CONSTANT FAMILIES OF $t$-STRUCTURES ON DERIVED CATEGORIES OF COHERENT SHEAVES

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ABSTRACT. We generalize the construction given in [1] of a “constant” $t$-structure on the bounded derived category of coherent sheaves $D(X \times S)$ starting with a $t$-structure on $D(X)$. Namely, we remove smoothness and quasiprojectivity assumptions on $X$ and $S$ and work with $t$-structures that are not necessarily Noetherian but are close to Noetherian in the appropriate sense. The main new tool is the construction of induced $t$-structures that uses unbounded derived categories of quasicoherent sheaves and relies on the results of [2]. As an application of the “constant” $t$-structures techniques we prove that every bounded nondegenerate $t$-structure on $D(X)$ with Noetherian heart is invariant under the action of a connected group of autoequivalences of $D(X)$. Also, we show that if $X$ is smooth then the only local $t$-structures on $D(U)$ for all open $U \subset X$, are the perverse $t$-structures considered in [4].

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INTRODUCTION

Originally $t$-structures appeared in the context of derived categories of constructible sheaves on a stratified space in the definition of perverse sheaves given in [3]. More recent studies led to interesting examples of $t$-structures on bounded derived categories of coherent sheaves on algebraic varieties (for example, in connection with the theory of stability conditions introduced by Bridgeland in [8]; see also [5] for examples relevant for representation theory). The present work is a continuation of [1] where we gave a construction of a $t$-structure on $D(X \times S)$, the bounded derived category of coherent sheaves on $X \times S$, starting with a $t$-structure on $D(X)$. This $t$-structure on $X \times S$ should be thought of as a constant family of $t$-structures over $S$ (we will often refer to it as a “constant $t$-structure”). Hopefully, it should serve as the first step towards constructing nice moduli spaces for stable...
objects with respect to a stability condition on $D(X)$ (see [1] for a discussion of this problem).

The main goal of this paper is to remove the smoothness assumption that was imposed on $X$ and $S$ in [1]. Moreover, we actually give an alternative construction even in the smooth case and remove the assumption of boundedness with respect to the standard $t$-structure in the results of [1], Sec. 2.7. We still need the most nontrivial ingredient from [1] that gives the required $t$-structures in the case $S = \mathbb{P}^r$. However, the remaining part of the construction is replaced by a new method based on the general procedure of “inducing” a $t$-structure with respect to a “nice” functor (see Theorem 2.1.2). The geometric example of such a functor relevant for the construction of constant $t$-structures is the push-forward with respect to a finite morphism of finite Tor dimension. The key idea is that it is much easier to construct $t$-structures in the unbounded derived categories of quasicoherent sheaves $D_{qc}(X)$ because one can use arbitrary small coproducts. This idea was employed effectively in Theorem A.1 of [2] that shows that any pre-aisle stable under all small coproducts and generated by a set of objects, extends to a $t$-structure (see Section 1.2 for terminology). Of course, a random $t$-structure on $D_{qc}(X)$ will not restrict to a $t$-structure on $D(X)$. However, if the two such categories are related by a “nice” functor $D_{qc}(X) \to D_{qc}(Y)$ then knowing that the $t$-structure on $D_{qc}(Y)$ restricts to $D(Y)$ allows to deduce the same about the $t$-structure on $D_{qc}(X)$. Applying this approach we construct the constant $t$-structure on $D(X \times S)$ for arbitrary $X$ and $S$ of finite type over a field (see Theorem 3.3.6).

We also come up with several other improvements to [1]. First of all, in loc. cit. we considered only Noetherian $t$-structures (i.e., $t$-structures with Noetherian heart). In this paper we introduce close to Noetherian $t$-structures that are obtained from Noetherian $t$-structures by tilting, and show that the construction of constant $t$-structures goes through for them as well. A technical observation that facilitates such a generalization is Theorem 1.2.1 stating that every pre-aisle, close to a Noetherian $t$-structure, automatically extends to a $t$-structure. It is easy to see that in a reasonable situation all $t$-structures associated with stability conditions are close to Noetherian (see Example in Section 1.2). However, it is important to observe that in the non-Noetherian case the constant $t$-structures will usually lack some important features established in [1] (such as the open heart property, see Proposition 2.3.7).

Next, we develop a little bit further the techniques of sheaves of $t$-structures by considering an arbitrary morphism $f: X \to S$ and defining $t$-structures on $D(X)$, local over $S$. In the case of a flat morphism we are able to define pull-backs of such $t$-structures under finite base changes of finite Tor dimension (see Theorem 2.3.5). As a corollary we show that if $X$ is smooth then local (over $X$) $t$-structures on $D(X)$ are exactly the perverse $t$-structures constructed in [4] (see Corollary 2.3.6). Another application of this technique gives a description of the heart of the constant $t$-structure on $D(X \times S)$ for affine scheme $S = \text{Spec}(A)$ in terms of $A$-modules in the heart of the corresponding $t$-structure on $D_{qc}(X)$ (see Proposition 3.3.7). We also show that if $L$ is an ample line bundle on $S$ then for a $t$-structure on $X$ such
that $f^*L \otimes D^{\leq 0}(X) \subset D^{\leq 0}(X)$ there exists a new $t$-structure $(D^{\leq 0}_l(X), D^{\geq 0}_l(X))$, local over $S$, with $D^{\geq 0}_l(X) = \bigcap_{n \geq 0} f^*L^{-n} \otimes D^{\geq 0}(X)$ (see Theorem 3.4.1).

