A COUNTEREXAMPLE TO A MULTIDIMENSIONAL VERSION OF THE WEAKENED HILBERT’S 16th PROBLEM

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Abstract. In the weakened 16th Hilbert’s Problem one asks for a bound on the number of limit cycles which appear after a polynomial perturbation of a planar polynomial Hamiltonian vector field. It is known that this number is finite for an individual vector field. In the multidimensional generalization of this problem one considers polynomial perturbation of a polynomial vector field with an invariant plane supporting a Hamiltonian dynamics. We present an explicit example of such perturbation with an infinite number of limit cycles which accumulate at some separatrix loop.

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1. The Result

Yu. Ilyashenko [II] and J. Ecalle [E] proved that an individual planar polynomial vector field can have only a finite number of limit cycles.

On the other hand, multi-dimensional vector fields with chaotic dynamics have an infinite number of periodic trajectories. The Lorentz system [MiMr] and the Duffing system [GuHo] provide best known examples. In the chaotic systems periodic orbits are usually encoded by periodic sequences in a suitable symbolic dynamical system. The existence of such an encoding is proved using topological methods (like the Lefschetz–Conley index or Smale’s horseshoe). This means that:

(1) Periods of the periodic trajectories tend to infinity in a rather irregular way.
(2) The 1-cycles represented by different periodic trajectories have different “topology”, i.e., they are linked between themselves.

In particular, these cycles do not form a continuous family (a so called center).

In the main theorem below we give an example of a polynomial 4-dimensional differential system with an infinite number of periodic solutions \( \gamma_1, \gamma_2, \ldots \) such that

- the periods of \( \gamma_j \) increase monotonically with \( j \);
all the corresponding 1-cycles have the same “topology”: they are concentric cycles on an embedded invariant 2-dimensional disc of class $C^1$;
- the cycles $\gamma_j$ are isolated (they are limit cycles).

Remark 1.1. A phenomenon similar to ours was described by D. Turaev in the paper [Tu]. He states sufficient conditions for a $C^r$ ($r \geq 3$) vector field in $\mathbb{R}^n$ ($n \geq 4$) which guarantee the existence of a sequence of limit cycles located on an invariant surface and accumulating at a homoclinic loop. He does not give any explicit example. Moreover, our example is not a special case of the Turaev construction; our system is not normally hyperbolic in the usual sense (see pages 12–13 below), and so the condition (A) from [Tu] is not satisfied.

To construct the example we begin with the Hamiltonian planar system
\[ \dot{x} = X_H = (H_{x_2}, -H_{x_1}), \quad (x_1, x_2) \in \mathbb{R}^2_x, \quad H = x_1^3 - 3x_1 - x_2^2 + 2 \tag{1.1} \]
and the 2-dimensional linear system
\[ \dot{y} = ay, \quad y = y_1 + iy_2 \in \mathbb{C} \equiv \mathbb{R}^2_y, \tag{1.2} \]
where $a = -\rho + i\omega$. Later we will put $\rho = \omega = \sqrt{3}$.

The Hamiltonian function from (1.1) is elliptic with the critical points $x = (-1, 0)$ (center) and $x = (1, 0)$ (saddle). The phase portrait of the field $X_H$ is shown on Figure 1.1.

![Figure 1.1. Phase portrait of the Hamiltonian vector field $X_H$ with ovals $\gamma_n$ generating limit cycles](image)

We consider the following coupling of the system (1.1) and (1.2):
\[ \begin{cases} \dot{x} = X_H + \text{Re}(\kappa y) e_2, \\ \dot{y} = ay + \varepsilon H^4(x)(1 - x_1), \end{cases} \tag{1.3} \]
where $\varepsilon > 0$ is a small parameter, $e_2 = (0, 1)$ is a vector in $\mathbb{R}_x^2$ and $\kappa \in \mathbb{C}$. 
Theorem 1.2 (Main). Suppose that \( \rho = \omega = \sqrt{3} \) and
\[
\kappa = 4\sqrt{3}i + \frac{(3-3i)\sqrt{6}}{\sqrt{\pi}} \left( 1 + 2i - \psi' \left( \frac{1-i}{2} \right) \right),
\]
where \( \psi(z) \) (Euler’s psi-function) is logarithmic derivative of Euler’s gamma-function \( \psi = \Gamma' \Gamma \).

Then there exists an \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) the system (1.3) has a sequence of limit cycles \( \gamma_n, n = 1, 2, \ldots \) which accumulate at the separatrix loop
\[
\gamma_0 = \{(x, y) : y = 0, H(x) = 0, x_1 < 1\}
\]
of the singular point \( (x = (1, 0), y = 0) \) and lie on an invariant surface \( y = \varepsilon G(x, \varepsilon) \) of class \( C^1 \).

Remark 1.3. The approximate numerical value of \( \kappa \) in formula (1.4) is
\[
\kappa \approx -0.56 + 4.57i.
\]

Systems of the form
\[
\begin{align*}
\dot{x} &= X_H + F(x)y + \varepsilon G(x) + \ldots, \\
\dot{y} &= A(x)y + \varepsilon b(x) + \ldots,
\end{align*}
\]
\( x \in \mathbb{R}^2, y \in \mathbb{R}^\nu \), i.e., systems like (1.3), appear in the so-called multidimensional generalization of the weakened Hilbert’s 16th problem (see [Bo], [BŽ1], [BŽ2], [LŽ]). Before perturbation, i.e. for \( \varepsilon = 0 \), we have an invariant plane \( y = 0 \) with the Hamiltonian vector field \( X_H(x) \). The ovals \( H(x) = h \) form a 1-parameter family of its periodic trajectories. One asks how many of these trajectories survive the perturbation. In the 2-dimensional case (\( \nu = 0 \)) the linearization of the problem leads to the problem of real zeroes of an Abelian integral \( I(h) = \int_{H(x)=h} \omega \); it is called weakened Hilbert’s 16th problem (see [AI], [Il]).

If \( \nu \geq 1 \), then the corresponding Pontryagin–Melnikov integrals (see [M], [P]), denoted by \( J(h) \), were found in [LŽ] and [BŽ1]. We call them generalized Abelian integrals.

The Abelian integrals \( I(h) \) satisfy ODEs of the Fuchs type and have regular singularities with real spectrum (see [Yak]). Using this fact, S. Yakovenko and others have found some effective estimates for the number of zeroes of \( I(h) \). However, generalized Abelian integrals do not satisfy a simple differential equation (see [Bo]) and sometimes have irregular singularities (e.g. at \( h = \infty \)). Moreover, even if the singularities are regular, their spectra can be non-real.

Actually, non-reality of the spectrum of \( J(h) \) at the singularity \( h = 0 \) is responsible for accumulation of zeroes of \( J \). Below we find the asymptotics
\[
J(h) \sim C \, h^{3/2} \sin(\log \sqrt{h}), \quad h \to 0^+.
\]
It turns out that the zeroes \( h_n \to 0^+ \) of \( J \) correspond to the limit cycles \( \gamma_n \) of the system (1.3); the cycle \( \gamma_n \) bifurcates from the oval \( H^{-1}(h_n) \) (see Figure 1.1).

Therefore the system (1.3) can be treated as a counterexample to the multidimensional weakened Hilbert’s problem.
The remaining part of the paper is devoted to the proof of the main theorem. In Section 2.1 we investigate generalized Abelian integrals and their zeroes. In Section 2.2 we find the estimates needed for existence of genuine limit cycles.

