At the parabolic points a positive half-space is therefore also defined.

This observation has strong topological consequences. For example:

**Theorem 7.** The elliptic domain of any smooth surface in the 3-space (Euclidean, affine or projective) can not contain a Möbius strip.

**Proof.** If a Möbius strip $M$ were contained in the elliptic domain $E$ of a surface, then it would be contained in a connected component of $E$, since $M$ is connected. Now, the theorem follows since each connected component of $E$ has a natural co-orientation (and $M$ is not co-orientable).

\[ \square \]

In the neighbourhood of a godron $g$ of a smooth surface $S$, we can distinguish explicitly which branch of the flecnodal curve is the right branch and which is the left one. For this, we need only to know the index of $g$ and the natural co-orientation of $S$ (given by the positive half-space at $g$):

Let $g$ be a godron, with $\rho \neq 1$, of a smooth surface $S$. Take an affine coordinate system $x$, $y$, $z$ such that the $(x, y)$-plane is tangent to $S$ at $g$, and the $x$-axis is tangent to the parabolic curve at $g$ (thus also tangent to $F$ at $g$). Direct the positive $z$-axis to the positive half-space at $g$. Direct the positive $y$-axis towards the neighbouring hyperbolic domain. Finally, direct the positive $x$-axis in such way that any basis $(e_x, e_y, e_z)$ of $x$, $y$, $z$ form a positive frame for the fixed orientation of $\mathbb{R}^3$ (or of $\mathbb{R}P^3$).

So one can locally parametrise the flecnodal curve at $g$ by projecting it to the $x$-axis.
Theorem 8. Under the above parametrisation, the left and right branches of the flecnodal curve at \( g \) correspond locally to the negative and positive semi-axes of the \( x \)-axis, respectively, if and only if \( g \) is a positive godron. The opposite correspondence holds for a negative godron.

In other words, if you stand on the tangent plane of \( S \) at \( g \) in the positive half-space and you are looking from the elliptic domain to the hyperbolic one, then you see the right (left) branch of the flecnodal curve on your right hand side if and only if \( g \) is a positive (resp. negative) godron, assuming the space is oriented with the usual right hand side rule. So the index of \( g \) determines and is determined by the side on which the right branch of \( F \) is located.

Remark. Theorems 2 and 8 (which are local theorems) together with the natural co-orientation of the elliptic domain, are the key elements to prove the global theorem (Theorem 10) of Section 3.6. They imply that some (global) configurations of the flecnodal curve are forbidden. So, for example, there is no surface having a hyperbolic disc without hyperbonodes.

3.4. The flec-godrons: Degenerated godrons with \( \rho = 0 \). After the preceding sections, a natural question arises: What happens if the \( \rho \)-invariant equals 0 or 1? The godrons for which the \( \rho \)-invariant equals 0 or 1 are degenerated godrons. We will explain the meaning of these degeneracies and describe the behaviour of such degenerated godrons under a small perturbation of the surface inside a generic one parameter family of smooth surfaces.

The case \( \rho = 0 \). If \( \rho = 0 \) then the 4-jet given by the (Platonova) normal form, used above, defines just a cubic surface, which is absolutely not generic: The asymptotic tangent line at that godron has 'infinite point-contact' with the surface and coincides with the flecnodal curve. In order to understand the behaviour of the flecnodal curve (and the geometric properties of the surface) at the godrons with \( \rho = 0 \), we need to add some terms of degree 5, which in this case are relevant and break the symmetry. In fact, we have the

Proposition 4. At a godron of a smooth surface the asymptotic tangent line has 4-point contact with the surface if and only if \( \rho \neq 0 \). At a godron with \( \rho = 0 \) the asymptotic tangent line and the surface have at least 5-point contact.

Proof. Consider the godron \( g \) with \( \rho \)-invariant \( \rho \) of the surface given by the graph of a function \( z = f(x, y) \). Let \( z = \frac{\partial^4}{\partial x^4} - x^2 y + \frac{\varphi_5}{2} x^4 + \varphi_6(x, y) + \cdots \) its Taylor decomposition in homogeneous forms. Since the asymptotic tangent line \( \ell \) at \( g \) is the \( x \)-axis, its order of contact with \( S \) at \( g \) is the multiplicity of the zero of the function \( h \circ \gamma \) at \( t = 0 \), where \( h(x, y, z) = -z + \frac{\varphi_5}{2} - x^2 y + \frac{\varphi_5}{2} x^4 + \varphi_6(x, y) + \cdots \) and \( \gamma(t) = (x(t), y(t), z(t)) = (t, 0, 0) \).

Since \( h \circ \gamma \) has the form \( (h \circ \gamma)(t) = \frac{\varphi_5}{2} t^4 + a t^5 + \cdots \), (where \( a \) is the coefficient of \( x^5 \) in \( \varphi_5(x, y) \)), the asymptotic line \( \ell \) has 4-point contact with \( S \) at \( g \) if and only if \( \rho \neq 0 \), and 5-point contact if and only if \( \rho = 0 \) (and \( a \neq 0 \)). \( \Box \)
**Definition.** A flec-godron of a smooth surface is a godron at which the asymptotic tangent line and the surface have at least 5-point contact (being exactly 5-point contact for a simple flec-godron).

It follows that a smooth surface in general position has no flec-godron: Under any small generic deformation of the surface the flec-godron condition of 5-point contact of the surface with the asymptotic line at \( g \) (or, equivalently, \( \rho = 0 \)) is destroyed. The smooth surfaces in the 3-space having one simple flec-godron form a discriminant hypersurface in the space of smooth surfaces. Thus, a generic 1-parameter family of smooth surfaces (that is, transversal to this discriminant hypersurface) can have, at isolated parameter values, a surface having one simple flec-godron. We shall describe the local bifurcation of the surface (of its tangential singularities) when a generic 1-parameter family crosses this discriminant hypersurface.

**Example.** The godron of the surface \( z = \frac{y^2}{2} - x^2 y + ax^5 \), with \( a \neq 0 \), is a simple flec-godron. We will perturb this surface inside a generic one parameter family of smooth surfaces in which the parameter is the cr-invariant \( \rho \):

\[
z = \frac{y^2}{2} - x^2 y + \frac{\rho}{2} x^4 + ax^5.
\]

The flecnodal, conodal and parabolic curves of this surface (with \( a > 0 \)) are depicted in Fig. 5 for \( \rho < 0 \), \( \rho = 0 \) and \( \rho > 0 \) (compare with Fig. 2).

![Figure 5](image)

**Figure 5.** The transition of the flecnodal curve (half-black half-white curve) and of the conodal curve (thick curve) at a flec-godron transition: \( \rho = 0 \).

Indeed, the power series expansions of the curves \( \overline{F} \) and \( \overline{D} \) start by

\[
y = \rho(2\rho - 1)x^2 + 10a(2\rho - 1)x^3 + \cdots \quad \text{and} \quad y = \rho x^2 + 2ax^3 + \cdots,
\]

respectively. So, for \( \rho = 0 \) we have, \( \overline{F} : y = -10ax^3 + \cdots \) and \( \overline{D} : y = 2ax^3 + \cdots \) (whence the bifurcation of Fig. 5).

Let \( S_t, \ t \in \mathbb{R} \), a generic one parameter family of smooth surfaces such that for \( t = 0 \) the surface \( S_0 \) has a flec-godron \( y_0 \) (for instance the above family).

As \( t \) is increasing and passing through 0, a biflenode \( b_t \) of \( S_t \) is moving along the flecnodal curve, passing, at \( y_0 \), from one branch to the other of the flecnodal curve. Roughly speaking, ‘a flec-godron is the superposition of a biflenode and a godron’:
Theorem 9. At the flec-godron (transition) the following happens: For any $t \neq 0$, sufficiently close to 0, the surface $S_t$ possesses an ordinary godron and a neighbour-
ing biflecnode. The biflecnode being left for all (small) $t$ of a given sign and being right for all (small) $t$ of the opposite sign. Moreover, for $t = 0$ both the flecnodal curve and the conodal curve of $S_0$ have an inflection at $g_0$.

Note that the index of the flec-godron of $S_0$ (and of the godron of $S_t$, for all $|t|$ sufficiently small) is $-1$, since $\rho < 1$.

The corresponding bifurcation of the tangential singularities on the dual surface is described in Section 3.8.1 (see Fig. 9).

3.5. The bigodrons: Degenerated godrons with $\rho = 1$

The case $\rho = 1$. If $\rho = 1$, then we also have a degenerate godron, which we name bigodron: it is the collapse (or the birth) of two godrons with opposite indices (it is not a simple godron). When $\rho = 1$ the normal form of the 4-jet that we used above is not convenient since it is degenerate: $z = \frac{1}{2}(y - x^2)^2$. For this reason the parabolic and flecnodal curves coincide with the curve $y = x^2$ in the $(x, y)$-plane (this curve is sent to a point under the tangential map of $S$). In order to have a generic polynomial of degree four, one must add another term of degree four: $z = \frac{1}{2}(y - x^2)^2 \pm x^3y$. Now, the bigodron obtained is generic (among the bigodrons: $\rho = 1$): the parabolic and flecnodal curves have 4-point contact and the whole flecnodal curve is either left or right, according to the sign $+$ or $-$ of the term $\pm x^3y$, respectively (see the central part of Fig. 6). To understand better the geometry of a bigodron, we will perturb this surface inside a generic one parameter family of smooth surfaces:

$$z = \frac{1}{2}(y - x^2)^2 \pm x^3y + \epsilon x^3.$$

The flecnodal and parabolic curves of this surface are depicted in Figure 6 for $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$. When the parameter $\epsilon$ is negative the flecnodal curve is left and does not touch the parabolic curve, while when $\epsilon$ is positive the flecnodal curve touches the parabolic curve at two neighbouring godrons with opposite indices, and a small segment of the right flecnodal curve has appeared between these godrons.