Finally, we present one application of constant $t$-structures that seems to underscore once again the role of the Noetherian property. Namely, we prove that every bounded nondegenerate Noetherian $t$-structure on $D(X)$ is invariant under the action of a connected group of autoequivalences of $D(X)$ (under a certain natural assumption on this action).

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Notation. All our schemes are always assumed to be Noetherian of finite Krull dimension. Starting from Section 3.2 they are assumed to be of finite type over a fixed field $k$. We denote by $D(X)$ the bounded derived category of coherent sheaves on a scheme $X$, and by $D_{qc}(X)$ the unbounded derived category of quasicoherent sheaves on $X$. We denote the derived functor of tensoring simply by $\otimes$. For a morphism of schemes $f: X \to Y$ we denote by $f_*$ and $f^*$ the derived functors of the push-forward and the pull-back, respectively. In the case of a locally closed embedding $i: Y \hookrightarrow X$ we also use notation $F|_Y = i^*F$. If $A \subset C$ is a subcategory in an additive category $C$ and $X \in C$ is an object then we write $\text{Hom}(X, A) = 0$ (resp., $\text{Hom}(A, X) = 0$) if $\text{Hom}(X, Y) = 0$ (resp., $\text{Hom}(Y, X) = 0$) for all $Y \in A$. We define left and right orthogonals $A^{-\perp} = \{X \in C: \text{Hom}(X, A) = 0\}$ and $A^{+} = \{X \in C: \text{Hom}(A, X) = 0\}$, respectively.

1. $t$-Structures that are Close to Noetherian Ones

1.1. Preliminary remarks on $t$-structures and tiltings. Our main reference for the theory of $t$-structures is Section 1 of [3]. Below we recall some basic definitions.

Let $\mathcal{D}$ be a triangulated category. A $t$-structure on $\mathcal{D}$ is a pair of full subcategories $(D^{\leq 0}, D^{\geq 0})$ satisfying the conditions (i) and (ii) below. We denote $\mathcal{D}^{\leq n} = D^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = D^{\geq 0}[-n]$ for every $n \in \mathbb{Z}$. Then the conditions are:

(i) $\text{Hom}(X, Y) = 0$ for every $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$;

(ii) every object $X \in \mathcal{D}$ fits into an exact triangle

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \to \ldots$$

with $\tau^{\leq 0}X \in D^{\leq 0}$, $\tau^{\geq 1}X \in D^{\geq 1}X$.

It is easy to see that $D^{\geq 1}$ is exactly the right orthogonal of $D^{\leq 0}$ (resp., $D^{\leq 0}$ is the left orthogonal of $D^{\geq 1}$), and the terms of the above triangle are determined functorially (due to condition (i)). Similarly, one defines other truncation functors $\tau^{\leq n}$, $\tau^{\geq n}$ for $n \in \mathbb{Z}$. The heart of the $t$-structure is $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$. It is an abelian category. The associated cohomology functors are defined by $H^0 = \tau^{\leq 0} \tau^{\geq 0}$, $H^1(X) = H^0(X)[1]$. We will also use the notation $\mathcal{D}^{[a, b]} = D^{\geq a} \cap D^{\leq b}$, where $[a, b] \subset \mathbb{Z}$ is a (possibly infinite on one side) interval.
Following [1] we will say that a $t$-structure is \textit{nondegenerate} if
t\bigcap_{n} D^{\leq n} = \bigcap_{n} D^{\geq n} = 0 \quad \text{and} \quad \bigcup_{n} D^{\leq n} = \bigcup_{n} D^{\geq n} = D.

Note that this terminology is not standard—in [3] such a $t$-structure is called \textit{bounded and nondegenerate}.

Let $D_1$ and $D_2$ be a pair of triangulated categories equipped with $t$-structures. An exact functor $\Phi: D_1 \to D_2$ is called \textit{left} (resp., \textit{right}) \textit{$t$-exact} if $\Phi(D_1^{\leq 0}) \subset D_2^{\geq 0}$ (resp., $\Phi(D_1^{\leq 0}) \subset D_2^{\leq 0}$). A \textit{$t$-exact} functor is a functor that is both left and right $t$-exact. We will use later the following simple observation.

\textbf{Lemma 1.1.1.} Let $D_1$ and $D_2$ be a pair of triangulated categories equipped with $t$-structures.

