The characterization problem for one class of second order operator pencil with complex periodic coefficients

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Abstract. The purpose of the present work is solving the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class.


Key words and phrases. Inverse problem, characterization problem, scattering data, transformation operator.

1. Introduction

The purpose of the present work is solving the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class. In our case $Q^2$ is the class of all $2\pi$ periodic complex valued functions on the real axis $\mathbb{R}$, belonging to $L^2[0, 2\pi]$ and $Q^2_+$ is its subclass consisting of functions

$$q(x) = \sum_{n=1}^{\infty} q_n \exp(inx).$$

(1.1)

The object under consideration is the operator $L$ generated by the differential expression

$$l \left( \frac{d}{dx}, \lambda \right) \equiv -\frac{d^2}{dx^2} + 2\lambda p(x) + q(x) - \lambda^2$$

(1.2)

in the space $L_2(-\infty, \infty)$ with potentials $p(x) = \sum_{n=1}^{\infty} p_n e^{inx}, q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$, for which $\sum_{n=1}^{\infty} n \cdot |p_n| < \infty$, $\sum_{n=1}^{\infty} |q_n| < \infty$ are fulfilled, $\lambda$ is a spectral parameter.

The inverse problem for the potentials (1.1) was formulated and solved in the papers [3], [4], where it was shown, that the equation $Ly = 0$, has the solution

$$e_{\pm}(x, \lambda) = e^{\pm i\lambda x} \left( 1 + \sum_{n=1}^{\infty} V_n^\pm e^{inx} + \sum_{n=1}^{\infty} \sum_{\alpha=0}^{\infty} \frac{V_{n\alpha}^\pm}{n \pm 2\lambda} e^{i\alpha x} \right),$$

(1.3)

and Wronskian of the system of solutions $e_{\pm}(x, \lambda)$ being equal to $2i\lambda$.

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The limit

\[ e_{n}^{\pm}(x) = \lim_{\lambda \to \mp n/2} (n \pm 2\lambda) e_{\pm}(x, \lambda) = \sum_{\alpha=0}^{\infty} V_{n\alpha}^{\pm} e^{i\alpha x} e^{-i\frac{n}{2}z}, \quad n \in \mathbb{N}, \]

is also the solution of the equation \( l(y) = 0 \), but already linearly dependent on \( e_{\pm}(x, \pm \frac{n}{2}) \). Therefore, there exist the numbers \( \hat{S}_{n}^{\pm}, n \in \mathbb{N} \), for which the conditions

\[ e_{n}^{\pm}(x) = \hat{S}_{n}^{\pm} (x, \mp \frac{n}{2}), \quad n \in \mathbb{N}, \quad (1.4) \]

are fulfilled.

From the last relation one may obtain that \( \hat{S}_{n}^{\pm} = V_{nn}^{\pm} \).

In [3] the spectral analysis of the operator pencil \( L \) was carried out and sufficient condition for reconstruction of \( p(x), q(x) \in Q_{2}^{+} \) using the values \( \hat{S}_{n}, n \in \mathbb{N} \), was found.

Note that some of the characterizations for the Sturm–Liouville operator with the real-valued potentials belonging to \( L_{1}^{1}(\mathbb{R}) \) (\( L_{1}^{+}(\mathbb{R}) \) is the class of measurable potentials satisfying the condition \( \int_{\mathbb{R}} dx (1 + |x|)^{\alpha} |p_{r}(x)| < \infty \)), have been given by Melin [9] and Marchenko [8]. (More details review can be found in the papers [1], [2], [7].) For the potentials \( p(x) = 0, q(x) \in Q_{2}^{+} \), which in the nontrivial cases are complex valued, the inverse problem was first formulated and solved by Gasymov [5]. Later the complete solution of the inverse problem for the cases \( p(x) = 0, q(x) \in Q_{2}^{+} \), was found by Pastur and Tkachenko [10].

Now let us formulate the basic result of the present work.

**Definition.** The sequence \( \{\hat{S}_{n}^{\pm}\}_{n=1}^{\infty} \) constructed by means of the formulae \( (1.4) \), is called a set of spectral data of the operator \( (1.2) \) with potentials \( p(x), q(x) \in Q_{2}^{+} \).