So, in the oriented 3-space (Euclidean, affine or projective), there are two types of bigodrons: a bigodron is said to be left (right) if it corresponds to a bifurcation in which a small segment of the left (resp. right) branch of the flecnodal curve is born or vanishes.
Remark. The case $\rho = 1$ (a bigodron) corresponds to an $A_4$ contact with the tangent plane (cf. [9], [10], [4]). That is, for the dual surface it corresponds to the $A_4$ bifurcation of wave fronts (two swallowtails are born or dying) occurring in generic one parameter families of fronts [2]. Thus, in the oriented 3-space $\mathbb{RP}^3$ (or $\mathbb{R}^3$), there are two types of $A_4$ singularities of wave fronts.

Remark. Strictly speaking, the $cr$-invariant is not defined at the bigodron (or $A_4$) singularity, since there is no conodal curve. However, the limit value of the $cr$-invariant for both dying (or born) godrons, at the bigodron bifurcation moment, is equal to 1.

3.6. Elliptic discs and hyperbolic discs of smooth surfaces. The following global theorem holds for any generic smooth surface:

Theorem 10. In any hyperbolic disc bounded by a Jordan parabolic curve, there is an odd number of hyperbonodes (hence at least one).

Remark. The hyperbonodes and the ellipnodes of a smooth surface are the points at which that surface is better approximated by a quadric. Indeed, a surface in $\mathbb{RP}^3$, different from a plane, is a one sheet hyperboloid (or an ellipsoid), if and only if all its points are hyperbonodes (resp. ellipnodes). These points play an essential role in (and are necessary for) several bifurcations of the parabolic curve and of wave fronts [27].

The cubic surfaces in $\mathbb{RP}^3$ provide examples of surfaces having elliptic discs whose bounding parabolic curves have $0$, $1$, $2$ or $3$ negative godrons: According to Segre [23], a generic cubic surface diffeomorphic to the projective plane contains four parabolic curves (each one bounding an elliptic disc) and six godrons. According to [5], Shustin has proved that the distribution of the godrons among the four parabolic curves is $6 = 0 + 1 + 2 + 3$. By Corollary 3, all these godrons are negative.

There exist smooth surfaces having an elliptic disc whose bounding parabolic curve has $4$ negative godrons:

Example. The algebraic surface given by the equation
$$z = (x^2 - 1)(y^2 - 1)$$
has an elliptic disc whose bounding parabolic curve contains $4$ godrons, all negatives.

Problem. Exist there smooth surfaces in the 3-space (affine or projective) having an elliptic disc whose bounding parabolic curve has more than $4$ godrons, all of them negative?

For a parabolic curve bounding a hyperbolic disc the situation is more restrictive:

Proposition 5. The sum of the indices of the godrons on the parabolic curve bounding a hyperbolic disc (of a generic surface) equals two. In particular, such parabolic curve contains a positive even number of godrons.

Proof. Write $H$ for the closure of the hyperbolic disc. The asymptotic double $A$ is a sphere. Its Euler characteristic equals 2. By Poincaré Theorem, the sum of indices of all singular points of the direction field on $A$ equals 2. □
In fact, for an immersed surface in general position in $\mathbb{RP}^3$ $(\mathbb{R}^3)$ Theorem 2 implies:

**Theorem 11.** For each connected component of the hyperbolic domain, whose boundary is contained in the parabolic curve, the flecnodal curve is the union of closed curves each of them having an even number (possibly zero) of godrons (that is, of contact points with the boundary of that domain). The godrons decompose these closed curves into left and right segments.

**Corollary 4.** The boundary of each connected component of the hyperbolic domain of a generic surface has an even number of godrons.

The statement of Corollary 4 belongs to Thom and Banchoff, [8]. Unfortunately their proof is not exact, since it is based in a wrong statement: *The Euler characteristic of a connected component $H$ of the hyperbolic domain equals the number of godrons in $\partial H$ at which the hyperbolic domain is locally convex (the asymptotic line has contact with $\partial H$ interior to $H$) minus the number of godrons in $\partial H$ at which the elliptic domain is locally convex (the asymptotic line has contact with $\partial H$ exterior to $H$).* This is wrong: 1) In the bigodron bifurcation (see Section 3.5) two godrons are born (or killed); 2) At both godrons the contact of the asymptotic line with $\partial H$ is exterior to $H$ and 3) This bifurcation does not change the Euler characteristic of $H$.

### 3.7. Godrons and Swallowtails

**Tangential Map and Swallowtails.** It is well known (cf. [22]) that under the tangential map of $S$ the parabolic curve of $S$ corresponds to the cuspidal edge of $S^\vee$, the conodal curve of $S$ corresponds to the self-intersection line of $S^\vee$ (this follows from the definitions of dual surface and conodal curve) and a godron corresponds to a swallowtail point.

**Legendrian Remark.** The most natural approach to the singularities of the tangential map is via Arnold’s theory of Legendrian singularities [6]. The image of a Legendrian map is called the *front* of that map. The tangential map of a surface is a Legendre map, and so it can be expected to have only Legendre singularities. Thus for a surface in general position, the only singularities of its dual surface (i.e. of its front) can be: self-intersection lines, cuspidal edges and swallowtails. So the godrons are the most complicated singularities of the tangential map of a generic surface.

**Definition of Front.** In this paper, a *front in general position* is a surface $S$ whose singularities, and the singularities of its dual surface $S^\vee$, are at most: self-intersection lines, semi-cubic cuspidal edges and swallowtails. Moreover, we require that the parabolic curve never passes through a swallowtail point (the same requirement for the dual front). In other words, we are requiring the Legendrian manifold $L_S = L_{S^\vee}$ in $PT^*\mathbb{RP}^3 = PT^*(\mathbb{RP}^3)^\vee$ (of the contact elements of $\mathbb{RP}^3$ tangent to $S$) to be in general position with respect to both natural Legendrian fibrations $\pi: PT^*\mathbb{RP}^3 \to \mathbb{RP}^3$ and $\pi^\vee: PT^*(\mathbb{RP}^3)^\vee \to (\mathbb{RP}^3)^\vee$. Thus, for a front in general position all godrons are simple.
Example (the standard swallowtail). The standard swallowtail is the discriminant of the vector space of polynomials of the form $x^4 + ax^2 + bx + c$ consisting of all those points $(a, b, c) \in \mathbb{R}^3$ for which the polynomial has a multiple root. This discriminant is the tangent developable of the curve $\gamma: t \mapsto (-6t^2, 8t^3, -3t^4)$ consisting of such polynomials having a triple real root. So, as any developable surface, it is the envelope of a 1-parameter family of planes (the osculating planes of the above curve $\gamma$, in this case), and hence its dual “surface” (that is, its image under the tangential map) is just a curve. This shows that, from the point of view of projective (or affine) differential geometry, the standard swallowtail is absolutely not in general position (in particular, all its points exterior to its cuspidal edge $\gamma$ are parabolic).

The cr-invariant of a swallowtail. We can associate a projective invariant (a number) to a swallowtail point $s$ of a generic front $S$: We apply the tangential map of $S$ (in a neighbourhood of $s$) to obtain a locally smooth surface $S^\vee$ having a godron with cr-invariant $\rho$. The number $\rho(s) := \rho$ is associated to the swallowtail $s$.

The tangential map of $S$ sends the elliptic (hyperbolic) domain of $S$ to the elliptic (resp. hyperbolic) domain of $S^\vee$. Thus the hyperbolic and elliptic domains of a front in general position are separated by the cuspidal edge (and by the parabolic curve). This implies that there are two types of swallowtails:

Definition. A swallowtail point of a generic front is said to be hyperbolic (elliptic) if, locally, the self-intersection line of that front is contained in the hyperbolic (resp. elliptic) domain.

The proofs of the following theorems show that the configurations of the curves $F$, $P$ and $D$ at a godron have a relevant meaning for the local (projective, affine or Euclidean) differential properties of the swallowtails.

Theorem 12. The dual of a surface at a positive godron is an elliptic swallowtail. The dual of a surface at a negative godron is a hyperbolic swallowtail.

Proof. By Proposition 2, a godron $g$ is positive (negative) if and only if its cr-invariant satisfies $\rho(g) > 1$ (resp. $\rho(g) < 1$).

By Theorem 5, $\rho(g) > 1$ (resp. $\rho(g) < 1$) if and only if the conodal curve at $g$ lies locally in the elliptic (hyperbolic) domain.

Finally, since the tangential map sends the elliptic (hyperbolic) domain to the elliptic (resp. hyperbolic) domain of the dual surface, it is evident that the conodal curve at $g$ lies locally in the elliptic (hyperbolic) domain if and only if the dual surface is an elliptic (resp. hyperbolic) swallowtail. □

Theorem 13. In the neighbourhood of a swallowtail point $s$ of a front $S$ in general position, the flecnodal curve $F$ has a cusp whose tangent direction coincides with that of the cuspidal edge. The point $s$ separates $F$ locally into its left and right branches. There are four possible generic configurations of $F$ in the neighbourhood of $s$ (see Fig. 7):

(e) For an elliptic swallowtail the flecnodal curve is a cusp lying in the small domain bounded by the cuspidal edge ($\rho(s) \in (1, \infty)$).
There are 3 different generic types of hyperbolic swallowtails.

(h₁) Each branch of the cuspidal edge is separated from the self-intersection line by one branch of the flecnodal curve ($\rho(s) \in (0, 1)$).

(h₂) The self-intersection line lies between the two branches of the flecnodal curve and separates them from the branches of the cuspidal edge. The cusp of the flecnodal curve points in the same direction as the cusp of the cuspidal edge ($\rho(s) \in (-\frac{1}{2}, 0)$).