(i) Let $\Phi: D_1 \to D_2$ be a $t$-exact functor with $\ker \Phi = 0$, i.e., for any $F \in D_1$ such that $\Phi(F) = 0$ one has $F = 0$. Then for any interval $[a, b]$ (possibly infinite on one side) one has

$$D_1^{[a, b]} = \{ F \in D_1 : \Phi(F) \in D_2^{[a, b]} \}.$$

(ii) Let $(\Phi_n : D_1 \to D_2)_{n \in \mathbb{Z}}$ be a family of exact functors such that for every $F \in D_1^{[a, b]}$ there exists an integer $N$ such that $\Phi_n(F) \in D_2^{[a, b]}$ for $n > N$. Assume also that for any $F \in D_1$ such that $\Phi_n(F) = 0$ for $n \gg 0$ one has $F = 0$. Then

$$D_1^{[a, b]} = \{ F \in D_1 : \Phi_n(F) \in D_2^{[a, b]} \text{ for } n \gg 0 \}.$$

\textbf{Proof.} (i) Let us check that for $F \in D_1$ such that $\Phi(F) \in D_2^{\leq 0}$, one has $F \in D_1^{\leq 0}$. Indeed, by $t$-exactness of $\Phi$ we have

$$\Phi(\tau^{\geq 1} F) = \tau^{\geq 1} \Phi(F) = 0.$$

Hence, $\tau^{\geq 1} F = 0$ by our assumption that $\ker \Phi = 0$. It follows that $F \in D_1^{\leq 0}$. Similarly, if $\Phi(F) \in D_2^{\geq 0}$ then $F \in D_1^{\geq 0}$.

(ii) The proof is completely analogous to (i). \hfill \Box

We refer to [10] for basic facts about tilting with respect to a torsion theory. Let $\mathcal{C}$ be an abelian category. Recall that a \textit{torsion pair} $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ consists of two full subcategories such that $\Hom(T, F) = 0$ for every $T \in \mathcal{T}$, $F \in \mathcal{F}$, and such that every object $X \in \mathcal{C}$ fits into an exact sequence

$$0 \to T \to X \to F \to 0$$

with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

Now assume that we have a $t$-structure $(D^{\leq 0}, D^{\geq 0})$ on a triangulated category $D$ and a torsion pair $(\mathcal{T}, \mathcal{F})$ in the heart $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$. Then one can define a new $t$-structure $(D_{\tau}^{\leq 0}, D_{\tau}^{\geq 0})$ on $D$ by setting

$$D_{\tau}^{\leq 0} = \{ X \in D^{\leq 1} : H^1 X \in \mathcal{T} \},$$
$$D_{\tau}^{\geq 0} = \{ X \in D^{\geq 0} : H^0 X \in \mathcal{F} \}.$$

We say that this $t$-structure is obtained from $(D^{\leq 0}, D^{\geq 0})$ by \textit{tilting with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$}. The tilted heart $\mathcal{C}_{\tau} = D_{\tau}^{\leq 0} \cap D_{\tau}^{\geq 0}$ is equipped with a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$. Moreover, performing tilting with respect to this torsion
pair will bring us back to the original $t$-structure (up to a shift).\footnote{Since we do not require $D$ to be equivalent to the derived category of $C$, the new heart $C_t$ does not have to be equivalent to the abelian category obtained by the tilting in the derived category of $C$, cf. Example 3.7 of \cite{[9]}.} Note that we have $D_i^{\leq 0} \subset D_i^{\leq 0} \subset D_i^{\leq 1}$. The following lemma shows that this property characterizes pairs of $t$-structures related by tilting.

\textbf{Lemma 1.1.2.} Let $(D_i^{\leq 0}, D_i^{\geq 0})$, $i = 1, 2$ be a pair of $t$-structures such that

$$D_i^{\leq 0} \subset D_i^{\leq 0} \subset D_i^{\leq 1}$$

Let us denote by $C_i$ the heart of $(D_i^{\leq 0}, D_i^{\geq 0})$ for $i = 1, 2$. Then $(D_i^{\leq 0}, D_i^{\geq 0})$ is obtained from $(D_i^{\leq 0}, D_i^{\geq 0})$ by tilting with respect to the torsion pair $(C_2[1] \cap C_1, C_2 \cap C_1)$ in $C_1$.

\textbf{Proof.} By passing to right orthogonals in (1.1.1) (shifted by $[1]$) we find that

$$D_i^{\geq 1} \subset D_i^{\geq 0} \subset D_i^{\leq 0}.$$  

Hence, $C_1 \subset D_i^{[-1,0]}$ and $C_2 \subset D_i^{[0,1]}$. Let us denote by $\tau_i^*$ (resp., $H_i^*$) the truncation (resp., cohomology) functors associated with $(D_i^{\leq 0}, D_i^{\geq 0})$ for $i = 1, 2$. For any $X \in C_1$ consider the exact triangle

$$A = \tau_2^{\leq -1} X \rightarrow X \rightarrow \tau_2^{\geq 0} X = B \rightarrow \ldots$$

(1.1.2)