2. Proof of the Main Theorem

2.1. Generalized Abelian integrals. The generalized Abelian integral is defined in two steps. First one solves the so-called normal variation equation

\[ X_H(g) = ag + (1 - x_1). \]  

Its solution \( x \mapsto g(x) \in \mathbb{C} \) appears in the first (linear in \( \varepsilon \)) approximation of the invariant surface (see the next section for more details):

\[ y = \varepsilon H^4(x) g(x) + O(\varepsilon^2). \]

We consider (2.1) only in the basin \( D = \{ x : H(x) \geq 0, x_1 \leq 1 \} \) of the center \( x = (-1, 0) \), filled by the periodic solutions \( \gamma_h(t) = \gamma(t) \subset \{ H^{-1}(h) \} \), each of period

\[ T_{\gamma}(h) = \int_{\gamma_h} \frac{dx_1}{2x_2} = \int_{\gamma_h} dt. \]  

(2.2)

We assume that the Hamiltonian time is chosen in such a way that for \( 0 < h < 4 \) one has \( x(0) = (x_1^{(1)}, 0) \), where \( x_1^{(1)}, x_1^{(2)}, x_1^{(3)} \) are roots of the equation \( H(x_1, 0) - h = 0 \) (see Figure 2.1). When restricted to \( \gamma_h \), equation (2.1) is treated as the ODE \( \dot{g} = ag + (1 - x_1) \) with a periodic boundary condition. Its unique solution is given in the integral form

\[ g(t, h) = (e^{-aT_{\gamma}} - 1)^{-1} \int_{t}^{t+T_{\gamma}} e^{a(t-s)}(1 - x_1)(s, h) ds. \]  

(2.3)

\[ \xi \]

\[ \delta_h \]

\[ \gamma_0 \]

\[ \gamma_h \]

\[ \gamma^{(1)} \]

\[ \gamma^{(2)} \]

\[ \gamma^{(3)} \]

Figure 2.1. The basin \( D \) filled with ovals \( \gamma_h \subset H^{-1}(h) \)
Substituting the invariant surface equation (see Section 2.2) \( y = \varepsilon H^4 g + O(\varepsilon^2) \) into the right hand side of the expression for \( \dot{x} \) from (1.3), we get the following perturbation of the planar Hamiltonian system:

\[
\begin{align*}
\dot{x}_1 &= -2x_2, \\
\dot{x}_2 &= 3(1 - x_1^2) + \varepsilon H^4 \text{Re}(\pi g) + O(\varepsilon^2).
\end{align*}
\]

The generating function for limit cycles is given by the integral

\[
J(h) = h^4 \int_{\gamma_h} \text{Re}(\pi g(x)) \, dx.
\]

Let us denote the “basic” generalized Abelian integral (see [BŻ1]) by

\[
\Psi_\gamma(h) = \int_{\gamma_h} g(t)(1 - x_1(t)) \, dt = (e^{-at} - 1)^{-1} \int_0^{T_\gamma} \int_t^{t+T_\gamma} ds \, e^{a(t-s)}(1 - x_1)(s)(1 - x_1)(t).
\]

It is related to the generating function via the following

**Lemma 2.1.** We have

\[
J(h) = h^4 \text{Re} \left[ \pi \left( a \Psi_\gamma + 2 \int_{\gamma_h} (1 - x_1) \, dt \right) \right].
\]

**Proof.** In this proof we denote by dot the differential with respect to the Hamiltonian time \( t \) (for example, \( \dot{f} = \frac{df}{dt} = X_H(f) \)). We have

\[
J(h) = -h^4 \int_{\gamma} \text{Re}(\pi g) \frac{d}{dt}(1 - x_1) \, dt = h^4 \text{Re} \left( \pi \int_{\gamma} \dot{g}(1 - x_1) \, dt \right).
\]

Next, the equation \( \dot{g} = a g + (1 - x_1) \) implies

\[
J(h) = h^4 \text{Re} \left[ \pi \left( a \Psi_\gamma + \int_{\gamma} (1 - x_1)^2 \, dt \right) \right] = h^4 \text{Re} \left[ \pi \left( a \Psi_\gamma + \int_{\gamma} (2 - 2x_1 - \frac{1}{2} \dot{x}_2) \, dt \right) \right]
\]

\[= h^4 \text{Re} \left[ \pi \left( a \Psi_\gamma + 2 \int_{\gamma_h} (1 - x_1) \, dt \right) \right]. \hspace{1cm} \Box
\]

Our next aim is to determine the leading terms in the asymptotic expansion of the integrals \( \int_{\gamma} (1 - x_1)(t) \, dt, T_\gamma, \) and \( \Psi_\gamma, \) as \( h \to 0^+ \). We begin with the Abelian integrals. It is known [Z] that these integrals extend to multivalued holomorphic functions with logarithmic singularities. We will need the explicit form of the leading terms.

**Lemma 2.2.** There exists an open neighborhood \( 0 \in U \subset \mathbb{C} \) in the complex domain and holomorphic functions \( \eta_0, \eta_1, \zeta_0, \zeta_1 \in \Omega(U) \) such that

\[
T_\gamma = \eta_0(h) + \zeta_0(h) \log h = -\frac{1}{2\sqrt{3}} \log h + \frac{\sqrt{3}}{2} \log 12 + O(h \log h), \quad (2.7)
\]

\[
\int_{\gamma} (1 - x_1) \, dt = \eta_1(h) + \zeta_1(h) \log h = 2\sqrt{3} + O(h \log h). \quad (2.8)
\]
Proof. We consider the pair of basis elliptic Abelian integrals

\[ I_0(h) = T_\gamma = \int_{\gamma} \frac{-dx_1}{2x_2}, \quad I_1(h) = \int_{\gamma} \frac{-x_1 dx_1}{2x_2}. \]

Note that \( \int_\gamma (1 - x_1) dt = I_0 - I_1 \) and that \( I_0 = \frac{d}{dh} (\text{area of } \{H > h\}) \).

These functions \((I_0, I_1)\) satisfy the Picard–Fuchs equations

\[
6h(h - 4)I_0'' = -(h - 2)I_0 - 2I_1, \quad 6h(h - 4)I_1' = 2I_0 + (h - 2)I_1.
\]

(2.9)

An independent solution of this system is given by the pair \((K_0, K_1)\), where

\[
K_0(h) = \int_{\delta_h} \frac{-dx_1}{2x_2}, \quad K_1(h) = \int_{\delta_h} \frac{-x_1 dx_1}{2x_2}
\]

are integrals along another cycle \(\delta_h\) in the complex curve \(E_h = \{H(x) = h\} \subset \mathbb{C}^2\). If \(h \in (0, 4)\), then the polynomial \(x_1^2 - 3x_1 + 2 - h\) has three real roots \(x_1^{(1)} < x_1^{(2)} < x_1^{(3)}\) (see Figure 2.1). The cycle \(\gamma_h\) (respectively \(\delta_h\)) is represented as the lift to the Riemann surface \(E_h\) of a loop in the complex \(x_1\)-plane surrounding the roots \(x_1^{(1)}\) and \(x_1^{(2)}\) (respectively \(x_1^{(2)}\) and \(x_1^{(3)}\)). Note the following integral formulas for \(T_\gamma(h)\):

\[
T_\gamma = \int_{x_1^{(1)}}^{x_1^{(3)}} \frac{dx_1}{\sqrt{x_1^2 - 3x_1 + 2 - h}} = \int_{x_1^{(3)}}^{\infty} \frac{dx_1}{\sqrt{x_1^2 - 3x_1 + 2 - h}}, \quad h \in (0, 4).
\]

(2.10)

The second equality corresponds to an unobstructed deformation of the integration contour \(\gamma_h\) to the loop surrounding \(x_1^{(3)}\) and \(\infty\).

The system (2.9) has a resonant singular point \(h = 0\). Each of its solutions is either analytic near \(h = 0\) (like \((K_1, K_2)\)) or it is of the form like \(I_0, I_1\):

\[
I_0(h) = (a_0 + a_1 h + \ldots) + \frac{1}{2\pi i} K_0 \log h, \quad I_1(h) = (b_0 + b_1 h + \ldots) + \frac{1}{2\pi i} K_1 \log h.
\]

(2.11)

This representation follows from the Picard–Lefschetz formula

\[
\gamma_h \to \gamma_h \cdot \delta_h, \quad \delta_h \to \delta_h,
\]

(2.12)

which describes the monodromy transformations of the generators of \(\pi_1(E_h, *)\) as \(h\) winds around the critical value \(0\); here \(*\) denotes a basepoint.