**Theorem 1.** For a given sequence of complex numbers \( \{\hat{S}_{n}^{\pm}\}_{n=1}^{\infty} \) to be a set of spectral data of the operator \( L \) generated by the differential expression \( (1.2) \) and potentials \( p(x), q(x) \in Q_{2}^{+} \), it is necessary and sufficient that the following conditions are fulfilled:

1) \( \{n^{2}\hat{S}_{n}^{\pm}\}_{n=1}^{\infty} \in l_{1}; \quad (1.5) \)

2) Infinite determinant

\[ D(z) \equiv \det \left\| \delta_{nm} - \sum_{k=1}^{\infty} \frac{4\hat{S}_{m}^{\pm} \hat{S}_{n}^{\pm}}{(m+k)(n+k)} e^{i\frac{m+k}{2}z} e^{i\frac{n+k}{2}z} \right\|_{n,m=1}^{\infty} \quad (1.6) \]

exists, is continuous, not equal to zero in the closed half-plane \( \overline{\mathbb{C}}_{+} = \{z: \text{Im} z \geq 0\} \) and analytical inside of the open half-plane \( \mathbb{C}_{+} = \{z: \text{Im} z > 0\} \).

2. **On an Inverse Problem of the Scattering Theory on the Semiaxis**

On the base of the proof of Theorem 1, we will study the equation \( Ly = 0 \).

Denoting

\[ x = it, \quad \lambda = -i\mu, \quad y(x) = Y(t) \quad (2.1) \]
we obtain the equation
\[-Y''(t) + 2\mu \bar{p}(it)Y(t) + \bar{q}(it)Y(t) = \mu^2 Y(t)\] (2.2)
in which
\[
\bar{p}(t) = ip(it) = i \sum_{n=1}^{\infty} p_n e^{-nt}, \quad \bar{q}(t) = -q(it) = - \sum_{n=1}^{\infty} q_n e^{-nt}.
\] (2.3)

As a result we obtain the equation (2.2), whose potentials exponentially decrease as \(t \to \infty\).

The specification of the considered inverse problem is defined by the fact that the potentials belong to the class \(Q^2\). In this section we suppose \(t \in \mathbb{R}^+\).

The procedure of analytic continuation that allows to get corresponding results for the equation (1.2) from the result for the (2.2) will be investigated in the next section.

The equation (2.2) with potentials (2.3) has the solution
\[
f_{\pm}(t, \mu) = e^{\pm i\mu t} \left(1 + \sum_{n=1}^{\infty} V_{n}^{\pm} e^{-nt} + \sum_{n=1}^{\infty} \sum_{\alpha=n+1}^{\infty} \frac{V_{n\alpha}^{\pm}}{in \pm 2\mu} e^{-\alpha t}\right)
\] (2.4)
and the numbers \(V_n^{\pm}, V_{n\alpha}^{\pm}\) are defined by the following recurrent formulae:
\[
\alpha^2 V_{n}^{\pm} + \alpha \sum_{s=1}^{\alpha-1} V_{ns}^{\pm} \sum_{s=1}^{\alpha-1} \left(q_{\alpha-s} V_{s}^{\pm} \pm p_{\alpha-s} \sum_{n=1}^{s} V_{ns}^{\pm}\right) + q_\alpha = 0,
\] (2.5)
\[
\alpha (\alpha - n) V_{n}^{\pm} + \sum_{s=n}^{\alpha-1} (q_{\alpha-s} \mp n \cdot p_{\alpha-s}) V_{ns}^{\pm} = 0,
\] (2.6)
\[
\alpha V_{\alpha}^{\pm} \pm \sum_{s=1}^{\alpha-1} V_{s}^{\pm} p_{\alpha-s} \pm p_\alpha = 0
\] (2.7)
and the sequence (2.4) admits double termwise differentiation. Then with the help of the condition (2.3) we obtain
\[
f_{\pm}(t, \mu) = \Psi_{\pm}(t) e^{\pm i\mu t} + \int_{t}^{\infty} K_{\pm}(t, u) e^{\pm i\mu u} du,
\] (2.8)
where \(K_{\pm}(t, u), \Psi_{\pm}(t)\) have the form
\[
K_{\pm}(t, u) = \frac{1}{2i} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{\pm} e^{-\alpha t} \cdot e^{-(u-t)n/2}, \quad \Psi_{\pm}(t) = 1 + \sum_{n=1}^{\infty} V_{n}^{\pm} e^{-nt},
\] (2.9)