Then $A \in C_2[1]$ and $B \in C_2$. Therefore, we have $H_i^1 A = 0$ for $i \neq -1, 0$ and $H_i^1 B = 0$ for $i \neq 0, 1$. The long exact cohomology sequence associated with exact triangle (1.1.2) shows that $H_1^1 A = 0$ and $H_1^1 B = 0$. Hence, both $A$ and $B$ belong to $C_1$. This proves that $(C_2[1] \cap C_1, C_2 \cap C_1)$ is a torsion pair in $C_1$. Switching the roles of $C_1[-1]$ and $C_2$ we derive that any object $Y \in C_2$ fits into an exact triangle

$$A \rightarrow Y \rightarrow B \rightarrow A[1]$$

where $A = \tau_1^{\leq 0} Y = H_1^0 Y \in C_1 \cap C_2$ and $B = \tau_2^{\geq 1} Y = (H_1^1 Y)[-1] \in C_1[-1] \cap C_2$. This implies that

$$C_2 = \{ Y \in D_1^{[0,1]} : H_1^0 Y \in C_2 \cap C_1, H_1^1 Y \in C_2[1] \cap C_1 \}$$

as required. \hfill $\square$

It is especially easy to construct torsion pairs in Noetherian abelian categories because of the following simple observation.

\textbf{Lemma 1.1.3.} Let $C$ be a Noetherian abelian category. Then any full subcategory $T \subset C$ closed under quotients and extensions is contained in a torsion pair $(T, F)$.

\textbf{Proof.} For every object $X \in C$ there is a unique maximal subobject of $X$ that belongs to $T$. \hfill $\square$

\textbf{Example.} If we have an increasing chain $T_1 \subset T_2 \subset \ldots$ of full subcategories closed under quotients and extensions then the same is true for $T = \bigcup_n T_n$. 
1.2. Pre-aisles that are close to Noetherian aisles. Recall (see [2]) that a full subcategory \( \mathcal{P} \subset \mathcal{D} \) is called a pre-aisle if \( \mathcal{P} \) is closed under extensions and the shift functor \( X \to X[1] \) (but not with respect to \( X \to X[-1] \)). A subcategory \( \mathcal{P} \subset \mathcal{D} \) is called an aisle if \( \mathcal{P} = \mathcal{D}^{\leq 0} \) for some \( t \)-structure on \( \mathcal{D} \). Clearly, every aisle is a pre-aisle. The converse is not true in general (see Remark after Theorem 2.1.2 below).

For a collection of subcategories \( S_1, \ldots, S_n \subset \mathcal{D} \) we denote by \( \text{p-a}[S_1, \ldots, S_n] \) the smallest pre-aisle containing all \( S_i \)'s. We call it the pre-aisle generated by \( S_1, \ldots, S_n \).

**Definition.** We say that a \( t \)-structure (or the corresponding aisle) is Noetherian if its heart is Noetherian.

**Theorem 1.2.1.** Let \( (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \) be a Noetherian \( t \)-structure on \( \mathcal{D} \). Then any pre-aisle \( \mathcal{P} \subset \mathcal{D} \) such that \( \mathcal{D}^{\leq -1} \subset \mathcal{P} \subset \mathcal{D}^{\leq 0} \), is an aisle, i.e., \( \mathcal{P} = \mathcal{D}^{\leq 0} \) for some \( t \)-structure on \( \mathcal{D} \).

**Proof.** Consider the heart \( \mathcal{C}_0 = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \). Set \( T = \mathcal{C}_0 \cap \mathcal{P} \). Clearly, \( T \) is stable under extensions. We claim that \( T \) is also stable under taking quotients. Indeed, let

\[
X \to Y \to Z \to X[1]
\]

be an exact triangle with \( X, Y, Z \in \mathcal{C}_0 \) and with \( Y \in \mathcal{T} \). Then \( Y \in \mathcal{P} \) and

\[
X[1] \in \mathcal{D}^{\leq 0}[1] = \mathcal{D}^{\leq -1} \subset \mathcal{P}.
\]

Hence, \( Z \in \mathcal{P} \) and therefore \( Z \in \mathcal{T} \). Since \( \mathcal{C}_0 \) is Noetherian, by Lemma 1.1.3 \( T \) extends to a torsion pair \((\mathcal{T}, \mathcal{F})\). We claim that

\[
\mathcal{P} = \text{p-a}[\mathcal{D}^{\leq -1}, \mathcal{T}] = \{ X \in \mathcal{D}^{\leq 0} : H^0 X \in \mathcal{T}\},
\]

where \( H^0 \) is taken with respect to the \( t \)-structure \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\). Indeed, it suffices to check that for \( X \in \mathcal{P} \) one has \( H^0 X \in \mathcal{T} \). Consider the exact triangle

\[
\tau^{\leq -1} X \to X \to H^0 X \to \tau^{\leq -1} X[1].
\]

We have \( \tau^{\leq -1} X[1] \in \mathcal{D}^{\leq -1}[1] = \mathcal{D}^{\leq -2} \subset \mathcal{P} \). Therefore, \( H^0 X \in \mathcal{P} \) and hence \( H^0 X \in \mathcal{T} \). Thus, \( \mathcal{P} \) coincides with the aisle \( \mathcal{D}^{\leq -1} \) of the tilted \( t \)-structure associated with \((\mathcal{T}, \mathcal{F})\).