We need to compute the expansions of \(I_0, I_1\). As we will see, it is enough to find \(K_0(0)\) and \(a_0\); all other coefficients can be found recursively from the system (2.9).

Indeed, to compensate for the terms with \(\log h\) in (2.9) we must have

\[
K_1(0) = K_0(0).
\]

(2.13)

The terms with \(h^0\) give

\[
b_0 = a_0 + \frac{12}{2\pi i} K_0(0),
\]

(2.14)

and so forth.
To determine $K_0(0)$ and $a_0$ simultaneously, we make a coordinate change $u = (x_1 - 1)/(x_3^{(3)} - 1)$ in the integral (2.10); we put also $p = (x_3^{(3)} - 1)$. Since $p = \sqrt{h/3} + O(h)$ as $h \to 0^+$, we see that

$$
\int_1^\infty du \left[ \frac{1}{\sqrt{u^2(3 + pu) - h/p^2}} - \frac{1}{\sqrt{u^2(3 + pu)}} - \frac{1}{\sqrt{3u^2 - 3}} + \frac{1}{\sqrt{3}} \right] \sim 0.
$$

We evaluate:

$$
\int_1^\infty du \left[ \frac{1}{\sqrt{3u^2 - 3}} - \frac{1}{\sqrt{3}} \right] = \log \frac{2}{\sqrt{3}},
$$

$$
\int_1^\infty du \left[ \frac{1}{u\sqrt{3 + pu}} \right] = \frac{2}{\sqrt{3}} \log \left( \frac{2\sqrt{3}}{\sqrt{p}} + o(1) \right) = -\frac{1}{2\sqrt{3}} \log h + \frac{\log(12\sqrt{3})}{\sqrt{3}} + o(h^{1/2}).
$$

Thus, $a_0 = \sqrt{2}\log 12$, $K_0(0) = -\frac{2\pi}{2\sqrt{3}}$. Substituting these values into the relations (2.13), (2.14) and using the expansion (2.11) we get the leading terms of the expansions as in formulas (2.7) and (2.8).

Let us pass to the expansion of $\Psi_\gamma$.

**Proposition 2.3.** Suppose that $-2\sqrt{3} < \text{Re}(a) < 0$. Then there exists an open neighborhood $0 \in \mathcal{U} \subset \mathbb{C}$ in the complex domain and holomorphic functions $\varphi_1, \varphi_2, \varphi_3$ such that

$$
\Psi_\gamma(z) = \varphi_1(h) + \varphi_2(h) \log h + \varphi_3(h) \left( e^{-a\gamma} - 1 \right)^{-1} = C_0 + C_1 h^{-a/2\sqrt{3}} + \ldots,
$$

where

$$C_1 = \frac{(\pi u)^2}{\sin^2(\pi u/2\sqrt{3})},
$$

$$C_0 = \frac{3\sqrt{2}}{\sqrt{\pi}} (-1 + 2w + 2w^2 \psi'(-w)), \quad w = \frac{a}{2\sqrt{3}}.
$$

**Remark 2.4.** One can easily observe that the value of $\kappa$ given by formula (1.4) satisfies the relation

$$
\kappa = i(4\sqrt{3} + aC_0).
$$

This value is chosen so as to kill the leading term ($\sim h^4$) of $J(h)$ and to reveal the term with the infinite sequence of zeroes—see Corollary 2.5 and its proof below.

**Corollary 2.5.** If the values of parameters $(\rho, \omega, \kappa)$ as in the main theorem, then the integral $J(h)$ has a sequence $h_\alpha$, $n = 1, 2, \ldots$, of simple zeroes accumulating at $h = 0$.

**Proof.** We compute the leading term of the expansion of $J(h)$ using Lemmas 2.1 and 2.2 and Remark 2.4:

$$
J(h) = h^4 \text{Re}\left[ \pi (aC_0 + aC_1 h^{1/2 - 1/2} + 4\sqrt{3} + o(h^{3/4})) \right] =
$$

$$
= h^4 \text{Re}\left[ \pi (aC_0 + 4\sqrt{3}) \right] + h^4 \text{Re}\left[ \pi a C_1 h^{1/2 - 1/2} \right] + o(h^{4+3/4}) =
$$

$$
= R h^{4+1/2} \cos(\log \sqrt{h} - a_0) + o(h^{4+3/4}),
$$

where $R$ is a constant.
The existence of a 2-dimensional Jordan cell in the Jordan form of \( \psi \) by the non-vanishing of the function \( \Psi \). Thus, by the implicit function theorem the zeroes \( (h_n) \) of \( J(h) \) approximate the simple zeroes \( h_n(0) \) of the function \( \cos(\log h - \alpha_0) \).

The remaining part of this section is devoted to the proof of Proposition 2.3. We will proceed in two steps. In the first step we show that the function \( \Psi(h) \) tends to \( C_0 \) as \( h \to 0^+ \) (hence, it is bounded).

In the second step we determine the monodromy of the generalized Abelian integral \( \Psi(h) \) as \( h \) winds around \( h = 0 \). We know that this monodromy replaces \( \gamma \) by \( \mathrm{Mon}_0 \gamma = \gamma \cdot \delta \), and that \( \mathrm{Mon}_0 \delta = \delta \). We would like to express \( \Psi(h) \) in simple terms, in order to determine the singularity of \( \Psi(h) \) at \( h = 0 \). Rather complicated formulas for \( \Psi(h) \) are given in [BŻ1] and [BŻ2]. In [Bo] these formulas were simplified using an upper triangle representation \( \rho \) of the fundamental group \( \pi_1(E_h, *) \). We recall this construction below.

Fix the following notation:

\[
\Psi_\gamma(h) = (e^{-aT\gamma} - 1)^{-1} \int_0^{T_\gamma} dt \int_t^{t+T_\gamma} ds \, e^{a(t-s)}(1-x_1)(s) (1-x_1)(t),
\]

\[
\lambda_\gamma(h) = e^{-aT\gamma/2},
\]

\[
\phi_\gamma(h) = \lambda_\gamma \int_0^{T_\gamma} dt \int_0^t ds \, (1-x_1)(t) (1-x_1)(s) \cdot e^{a(t-s)} = \lambda_\gamma \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} ds \, (1-x_1)(t+T/2) (1-x_1)(s+T/2) e^{a(t-s)},
\]

\[
\theta^+_\gamma = \lambda_\gamma \int_0^{T_\gamma} dt \, (1-x_1)(t) e^{at} = \int_{-T_\gamma/2}^{T_\gamma/2} dt \, (1-x_1)(t+T/2) e^{at},
\]

\[
\theta^-_\gamma = \lambda_\gamma^{-1} \int_0^{T_\gamma} dt \, (1-x_1)(t) e^{-at} = \int_{-T_\gamma/2}^{T_\gamma/2} dt \, (1-x_1)(t+T/2) e^{-at}.
\]

Here the subscript \( \gamma \) reminds us that the above functions depend on the loop \( \gamma = \gamma_h \).