So, it is proved the following

**Lemma 1.** The function \(\Psi_{\pm}(t)\) and the kernel of the transformation operator of the equation (2.2) \(K(t, u), \ u \geq t, \ attached\ to +\infty, \ with\ the\ potentials\ (2.3)\ permits\ the\ representation\ (2.9),\ in\ which\ the\ series
\[
\sum_{n=1}^{\infty} n^2 |V_{n}^{\pm}|; \quad \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha (\alpha - n) |V_{n\alpha}^{\pm}|; \quad \sum_{n=1}^{\infty} n \cdot |V_{n\alpha}^{\pm}|
\]
are convergent.
Remark. In our case the kernel of the operator of transformation $K^\pm(t, u), u \geq t,$ at $+\infty,$ and the function $\Psi^\pm(t)$ are constructed effectively.

Then it is possible to get the equality

$$f_\pm^\pm(t) = S_n^\pm \left( t, \mp i \frac{n}{2} \right), \quad (2.10)$$

where

$$f_\pm^\pm(t) = \lim_{\mu \to \mp \infty} (in \pm 2\mu) f_\pm(t, \mu).$$

Rewriting the equality (2.10) in the form

$$\sum_{\alpha=n}^{\infty} V_{n\alpha}^- e^{-\alpha t} \cdot e^{nt/2} = S_n^\pm e^{-nt/2} \left( 1 + \sum_{m=1}^\infty V^-_{m} e^{-mt} + \sum_{m=1}^\infty \sum_{\alpha=m}^\infty \frac{V^-_{m\alpha} e^{-\alpha t}}{i(m + n)} \right) \quad (2.11)$$

and denoting by

$$z^\pm(t + s) = \sum_{m=1}^\infty S_m^\pm e^{-(t+s)m/2} \quad (2.12)$$

we obtain the Marchenko type equation

$$K^\pm(t, s) = \Psi^\pm(t) z^\pm(t + s) + \int_t^\infty K^\mp(t, u) z^\pm(u + s) \, du. \quad (2.13)$$

So, it is proved the following

**Lemma 2.** If the coefficients $\bar{p}(t)$ and $\bar{q}(t)$ of the equation (2.2) have the form (2.3), then at every $t \geq 0,$ the kernel of the transformation operator (2.9) satisfies to the equation of the Marchenko type (2.13) in which the transition function $z^\pm(t)$ has the form (2.12) and the numbers $S_m^\pm$ are defined by the equality (2.10), from which it is obtained, that $S_m^\pm = V_m^\pm.$

Note that from relation (2.7) one may easily obtain the formulae $\Psi^+(t) \cdot \Psi^-(t) = 1$ and $\lim_{x \to -\infty} \Psi^+(x) = 1$ useful later on. The potentials are reconstructed by the kernel of the transformation operator and the function $\Psi^\pm(t)$ with the help of the formulae

$$\Psi^\pm(t) = J \pm i \int_t^\infty \bar{p}(u) \Psi^\pm(u) \, du, \quad (2.14)$$

$$K^\pm(t, t) = \pm \frac{1}{2} \int_t^\infty \bar{q}(u) \Psi^\pm(u) \, du \mp i \bar{p}(t) \Psi^\pm(t) \pm i \int_t^\infty \bar{p}(u) K^\mp(u, u) \, du \quad (2.15)$$

Hence the basic equation (2.13) and the form of the transition function (2.12) make natural the formulation of the inverse problem about reconstruction of the potentials of the equation (2.2) by numbers $S_n^\pm.$

In this formulation, which employs the transformation operator, an important moment is a proof of unique solvability of the basic equation (2.13).

**Lemma 3.** The homogenous equation

$$g^\pm(s) - \int_0^\infty z^\pm(u + s) g^\mp(u) \, du = 0, \quad (2.16)$$
corresponding to the potential $\bar{p}, \bar{q} \in Q_{2}^{2}$ has only trivial solution in the space $L_{2}(\mathbb{R}^{+})$.

The proof of Lemma 3 is similar to [7, p. 198].