\( \square \)

**Corollary 1.2.2.** Let \((\mathcal{D}^{\leq 0}_n, \mathcal{D}^{\geq 0}_n)_{n \geq 0}\) be a sequence of \( t \)-structures in \( \mathcal{D} \) such that

\[
\mathcal{D}^{\leq 0}_0 \subset \mathcal{D}^{\leq 0}_1 \subset \mathcal{D}^{\leq 0}_2 \subset \cdots \subset \mathcal{D}^{\leq 0}_\infty.
\]

Assume in addition that \( \mathcal{D}^{\leq 0}_0 \cap \mathcal{D}^{\geq 0}_0 \) is Noetherian. Then there exists a \( t \)-structure \((\mathcal{D}^{\leq 0}_\infty, \mathcal{D}^{\geq 0}_\infty)\) with \( \mathcal{D}^{\leq 0}_\infty = \bigcup_n \mathcal{D}^{\leq 0}_n \).

**Definition.** We say that a \( t \)-structure \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\) on a triangulated category \( \mathcal{D} \) is close to Noetherian if there exists a Noetherian \( t \)-structure \((\mathcal{D}^{\leq 0}_0, \mathcal{D}^{\geq 0}_0)\) on \( \mathcal{D} \) such that \( \mathcal{D}^{\leq -1}_0 \subset \mathcal{D}^{\leq 0}_0 \subset \mathcal{D}^{\leq 0}_\infty \).

In other words, close to Noetherian \( t \)-structures are precisely \( t \)-structures obtained by tilting from Noetherian \( t \)-structures.
Example. Let $\mathcal{D}$ be a numerically finite triangulated category (see [8], sec. 1.3). With every connected component $\Sigma$ of the space of numerical stability conditions Bridgeland [8] associates a subspace $V(\Sigma) \subseteq (\mathcal{N}(\mathcal{D}) \otimes \mathbb{C})^*$ such that the map sending a stability to its central charge gives a local homeomorphism $\Sigma \rightarrow V(\Sigma)$. Assume that $V(\Sigma)$ is defined over $\mathbb{Q}$ (this is true in all the known examples). Then for a dense subset $\Sigma_Q \subset \Sigma$ the central charge has the image in $\mathbb{Q} + i\mathbb{Q}$.

By Proposition 5.0.1 of [1] for $(\mathcal{P}, Z) \in \Sigma_Q$ the abelian category $\mathcal{P}(t, t+1)$ will be Noetherian for a dense set of $t \in \mathbb{R}$. It follows that for every stability $(\mathcal{P}, Z) \in \Sigma$ the corresponding $t$-structure $(\mathcal{P}(0, +\infty), \mathcal{P}(-\infty, 1])$ is close to Noetherian. Indeed, if $(\mathcal{P}, Z)$ is sufficiently close to $(\mathcal{P}', Z') \in \Sigma_Q$ then $\mathcal{P}(0, 1] \subset \mathcal{P}'(-\epsilon, 1+\epsilon] \subset \mathcal{P}'(t, t+2]$ for some $t \in \mathbb{R}$ such that $\mathcal{P}'(t, t+1]$ is Noetherian.

2. Induced $t$-Structures

2.1. Abstract setting. Let $\mathcal{T}$ be a triangulated category in which all small coproducts exist. Recall that a subcategory of $\mathcal{T}$ is called cocomplete if it is closed under small coproducts.

Definition. For a subcategory $S \subset \mathcal{T}$ we define the cocomplete pre-aisle generated by $S$, denoted by $p-a[[S]] = p-a_{\mathcal{T}}[[S]]$, as the smallest cocomplete pre-aisle containing $S$.

Below we are going to use the following powerful theorem (Theorem A.1 of [2]):

for a small subcategory $S \subset \mathcal{D}$ the pre-aisle $p-a[[S]]$ is an aisle, i.e., $p-a[[S]] = \mathcal{T}^{\leq 0}$

for some $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $\mathcal{T}$. Here is the first immediate application.

Lemma 2.1.1. Let $\tilde{\mathcal{D}}$ be a triangulated category in which all small coproducts exist, and let $\mathcal{D} \subset \tilde{\mathcal{D}}$ be a full triangulated essentially small subcategory. Then for every $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ there exists a $t$-structure $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$ on $\tilde{\mathcal{D}}$ such that

$$\tilde{\mathcal{D}}^{\leq 0} = p-a_{\tilde{\mathcal{D}}}[[\mathcal{D}^{\leq 0}]],$$

$$\tilde{\mathcal{D}}^{\geq 0} = \{ F \in \tilde{\mathcal{D}} : \text{Hom}(\mathcal{D}^{\leq -1}, F) = 0 \}.$$ 

Furthermore, for every interval $[a, b]$ (possibly infinite on one side) one has $\tilde{\mathcal{D}}^{(a,b]} \cap \mathcal{D} = \mathcal{D}^{(a,b]}$.