We introduce the following space of triangular matrices:

\[
\mathbb{T} = \left\{ \begin{pmatrix} \lambda & \theta^- & \phi \\ 0 & \lambda^{-1} & \theta^+ \\ 0 & 0 & \lambda \end{pmatrix} \right\}; \quad \lambda \in \mathbb{C}^*, \quad \theta^+, \theta^-, \phi \in \mathbb{C};
\]

this space is a group. For \( W \in \mathbb{T} \) we put

\[
|W| = \det W = \lambda.
\]

The existence of a 2-dimensional Jordan cell in the Jordan form of \( W \) is detected by the non-vanishing of the function \( \psi(W) \):

\[
(W - |W|)(W - 1/|W|) = \psi(W) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};
\]

\[
|W|^2 - 1
\]
we have
\[ \psi(W) = \frac{\theta^+ \theta^-}{\lambda^2 - 1} + \phi \lambda \quad \text{ (2.22)} \]

**Theorem 2.6** [Bo]. The map \( \rho: \pi_1(E_h, \ast) \to \mathbb{T} \) defined by the formula
\[ \rho(\gamma) = \begin{pmatrix} \lambda \gamma & -\gamma \phi \\ \lambda^2 - 1 & \gamma \phi \end{pmatrix} \quad \text{ (2.23)} \]

where \( \lambda, \gamma, \phi \), are as in (2.19), is a representation of the fundamental group of \( E_h \). Moreover,
\[ \Psi_\gamma = \psi \circ \rho(h) \quad \text{ (2.24)} \]

**Sketch of proof.** We have \( \Psi_\gamma = (\lambda^2 - 1)^{-1} \int e^{a(t-s)}(1-x)(t)(1-x)(s) \), where the integration domain is \( \Sigma = \{(t, s): 0 \leq t \leq T_\gamma, t \leq s \leq T_\gamma + t\} \) (see (2.19)). We divide \( \Sigma \) into two “triangles” \( \Delta_1 = \{0 \leq t \leq T_\gamma, t \leq s \leq T_\gamma \} \) and \( \Delta_2 = \{0 \leq t \leq T_\gamma, T_\gamma \leq s \leq T_\gamma + t\} \), where \( \Delta_0 = \{0 \leq t \leq T_\gamma, 0 \leq s \leq t\} \).

Since \( \int_{\Delta_{1+\Delta_2}}(\cdot) = \int_{\Delta_{1+\Delta_0}}(\cdot) + \int_{\Delta_{2-\Delta_0}}(\cdot) \), where \( \int_{\Delta_{1+\Delta_0}}(\cdot) = \theta^+_\gamma \theta^-_\gamma \) and \( \int_{\Delta_{2-\Delta_0}}(\cdot) = (\lambda^2 - 1)\lambda^{-1}\phi_\gamma \), the formula (2.24) follows from (2.22).

The identity \( \rho(\gamma \delta) = \rho(\gamma) \rho(\delta) \), \( \gamma, \delta \in \pi_1(E_h, \ast) \) is proved similarly. One divides the line integrals in \( \theta^+_\gamma \phi_\gamma \) and the surface integral in \( \phi_\delta \phi_\gamma \) into parts where \( t \) or \( s \) lies in \( \gamma \) or in \( \delta \). One uses also the identity \( \lambda_{\gamma \delta} = \lambda_\gamma \lambda_\delta \).

**Proposition 2.7.** Suppose that \( \xi(t) = (1-x_0)(t) \) with the initial value \( \xi(0) = 1 - x_1^{(1)} \) (see Figure 2.1). Then the following integral formula holds for the generalized Abelian integral:
\[ \Psi_\gamma(h) = (e^{-aT_\gamma} - 1)^{-1} \left( \int_{-T_\gamma/2}^{T_\gamma/2} \xi(t) e^{at} dt \right)^2 + \int_{-T_\gamma/2}^{T_\gamma/2} dt \int_{-T_\gamma/2}^{\gamma} ds \xi(s) \xi(t) e^{a(t-s)} \quad \text{ (2.25)} \]

As \( h \to 0^+ \), these integrals have finite limits:
\[ \int_{-T_\gamma/2}^{T_\gamma/2} \xi(t) e^{at} dt \to \frac{\pi a}{\sin(\pi a/2\sqrt{3})} = \sqrt{C_1}, \quad \text{ (2.26)} \]
\[ \int_{-T_\gamma/2}^{T_\gamma/2} dt \int_{-T_\gamma/2}^{\gamma} ds \xi(s) \xi(t) e^{a(t-s)} \to C_0, \]

where \( C_0, C_1 \) are defined in Proposition 2.3.

**Proof.** The value of the generalized Abelian integral \( \Psi_\gamma \) does not depend on the shift of parametrization (e.g. \( t \mapsto t + \frac{T_\gamma}{2} \)), but the values of integrals \( \phi_\gamma \) and \( \theta^+_\gamma \) do.

We choose a Hamiltonian time parameter in such a way that \( x_1(0) = x_1^{(2)} \), where \( x_h^{(1)} < x_h^{(2)} < x_h^{(3)} \) are real roots of the polynomial \( x^3 - 3x + 1 = 0 \) (see Figure 2.1). Thus,
\[ (1-x_1)(t + T_\gamma/2) = \xi(t) \]
and so, using formula (2.22) and formulas (2.19), we obtain formula (2.25).
To determine the asymptotic expansion we observe that the singular curve $E_0 = \{ H(x) = 0 \} = \{ x^2 = (x - 1)^2(x + 2) \}$ is rational. The Hamiltonian parametrization of the limit loop $\gamma_0$ can be explicitly calculated:

$$
\xi(t) \to \xi_0(t) = \frac{3}{\cosh^3(\sqrt{3}t)}, \quad -\infty < t < \infty \quad (2.27)
$$
as $h \to 0^+$. Recall that $T_0(0) = \infty$. Substituting these values into the integrals in (2.25) we get the limit

$$
\int_{-T_0/2}^{T_0/2} \xi(t) e^{at} dt \to \int_{-\infty}^{\infty} \frac{3}{\cosh^3(\sqrt{3}t)} e^{i(a/\sqrt{3})} dt = \sqrt{2 \pi} \mathcal{F}(\xi_0)(a/\sqrt{3}),
$$
where $\mathcal{F}$ denotes the Fourier transform. Since $\mathcal{F}(\frac{1}{\cos^2})(k) = \sqrt{\frac{k}{2}} \frac{\sinh(k\pi/2)}{\sinh(k\pi)}$ (see, for example, [GR], Integral 3.982.1), we have

$$
\frac{i\pi a}{\sinh(i\pi a/2\sqrt{3})} = \frac{\pi a}{\sin(\pi a/2\sqrt{3})} = \sqrt{C_1}.
$$

To determine the limit of the second integral

$$
\int_{-T_0/2}^{T_0/2} dt \int_{-T_0/2}^T ds \xi(t)\xi(s) e^{a(t-s)} \to \int_{-\infty}^{\infty} dt \int_{-\infty}^T ds \xi_0(t)\xi_0(s) e^{a(t-s)}
$$
we substitute $u = t - s$, use the symmetry $\xi_0(-t) = \xi_0(t)$ and the Parseval identity; then we have

$$
\int_{-\infty}^{\infty} dt \int_{-\infty}^\infty du \xi_0(t)\xi_0(u-t) e^{au} = \int_{-\infty}^{\infty} dk \left( \mathcal{F}(\xi_0)(k) \right)^2 \mathcal{F}(e^{au} \chi_{[0,\infty)})(k) =
$$

$$
= \frac{3\sqrt{2\pi}}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{k^2}{\sinh^2 k (k - \pi i (a/2\sqrt{3}))} dk.
$$

To evaluate the latter integral, which has the form

$$
F(w) = \int_{-\infty}^{\infty} \frac{z^2}{\sinh^2 z} \frac{dz}{z - \pi i w}, \quad \text{Re } w < 0, \quad (2.28)
$$
we will use logarithmic derivative of Euler’s Γ function, i.e., the function $\psi = (\log \Gamma)' = \Gamma'/\Gamma$.