**Lemma 4.** At every fixed value $a$, $\text{Im} \, a \geq 0$, the homogenous equation
\[
g^{\pm}(s) - \int_{t}^{\infty} z^{\pm}(u + s - 2ai)g^{\mp}(u) \, du = 0,
\]
has only trivial solution in the space $L_{2}(\mathbb{R}^{+})$.

**Proof.** We substitute $x + a$ for $x$, where $\text{Im} \, a \geq 0$ in equation (1.2), and we obtain the same equation with the coefficients $p_{\alpha}(x) = p_{\alpha}(x + a)$, $q_{\alpha}(x) = q_{\alpha}(x + a)$ belonging to $Q_{2}^{2}$. Let us remark, that the functions $e^{\pm}(x, \lambda)$ are solutions of the equation
\[
- y'' + 2\lambda p_{\alpha}(x)y + q_{\alpha}(x)y = \lambda^2 y
\]
that as $x \to \infty$ have the form
\[
e^{\pm}(x + a, \lambda) = e^{\pm i\lambda} e^{\pm i\lambda x} + o(1).
\]

Therefore, the functions $e^{\alpha}(x, \lambda) = e^{\mp i\alpha} e^{\pm}(x + a, \lambda)$ are also solutions of type (1.3).

Then let us denote by $\{\hat{S}_{\alpha}^{\pm}(a)\}_{n=1}^{\infty}$ the spectral data of the operator $L$ with the potentials $p_{\alpha}(x)$, $q_{\alpha}(x)$
\[
L \equiv - \frac{d^2}{dx^2} + 2\lambda p_{\alpha}(x) + q_{\alpha}(x) - \lambda^2.
\]

According to (1.4), we have
\[
\hat{S}_{\alpha}^{\pm}(a) e^{\alpha}(x, \pm n/2) = \lim_{\lambda \to \pm n/2} (n \mp 2\lambda) e^{\alpha}(x, \lambda) =
\]
\[
= \lim_{\lambda \to \pm n/2} (n \mp 2\lambda) e^{\pm i\alpha} e^{\pm}(x + a, \lambda) = e^{\pm i\alpha/2} \hat{S}_{\alpha}^{\pm} e^{\pm}(x + a, \pm n/2) =
\]
\[
= e^{\pm i\alpha/2} \hat{S}_{\alpha}^{\pm} e^{\pm i\alpha/2} e^{\alpha}(x, \pm n/2) = \hat{S}_{\alpha}^{\pm} e^{\alpha}(x, \pm n/2).
\]

Hence
\[
\hat{S}_{\alpha}^{\pm}(a) = \hat{S}_{\alpha}^{\pm} e^{i\alpha}.
\]

Now arguing as above, we obtain the basic equation of the form (2.13) with the transition function
\[
Z^{\pm}_{a}(t) = \sum_{n=1}^{\infty} S^{\pm}_{a}(a) e^{-nt/2} = \sum_{n=1}^{\infty} S^{\pm}_{a} e^{i\alpha} e^{-nt/2} = Z^{\pm}(t - 2ia). \quad \Box
\]

The next theorem follows from Lemmas 3 and 4.

**Theorem 2.** The potentials $\bar{p}(t)$ and $\bar{q}(t)$ of the equation (2.2), satisfying the condition (2.3) are uniquely defined by the numbers $\hat{S}_{a}^{\pm}$. 
3. Proof of Theorem 1

Necessity: The necessity of the condition 1) is proved in [3].

To prove the necessity of the condition 2) of the Theorem 1 let us demonstrate first of all, that from the trivial solvability of the basic equation (2.13) at \( t = 0 \) in the class of functions satisfying to the inequality \( \|g(u)\| \leq Ce^{-\frac{r}{2}}, \, u \geq 0 \), it follows trivial solvability in \( l_2(\mathbb{R}^+) \) of the infinite system of equations

\[
g_n^\pm = -\sum_{m=1}^{\infty} \frac{2S_m^\pm}{m + n} g_m^\pm = 0
\]  

or

\[
g_n^\pm = -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{4S_m^\pm S_k^\pm}{(m + k)(n + k)} g_m^\pm = 0,
\]

where \( g_n^\pm \in l_2(\mathbb{R}) \), \( S_n^\pm \in l_1 \).