Proof. By Theorem A.1 of [2] quoted above there exists a $t$-structure $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$ on $\tilde{\mathcal{D}}$ with $\tilde{\mathcal{D}}^{\leq 0} = p-a_{\tilde{\mathcal{D}}}[[\mathcal{D}^{\leq 0}]]$. The formula for $\tilde{\mathcal{D}}^{\geq 0}$ is easy to deduce (cf. Lemma 3.1 of [2]). Note that since $\mathcal{D}$ is a full subcategory in $\tilde{\mathcal{D}}$, we have $\mathcal{D}^{\geq 0} \subset \tilde{\mathcal{D}}^{\geq 0}$. Therefore, the inclusion functor $\mathcal{D} \rightarrow \tilde{\mathcal{D}}$ is $t$-exact, and the last assertion follows from Lemma 1.1.1(i). \qed

Let $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ be a pair of triangulated categories in which all small coproducts exist, and let $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_1$ and $\mathcal{D}_2 \subset \tilde{\mathcal{D}}_2$ be full triangulated essentially small subcategories. Assume we have an exact functor $\Phi : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ commuting with small coproducts that admits a left adjoint functor $\Psi : \mathcal{D}_2 \rightarrow \mathcal{D}_1$. Assume that $\Phi(\mathcal{D}_1) \subset \mathcal{D}_2$ and $\Psi(\mathcal{D}_2) \subset \mathcal{D}_1$. 
2.1.1 Theorem 2.1.2. (i) Let \((D^\leq_2, D^\geq_2)\) be a \(t\)-structure on \(D_2\) such that the functor \(\Phi \Psi : D_2 \to D_2\) is right \(t\)-exact. Assume in addition that \(D_1 = \Phi^{-1}(D_2)\), i.e., for any object \(F \in D_1\) such that \(\Phi(F) \in D_2\) one has \(F \in D_1\). Then there exists a (unique) \(t\)-structure on \(D_1\) with

\[
D^\geq_1 = \{ F \in D_1 : \Phi(F) \in D^\geq_2 \}.
\]

Moreover, the functor \(\Phi\) is \(t\)-exact with respect to these \(t\)-structures.

(ii) Assume also that for any \(F \in D_1\) such that \(\Phi(F) = 0\) one has \(F = 0\). Then

\[
D^\leq_1 = \{ F \in D_1 : \Phi(F) \in D^\leq_2 \}.
\]

In this situation if \(C_2\) is Noetherian then so is \(C_1\), where \(C_i = D^\leq_i \cap D^\geq_0\).

Proof. (i) Let us extend the \(t\)-structure on \(D_2\) to a \(t\)-structure \((\tilde{D}^\leq_2, \tilde{D}^\geq_2)\) on \(\tilde{D}_2\) as in Lemma 2.1.1, so that \(\tilde{D}^\leq_2 = p-a_{\tilde{D}_2}([[D^\leq_2]])\). Now let us define the \(t\)-structure on \(\tilde{D}_1\) by setting

\[
\tilde{D}^\leq_1 = p-a_{\tilde{D}_1}([[D^\leq_2]])
\]

(this is possible by Theorem A.1 of \([2]\)). Then \(\tilde{D}^\geq_1\) is the right orthogonal to \(\Psi(D^\leq_2)\) in \(D_1\). Using adjointness of the pair \((\Psi, \Phi)\) we obtain that

\[
\tilde{D}^\geq_1 = \{ F \in D_1 : \Phi(F) \in D^\geq_2 \}.
\]

Now we claim that the functor \(\Phi : \tilde{D}_1 \to D_2\) is \(t\)-exact. Indeed, clearly we have \(\Phi(\tilde{D}^\geq_1) \subset \tilde{D}^\geq_2\). Also, we have to check that \(\Phi(\tilde{D}^\leq_1) \subset \tilde{D}^\leq_2\). Since \(\Phi\) is exact and commutes with small coproducts, this follows from our assumption that \(\Phi \Psi(D^\leq_2) \subset D^\leq_2\).

Next we claim that setting \(D^\leq[a,b] = \tilde{D}^[a,b] \cap D_1\) we get a \(t\)-structure on \(D_1\). For this we need to prove that if \(F \in D_1\) then \(\tau^\leq F \in D_1\) (apriori it lies in \(\tilde{D}_1\)). By \(t\)-exactness of \(\Phi\) we get that

\[
\Phi \tau^\leq F \simeq \tau^\leq \Phi F \in D_2.
\]

Since \(\Phi^{-1}(D_2) = D_1\), this implies that \(\tau^\leq F \in D_1\). Our formula for \(D^\leq_1\) follows from the above formula for \(\tilde{D}^\leq_1\) and from the fact that \(\tilde{D}^\geq_2 \cap D_2 = D^\geq_2\) (see Lemma 2.1.1).