(h₃) The cusp of the flecnodal curve and the cusp of the cuspidal edge are pointing in opposite directions ($\rho(s) \in (-\infty, -\frac{1}{2})$).

There are 7 possible generic configurations of $F$, $\Sigma$, and $D^\vee$ with respect to the tangent plane (see Fig. 8).

There are 3 different generic configurations for elliptic swallowtails:

(e₁) The surface lies locally on one side of the tangent plane at $s$ ($\rho > 4/3$).

3.8. Finer classification of swallowtails. Besides the description of the 4 generic configurations of the flecnodal curve in the neighbourhood of a swallowtail point (given in Theorem 13 and Fig. 7), it is interesting to know how the cuspidal edge, the flecnodal curve and the self-intersection line of the swallowtail surface are placed with respect to the tangent plane.

The following theorem is a refinement of Theorem 13, providing the local configurations of the cuspidal edge, the flecnodal curve and the self-intersection line with respect to the tangent plane, at a swallowtail.

Theorem 14. In the notations – and conclusions – of Theorem 13, write $\Sigma$ for the cuspidal edge of the front $S$, and $D^\vee$ for its self-intersection line.

If $s$ is elliptic and $1 < \rho < 4/3$ (resp. hyperbolic, $\rho < 1$), then the tangent plane to $S$ at $s$ intersects $S$ along two semi-cubic cusps $C^{-}_\Sigma$ and $C^+_\Sigma$, pointing in the same direction — as the cuspidal edge — (resp. in opposite directions) and lying one in the elliptic and the other in the hyperbolic domain, respectively (resp. lying both in the hyperbolic domain). There are 7 possible generic configurations of $F$, $\Sigma$, and $D^\vee$ with respect to the tangent plane (see Fig. 8).

There are 3 different generic configurations for elliptic swallowtails:

(e₁) The surface lies locally on one side of the tangent plane at $s$ ($\rho > 4/3$).
The cusp $C^\vee$ separates locally the cuspidal edge $\Sigma$ from the flecnodal curve $F$, and $C^\vee_+$ separates $\Sigma$ from the self-intersection line $D^\vee$ ($\rho \in (\sqrt{\frac{7}{2}}, \frac{1}{3})$).

The flecnodal curve $F$ separates locally $\Sigma$ from $C^\vee_+$, and again $C^\vee_-$ separates $\Sigma$ from $D^\vee$ ($\rho \in (1, \sqrt{\frac{7}{2}})$).

There are 4 different generic configurations for hyperbolic swallowtails:

1. The cusp $C^\vee_+$ separates locally $F$ from $D^\vee$ ($\rho \in (0, 1)$).
2. Again, the cusp $C^\vee_+$ separates locally $F$ from $D^\vee$ ($\rho \in (-\frac{1}{2}, 0)$).
3. The tangent plane separates locally $F$ from $\Sigma$ and $D^\vee$ ($\rho \in (-\sqrt{\frac{7}{2}}, -\frac{1}{2})$).
4. The curves $F$, $\Sigma$ and $D^\vee$ lie locally on the same side of the tangent plane to $S$ at $s$ ($\rho < -\sqrt{\frac{7}{2}}$).

Figure 8. The 7 generic configurations of the flecnodal curve, the self-intersection line, the cuspidal edge and the tangent plane at a swallowtail.

3.8.1. The local transition at swallowtails with $\rho = 0$. Now we can describe the local bifurcation of the tangential singularities occurring in a generic one parameter family of wave fronts at the moment of a swallowtail with $\rho$-invariant $\rho = 0$.

Let $S_t$, $t \in \mathbb{R}$, a generic one parameter family of fronts such that for $t = 0$ the front $S_0$ has a swallowtail point $s_0$ with $\rho = 0$ (we name it a flec-swallowtail).

Just by duality of the flec-godron transition, we have that as $t$ is increasing and passing through $0$, a biflenode $b_t$ of $S_t$ is moving along the flecnodal curve, passing, at $s_0$, from one branch to the other of the flecnodal curve. So, ‘a flec-swallowtail is the superposition of a biflenode and a swallowtail’.

In Fig. 9, we show the pictures describing the bifurcation of the flecnodal curve occurring in a generic one parameter family of wave fronts at a flec-swallowtail. They follow from Section 3.4, Theorem 14, and Figures 5 and 8.

At the moment with $\rho = 0$ both the self-intersection line $D^\vee$ and the flecnodal curve $F$ have higher order of contact with the tangent plane; in particular, for $\rho > 0$, $D^\vee$ lies locally in the opposite side of the tangent plane than for $\rho < 0$. The flecnodal curve is also locally passing from one side to the other of the tangent plane at $\rho = 0$. 
3.9. The local $q$-contour of a surface. We shall investigate whether any arbitrarily small neighbourhood of a godron $g$ (or of a swallowtail point $s$) of a front $S$ in general position has (or not) points $p \in S$ such that the tangent plane to $S$ at $p$ passes through $g$ (resp. through $s$).

Consider a point $q$ of a smooth surface or of a wave front $S$ in the 3-space (Euclidean, affine or projective).

**Definition.** The $q$-contour of $S$ is the set of points of tangency of the planes tangent to $S$ passing through $q$.

Equivalently, the $q$-contour of $S$ is the set of points of tangency of the lines tangent to $S$ passing through $q$. The following lemma is evident:

**Lemma.** The $q$-contour of $S$, without the point $q$, consist of the critical points of the “stereographic” projection of $S$, from $q$ to $\mathbb{RP}^2_q$, $\pi_q : S \setminus \{q\} \to \mathbb{RP}^2_q$, (associating to each point $p \in S$ the line joining it to $q$).

One usually considers the projection of a surface from a point exterior to it (for which the set of critical points is, generically, a possibly empty smooth curve), but here we shall consider the projection of a surface from a point belonging to it, and the critical points of such a projection.

Hence, given a point $q$ of a smooth surface (or of a wave front) $S$ in the 3-space, it is interesting to know whether any arbitrarily small neighbourhood of $q$ contain points of the $q$-contour of $S$ different from $q$ and, if it is the case, to know its behaviour at $q$.

**Remark.** For any point $q$ of a smooth surface (or of a wave front) $S$ the $q$-contour of $S$ is projectively invariant.

**Definition.** The local $q$-contour of a smooth surface (or of a wave front) $S$ is the germ at $q$ of the $q$-contour of $S$. We denote it by $C^q(S)$.

We will say that the local $q$-contour of a smooth surface (or of a wave front) is trivial if it consists just of the point $q$. 

Figure 9. Two points of view of the transition at a swallowtail with $\rho = 0$. 
Example. For the elliptic points of a surface (or of a front) $S$ the $q$-contour of $S$ is trivial. For the hyperbolic points the $q$-contour of $S$ consist of two transverse curves (each one being tangent to one of the asymptotic lines).

The following theorem shows, for example, that there are two essentially different kinds of positive godrons: For one kind the local $g$-contour is non-trivial and for the other it is trivial. This difference is more visible on the dual surface: The tangent plane traverses locally the surface or not (see Fig. 8).

**Theorem 15.** Let $g$ be a godron of a smooth surface $S$.

(a) There exists a neighborhood $U$ of $g$, in $S$, which contains no point of the set $C^g(S) \setminus \{g\}$ (i.e. no tangent plane to $U \setminus \{g\}$ passes through $g$) if and only if

$$\rho(g) > \frac{4}{3}.$$

(b) If $\rho(g) < \frac{4}{3}$, $\rho(g) \neq 1$, then, in any sufficiently small neighborhood of $g$, the $g$-contour of $S$ consists of two smooth curves tangent to the parabolic curve at $g$.

If the godron $g$ is positive ($1 < \rho < \frac{4}{3}$), then one of these two curves lies locally in the elliptic domain and the other in the hyperbolic domain. If $g$ is negative ($\rho < 1$), then both curves lie in the hyperbolic domain.

In case $b$ of Theorem 15, we write $C_-$ and $C_+$ for the tangent curves forming the local $g$-contour of $S$. Since $C_-$ and $C_+$ are tangent to the parabolic curve at $g$, their projections to the tangent plane can be written locally as $y = c_G x^2 + \cdots$ and $y = c_G^+ x^2 + \cdots$ (using Platonova’s normal form). The relative positions of $F$, $P$ and $D$ with respect to $C_-$ and $C_+$ are determined by the order, in the real line, of the coefficients $c_F$, $c_P$, $c_D$, $c_\sigma$ and $c_G$.

**Theorem 16.** Given a godron $g$ of a generic smooth surface, there are 7 possible configurations of the curves $F$, $P$ and $D$ with respect to the $g$-contour the surface at $g$ (they are represented in Fig. 10). The actual configuration at $g$ depends on which of the following 7 open intervals the $cr$-invariant $\rho(g)$ belongs to, respectively:

\[
\begin{align*}
\rho & \in \left(\frac{4}{7}, \infty\right) \quad \iff \quad 1 < c_D < c_P < c_F \quad (C^g(S) \text{ is trivial}); \\
\rho & \in \left(\frac{\sqrt{7}}{7}, \frac{4}{7}\right) \quad \iff \quad 1 < c_D < c_G^- < c_P < c_G^+ < c_F; \\
\rho & \in \left(1, \frac{\sqrt{7}}{7}\right) \quad \iff \quad 1 < c_D < c_G^- < c_P < c_F < c_G^+; \\
\rho & \in (0, 1) \quad \iff \quad c_P < c_F < c_G^- < c_D < 1 < c_G^+; \\
\rho & \in (-\frac{1}{2}, 0) \quad \iff \quad c_P < c_D < c_G^- < c_F < 1 < c_G^+; \\
\rho & \in (-\frac{\sqrt{7}}{7}, -\frac{1}{2}) \quad \iff \quad c_P < c_D < c_G^- < 1 < c_F < c_G^+; \\
\rho & \in (-\infty, -\frac{\sqrt{7}}{7}) \quad \iff \quad c_P < c_D < c_G^- < 1 < c_G^+ < c_F.
\end{align*}
\]

3.9.1. The local $q$-contour at a swallowtail point. It is also interesting to know whether the local $s$-contour at a swallowtail point $s$ of a front in general position is trivial or not, and, if it is non trivial, to know how it is placed with respect to the self-intersection line $D^y$, the cuspidal edge $\Sigma$ and the flecnodal curve $F$.