Really, if \( \{g_n\}_{n=1}^{\infty} \in l_2 \) is a solution of this system, then the function

\[
g^\pm(u) = -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m + k} e^{-ku/2} g_m^\pm
\]

is defined for all \( u \geq 0 \), satisfies the inequality

\[
|g^\pm(u)| \leq c \cdot e^{-u/2}, \quad u \geq 0,
\]

and it is a solution of equation (2.16)

\[
g^\pm(s) = \int_0^{\infty} \int_0^s z^\pm(u + s)z^\mp(u + s_1)g^\pm(s_1) ds_1 du =
\]

\[
-\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m + k} e^{-ks/2} g_m^\pm + \int_0^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} S_m^\pm S_k^\pm S_r^\pm e^{-(u + s)k/2} \cdot e^{-(u + s_1)m/2} \times
\]

\[
\left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{2S_n^\pm S_r^\pm}{n + r} e^{-rs_1/2} g_n^\pm \right) ds_1 du = -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \frac{2S_m^\pm S_k^\pm S_r^\pm}{m + k} e^{-ks/2} g_m^\pm +
\]

\[
+ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m + k} e^{-ks/2} \left( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{4S_n^\pm S_r^\pm}{(n + r)(m + r)} g_n^\pm \right) =
\]

\[
= -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m + k} e^{-ks/2} g_m^\pm - \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{2S_n^\pm S_r^\pm}{(n + r)(m + r)} g_n^\pm = 0.
\]

Since \( g^\pm(u) = 0 \) then, \( S_m^\pm S_k^\pm g_m^\pm = 0 \) for all \( m \geq 1, \, k \geq 1, \) and \( g_n^\pm = 0, \, m \geq 1 \) according to (3.2). Let us introduce in the space \( l_2 \) operator \( F_2^\pm(t) \), given by the matrix

\[
F_{mn}^\pm(t) = \sum_{k=1}^{\infty} \frac{4S_n^\pm S_k^\pm}{(n + k)(m + k)} e^{-(m+k)t/2} e^{-(n+k)t/2}, \quad n, \, m \in \mathbb{N}.
\]  

Then, we obtain from \( n^2 S_n^\pm \in l_1 \), that \( \sum_{n,k=1}^{\infty} |(F_2^\pm \varphi_j, \varphi_k)_2| < \infty \), i.e. \( F(t) \) is a kernel operator [11]. Therefore, there exists the determinant \( \Delta^\pm(t) = \)
det\((E - F^\pm_2(t))\) of the operator \(E - F^\pm_2(t)\) connected, as easy to see, with the determinant \(D^\pm(z)\) from the condition 2) of Theorem 1, with relation \(\Delta^\pm(-iz) = \det(E - F^\pm_2(-iz)) \equiv D^\pm(z)\).

The determinant of system (3.1) is \(D^\pm(0)\), and the determinant of the similar system corresponding to the potentials \(p_z(x) = p(x + z), q_z(x) = q(x + z), \text{Im } z > 0\) is

\[
D^\pm(z) = \det \left| \delta_{mn} - \sum_{k=1}^{\infty} \frac{4S^\pm_n(z)S^\pm_k(z)}{(m+k)(n+k)} \right|_{m,n=1}^\infty = \left| \delta_{mn} - \sum_{k=1}^{\infty} \frac{4S^\pm_nS^\pm_k}{(m+k)(n+k)} e^{i \frac{m+n+k}{2} z} e^{i \frac{m+n+k}{2} z} \right|_{m,n=1}^\infty.
\]

Therefore in order to prove the necessity of the condition 2) of Theorem 1 one should check that \(\Delta^\pm(0) = D^\pm(0) \neq 0\). System (3.1) can be written in \(l_2\) as the equation

\[
g^\pm - F^\pm_2(0)g^\pm = 0.
\]

As \(F^\pm_2(0)\) is a kernel operator, we can apply the Fredholm theory to this equation, according to which its trivial solvability is equivalent to the condition that \(\det(E + F^\pm_2(0))\) is not equal to zero [11]. Necessity of the condition 2) is proved.

**Sufficiency:** Let us study (2.13) in detail. It is known [7] that \(K^\pm(t, s)\) can be expressed by \(\Psi^\pm(t)\) and solutions \(P^\pm(t, s), Q^\pm(t, s)\) of the Marchenko type equations (2.13) by the replacement of \(\Psi^\pm(t)\) by 1 and \(\pm i\).