(ii) The first assertion follows from Lemma 1.1.1(i). Now assume that \(C_2\) is Noetherian. Since \(\Phi\) induces an exact functor with zero kernel from \(C_1 = D^\leq_1 \cap D^\geq_0\) to \(C_2\), this immediately implies that \(C_1\) is also Noetherian.

\[\square\]

Remark. According to Theorem A.1 of \([2]\) used above it is very easy to construct \(t\)-structures in the unbounded derived category of quasicoherent sheaves \(D_{qc}(X)\) by taking \(D_{qc}^\leq\) to be the cocomplete pre-aisle generated by some set of objects. However, one should keep in mind that these \(t\)-structures rarely induce a \(t\)-structure on \(D(X)\). Here is the simplest example. Let \(X\) be a smooth curve. Fix a point \(p\) and define \(D_{qc}^\leq = p-a[[O_p]]\). Then \(D_{qc}^\geq\) consists of \(F\) such that \(\text{Hom}^i(O_p, F) = 0\) for \(i \leq 0\). Consider the exact triangle in \(D_{qc}(X)\)

\[
(j_* O_{X-p}/O_X)[-1] \to O_X \to j_* O_{X-p} \to j_* O_{X-p}/O_X,
\]
where $j: X - p \to X$ is the natural open embedding. It is easy to check that $j_*\mathcal{O}_{X-p} \in D_{qc}^{>2}$ while $(j_*\mathcal{O}_{X-p}/\mathcal{O}_X)[-1] \in D_{qc}^{\leq 1}$. It follows that with respect to our t-structure one has $\tau^{>2}(\mathcal{O}_X) = j_*\mathcal{O}_{X-p}$, so the coherence is not preserved. In other words, $D_{qc}^{\leq 0} \cap D(X)$ is a pre-aisle, but not an aisle.

2.2. Applications to coherent sheaves: first examples. The above abstract theorem can be applied in the case when $D_1$ and $D_2$ are bounded derived categories of coherent sheaves on some schemes and $\mathcal{D}$ are corresponding unbounded derived categories of quasicoherent sheaves (where $(\Psi, \Phi)$ is a pair of adjoint exact functors of geometric origin). The simplest case when $\Phi$ is the push-forward functor gives the following result.

**Proposition 2.2.1.** Let $f: X \to Y$ be a finite morphism of finite Tor dimension. Assume that we have a t-structure $(D^{\leq 0}(Y), D^{\geq 0}(Y))$ on $D(Y)$ such that tensoring with $f_*\mathcal{O}_X$ is a right t-exact functor. Then there exists a t-structure on $D(X)$ with

$$D^{[a,b]}(X) = \{ F \in D(X): f_*F \in D^{[a,b]}(Y) \}$$

If $D^{\leq 0}(Y) \cap D^{\geq 0}(Y)$ is Noetherian then so is $D^{\leq 0}(X) \cap D^{\geq 0}(X)$.

**Proof.** We simply have to apply Theorem 2.1.2 to $\Phi = f_*$. Note that $f_*$ commutes with small coproducts (see Lemma 1.4 of [12]). The left adjoint is $\Psi = f^*$, and we have $\Phi\Psi(F) \simeq f_*f^*F \simeq F \otimes f_*\mathcal{O}_X$. The assumption that $f$ has finite Tor dimension ensures that $f^*$ preserves boundedness of cohomology. □

We have the following corollary for the theory of stability conditions (see [8]).

**Corollary 2.2.2.** Let $f: X \to Y$ be a finite morphism of finite Tor dimension. Assume that we have a stability condition $(\mathcal{P}, Z)$ on $D(Y)$ such that $f_*\mathcal{O}_X \otimes \mathcal{P}(t) \subset \mathcal{P}(t, +\infty)$ for every $t \in \mathbb{R}$. Then there exists an induced stability condition $(\mathcal{P}', Z')$ on $D(X)$ with central charge $Z' = Z \circ f_*$ and $\mathcal{P}'(t) = \{ F: f_*F \in \mathcal{P}(t) \}$.

**Proof.** By Proposition 2.2.1 we have a t-structure on $D(X)$ with the heart $\mathcal{C}' = \mathcal{P}'(0, 1) = \{ F: f_*F \in \mathcal{P}(0, 1) \}$ such that $Z'$ is the centered slope-function on $\mathcal{C}'$. Similarly, for every $t \in \mathbb{R}$ we can define a t-structure $(\mathcal{P}'(t), \mathcal{P}'(\leq t+1))$ on $D(X)$. It remains to prove that the pair $(\mathcal{C}', Z')$ satisfies the Harder–Narasimhan property. Note that if $f_*F$ is semistable of phase $t \in [0, 1]$ then so is $F$. Now for any $F \in \mathcal{C}'$ let $G_1 \subset G_2 \subset \ldots \subset G_n = f_*F$ be the Harder–Narasimhan filtration of $f_*F$, where $G_i/G_{i-1}$ is semistable of phase $t_i$. Using the truncations with respect to the t-structures on $D(X)$ associated with $t_i$’s we can construct a filtration $F_1 \subset F_2 \subset \ldots \subset F_n = f_*F$, such that $G_i = f_*F_i$. Since $f_*(F_i/F_{i-1}) = G_i/G_{i-1}$ is semistable of phase $t_i$, the same is true for $F_i/F_{i-1}$. □

**Example.** The assumptions of the above corollary are satisfied if $f_*\mathcal{O}_X = \bigoplus_i L_i$, where $L_i \in \text{Pic}^0(Y)$, and the stability condition on $Y$ is stable under tensoring with $\text{Pic}^0(Y)$. The latter condition can often be checked (see Corollary 3.5.2).