The following theorem is a refinement of Theorem 13, providing the required local configurations.
Theorem 17. In the notations and conclusions of Theorem 13:
(a) The local $s$-contour of $S$ is trivial (consisting just of $s$) if and only if $s$ is an elliptic swallowtail ($\rho > 1$).
(b) If $s$ is hyperbolic, then the local $s$-contour of $S$ consists of two cusps $T_+^\nu$, $T_-^\nu$, pointing in opposite directions. In this case, there are 6 possible generic configurations of the local $s$-contour with respect to $F$, $\Sigma$ and $D^\nu$ (see Fig. 11):

\begin{enumerate}
  \item ($h_{1,1}$) $T_+^\nu$ lies in the elliptic domain ($\rho \in (\frac{\sqrt{3}}{2}, 1)$).
  \item ($h_{1,2}$) $T_-^\nu$ separates locally $\Sigma$ from $F$ ($\rho \in (\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3})$).
  \item ($h_{1,3}$) $T_-^\nu$ separates locally $F$ from $D^\nu$ ($\rho \in (0, \frac{\sqrt{3}}{2})$).
  \item ($h_{2,1}$) $T_+^\nu$ separates locally $D^\nu$ from $F$ ($\rho \in (-\frac{1}{2}, 0)$).
  \item ($h_{3,1}$) $T_-^\nu$ separates $D^\nu$ from $F$. $T_-^\nu$ and $T_+^\nu$ are separated by $F$ ($\rho \in (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$).
  \item ($h_{3,2}$) $T_-^\nu$ and $F$ are separated locally by $T_+^\nu$ ($\rho \in (-\infty, -\frac{\sqrt{3}}{2})$).
\end{enumerate}

3.10. The tangent section of a surface or of a front. The intersection of a surface $S$ with its tangent plane at a point $q$ will be called the $q$-tangent section of $S$ or the tangent section of $S$ at $q$.

Definition. The local tangent section of a smooth surface (or of a front) $S$ at its point $q$ is the germ at $q$ of the $q$-tangent section of $S$. The local tangent section at $q$ is said to be trivial if it consists just of the point $q$.

Lemma 1. Consider a simple godron $g$ of a smooth surface $S$.

(a) The local tangent section of $S$ at $g$ is trivial (consisting just of $g$) if and only if $g$ is a positive godron ($\rho > 1$).

(b) If $g$ is negative ($\rho < 1$), then the local tangent section of $S$ at $g$ consists of two smooth curves (simply) tangent to the parabolic curve at $g$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{The 7 generic configurations of the curves $F$ (half-white half-black curves), $P$ (boundary between white and gray domains), $D$ (thick curves), the separating 2-jet (doted curves) and the $g$-contour (gray curves) at a godron $g$.}
\end{figure}
Figure 11. The 7 generic configurations of the flecnodal curve, the self-intersection line, the cuspidal edge and the local s-contour (gray cusps) at a swallowtail s.

Write \( T_- \) and \( T_+ \) for the tangent curves of the case \( b \) of Lemma 1, forming the local tangent section of \( S \) at \( g \). Since \( T_- \) and \( T_+ \) are tangent to the parabolic curve at \( g \), their projections to the tangent plane can be written locally as \( y = c_T x^2 + \cdots \) and \( y = c_T x^2 + \cdots \) (using Platonova’s normal form). The local configurations of \( F, P \) and \( D \) with respect to \( T_- \) and \( T_+ \) are determined by the relative positions of the coefficients \( c_F, c_P, c_D, c_\sigma = 1 \), with respect to \( c_T \) and \( c_T^+ \), in the real line.

**Theorem 18.** Given a simple godron \( g \) of a smooth surface, there are 7 possible configurations of the curves \( F, P \) and \( D \) with respect to the separating 2-jet and the local tangent section at \( g \) (they are represented in Fig. 12). The actual configuration at \( g \) depends on which of the following 7 open intervals the cr-invariant \( \rho(g) \) belongs to, respectively:

\[
\begin{align*}
\rho &\in (1, \infty) \quad \iff \quad 1 < c_D < c_P < c_F \quad (T^0(S) \text{ is trivial}); \\
\rho &\in \left( \frac{5}{9}, 1 \right) \quad \iff \quad c_T^- < c_P < c_F < c_D < 1 < c_T^+; \\
\rho &\in \left( \frac{\sqrt{3}}{2}, \frac{5}{9} \right) \quad \iff \quad c_P < c_T^- < c_F < c_D < 1 < c_T^+; \\
\rho &\in \left( 0, \frac{\sqrt{3}}{2} \right) \quad \iff \quad c_P < c_F < c_T^- < c_D < 1 < c_T^+; \\
\rho &\in \left( -\frac{1}{2}, 0 \right) \quad \iff \quad c_P < c_D < c_T^- < c_F < c_T^+; \\
\rho &\in \left( -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \quad \iff \quad c_P < c_D < c_T^- < 1 < c_F < c_T^+; \\
\rho &\in (-\infty, -\frac{\sqrt{3}}{2}) \quad \iff \quad c_P < c_D < c_T^- < 1 < c_F < c_T^+.
\end{align*}
\]

**Note.** By Theorem 5 (Fig. 2) the curves \( P \) and \( F \) change their local convexity at the values \( \rho = \frac{5}{9}, \rho = \frac{1}{2} \) in the interval \( 0 < \rho < \frac{\sqrt{3}}{2} \), respectively. So, in this interval we can choose 3 pictures with the same configuration but with different local convexities of \( P \) and \( F \). In Fig. 12 we have chosen the picture corresponding to the subinterval \( (0, \frac{1}{2}) \subset \left( 0, \frac{\sqrt{3}}{2} \right) \). In Fig. 10 we had the same situation and we chosen the same subinterval \( (0, \frac{1}{2}) \subset (0, 1) \).
Remark. The above theorems show that the value of the cr-invariant of a simple godron $g$ of a smooth surface $S$ determines completely the local configuration of the parabolic, flecnodal and conodal curves with respect to the local tangent section at $g$ and to the local $g$-contour of $S$ (determining also their local convexities). Similarly, the local configurations obtained for the swallowtails are also determined by the cr-invariant.

Remark. In [9], it was observed that there are two types of godrons, called elliptic and hyperbolic, corresponding to the sign of a ‘discriminant’ relating some coefficients of the 4-jet of the surface. In fact, they are the positive and negative godrons of Section 3.2 (definition of index), which are distinguished geometrically by the behaviour of the asymptotic directions along the parabolic curve (see Fig. 3). Here, we have provided several geometric characterisations of the godrons of indices +1 and −1: In terms of the cr-invariant (Proposition 2), in terms of the left and right flecnodal curves (Theorem 8), in terms of the geometry of the dual surface − elliptic and hyperbolic swallowtails (Theorem 12) and in terms of the asymptotic double (Figure 4). Of course, Lemma 1 also distinguishes these points (in terms of the local tangent section).

4. The Proofs of the Theorems

Preparatory conventions and results. In the sequel, we will consider the surface $S$ as the graph of a smooth function $z = f(x, y)$, where $x, y, z$ form an affine coordinate system. The asymptotic directions satisfy the equation:

$$f_{xx}(dx)^2 + 2f_{xy} dx dy + f_{yy}(dy)^2 = 0.$$
For $\frac{dy}{dx} = p$, this equation takes the form
\[ A^f(x, y, p) = f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0. \] (1)

Equation (1) is called the asymptote-equation of $f$.

In what follows, we will assume without loss of generality that the point under consideration in the $(x, y, p)$-space is the origin: by a translation and a rotation in the $(x, y)$-plane, we can take $(x, y) = (0, 0)$ and $p = 0$, respectively.

Moreover, we will take an affine coordinate system $x, y, z$ such that the $(x, y)$-plane is tangent to $S$ at the point under consideration. Thus we will have the conditions
\[ f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0. \] (2)

The parabolic curve of the surface $z = f(x, y)$ is the restriction of the graph of $f$ to the discriminant curve (in the $(x, y)$-plane) of equation (1). That is, the parabolic curve is determined by the equations
\[ A^f(x, y, p) = 0 \quad \text{and} \quad A^f_p(x, y, p) = 0. \] (*)

The fact that a godron is a folded singularity of eq. (1) implies that
\[ A^f_{xx}(0, 0, 0) = 0. \] (**) 

The conditions (*) and (**), at the origin in the $(x, y, p)$-space, imply that
\[ f_{xx} = f_{xy} = f_{xxx} = 0 \] at the origin in the $(x, y)$-plane.

The choice of a coordinate system such that the $x$-axis is an asymptotic direction of $S$ at the origin is equivalent to our assumption that the point under consideration in the $(x, y, p)$-space is the origin.

So the $x$-axis is tangent to the parabolic curve at the godron.

4.1. Proof of Theorem 3. Let $\gamma(t) = (x(t), y(t), z(t))$ be a curve on $S$, where $z(t) = f(x(t), y(t))$, which is tangent to the parabolic curve at the origin, that is,
\[ \dot{y}(0) = 0. \] (4)

Since all our calculations and considerations take place at the origin $(x, y) = (0, 0)$ and at $t = 0$, we will omit to write this explicitly.