Integrating by parts we find that

$$
F(w) = \lim_{R \to \infty} \int_{-R}^{-R-1} + \int_{R-1}^R (\text{sgn} z - \coth z) f \frac{z^2}{z - \pi i w} dz =
$$

$$
= -2\pi i w + \pi^2 w^2 \lim_{R \to \infty} \int_{-R}^{-R-1} + \int_{R-1}^R \coth z \frac{z^2}{z - \pi i w} dz.
$$

Next, we integrate the function $\frac{\coth z}{(z - \pi i w)^2}$ along the contour consisting of the segment $[-R, -R^{-1}]$ followed by the semicircle $R^{-1} e^{i\varphi}, \varphi \in [\pi, 2\pi]$ and segments: $[R^{-1}, R], [R, R + i(N + \frac{1}{2})\pi], [R + i(N + \frac{1}{2})\pi, -R + i(N + \frac{1}{2})\pi], [-R, -R + i(N + \frac{1}{2})\pi]$,
where \( N \in \mathbb{N} \). Using the residue formula and passing to the limit as \( R, N \to \infty \) we infer that
\[
\lim_{R \to \infty} \left( \int_{-R}^{-1} \int_{R}^{1} \frac{2\pi i}{\pi iw} \, dz + \frac{\pi i}{\pi iw} \int_{R}^{1} \frac{\coth z}{(z - \pi iw)^2} \, dz \right) = \frac{2}{\pi} \sum_{n=0}^{\infty} \text{Res}_{i\pi n} \left( \frac{\coth z}{(z - \pi iw)^2} \right) = -2 \psi(-w);
\]
the latter sum was evaluated using [GR], formula 8.363.8. Finally we have
\[
F(w) = \pi i \left( 1 - 2w - 2w^2 \psi'(-w) \right) \quad \text{for } \text{Re}(w) < 0,
\]
which yields the second of the limits (2.26).

Now we investigate the monodromy properties of the generalized Abelian integral \( \Psi_{\gamma} \). We shall need the following

**Lemma 2.8 [Bo].** For \( W, W' \in \mathbb{T} \) we have
\[
\psi(W \cdot W') = \psi(W) + \psi(W') + \frac{|W|^2 |W'|^2}{(|W|^2 - 1)(|W'|^2 - 1)(|W|^2 |W'|^2 - 1)} \tilde{\psi}([W, W']),
\]
where \([W, W'] = W W' W^{-1} (W')^{-1}\) is the commutant and
\[
\tilde{\psi}(W) = (|W|^2 - 1) \psi(W);
\]
(for \(|W| = 1\) we have \(\tilde{\psi}(W) = \theta^+ \theta^-\) in terms of (2.20)).

**Proof.** The proof consists of direct calculations. \(\square\)

**Corollary 2.9.** The function \( \Psi_{\gamma}(h) \) near \( h = 0 \) has the form
\[
\Psi_{\gamma}(h) = \varphi_1(h) + \frac{1}{2\pi i} \Psi_\delta(h) \log h - \frac{\lambda_\delta^2(h)}{(\lambda_\delta^2(h) - 1)^2} \tilde{\Psi}_{[\gamma, \delta]}(h) \cdot \frac{1}{\lambda_\delta(h) - 1}, \quad (2.29)
\]
where \( \delta \) is the second cycle in \( \pi_1(E_h, +) \) (see the proof of Lemma 2.2) and \( \tilde{\Psi}_{[\gamma, \delta]}(h) = \tilde{\psi}(\rho([\gamma, \delta])) \). The functions \( \varphi_1, \Psi_\delta \) and \( \tilde{\Psi}_{[\gamma, \delta]} \) are holomorphic near \( h = 0 \).

**Remark 2.10.** One can prove that the function \( \tilde{\Psi}_{[\gamma, \delta]} \) is constant:
\[
\tilde{\Psi}_{[\gamma, \delta]}(h) = (2\pi a)^2.
\]
Indeed, since the contour \([\gamma, \delta]\) is monodromy invariant (see proof of Corollary 2.9 below), the function \( \tilde{\Psi}_{[\gamma, \delta]} \) is meromorphic on the whole \( \mathbb{C} \) with possible poles at \( h = 0, 4 \). We know, by Proposition 2.7, that it is bounded as \( h \to 0 \). Similarly one shows that it is bounded as \( h \to 4 \). The computations are similar to the computation of the first limit in (2.26). One also checks that \( \tilde{\Psi}_{[\gamma, \delta]} \) is bounded as \( h \to \infty \). Thus, this function has to be constant; one evaluates it by passing to the limit as \( h \to 0 \) and comparing the terms in (2.25) and (2.29) corresponding to each other.
Proof of Corollary 2.9. The Picard–Lefschetz formula (2.12), Theorem 2.6 and Lemma 2.8 imply that
\[ \mathcal{M}_{\gamma} \psi = \psi(\rho(\gamma) \cdot \rho(\delta)) = \psi_{\gamma} + \psi_{\delta} + \frac{\lambda_{\gamma}^2 \lambda_{\delta}^2}{(\lambda_{\gamma}^2 - 1)(\lambda_{\delta}^2 - 1)(\lambda_{\gamma}^2 \lambda_{\delta}^2 - 1)} \tilde{\psi}_{[\gamma, \delta]} \] (2.30)
and \( \mathcal{M}_{\gamma} \psi_{\delta} = \psi_{\delta} \). Moreover, the following monodromy relations follow from the Picard–Lefschetz formula (2.12):
\[ \mathcal{M}_{\gamma} \lambda_{\gamma} = \lambda_{\gamma} \lambda_{\delta}, \quad \mathcal{M}_{\gamma} \lambda_{\delta} = \lambda_{\delta}, \]
\[ \mathcal{M}_{[\gamma, \delta]} \lambda = (\gamma \delta) \cdot \delta^{-1} \cdot \delta^{-1} = [\gamma, \delta]. \]
Therefore \( \psi_{\delta}, \tilde{\psi}_{[\gamma, \delta]} \) and \( \lambda_{\delta} \) are locally single-valued functions of \( h \). Since they are bounded (see Proposition 2.7), they must be holomorphic. Now
\[ \mathcal{M}_{\gamma} \left( \frac{1}{2\pi i} \psi_{\delta} \log h \right) = \frac{1}{2\pi i} \psi_{\delta} \log h + \psi_{\delta}, \]
\[ \mathcal{M}_{\gamma} \left( -\frac{\lambda_{\gamma}^2 \tilde{\psi}_{[\gamma, \delta]}}{(\lambda_{\gamma}^2 - 1)^2 (\lambda_{\delta}^2 - 1)} \right) = -\frac{\lambda_{\gamma}^2 \tilde{\psi}_{[\gamma, \delta]}}{(\lambda_{\gamma}^2 - 1)^2 (\lambda_{\delta}^2 - 1)} + \frac{\lambda_{\gamma}^2 \lambda_{\delta}^2 \tilde{\psi}_{[\gamma, \delta]}}{(\lambda_{\gamma}^2 - 1)(\lambda_{\delta}^2 - 1)(\lambda_{\gamma}^2 \lambda_{\delta}^2 - 1)}. \]
Therefore the function \( \varphi_1 \) defined by (2.29) is single-valued. Since \( (\lambda_{\gamma}^2 - 1) \) and \( (\lambda_{\delta}^2 - 1) \) are separated from zero and the function \( \psi_{\gamma} \) is bounded (see Proposition 2.7), the function \( \varphi_1 \) is holomorphic. \( \square \)

Corollary 2.9 allows one to finish the proof of Proposition 2.3. The holomorphic function \( \varphi_1 \) is defined in this corollary; \( \varphi_2 \) and \( \varphi_3 \) can be read from (2.29):
\[ \varphi_2 = \frac{1}{2\pi i} \psi_{\delta}, \]
\[ \varphi_3 = -\frac{\lambda_{\delta}^2 \tilde{\psi}_{[\gamma, \delta]}}{(\lambda_{\delta}^2 - 1)^2} = -\left( \frac{2\pi a \lambda_{\delta}}{\lambda_{\delta}^2 - 1} \right)^2 \]
(the latter equation follows from Remark 2.10).