Then

\[
K^\pm(t, s) = \Psi^\mp(t)\alpha^\pm(t, s) + \Psi^\pm(t)\beta^\mp(t, s)
\]

where

\[
\alpha^\pm(t, s) = \frac{1}{2} [P^\pm(t, s) \mp iQ^\pm(t, s)],
\]

\[
\beta^\mp(t, s) = \frac{1}{2} [P^\pm(t, s) \pm iQ^\pm(t, s)],
\]

\[
[\Psi^\pm(t)]^2 = \frac{1 - \int_t^\infty [\alpha^\pm(t, u) - \beta^\pm(t, u)] du}{1 - \int_t^\infty [\alpha^\mp(t, u) - \beta^\mp(t, u)] du},
\]

from which we uniquely define \(\Psi^\pm(t)\). We also take into account that the sign of \(\Psi^\pm(t)\) is fixed from condition \(\lim_{t \to \infty} \Psi^\pm(t) = 1\).

Thus for further studies we should consider the following equations

\[
P^\pm(t, s) = z^\pm(t + s) + \int_t^\infty P^\mp(t, u)z^\pm(u + s) du,
\]

\[
Q^\pm(t, s) = \pm iz^\pm(t + s) + \int_t^\infty Q^\mp(t, u)z^\pm(u + s) du.
\]
Rewriting (3.9) in the form

\[ P^\pm(t, s) = z^\pm(t + s) + \int_t^\infty z^\mp(u + t)z^\pm(u + s)\,du + \int_t^\infty \int_t^\infty P^\pm(t, \tau)z^\mp(u + \tau)z^\pm(u + s)\,du\,d\tau. \]  \hfill (3.11)

in the space \( l_2 \) we introduce the operator \( F^\pm_1(t) \) given by the matrix

\[ F^\pm_{mn} = \frac{2S^\pm_n}{m + n} e^{-(m+n)t}/2; \quad \Re t > 0. \]  \hfill (3.12)

Let’s multiply the equation (3.11) by \( e^{-nu/2} \) and integrate it over \( s \in [t, \infty) \). Then we obtain

\[ p^\pm(t) = F^\pm_1(t)e(t) + F^\pm_2(t)e(t) + p^\pm(t)F^\pm_2(t), \]  \hfill (3.13)

in which the operators \( F^\pm_1(t), F^\pm_2(t) \) are defined by the matrix (3.4), (3.12),

\[ e(t) = \{e^{-nt}E\}^\infty_{n=1}; \quad p^\pm(t) = \left\{ \int_t^\infty P^\pm(t, u)e^{-nu/2}\,du \right\}^\infty_{n=1}. \]

As \( F^\pm_2(t) \) is the trace class for \( t \geq 0 \) and the condition \( \det(E - F^\pm_2(t)) \neq 0 \) holds, there exists the inverse operator \( R^\pm(t) = (1 - F^\pm_2(t))^{-1} \) bounded in \( l_2 \).

Since \( F^\pm_1(t)e(t), F^\pm_2(t)e(t) \in l_2 \), then from (3.13) we get

\[ p^\pm(t) = R^\pm(t)[F^\mp_1(t) + F^\mp_2(t)]e(t). \]  \hfill (3.14)

Now, denoting \( \langle f, g \rangle = \sum_{n=1}^\infty f_n g_n \), we find from (3.11) that

\[ P^\pm(t, s) = \langle e(t), B^\pm(s) \rangle + \langle e(t), A^\pm(s, t) \rangle + \langle p^\pm(t), A^\pm(s, t) \rangle = \\
= \langle e(t), B^\pm(s) \rangle + \langle e(t), A^\pm(s, t) \rangle + \langle R^\pm(t)(F^\pm_1(t) + F^\pm_2(t))e(t), A^\pm(s, t) \rangle = \\
= \langle e(t), B^\pm(s) \rangle + \langle (R^\pm(t)(F^\pm_1(t) + F^\pm_2(t)) + 1)e(t), A^\pm(s, t) \rangle, \]  \hfill (3.15)

where

\[ B^\pm(s) = \{B^\pm_m(s) = S_m^\pm e^{-ms/2}, \ s > 0\}^\infty_{m=1} \]

and

\[ A^\pm(s, t) = \left\{ A^\pm_m(s, t) = \frac{2S^\pm_m S^\pm_k}{m + k} e^{-ks/2} \cdot e^{-(m+k)t/2}, \ s, t > 0 \right\}. \]