Evidently conditions (2) imply $\ddot{z} = f_x\ddot{x} + f_y\ddot{y} = 0$. The equality
\[ \ddot{z} = f_x\ddot{x} + f_y\ddot{y} + (f_{xx}\dddot{x} + 2f_{xy}\ddot{x}\dddot{y} + f_{yy}\dddot{y}^2) \]
together with conditions (2), (3) and (4) imply that $\dddot{z} = 0$. This proves that the plane $z = 0$ is osculating.

Finally, the equality
\[ \dddot{z} = f_x\dddot{x} + f_y\dddot{y} + 3(f_{xx}\dddot{x} + f_{xy}\dddot{x}\dddot{y} + f_{yy}\dddot{y}) + f_{xxx}\dddot{x}^3 + 3f_{xyy}\dddot{x}\dddot{y} + 3f_{yy}\dddot{y}^2 + f_{yyyy}\dddot{y}^3 \]
together with conditions (2), (3) and (4), imply that $\dddot{z} = 0$, proving that the first three derivatives of $\gamma$ at $t = 0$ are linearly dependent (all of them lie in the $(x, y)$-plane). So $\gamma$ has a flattening or an inflection at the origin, according to the linear independence or dependence, respectively, of its first two derivatives at $t = 0$. \hfill \Box
4.2. Preliminary remarks and computations. We recall that Platonova’s theorem \[21\] implies that at a godron $g$ of a generic smooth surface $S$, there is an affine coordinate system such that the 4-jet of $S$ at $g$ is given by

$$z = \frac{y^2}{2} - x^2 y + \lambda x^4 \quad \text{(for some } \lambda \neq \frac{1}{2}, 0).$$ (G1)

The information we need about $S$ (for the proofs of our theorems) is contained in its 4-jet. The generic configurations of the curves $F$, $P$, and $D$ (with respect to the other special curves), and the exceptional values of $\rho$, separating the different generic configurations, depend only on the values of the canonical coefficients of the curves $F$, $P$ and $D$ (and on the coefficients of the other special curves). It is not difficult to see (looking at the Newton diagram of those curves) that these coefficients do not change when we add higher order terms. Thus, we will systematically use Platonova’s normal form of the 4-jet of $S$. However, the terms of degree 5 are relevant for the study of the bifurcations occurring at the exceptional values of $\rho$ (for example, $\rho = 0$), separating the values corresponding to the generic configurations. Such bifurcations will be studied in another paper.

First we need to calculate the curves $F$, $P$ and $D$. For we need the second partial derivatives of the functions $f(x, y; \lambda) = \frac{y^2}{2} - x^2 y + \lambda x^4$:

$$f_{xx} = -2y + 12\lambda x^2, \quad f_{xy} = -2x, \quad f_{yy} = 1.$$ (H)

The asymptote-equations of the surfaces $z = \frac{y^2}{2} - x^2 y + \lambda x^4$ are therefore given by

$$A^f(x, y, p; \lambda) = (12\lambda x^2 - 2y) - 4xp + p^2 = 0.$$ (5)

We are interested in the configurations of the curves $F$, $P$ and $D$, at the godron $g$. According to Theorem 2, these curves have at least 4-point contact with the $(x, y)$-plane. We will thus consider the curves $\overline{F}$, $\overline{P}$ and $\overline{D}$, on the $(x, y)$-plane, whose images by $f$ are $F$, $P$ and $D$, respectively. These plane curves have the same 2-jet as $F$, $P$ and $D$, respectively.

The parabolic curve. The equations (*) of Section 4.1 imply that $\overline{F}$ is given by the Hessian of $f$, $f_{xy}^2 - f_{xx}f_{yy} = 0$. From (H), one obtains that $\overline{P}$ is a parabola:

$$y = 2(3\lambda - 1)x^2.$$

The flecnodal curve. According to [24], [27], the curve $\overline{F}$ associated to the surface $z = f(x, y)$ is obtained from the intersection of the surfaces

$$A^f(x, y, p) = 0 \quad \text{and} \quad I^{A^f}(x, y, p) := (A^f_x + pA^f_p)(x, y, p) = 0,$$

in the $(x, y, p)$-space, by the projection of this intersection to the $(x, y)$-plane, along the $p$-direction. From eq. (5) one obtains

$$I^{A^f}(x, y, p) = 6(4\lambda x - p).$$

Combining the equation $p = 4\lambda x$ with eq. (5) one obtains that $\overline{F}$ is a parabola:

$$y = 2\lambda(4\lambda - 1)x^2.$$
The conodal curve. Since Platonova’s normal form is symmetric with respect to the $x$-direction, the bitangent planes in the neighbourhood of $g$ are invariant under the reflection $(x, y, z) \mapsto (-x, y, z)$. Thus the points of the conodal curve satisfy $f_\lambda(x, y; \lambda) = 0$. That is, $-2x(y - 2\lambda x^2) = 0$. Thus the curve $D$ is a parabola:

$$y = 2\lambda x^2.$$ 

4.3. Proof of Theorem 4. We consider the parabolas $F$, $P$ and $D$ as graphs of functions $y = y(x)$. The Legendrian curves $L_F$, $L_P$ and $L_D$ in the $(x, y, p)$-space $J^1(\mathbb{R}, \mathbb{R})$ (which is the space of 1-jets of the real functions $y(x)$ of one real variable) are tangent to the contact plane $\Pi$ at the origin (parallel to the plane $y = 0$). The slope of the tangent line at the origin, of each of these Legendrian curves, equals twice the second derivative at zero of the function $y = y(x)$ associated to the corresponding parabola, that is, equals twice the coefficient of that parabola (note that the higher order terms produce no changes on these coefficients).

The Legendrian curve consisting of the contact elements tangent to the origin is vertical. Write $\ell_g$ for its tangent line. The cross-ratio of the tangent lines $\ell_F$, $\ell_P$, $\ell_D$ and $\ell_g$ is given in terms of the coefficients $c$ of the parabolas $F$, $P$ and $D$ by

$$\rho(g) = (\ell_F, \ell_P, \ell_D, \ell_g) = 2\lambda(4\lambda - 1) - 2\lambda = 2\lambda.$$

This proves Theorem 4. □

Rewriting the equations in terms of $\rho$. After Theorem 4, we rewrite Platonova’s normal forms of the 4-jet of $S$ at a godron and the equations of the curves $F$, $P$ and $D$ in terms of the $c$-invariant $\rho$:

\[
\begin{align*}
(z &= \frac{y^2}{2} - x^2y + \rho \frac{x^4}{2} \quad (\rho \neq 1, 0); \quad \text{(R)} \\
y &= (3\rho - 2)x^2; \quad \text{(P)} \\
y &= \rho(2\rho - 1)x^2; \quad \text{(F)} \\
y &= \rho x^2. \quad \text{(D)}
\end{align*}
\]

4.4. Proof of the Separating 2-jet Lemma. An easy way to compute (and to see) the dual surface of $S \subset \mathbb{R}^3$, viewed as a surface in the same space $\mathbb{R}^3$ and with the same coordinate system, is by the ‘polar duality map’ with respect to a quadric. The calculations are simpler if the quadric (considered for this map) is a paraboloid of revolution (see [25]). Moreover, if the surface $S$ is the graph of a function $z = f(x, y)$, then the polar duality map with respect to the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ coincides with the classical Legendre transform of $f$. So, the dual surface of the graph $\{(x, y, f(x, y))\}$ has the following parametrisation:

$$\tau_f: (x, y) \mapsto (f_x(x, y), f_y(x, y), xf_x(x, y) + yf_y(x, y) - f(x, y)).$$

In the case of the surfaces $S_\rho$ given in eq. (R), one obtains

$$\tau_\rho: (x, y) \mapsto \left(-2xy + 2\rho x^3, y - x^2, \frac{y^2}{2} - 2x^2y + 3\rho \frac{x^4}{2}\right). \quad \text{(R’)}$$
The images of our plane curves \( \mathcal{F}, \mathcal{P} \) and \( \mathcal{D} \), under \( \tau_\rho \), are exactly the flecnodal curve, the cuspidal edge and the self-intersection line of the dual surface \( S_\rho^\vee \), respectively. Since \( \mathcal{F}, \mathcal{P} \) and \( \mathcal{D} \) are parabolas, we state the

**Lemma 2.** The image of the parametrised parabola \( t \mapsto (t, ct^2) \), under \( \tau_\rho \), is the parametrised space curve (lying on \( S_\rho^\vee \)):

\[
\alpha_\rho^c: t \mapsto \left(2(\rho - c)t^3, (c - 1)t^2, \left(\frac{c^2}{2} - 2c + \frac{3\rho}{2}\right)t^4\right).
\]

**Proof.** This is a direct application of the above Legendre duality map \( \tau_\rho \). \( \square \)

The above parametrisation implies that the curves \( \alpha_\rho^c \) have at least 4-point contact with the \((x, y)\)-plane at \( t = 0 \). In order to study the behaviour of the curves \( \alpha_\rho^c \) for different values of \( c \) (for a fixed value of the cr-invariant \( \rho \)), we will consider their projection to the \((x, y)\)-plane along the \( z \)-direction:

\[
\gamma_\rho^c: t \mapsto (2(\rho - c)t^3, (c - 1)t^2).
\]  
(6)

Clearly, \( \gamma_\rho^c(t) - \alpha_\rho^c(t) = O(t^4) \).

**Lemma 3.** Fix a value of the godron invariant \( \rho \). The images of all parabolas \( y = cx^2, c \neq 1 \), under the composition of \( \tau_\rho \) with the projection \((x, y, z) \mapsto (x, y)\), are cusps pointing down if \( c > 1 \) and pointing up if \( c < 1 \). These cusps are semi-cubic if \( c \neq \rho \) and (very) degenerate if \( c = \rho \).