Since \( T_\gamma = -\frac{1}{2\sqrt{a}} \log h + O(1) \), we have the following leading term of the expansion:
\[ (\lambda_{\gamma}^2 - 1)^{-1} = (e^{-aT_\gamma} - 1)^{-1} = h^{-a/2\sqrt{a}} + \ldots \] \( \square \)

2.2. Estimates. In this subsection we show that the zeroes \( h_n \) of a generalized Abelian integral (see Corollary 2.5) generate the corresponding limit cycles of the system (1.3), provided that \( \varepsilon \) is sufficiently small.

Recall that the problem of limit cycles of (1.3) is reduced to the problem of limit cycles of the planar system
\[ \dot{x} = X_H(x) + \varepsilon \text{Re}(\pi G(x, \varepsilon)) e_2, \] (2.31)
where the function \( G(x, \varepsilon) \) is defined via the invariant surface \( L_\varepsilon = \{ y = \varepsilon G(x, \varepsilon) \} \), which is a graph of \( G(\cdot, \varepsilon) \).

At the moment we do not even know whether an invariant surface exists. Indeed, the normal hyperbolicity conditions are not satisfied: the eigenvalues in the normal
direction are \( \lambda_{3,4} = -\sqrt{3} \pm i\sqrt{3} \), whereas the eigenvalues at the saddle point \( x = (1, 0), \ y = 0 \) in the \( x \)-direction are \( \pm 2\sqrt{3} \) (cf. [BŻ1], [BŻ2], [HPS], [Ni]). We are to do two things:

1. prove the existence of an invariant surface,
2. estimate the discrepancy \( G(x, \varepsilon) - H^4 g(x) \), where \( g(x) \) is the solution to the normal variation equation given in (2.1).

In both tasks the crucial role is played by the following Lemma. Let us recall the notation related to the elliptic Hamiltonian \( H(x) = x_1^4 - 3x_1 - x_2^2 + 2 \). The basin \( D \subset \mathbb{R}^2 \) (see Figure 2.1) is filled with closed orbits of the Hamiltonian vector field \( X_H \).

**Lemma 2.11.** Let \( U \supset D \times \{0\} \) be an open neighborhood in \( \mathbb{R}^2 \times \mathbb{C} \) and \( V_\varepsilon \) be the following vector field in \( U \):

\[
V_\varepsilon = \begin{cases} 
\dot{x} = X_H + H^k \Re(\pi y)v_0 + Q(x, y; \varepsilon), \\
\dot{y} = ay + B(x, y; \varepsilon),
\end{cases}
\]

(2.32)

where \( k \geq 0, \ a = -\sqrt{3} + i\sqrt{3}, \ \kappa \in \mathbb{C}, \ v_0 \in \mathbb{R}^2 \) and \( Q, B \) are functions of class \( C^2(U) \) satisfying the following inequalities:

\[
|Q| \leq \text{Const} \cdot \epsilon|H(x)|^2,
\]

\[
|B| \leq \text{Const} \cdot \epsilon|H(x)|^2.
\]

Then for a sufficiently small \( \varepsilon \) there exists a unique invariant surface of \( V_\varepsilon \):

\[
L_\varepsilon = \{(x, y): x \in D, \ y = \varepsilon G(x, \varepsilon)\}.
\]

The function \( G(\cdot, \varepsilon) \) extends by zero to a \( C^1 \) function on a neighborhood of \( D \) in \( \mathbb{R}^2 \).

**Proof.** We will prove that the Poincaré return map associated to vector field \( V_\varepsilon \) satisfies the normal hyperbolicity condition.

In a neighborhood of the center critical point \( (x = (-1, 0), \ y = 0) \) for unperturbed vector field \( V_0 \), the system is normally hyperbolic and so an invariant surface exists. In the remaining part of the proof we shall concentrate on a neighborhood of the separatrix \( \gamma_0 = \{(x, y): y = 0, \ H(x) = 0\} \) (see Figure 2.2).

The non-degenerate critical point \( p_0 = ((1, 0), 0) \) situated on this separatrix is preserved after the perturbation. Let us choose a 3-dimensional hypersurface \( S \) that is transversal to \( \gamma_0 \) and close to the singular point \( p_0 \). Let

\[
\Sigma_{\alpha,d} = S \cap \{(x, y): |y|^2 \leq \alpha^2(H(x))^2, \ H(x) < d\}
\]

be a sector in \( S \) with vertex

\[
\sigma_0 = \Sigma_{\alpha,d} \cap \gamma_0 = \{(x, 0) \in \Sigma_{\alpha,d}: H(x) = 0\}.
\]

**Lemma 2.12.** If \( \alpha > 0 \) is sufficiently small, \( d_2 > d_1 > 0 \) and \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), then the Poincaré return map

\[
P_\varepsilon: \Sigma_{\alpha,d_1} \to \Sigma_{\alpha,d_2}
\]

(2.33)

is a diffeomorphism onto its image and extends to a map of class \( C^1 \) at the point \( \sigma_0 \).
Now we finish the proof of Lemma 2.11. The unperturbed Poincaré map \( P_0 \) is identity on the invariant segment \( I_0 = \Sigma_{\alpha,d_1} \cap \{ y = 0 \} \). Thus \( P_0 \) is normally hyperbolic on \( I_0 \), since we have strong contraction in the normal direction. By virtue of the Hirsh-Pugh-Shub-Theorem [HPS], for a sufficiently small \( \varepsilon \) there exists a unique invariant embedded interval \( I_{\varepsilon} \) close to \( I_0 \); it is of class \( C^1 \). Considering Hamiltonian as the parameter on \( I_0 \), we get

\[
I_{\varepsilon} = \{ y = \varepsilon F(h, \varepsilon), h \in [0, \delta) \}, \quad F \in C^1([0, \delta) \times (-\varepsilon_0, \varepsilon_0)).
\]

The surface \( S_{\varepsilon} \) spanned by trajectories of \( V_{\varepsilon} \) passing through \( I_{\varepsilon} \) is \( V_{\varepsilon} \)-invariant due to the invariance of \( I_{\varepsilon} \) under the Poincaré map \( P_{\varepsilon} \). The forms of the invariant interval \( I_{\varepsilon} \) and the vector field \( V_{\varepsilon} \) imply that the surface \( S_{\varepsilon} \) is the graph of a \( C^1(D) \) function:

\[
L_{\varepsilon} = \{ (x, y) : x \in D, y = \varepsilon G(x, \varepsilon) \}.
\]

We prove that \( G \) can be extended by zero outside \( D \).

We claim that if the assumptions of Lemma 2.11 hold, then the set

\[
\{(x, y) : x \in D, |y|^2 \leq R^2 \varepsilon^2 |H(x)|^4\}
\]

is invariant with respect to \( V_{\varepsilon} \) for big enough \( R \). Indeed, denoting all constants by \( C \), we have

\[
V_{\varepsilon}( |y|^2 - R^2 \varepsilon^2 |H|)^2 \big|_{|y| = R \varepsilon |H|} =
\]

\[
= 2\text{Re}[y (a y + B)] - 4 R^2 \varepsilon^2 H^3 \text{Re}(\pi y \langle dH, v_0 \rangle + \langle dH, Q \rangle)
\]

\[
\leq 2 R^2 \varepsilon^2 H^4 \left( -\rho + \frac{C}{R} + 2 R \varepsilon C |H|^{1+k} + \varepsilon C |H| \right). \quad (2.34)
\]
Since the latter expression is \( \leq 0 \) (for sufficiently large \( R \)), the subset in question is \( V_\varepsilon \)-invariant.