Now assume that the conditions of the theorem are fulfilled. Let us define the function \( P^\pm(t, s) \) by the equality (3.15) at \( 0 \leq t \leq u \) according to the given above
considerations. Then at $u \geq t$ we have

$$P^\pm(t, s) - \int_t^\infty \int_t^\infty P^\pm(t, \tau)z^\mp(u + \tau)z^\pm(u + s)\,du\,d\tau =$$

$$= \langle e(t), B^\pm(s) \rangle + \langle (R^\pm(t)(F_1^\pm(t) + F_2^\pm(t)) + 1)e(t), A^\pm(s, t) \rangle -$$

$$- \int_t^\infty \langle (e(t), B^\pm(\tau)) + \langle (R^\pm(t)(F_1^\pm(t) + F_2^\pm(t)) + 1)e(t), A^\pm(\tau, t) \rangle \times$$

$$\times ((e(\tau), A^\pm(s, t)) d\tau =$$

$$= \langle e(t), B^\pm(s) \rangle + \langle e(t), A^\pm(s, t) \rangle + \langle R^\pm(t)(F_1^\pm(t) + F_2^\pm(t))e(t), A^\pm(s, t) \rangle -$$

$$- \langle F_1^\pm(t)e(t), A^\pm(s, t) \rangle - \langle F_2^\pm(t)e(t), A^\pm(s, t) \rangle -$$

$$- \langle (R^\pm(t)F_2^\pm(t)(F_1^\pm(t) + F_2^\pm(t))e(t), A^\pm(s, t) \rangle =$$

$$= \langle e(t), B^\pm(s) \rangle + \langle e(t), A^\pm(s, t) \rangle = z^\pm(t + s) + \int_\infty^\infty z^\mp(u + t)z^\pm(u + s)\,du$$

For $Q^\pm(t, s)$ we similarly obtain, that

$$Q^\pm(t, s) = \pm i\langle e(t), B^\pm(s) \rangle \mp i\langle e(t), A^\pm(s, t) \rangle + \langle Q^\pm(t), A^\pm(s, t) \rangle$$

where

$$Q^\pm(t) = \pm iR^\pm(t)[F_1^\pm(t) - F_2^\pm(t)]e(t).$$

So, we have

**Lemma 5.** For any $t \geq 0$ the kernel $K^\pm(t, s)$ of the transformation operator and the function $\Psi^\pm(t)$ satisfies the basic equation

$$K^\pm(t, s) = \Psi^\pm(t)z^\pm(t + s) + \int_t^\infty K^\mp(t, u)z^\pm(u + s)\,du.$$  

A unique solvability of the basic equation follows from Lemma 3. By the direct substitution it is easy to calculate that the solution of the basic equation is

$$K^\pm(t, u) = \frac{1}{2i} \sum_{n=1}^\infty \sum_{\alpha=\pm} V^{\pm}_{n, \alpha} e^{-nt}, e^{-(u-t)n/2}, \quad \Psi^\pm(t) = 1 + \sum_{n=1}^\infty V_n^\pm e^{-nt},$$

where the numbers $V^{\pm}_{n, \alpha}, V^\pm_n$ are defined by the recurrent relations

$$V^{\pm}_{m, m} = S^\pm_m, \quad V^{\pm}_{m, m+\alpha} = S^\pm_{m+\alpha} \left( V^\mp + \sum_{n=1}^\alpha \frac{V^n_{m, n}}{n + m} \right).$$

Passing to the proof of the basic statement of the theorem, that the potentials $\tilde{p}(t)$ and $\tilde{q}(t)$ have form (2.3) let us first establish the estimations for the matrix elements $R_{mn}(t)$ of the operator $R(t)$

$$\|R^\pm_{mn}(t)\| \leq \delta_{mn} + C_0|S^n_m|, \quad \left(3.16\right)$$

$$\left| \frac{d^j}{dt^j} R^\pm_{mn}(t) \right| \leq C_j|S^n_m|, \quad j = 1, 2, \quad \left(3.17\right)$$

where $S^n_m = \max(S^+_m, S^-_m)$ and $C_n, n \in \mathbb{N}$, are constants.
Indeed, from the identity $R_{mn}^\pm(t) = E + R_{mn}^\pm(t)F_{2}^\pm(t)$ it follows that