The image of the parabola \( y = x^2 \) \((c = 1)\) under the above composition is the \( x \)-axis if \( \rho \neq 1 \) and it is the origin if \( \rho = 1 \).

**Proof.** Lemma 3 and Separating Lemma follow from parametrisation (6). \( \square \)

**Remark.** It is clear from Lemma 3 that the behaviour of the curve \( \tau_\rho(\mathcal{F}), \tau_\rho(\mathcal{P}) \) or \( \tau_\rho(\mathcal{D}) \) in \( S_\rho^\vee \), changes drastically when the coefficient \( c_F(\rho), c_P(\rho) \) or \( c_D(\rho) \), respectively, passes through the value 1.

### 4.5. Proof of Theorem 5

The projection of \( S_\rho \) to the \((x, y)\)-plane, along the \( z \)-axis, is a local diffeomorphism. So the configuration of the curves \( F, P \) and \( D \) with respect to the asymptotic line and the separating 2-jet at \( g \), on the surface \( S \), is equivalent to the configuration of the parabolas \( \mathcal{F}, \mathcal{P} \) and \( \mathcal{D} \) with respect to the parabolas \( y = 0 \cdot x^2 = 0 \) and \( y = 1 \cdot x^2 \) on the \((x, y)\)-plane (see Remark of Section 3.1).

Given a value of \( \rho \), this configuration is determined by the order, in the real line, of the coefficients of these five parabolas:

\[
c_F = \rho(2\rho - 1), \quad c_P = (3\rho - 2), \quad c_D = \rho, \quad c_{al} = 0, \quad c_s = 1.
\]

The graphs of these coefficients, as functions of \( \rho \), are depicted in Fig. 13.
Using the formulas of the canonical coefficients $c_F$, $c_P$ and $c_D$ (or from Fig. 13) one obtains by straightforward and elementary calculations that:

- $\rho \in (1, \infty) \iff 1 < c_D < c_P < c_F$;
- $\rho \in \left(\frac{2}{3}, 1\right) \iff 0 < c_P < c_F < c_D < 1$;
- $\rho \in \left(\frac{2}{3}, \frac{2}{3}\right) \iff c_P < 0 < c_F < c_D < 1$;
- $\rho \in (0, \frac{1}{2}) \iff c_P < c_F < 0 < c_D < 1$;
- $\rho \in (-\frac{1}{2}, 0) \iff c_P < c_D < 0 < c_F < 1$;
- $\rho \in (-\infty, -\frac{1}{2}) \iff c_P < c_D < 0 < 1 < c_F$.

This proves Theorem 5.

4.6. Proof of Proposition 2. Consider the family of surfaces $S_\rho$ given by eq. (R). By eq. (P), the slope $m$ of the tangent lines of the curve $\mathcal{P}$ is given by:

$$m(x) = 2(3\rho - 2)x.$$

The slope $p$ of the (double) asymptotic lines along the parabolic curve, projected to the $(x, y)$-plane, is given by the equation $A_0^\dagger(x, y, p; \frac{R}{2}) = 0$, that is,

$$p(x) = 2x.$$

The points of the positive $y$-axis, near the origin, are hyperbolic points of the surface $S_\rho$ of eq. (R). So the hyperbolic domain of $S_\rho$ lies locally in the upper side of the parabolic curve. Therefore $g$ is a positive (negative) godron if and only if the difference of slopes $(p - m)$ is a decreasing (resp. increasing) function of $x$, at $x = 0$.

Consequently, the equation $(p - m)'(0) = -6(\rho - 1)$ implies that the godron $g$ is positive for $\rho > 1$ and negative for $\rho < 1$, proving Proposition 2.
4.7. Proof of Theorem 2 and of Theorem 8

Preliminary remarks on the asymptotic-double (see Section 3.2). The asymptotic double $\mathcal{A}$ of the surface $S$ is foliated by the integral curves of the asymptotic lifted field of directions. By definition of the lifted field, the asymptotic curves of $S$ are the images of these integral curves under the natural projection $PT^*S \rightarrow S$ (sending each contact element to its point of contact) and, under this projection, the asymptotic double $\mathcal{A}$ (of $S$) doubly covers the hyperbolic domain with a fold singularity over the parabolic curve.

Write $\bar{P}$ for the curve of $\mathcal{A}$ which projects over $P$ (that is, the curve formed by the fold points in $\mathcal{A}$ of the above projection). The surface $\mathcal{A} \setminus \bar{P}$, has two (not necessarily connected) components, noted by $\mathcal{A}_l$ and $\mathcal{A}_r$, separated by $\bar{P}$. The integral curves on the component $\mathcal{A}_l$, are projected over the left asymptotic curves and the integral curves on the component $\mathcal{A}_r$ are projected over the right ones. We call these components the left component and the right component, respectively, of $\mathcal{A} \setminus \bar{P}$.

Now, consider the surface $S$ as the graph of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $z = f(x, y)$, and take the projection $\pi : (x, y, z) \mapsto (x, y)$, along the $z$-axis. The derivative of $\pi$ sends the contact elements of $S$ onto the contact elements of $\pi(S) \subseteq \mathbb{R}^2$ and it induces a contactomorphism $PT^*S \rightarrow PT^*\mathbb{R}^2$ sending $\mathcal{A}$ to a surface $\bar{A}$ in $PT^*\mathbb{R}^2$, which doubly covers (under the natural projection $PT^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$) the image in $\mathbb{R}^2$ of the hyperbolic domain. We still call the surface $\bar{A} \subset PT^*\mathbb{R}^2$ the asymptotic-double of $S$. This surface consists of the contact elements of the $(x, y)$-plane satisfying the following equation:

$$f_{xx} \, dx^2 + 2f_{xy} \, dx \, dy + f_{yy} \, dy^2 = 0. \quad (*)$$

In order to handle the asymptotic double $\bar{A}$, we take an ‘affine’ chart of $PT^*\mathbb{R}^2$. The space of 1-jets of the real functions of one real variable $J^1(\mathbb{R}, \mathbb{R})$ (with coordinates $x, y, p$) has a natural contact structure (defined by the 1-form $\alpha = dy - p \, dx$) and it parameterises almost all contact elements of $\mathbb{R}^2$: The contact element with slope $p_0 \neq \infty$ at the point $(x_0, y_0)$ of the plane of the variables $(x, y)$ is represented by the point $(x_0, y_0, p_0)$ in $J^1(\mathbb{R}, \mathbb{R})$. The asymptotic-double $\bar{A}$ is the surface in $J^1(\mathbb{R}, \mathbb{R})$ given by the equation

$$A^f(x, y, p) := f_{xx} + 2f_{xy}p + f_{yy}p^2 = 0, \quad (7)$$

(obtained from eq. (*)) by taking $p = dy/dx$). Moreover, the solutions of the implicit differential equation (7) are the images (by $\pi$) of the asymptotic curves of $S$. Equation (7) is called the asymptote-equation of $f$.

The curve $\bar{P}$ is the crinimant curve (see [3]) of the implicit differential equation $A^f(x, y, p) = 0$ and it is determined by the pair of equations $A^{f_1}(x, y, p) = 0$ and $A^{f_2}(x, y, p) = 0$.

Below, the images on the plane, under the map $\pi : (x, y, z) \mapsto (x, y)$, of the parabolic and flecnodal curves, of the godrons and of the hyperbodons and ellipnodes of $S$, will be called with the same name, that is, parabolic curve, etc. One obtains the original objects by applying the function $f$ and taking the graph.
Proof of Theorem 2. Write $\tilde{F}$ for the intersection of $\tilde{A}$ ($A^f(x, y, p) = 0$) with the surface given by the equation $I^f(x, y, p) = 0$. As we mentioned in §4.2, the flecnodal curve in the $(x, y)$-plane is the image of the curve $\tilde{F}$ under the projection $(x, y, p) \mapsto (x, y)$. The points of (transverse) intersection of the curves $\tilde{F}$ and $\tilde{P}$ project over the godrons of $S$. So, over a godron the curve $\tilde{P}$ locally separates $\tilde{F}$ (see Fig. 14).

![Figure 14. The projection $\pi: A \to S$, the curves $\tilde{P}$, $\tilde{F}$, $P$ and $F$.](image)

That is, $\tilde{F}$ has one branch on the left component of $\tilde{A}$ and other branch on the right component. This implies that a godron separates locally the left and right branches of the flecnodal curve, proving Theorem 2.

Proof of Theorem 8. Consider a godron $g$ with cr-invariant $\rho$ of a smooth surface. To prove Theorem 8 we need to know the values of $\rho$ for which the curves $\tilde{P}$ and $\tilde{F}$ are tangent. Of course, such non-generic values correspond to godrons of non-generic surfaces. To find these values we only need to know the tangent directions of these curves over $g$. The tangent lines of these curves belong to the tangent plane of $\tilde{A}$ at $\tilde{g}$ (the point over $g$ in the $(x, y, p)$-space), which is also the contact plane at $\tilde{g}$. So it suffices to take the 4-jet of $S$ at $g$.

Take the normal form considered above

$$z = \frac{y^2}{2} - x^2y + \rho \frac{x^4}{2}, \quad (R)$$

The coordinates $(x, y, z)$ of this normal form satisfy the conditions considered in Theorem 8.

Since the point $\tilde{g}$ is the origin, the tangent lines to the curves $\tilde{P}$ and $\tilde{F}$ belong to the $(x, p)$-plane. The surface $A^f(x, y, p) = 0$ is the plane given by the equation $p = 2x$, which is independent of $\rho$. The surface $I^f(x, y, p) = 0$ is the plane given by the equation $p = 2\rho x$. So the curves $\tilde{P}$ and $\tilde{F}$ are tangent only for $\rho = 1$ (in this case we have the collapse of two godrons).
By Proposition 2, this implies that the side on which the right branch of the flecnodal curve will lie depends only on the index of the godron.