Thus,

\[
|G(x, \varepsilon)| \leq \text{Const} \cdot |H(x)|^2
\]

and the function \( G(x, \varepsilon) \) can be extended by zero to a function of class \( C^1 \). \( \square \)

**Proof of Lemma 2.12.** Calculations similar to those used in (2.34) show that

\[
V_\varepsilon((|y|^2 - \alpha^2 H^2)|_{|y|=\alpha H}) \leq 2\alpha^2 H^2 \left( -\rho + \frac{\varepsilon |H|}{\alpha} + \alpha C |H|^k + C \varepsilon |H| \right).
\]

Hence, for \( \alpha \) and \( \varepsilon \) sufficiently small the subset

\[
\{(x, y) : x \in D, |y|^2 \leq \alpha^2 (H(x))^2 \}
\]

is \( V_\varepsilon \)-invariant. Moreover, the separatrix \( \gamma_0 \) is also \( V_\varepsilon \)-invariant. This proves that the Poincaré return map defines the diffeomorphism (2.33), which is of class \( C^1 \) outside the border.

Thus it remains to show that \( \mathcal{P}_\varepsilon \) can be extended to a map that is \( C^1 \) at the point \( \sigma_0 \). We choose another auxiliary 3-dimensional hypersurface \( \tilde{S} \) transversal to \( \gamma_0 \) and close to \( p_0 \) which lies in the opposite direction from the point \( p_0 \) (see Figure 2.2). The return map \( \mathcal{P}_\varepsilon \) is the composition \( \mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon^\tau \circ \mathcal{P}_\varepsilon^\tau \) of the correspondence maps

\[
\mathcal{P}_\varepsilon^\tau : S \to \tilde{S} \quad \text{and} \quad \mathcal{P}_\varepsilon^\tau : \tilde{S} \to S,
\]

defined by trajectories near the singular point \( p_0 \) and trajectories near the regular part of \( \gamma_0 \) respectively. The regular map naturally extends to a \( C^1 \) map (even \( C^2 \), in fact) as the flow of the non-vanishing, \( C^2 \) vector field \( V_\varepsilon \). To analyze the singular part \( \mathcal{P}_\varepsilon^\tau \), we use the following theorem of H. Belitskii.

**Theorem 2.13 [Be].** Let \( \Lambda \in \text{End}({\mathbb{R}}^n) \) be a linear endomorphism whose eigenvalues \((\lambda_1, \ldots, \lambda_n)\) satisfy the inequalities

\[
\text{Re} \lambda_i \neq \text{Re} \lambda_j + \text{Re} \lambda_k
\]

for all \( i \) and \( j, k \) such that \( \text{Re} \lambda_j \leq 0 \leq \text{Re} \lambda_k \). Then any \( C^2 \)-differential system

\[
\frac{dx}{dt} = \Lambda x + f(x), \quad f(0) = 0 = f'(0)
\]

is \( C^1 \)-equivalent to its linearization in a neighborhood of 0.

The eigenvalues of the linearization of our vector field \( V_\varepsilon \) in \( p_0 \) are \( \pm 2\sqrt{3}, \ -\sqrt{3} \pm i\sqrt{3} \), so \( V_\varepsilon \) satisfies the hypothesis of the Belitskii theorem. In suitable coordinates \((u, v)\) associated with the linearization of \( V_\varepsilon \) in a neighborhood of \( p_0 \), the correspondence map \( \mathcal{P}_\varepsilon^\tau \) has the form

\[
\mathcal{P}_\varepsilon^\tau(u, v) = (u, Cu^\beta v), \quad u \in {\mathbb{R}}_+, \quad v \in (\mathbb{C}, 0), \quad \beta = \frac{1}{2} - \frac{i}{2}.
\]

The restriction to \( \Sigma_{\alpha, \beta} \) corresponds to the restriction to the set \( \{|v|^2 \leq \tilde{\alpha}(u) u, \ u \in {\mathbb{R}}_+, \ \tilde{\alpha}(0) > 0\} \). In this region the map \( \mathcal{P}_\varepsilon^\tau \) is of the class \( C^1 \) at \( \sigma_0 \); we have

\[
(u, v)(\sigma_0) = (0, 0) \quad \text{and} \quad (\mathcal{P}_\varepsilon^\tau)'(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Thus, the lemma follows. \( \square \)
Now we can prove the existence of an invariant surface and estimate the distance to its linear approximation $H^4g$.

**Proposition 2.14.** For a sufficiently small $\varepsilon$, there exists an invariant surface

$$L_\varepsilon = \{y = \varepsilon G(x, \varepsilon), x \in D\} \quad (2.35)$$

of the system (1.3). The function $G$ is of class $C^1$ and the distance to the linearization $(H^4g)(x) = G(x, 0)$ satisfies the estimates

$$|G - H^4g| \leq C|\varepsilon||h|^5, \quad (2.36)$$

$$|G' - (H^4g)'| \leq C|\varepsilon||h|^4, \quad (2.37)$$

where $G' = G_x$ is derivative with respect to $x$.

**Proof.** The existence of an invariant surface of class $C^1$ having the form (2.35) is a direct consequence of Lemma 2.11.

To justify the bounds (2.36), (2.37), we make a coordinate change $(x, y) \mapsto (x, z)$, $z = y - \varepsilon H^4g(x)$. The system (1.3) takes the form

$$\begin{cases}
\dot{x} = X_H + H^4\text{Re}(\overline{z}) \varepsilon e_2 + \varepsilon H^4\text{Re}(\overline{g}) e_2,
\dot{z} = a(z + 4\varepsilon H^3(z + g)\text{Re}(\overline{z} + g)) 2x_2 - \varepsilon H^4 \frac{\partial g}{\partial X_2} \text{Re}(\overline{z} + g)).
\end{cases} \quad (2.38)$$

Using the integral formula (2.3) for the function $g$ we deduce that it is bounded: $g \leq C$. Since the function $g$ extends to a (multivalued) holomorphic function ramified along the singular curve $\gamma_0$, the following bounds for derivatives of $g$ hold:

$$|g^{(k)}| \leq C_k |H|^{-k}. \quad (2.39)$$

Using this one can check that the system (2.38) satisfies the hypothesis of Lemma 2.11. Hence, the invariant surface has the form

$$z = \frac{\varepsilon G - \varepsilon H^4g}{\varepsilon H^4} = \varepsilon U(x, \varepsilon)$$

and the function $U$ extends by zero to a $C^1$ function in a neighborhood of $D$. Thus, the function $U$ satisfies the estimates

$$|U| \leq C|H|, \quad |U'| \leq C$$

which are equivalent to (2.36) and (2.37). \qed

Now we show that the generalized Abelian integral $J(h)$ is a good approximation of the Poincaré return map and so the zeroes of $J(h)$ generate limit cycles for a sufficiently small $\varepsilon$.

**Proposition 2.15.** Suppose that $\Delta H(h, \varepsilon)$ is the increment of the Hamiltonian after the first return of the system $V_\varepsilon$ restricted to the invariant surface $L_\varepsilon$. Then there exists a constant $C$ such that

$$|\Delta H - \varepsilon J| \leq C \varepsilon |h|^5, \quad (2.40)$$

$$|\partial_h(\Delta H) - \varepsilon J'(h)| \leq C \varepsilon |h|^4 |\log h|. \quad (2.41)$$
Proof. Here we study the phase curves of the 2-dimensional vector field (2.31), i.e.,
\[ W_\varepsilon := V_\varepsilon|_{L_\varepsilon} = X_H + \varepsilon \text{Re}(\pi G) \partial x_2. \]

We fix the segment \( I = \{(x_1, 0) : x_1 \in [-2, -1]\} \) transversal to the Hamiltonian flow. Denote by \( \beta_\varepsilon(t, h) \) the integral curves of \( W_\varepsilon \) which start and finish at \( I \). They satisfy
\[ \dot{\beta}_\varepsilon(t, h) = X_H + \varepsilon \text{Re}(\pi G) e_2, \]
\[ \beta_\varepsilon(0, h) \in I, \quad \beta_\varepsilon(T_\varepsilon, h) \in I, \]
\[ H(\beta_\varepsilon(0, h)) = h. \]

For \( \varepsilon = 0 \) the curve \( \beta_0 \) is the oval \( \{H = h\} \), and if \( \varepsilon \) is small but non-zero, \( \beta_\varepsilon \) is a small perturbation of \( \beta_0 \):
\[ \beta_\varepsilon(t, h) = \beta_0(t, h) + \varepsilon b(t, h; \varepsilon). \]

In the formulas above, \( T_\varepsilon = T_\varepsilon(h) \) is the time of the first return to the unit \( I \).