$$
\| \sum_{p=1}^{\infty} \| R_{mp}^\pm(t) \|^{2} \right)^{1/2} \leq \\
\leq \delta_{mn} + 2 \cdot \left( (R_{tn}^\pm(t)R_{tn}^\pm(t))_{pp} \sum_{p=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{4S_{k}^\pm S_{k}^\pm}{(p+k)(n+k)} \right) \right)^{1/2} \leq \\
\leq \delta_{mn} + 2 \cdot \left( (R_{tn}^\pm(t)R_{tn}^\pm(t))_{pp} \sum_{p=1}^{\infty} \frac{1}{(p+1)^{2}} \left( \sum_{k=1}^{\infty} |S_{k}^\pm| \right) \right)^{1/2} \leq \\
\leq \delta_{mn} + \| R(t) \|_{[2-\infty]} |S_{n}^*|.
$$

On the other hand, as it has been noted, the operator-function $R_{tn}^\pm(t)$ exists and is bounded in $l_{2}$ (because $F_{2}^\pm(t)$ is a kernel operator at $t \geq 0$ and $\Delta_{\pm}(t) = \det(E + F_{2}^\pm(t)) \neq 0$) that proves the first inequality (3.16).

In order to prove the second estimation (3.17) we use the identity

$$
\frac{d}{dt} R_{mn}^\pm(t) = R_{mn}^\pm(t)F_{2}^\pm(t)R_{mn}^\pm(t)
$$

and by means of the first estimation (3.16) obtain that

$$
\left\| \frac{d}{dt} R_{mn}^\pm(t) \right\| \leq \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} \| R_{mq}^\pm(t) \| \| F_{2}^\pm(q,t) \| \| R_{pn}^\pm(t) \|
\leq \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} (\delta_{mn} + C_{1}|S_{q}^*|)(|S_{n}^*|)(\delta_{pn} + C_{2}|S_{n}^*|)
\leq \left( 1 + C_{3} \sum_{p=1}^{\infty} |S_{p}^*| \right)^{2} |S_{n}^*| \leq C_{4}|S_{n}^*|.
$$

The estimation

$$
\left\| \frac{d^2}{dt^2} R_{mn}^\pm(t) \right\| \leq C_{5}|S_{n}^*|
$$

can be proved analogously.

Using these estimations, from (2.13) and (2.14) one can establish the correctness of the estimations

$$
\left\| \frac{d^2}{dt^2} P_{\pm}(t, s) \right\| \leq C_{6}, \quad \left\| \frac{d^2}{dt^2} Q_{\pm}(t, s) \right\| \leq C_{7}.
$$

Thus, the functions $K_{\pm}(t, s)$ and $\Psi_{\pm}(t)$ have the second derivatives over $t$. From this we conclude that the series $\sum_{n=1}^{\infty} \alpha^{2}|V_{\alpha}^{\pm}|$ and $\sum_{n=1}^{\infty} n^{-1} \sum_{\alpha=1}^{\infty} (\alpha + n)|V_{\alpha}^{\pm}| |V_{n}^{\pm}|$ are convergent. The forms of the coefficients $\tilde{p}(t)$ and $\tilde{q}(t)$ are directly determined from the form of the functions $K_{\pm}(t, s), \Psi_{\pm}(t)$ employing the formulas (2.14), (2.15). We obtain that for the numbers $p_{n}$ and $q_{n}$ the recurrent relations (2.5)–(2.7) are correct and hence the series $\sum_{n=1}^{\infty} n \cdot |p_{n}| < \infty, \sum_{n=1}^{\infty} |q_{n}| < \infty$ converges. Let, finally, $\{S_{n}^{\pm} \}_{n=1}^{\infty}$ be a set of spectral data of the operator $L$ with the constructed potentials $p(x), q(x) \in Q_{+}^{2}$. For completing the proof it remains to show, that
\[ \{ S_n^\pm \}_{n=1}^\infty \text{ coincides with the initial set } \{ \hat{S}_n^\pm \}_{n=1}^\infty. \] 
This follows from the equality 
\[ S_n^\pm = V_n^\pm = \hat{S}_n^\pm. \]
The theorem is proved. 

References


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