To see explicitly on which side of the $x$-axis the right branch of the flecnodal curve lies for a negative godron, it is enough to look at an example. We will take a godron of a cubic surface (whose index is $-1$, after Corollary 3).

The osculating plane of an asymptotic curve at a point of a surface is the tangent plane to the surface at that point. Using this fact, one defines the “osculating plane” of a straight line lying in a surface.

In this way, a segment of a straight line lying in a surface is said to be a left (right) curve, if the tangent plane to the surface along that segment twists like a left (resp. right) screw.

The $x$-axis is an asymptotic (and flecnodal) curve of the cubic surface $z = y^2/2 - x^2y$. One verify easily that the positive half axis is a left asymptotic curve. This proves Theorem 8. □

4.8. Proof of Theorem 9. Consider the surface as the graph of a function $z = f(x, y)$. In [27], it is proved that the projection of the flecnodal curve to the $(x, y)$-plane (the curve $F$ in the above notation) is also the image under the natural projection $\pi: A \subset J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2 (\pi: (x, y, p) \mapsto (x, y))$ of the critical points of the Legendre dual projection $\pi^\vee: A \ni (x, y, p) \mapsto (p, px - y)$ (the folds of $\pi^\vee$). Moreover, it is also proved that, in the hyperbolic domain, the biflecnodes correspond to the Whitney pleat singularities of $\pi^\vee: A \to (\mathbb{R}^2)^\vee$. Both, folds and Whitney pleats, are the only stable singularities of a map from a surface to the plane (Whitney).

It is easy to show that a flec-godron corresponds also to a Whitney pleat of $\pi^\vee$ (one shows that, at the point of $A$ over the flec-godron, the kernel of $\pi_{\ast}^\vee$ is tangent to the curve of fold points of the map $\pi^\vee: A \to (\mathbb{R}^2)^\vee$). Thus, after any small generic deformation of the surface the flec-godron splits into an ordinary godron and a neighbouring ordinary (left or right) biflecnode (since the Whitney pleat singularity is stable).

The fact that, at the flec-godron moment, both the flecnodal curve and the conodal curve have an inflection, follows from the vanishing of the canonical coefficients $c_D = \rho$ and $c_F = \rho(2\rho - 1)$, for $\rho = 0$. □

4.9. Proof of Theorem 10. First, we will prove Theorem 10 for the case in which the parabolic curve bounding the hyperbolic disc has only two godrons.

Lemma 4. If the parabolic curve bounding a hyperbolic disc $H$ (of a generic smooth surface) has exactly two godrons, then the disc $H$ contains an odd number of hyperbonodes.

Write $g_1$ and $g_2$ for the godrons lying on $\partial H$. By Proposition 5, both $g_1$ and $g_2$ are positive godrons.

Claim 1. If two vectors $v_1$ and $v_2$ are tangent to $F$ at $g_1$ and $g_2$, respectively, and both are pointing from $F_l$ to $F_r$, then $v_1$ and $v_2$ orient the parabolic curve $\partial H$ in the same way.
Proof. Since all neighbouring elliptic points of the parabolic curve $\partial H$ belong to the same connected component of the elliptic domain, they have the same "natural" co-orientation (given by the tangent plane). Since both godrons are positive, Claim 1 follows from Theorem 8. □

Proof of Lemma 4. Write $f_r$ for the connected component of $F_r$ which starts at $g_1$. Since there are only two godrons on $\partial H$, $f_r$ is a segment ending in $g_2$. This segment separates $H$ into two parts, which we name $A$ and $B$. The connected component of $F_l$ starting in $g_1$, $f_l$, is also a segment ending in $g_2$. Claim 1 implies that if in the neighbourhood of $g_1$ the segment $f_l$ lies in $A$, then, in the neighbourhood of $g_2$, it lies in $B$. Thus $f_l$ crosses $f_r$ an odd number of times.

If $H$ contains other connected components of $F_l$ and $F_r$, then there are (possibly) additional hyperbonodes in $H$. Apart from $f_l$ and $f_r$, the only connected components of $F_l$ and $F_r$ in $H$ are closed curves (possibly empty). But the number of intersection points of a closed curve of $F_r$ (lying in $H$) with $f_l$, or with a closed curve of $F_l$, is even. Thus the number of intersection points of $F_l$ with $F_r$ is odd. □

Proof of Theorem 10. To prove the general case of Theorem 10, we will consider the closure of the hyperbolic disc, the parabolic curve $\partial H$ and the connected components of $F_l$ and $F_r$ lying in $H$ as a diagram $\Delta$. We will prove in a purely combinatorial manner that the number of intersection points of $F_l$ with $F_r$ is odd. For this, we will transform the diagram $\Delta$ using two "moves", which are elementary changes (of two types) of local diagrams, that preserve the number of intersection points of $F_l$ with $F_r$ in the deformed diagram:

\begin{figure}
\centering
\includegraphics{figure15.png}
\caption{The two elementary moves of diagrams. The moves with opposite choice of colours of the flecnodal curve are also possible.}
\end{figure}

These moves are depicted in Fig. 15, where an intermediate singular diagram is marked by a dotted box.

Write $G^+$ and $G^-$ for the number of positive and negative godrons on $\partial H$, respectively. Since the asymptotic covering of $\overline{H}$ is a sphere, $G^+ - G^- = 2$.

If $G^- = 0$, the theorem is proved in Lemma 4. So suppose $G^- > 0$.

Consider a pair of godrons $g_+$ and $g_-$ of opposite index, which are consecutive on $\partial H$. Two vectors tangent to $\partial H$ and pointing from $F_l$ to $F_r$, one at $g_+$ and the other at $g_-$, provide different orientations of $\partial H$ (see Claim 1).
Consider the segment of parabolic curve joining $g_+$ to $g_-$, and which does not contain other godrons. The local diagram in the tubular neighbourhood of this segment of the parabolic curve is depicted in the left side of Fig. 16.

**Step 1.** In this tubular neighbourhood we deform the black curves starting in $g_+$ and $g_-$, in order to approach one to the other (the central diagram of Fig. 16). Now we apply a move of type I to this diagram in order to obtain a new diagram in which the connected component of $F_r$ starting at $g_+$ will be a segment ending at $g_-$ and lying in the tubular neighbourhood of the considered segment of the parabolic curve.

![Figure 16. A deformation of $F_r$ and a move of type I.](image)

(It must be clear that, in Fig. 15 and in Fig. 16, we are not claiming that the surface is being deformed in such a way that this evolution of the flecnodal curve happens. We are deforming the diagram, not the surface. However, to avoid new notations and symbols, we have kept the names: godron, parabolic curve, etc; and the notations: $g_+$, $g_-$, $F_l$, $F_r$, $\partial H$, etc.)

**Step 2.** Applying a move of type II to the local diagram obtained in Step 1, one obtains a new diagram without the pair of godrons $g_+$ and $g_-$. Applying $G^-$ times the above process, one obtains a final diagram having only two positive godrons. Theorem 10 is proved applying Lemma 4 to this final diagram (note that also the proof of Lemma 4 depends only on the combinatorial properties of the initial diagram). □

4.10. **Proof of Theorem 13.** To prove Theorem 13 we will use the fact that the tangential map of $S$ sends the flecnodal curve of $S$ onto the flecnodal curve of $S^\vee$.

The dual of a front $\tilde{S}$ in general position at a swallowtail point $s$ is a godron of a (locally) smooth surface. So Theorem 1 and the Separating Lemma imply that the flecnodal curve of $\tilde{S}$ has a cusp at $s$ having the same tangent line that the cuspidal edge of $\tilde{S}$. Now, by Theorem 2, the swallowtail point separates the flecnodal curve into its left and right branches.

The configuration formed by the flecnodal curve, the cuspidal edge and the self-intersection line of $\tilde{S}$ at the swallowtail point $s$, is determined by the configuration formed by the curves $F$, $P$, $D$ and the separating 2-jet on the (locally smooth) dual surface $\tilde{S}^\vee$, at its godron $g = s^\vee$.

Theorem 5 says that, at a godron $g$, there are six possible generic configurations of the curves $F$, $P$ and $D$, with respect to the separating 2-jet and to the asymptotic line at $g$. Since the asymptotic line is not considered in the concerned configurations, we can eliminate the number 0 (corresponding to the asymptotic line) from the
six inequalities of the proof of Theorem 5. One obtains four distinct inequalities, corresponding to four open intervals for the values of \( \rho \):

\[
\begin{align*}
\rho &\in (1, \infty) \quad \iff \quad 1 < c_D < c_P < c_F; \\
\rho &\in (0, 1) \quad \iff \quad c_P < c_F < c_D < 1; \\
\rho &\in (-\frac{1}{2}, 0) \quad \iff \quad c_P < c_D < c_F < 1; \\
\rho &\in (-\infty, -\frac{1}{2}) \quad \iff \quad c_P < c_D < 1 < c_F.
\end{align*}
\]

Using the Separating Lemma and the configurations of Theorem 5 (not considering the asymptotic line) one obtains that these four configurations correspond to the four configurations (of Theorem 13) for the flecnodal curve, the cuspidal edge and the self-intersection line in the neighbourhood of a swallowtail point of a front in general position. \( \Box \)

4.11. Proof of Theorems 14, 15 and 16. The following general lemma is the key to prove of Theorems 14, 15 and 17.

Let \( q \) be a point of a front \( S \) of \( \mathbb{RP}^3 \), in general position. Write \( q^\vee \) for the point of the dual surface \( S^\vee \subset (\mathbb{RP}^3)^\vee \), corresponding to \( q \) (that is, \( q^\vee \) is the tangent plane to \( S \) at \( q \)), and write \( \Pi \) for the plane tangent to \( S^\vee \) at \( q^\vee \).