Lemma 2.16. There exist a constant \( C \) and a small positive constant \( \nu \) such that the following estimates hold:
\[ |b| \leq C|h|^4 e^{2\sqrt{3}+\nu} t, \]
\[ |\partial_b b| \leq C|h|^3 e^{2\sqrt{3}+\nu} t, \]
\[ |T_\varepsilon - T_0| \leq C \varepsilon |h|^3. \]

Proof. We use the scalar product \( x \cdot x' = 3x_1 x'_1 + x_2 x'_2 \); we put \( |x| = \sqrt{x \cdot x} \).

It follows from the equation (2.42) that the function \( b \) is a solution of the initial value problem
\[ \left\{ \begin{array}{l}
\dot{b} = \bar{d}X_H b + \text{Re}(\pi G(\beta_0 + \varepsilon b)) e_2, \\
b(0, h; \varepsilon) = 0,
\end{array} \right. \]
(2.46)
where \( \bar{d}X_H b = \frac{1}{3}(X_H(\beta_0 + \varepsilon b) - X_H(\beta_0)) \). We have \( \bar{d}X_H = dX_H(\beta_0 + \theta \varepsilon b) \) for some \( \theta \in (0, 1) \). Hence,
\[ \bar{d}X_H = \left( \begin{array}{cc}
0 & -2 \\
-6x_1 & 0
\end{array} \right), \quad x_1 \in [-2, 1]. \]

Moreover, using the estimates (2.36), (2.37), and (2.39) we get \( |G(\beta_0 + \varepsilon b)| \leq C_1 h^4 \).

For the solution \( b \) of the equation (2.46) we have
\[ \frac{d}{dt} |b|^2 = 2|b| \frac{d}{dt} |b| = 2b \cdot \bar{d}X_H b + 2b \cdot \text{Re}(\pi G(\beta_0 + \varepsilon b)) e_2 = \\
= -12(x_1 + 1) b_1 b_2 + 2b_2 \text{Re}(\pi G(\beta_0 + \varepsilon b)) \leq 4\sqrt{3}|b|^2 + 2C_2 h^4 |b|. \]

Therefore \( \frac{d}{dt} (|b|) \leq 2\sqrt{3}|b| + C_2 h^4, |b|(0) = 0 \) and the Gronwall inequality [H] gives the bound (2.43).

Since the difference of flows \( \varepsilon b \) after the Hamiltonian period \( T_0 \) is \( \leq C|h|^3 \) and the “velocity” \( |V_\varepsilon| \sim 1 \), the difference of periods \( |T_\varepsilon - T_0| \) satisfies (2.45).

The derivative \( \frac{\partial b}{\partial h} \) satisfies the linear variation equation corresponding to (2.46):
\[ \frac{d}{dt} \left( \frac{\partial b}{\partial h} \right) = (\bar{d}X_H + \varepsilon (\ldots)) \frac{\partial b}{\partial h} + h^3 (\ldots) \left( \frac{\partial b_0}{\partial h} \right), \quad \frac{\partial b}{\partial h}(0, h; \varepsilon) = 0. \]
where we denoted by (...) the bounded terms (see the estimates (2.43), (2.45), (2.37), (2.39)). Since the flow variation $\frac{\partial h}{\partial t}$ of the Hamiltonian satisfies $|\frac{\partial h}{\partial t}| \leq C e^{2\sqrt{\delta}}$, the inequality (2.44) holds. This finishes the proof of Lemma 2.16. □

We continue the proof of Proposition 2.15. We split the difference between the Poincaré map and the linearization $\varepsilon J(h)$ in two integrals $R_1(h, \varepsilon)$ and $R_2(h, \varepsilon)$:

$$R_1(h, \varepsilon) = \int_{\gamma_\varepsilon} \text{Re}(\pi(G - H^4 g)) \, dx_1,$$

$$R_2(h, \varepsilon) = \int_{\gamma_\varepsilon} \text{Re}(\pi H^4 g) \, dx_1 - \int_{h_0} \text{Re}(\pi H^4 g) \, dx_1.$$

We will show that the estimates (2.40) and (2.41) hold for both $R_1$ and $R_2$.

The inequality (2.40) is a direct consequence of (2.36). The difference of the values of $R_1$ at two close $h$’s is

$$R_1(h + \delta) - R_1(h) = \int_{\gamma_{h+h+\delta}} \partial_x \text{Re}(\pi(G - H^4 g)) \, dx_2 \wedge dx_1 + O(\varepsilon|h - 2|),$$

where the integral is taken along the strip $\gamma_{h}(h + \delta)$ between $\gamma_{h}(h)$ and $\gamma_{h}(h + \delta)$. The area of this strip is of the same order as the area of the domain $\{X : h < H < h + \delta\}$, i.e., $\sim \delta \cdot I_0 \sim C \delta |\log h|$ (see proof of Lemma 2.2). So the estimate (2.41) follows from (2.37).

To prove the estimate for $R_2$ we use (2.42) and (2.39):

$$|R_2| \leq C_1 \int_0^{T_0} \text{Re}(\pi H^4 g)(\beta_\varepsilon + \varepsilon b) - \text{Re}(\pi H^4 g)(\beta_\varepsilon) + \int_{T_0}^{T_s} C_2 |h|^4 \leq \leq C_3 \varepsilon |h|^7 \int_0^{T_0} e^{(2\sqrt{\delta} + \delta) t} \, dt + C_4 |h|^4 |T_s - T_0| \leq C_5 \varepsilon |h|^6 - 2\nu \leq C \varepsilon |h|^5.$$

Similarly, using the following formula for differential of integral

$$\frac{\partial}{\partial h} \int_{\beta_\varepsilon} \omega = \int_{\beta_\varepsilon} i\partial_\varepsilon \omega \, dx + \omega \left(\frac{\partial \beta_\varepsilon}{\partial h} \bigg|_{t=0}\right) - \omega \left(\frac{\partial \beta_\varepsilon}{\partial h} \bigg|_{t=T_s}\right)$$

and the bounds (2.42), (2.39), we get

$$|R_2| \leq C \varepsilon |h|^5, \quad |\partial_h R_2| \leq C \varepsilon |h|^4.$$ □

Now we can finish the proof of the main theorem. The restriction of system (1.3) to its invariant surface $L_\varepsilon = \{y = \varepsilon G(x, \varepsilon)\}$ has the form (2.31). The increment $\Delta H = \mathcal{P}(h) - h$ associated with the Poincaré map $\mathcal{P}(h)$ (on a section transversal to $\gamma_0$), equals (see Proposition 2.15)

$$\Delta H(h) = \varepsilon J(h) + O(|\varepsilon| |h|^5),$$

$$\left(\Delta H\right)'(h) = \varepsilon J'(h) + o(|\varepsilon| |h|^{4-1/4}).$$

Since we also have (see proof of Corollary 2.5)

$$J(h) = R h^{4+1/2} \cos(\sqrt{h} \alpha_0) + o(h^{4-1/4}),$$

any simple zero of $J(h)$ which is sufficiently close to 0 gives rise, by the implicit function theorem, to a simple zero of $\Delta H$. Thus the existence of a sequence $h_n \to 0^+$
of simple zeroes of $J(h)$ (see Corollary 2.5) guarantees the existence of an infinite sequence $\tilde{h}_n \to 0^+$, $n \geq N_0$, of simple zeroes of the increment $\Delta H$. Any simple zero of $\Delta H$ corresponds to a limit cycle of (1.3).

The proof of the main theorem is now complete. □

REFERENCES


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