\textbf{q-Contour Lemma.} The image of the (total) \( q \)-contour of \( S \) under the tangential map \( S \to S^\vee \subset (\mathbb{RP}^3)^\vee \) is the tangent section \( \Pi \cap S^\vee \) of the dual surface \( S^\vee \) at \( q^\vee \).

\textit{Proof.} One needs to prove that any plane tangent to \( S \) and passing through \( q \) is a point of \( \mathbb{RP}^3 \) belonging to \( \Pi \cap S^\vee \), and that every point of \( \Pi \cap S^\vee \) is a plane tangent to \( S \) passing through \( q \).

Since the point \( q \in S \subset \mathbb{RP}^3 \) is precisely the plane \( \Pi \) of \( (\mathbb{RP}^3)^\vee \) tangent to \( S^\vee \) at \( q^\vee \), the points of \( \Pi \) are the planes of \( \mathbb{RP}^3 \) passing through \( q \). Consequently, the points of \( (\mathbb{RP}^3)^\vee \) belonging to \( \Pi \cap S^\vee \) (forming the tangent section of \( S^\vee \) at \( q^\vee \)) are the planes of \( \mathbb{RP}^3 \) tangent to \( S \) (since they belong to \( S^\vee \)) and passing through \( q \) (since they belong to \( \Pi \)). \( \Box \)

The \( q \)-Contour Lemma and the arguments given in the proof of Theorem 13 imply that Theorems 14 and 15 together are equivalent to Theorem 16. So we only need to prove Theorem 16.

\textit{Proof of Theorem 16.} Consider the surface \( S_\rho \) given by eq. \((R)\). We will use the parametrisation \((R^\vee)\) of the dual surface \( S_\rho^\vee \) given in Section 4.4.

So, in order to find the first terms of the power series expansion of the \( g \)-contour of \( S \), we need to find the zeros of the equation \( y^2/2 - 2x^2y + 3\rho x^4 = 0 \). Completing squares, one easily factorises the left hand side of this equation:

\[
\frac{1}{2} \left( y - (2 + \sqrt{4 - 3\rho})x \right) \left( y - (2 - \sqrt{4 - 3\rho})x \right) = 0.
\]

This implies that the projection of the \( g \)-contour of \( S \) to the \((x, y)\)-plane (in the \( z \)-direction) is given by two tangent curves:

\[
C_- : \quad y = c^-_C x + \cdots \quad \text{and} \quad C_+ : \quad y = c^+_C x + \cdots,
\]

where \( c^-_C = 2 - \sqrt{4 - 3\rho} \) and \( c^+_C = 2 + \sqrt{4 - 3\rho} \).
Figure 17. The coefficients \(c_F\), \(c_P\) and \(c_D\) as functions of the invariant \(\rho\), together with the coefficients \(c_C^-\), \(c_C^+\) (part C) and \(c_T^-\), \(c_T^+\) (part T).

Of course these two coefficients are defined as functions of \(\rho\) only for \(\rho < \frac{4}{3}\). The graphs of these coefficients, as functions of \(\rho\), are depicted in Fig. 17(C) (together with the graphs of \(c_F\), \(c_P\), \(c_D\) and \(c_\sigma = 1\)). Indeed, the union of these two graphs forms the parabola given by the equation:

\[
\rho = -\frac{1}{3}(c - 2)^2 + \frac{4}{3}.
\]

Using the above formulas of \(c_C^-\) and \(c_C^+\), and those of the canonical coefficients \(c_F\), \(c_P\), \(c_D\) and \(c_\sigma\):

\[
c_F = \rho(2\rho - 1), \quad c_P = (3\rho - 2), \quad c_D = \rho, \quad c_\sigma = 1,
\]
(or from Fig. 17(C)) one obtains the inequalities of Theorem 16 (and the corresponding exceptional values of \(\rho\): \(\frac{4}{3}, \pm \frac{\sqrt{7}}{2}, -\frac{1}{2}, 0, 1\)) by straightforward and elementary calculations. To make the calculations simpler, one can use the inequalities already obtained in Theorem 5 (eliminating the number 0).

Remark. One can also prove Theorem 16 by using the parametrisations (obtained from Lemma 2 of Section 4.4) of the self-intersection line \(D^\vee\), the flecnodal curve \(F^\vee\) and the cuspidal edge \(\Sigma = P^\vee\) of the dual swallowtail \(S^\vee\):

\[
D^\vee = \{0, (\rho - 1)t^2, \frac{1}{2}\rho(\rho - 1)t^4\},
\]

\[
F^\vee = \{4\rho(1 - \rho)t^3, (\rho(2\rho - 1) - 1)t^2, \frac{1}{2}\rho(\rho - 1)(4\rho^2 - 7)t^4\},
\]

\[
P^\vee = \{4(1 - \rho)t^3, 3(\rho - 1)t^2, 3(\rho - 1)(3\rho - 4)t^4\}.
\]

For one needs to find the values of \(\rho\) for which a component of one of these curves change its sign, and then to use the geometric meaning of that sign change.

For instance, the flecnodal curve of the swallowtail \(S^\vee\) has higher order of contact with the tangent plane (passing locally from one side to the other of it) when the
third component of \( F^r \), \( \frac{4}{5}p(\rho - 1)(4\rho^2 - 7)t^4 \), equals zero. In this case one finds the degenerate case \( \rho = 1 \), the flec-godron \( \rho = 0 \) and the exceptional values \( \pm \sqrt[5]{\frac{7}{2}} \).

4.12. Proof of Theorems 17 and 18. As in Section 4.11, the \( q \)-Contour Lemma and the arguments given in the proof of Theorem 13 imply that Lemma 1 and Theorem 18 together are equivalent to Theorem 17. So, we will prove only Lemma 1 and Theorem 18.

Proof of Lemma 1 and Theorem 18. Consider surface \( S_\rho \) given by the graph of a function \( z = f(x, y) \). Let \( z = \frac{w^2}{2} - x^2y + \frac{2}{5}x^3 + \varphi_5(x, y) + \varphi_8(x, y) + \cdots \) its Taylor decomposition in homogeneous forms. Using the Newton diagram of this decomposition, one has that the first terms of the power series expansion of the branches of the tangent section at \( g \) are given by the zeros of the equation \( \frac{y^2}{2} - x^2y + px^3/2 = 0 \). Completing squares, we factorise the left hand side of this equation:

\[
\frac{1}{2} \left( y - (1 + \sqrt{1 - \rho})x^2 \right) \left( y - (1 - \sqrt{1 - \rho})x^2 \right) = 0.
\]

Hence, the tangent section of \( S \) at \( g \) is given by two (simply) tangent curves:

\[
C_- : \quad y = c_T^-x^2 + \cdots \quad \text{and} \quad C_+ : \quad y = c_T^+x^2 + \cdots,
\]

where \( c_T^- = 1 - \sqrt{1 - \rho} \) and \( c_T^+ = 1 + \sqrt{1 - \rho} \) (proving Lemma 1).

The coefficients \( c_T^- \) and \( c_T^+ \) are defined as functions of \( \rho \) only for \( \rho < 1 \), and their graphs, as functions of \( \rho \), are depicted in Fig. 17(T) (together with the graphs of \( c_F, c_P, c_D \) and \( c_S \)). The union of these two graphs forms the parabola given by the equation:

\[
\rho = -(c - 1)^2 + 1.
\]

Using the above formulas of \( c_T^- \) and \( c_T^+ \), and those of the coefficients \( c_F, c_P, c_D \) and \( c_S \) (or from Fig. 17(T)) one obtains the inequalities of Theorem 18 (and the corresponding exceptional values of \( \rho \): \( \frac{8}{9}, \pm \sqrt[5]{\frac{7}{2}}, -\frac{1}{2}, 0, 1 \)) by straightforward and elementary calculations.

Remark. One can also prove Theorem 18 by using the parametrisations of the conical curve \( D \), the flecnodal curve \( F \), and the parabolic curve \( P \) of \( S \):

\[
D = (x, \rho x^2 + \cdots, \frac{1}{5}p(\rho - 1)x^4 + \cdots),
\]

\[
F = (x, \rho(2\rho - 1)x^2 + \cdots, \frac{1}{5}p(\rho - 1)(4\rho^2 - 3)x^4 + \cdots),
\]

\[
P = (x, (3\rho - 2)x^2 + \cdots, \frac{1}{5}p(\rho - 1)(9\rho - 8)x^4 + \cdots).
\]

For one needs to find the values of \( \rho \) for which the first term of a component of one of these curves change its sign, and then to use the geometric meaning of that sign change.

For example, the flecnodal curve of \( S \) has higher order of contact with the tangent plane (passing locally from one side to the other of it) when the term \( \frac{1}{5}p(\rho - 1)(4\rho^2 - 3)t^4 \), of the third component of \( F \), equals zero, providing the degenerate case \( \rho = 1 \), the flec-godron \( \rho = 0 \) and the exceptional values \( \pm \sqrt[5]{\frac{7}{2}} \).

Remark. When this paper was almost finished, I visited l’École Normale Supérieure de Lyon to give a talk about the results of [27] and of this paper. Few days
before my talk, E. Ghys and D. Serre have found the book [18] on the history of
thermodynamics in Netherlands. It describes a part of Korteweg’s work [14], [15]
about the godrons (called plaits in [18]), the parabolic curve and the conodal curve.
According to [18], Korteweg had also described the bifurcations of the parabolic and
conodal curves when two godrons are born or disappear, for an evolving surface.
Korteweg’s work on the theory of surfaces was motivated by thermodynamical
problems.

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