Here is another application of Theorem 2.1.2.

**Proposition 2.2.3.** Let $G$ be a finite (discrete) group acting on $X$. Then there is a bijection between t-structures on $D(X)$, invariant under $g^*: D(X) \to D(X)$.
for every $g \in G$, and $t$-structures on the derived category of equivariant coherent sheaves $D_G(X)$ with respect to which the functor $D_G(X) \to D_G(X) : F \mapsto F \otimes_G R$ is $t$-exact, where $R$ is the regular representation of $G$. Similarly, there is a bijection between stabilities on $D(X)$ invariant under $G$ and stabilities $(\mathcal{P}, Z)$ on $D_G(X)$ such that $\mathcal{P}(t) \otimes_G R \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$ and $Z(F \otimes_G R) = |G|Z(F)$ for all $F \in D_G(X)$.

**Proof.** Let $\Psi : D_G(X) \to D(X)$ denote the forgetful functor, and let $\Phi : D(X) \to D_G(X)$ be the functor sending a coherent sheaf $F$ to the $G$-equivariant sheaf $\bigoplus_{g \in G} g^*F$. Then both pairs $(\Phi, \Psi)$ and $(\Psi, \Phi)$ are adjoint and we have natural isomorphisms

$$\Phi \Psi F \simeq F \otimes_G R, \quad \Psi \Phi F \simeq \bigoplus_{g \in G} g^*F, \quad (2.2.1)$$

where $R$ is the regular representation of $G$. It remains to apply Theorem 2.1.2 to both functors.

The bijection between stability conditions is established in a similar way. If $(\mathcal{P}, Z)$ is a $G$-invariant stability on $D(X)$ then we define a stability $(\mathcal{P}', Z')$ on $D_G(X)$ by setting $\mathcal{P}'(t) = \Psi^{-1}(\mathcal{P}(t))$, $Z' = Z \circ \Psi$. We leave the details for the reader.

### 2.3. Locality

The following general result appears as Corollary 2 in [4]. For completeness we supply a proof (sketched in [4]).

**Lemma 2.3.1.** Let $X$ be a Noetherian scheme, $j : U \hookrightarrow X$ an open subscheme. Then the restriction functor $j^* : D(X) \to D(U)$ is essentially surjective.

**Proof.** We use the following fact about extensions of sheaves (that follows easily from [11], Ex. II.5.15). Let $F \to G$ be a surjective morphism of quasi-coherent sheaves on a Noetherian scheme $X$ and let $U \subset X$ be an open subset such that $G|_U$ is coherent. Then there exists a coherent subsheaf $F' \subset F$ such that the induced morphism $F'|_U \to G|_U$ is surjective.

Now let $F^*$ be a bounded complex of quasi-coherent sheaves such that $F^*|_U$ has coherent cohomology. Let us denote $Z^i = \ker(d : F^i \to F^{i+1})$, $B^i = d(F^{i-1}) \subset Z^i$, $H^i = Z^i / B^i$. Then using the above observation we can construct a subcomplex of coherent sheaves $F^*_c \subset F^*$ such that

1. $d(F^*_c) = (F^*_c)^{i+1} \cap B^{i+1}$
2. the natural map $(F^*_c \cap Z^i)|_U \to H^i|_U$ is surjective.

It is easy to see that (i) and (ii) imply that $F^*_c|_U$ is quasiisomorphic to $F^*|_U$. The subsheaves $F^*_c \subset F^*$ are constructed by descending induction in $i$. Namely, assuming that $F^*_c|_U$ is already constructed we first construct a coherent subsheaf $Z^i_c \subset Z^i$ such that $Z^i_c|_U$ surjects onto $H^i|_U$ (by applying the above fact to the morphism $Z^i \to H^i$). This will guarantee condition (ii) for any $F^*_c$ containing $Z^i_c$. Next, applying the above fact to the morphism $d^{-1}(F^*_c|_U)/Z^i_c \to F^*_c \cap B^{i+1}$ induced by $d$, we construct a subsheaf $G_c \subset d^{-1}(F^*_c|_U)/Z^i_c$ such that $d(G_c) = F^*_c \cap B^{i+1}$. Finally, we let $F^*_c \subset d^{-1}(F^*_c|_U)$ to be the preimage of $G_c$ under the natural projection.

Now given $G \in D(U)$ we take $F = j_*G$, so that $F|_U \simeq G$, and construct $F_c \subset F$ as above. Then $F_c \in D(X)$ and $F_c|_U \simeq F|_U \simeq G$. \qed
**Definition.** Let $f : X \to S$ be a morphism of schemes. We say that a $t$-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ on $D(X)$ is local over $S$ if for every open $U \subset S$ there exists a $t$-structure on $D(f^{-1}(U))$ such that the restriction functor $D(X) \to D(f^{-1}(U))$ is $t$-exact. A $t$-structure on $D(X)$ is called local if it is local over $X$ (with $f = \text{id}$).

By Lemma 2.3.1 the induced $t$-structure on $D(f^{-1}(U))$ is uniquely defined for every open $U \subset S$. It is easy to see that for $F \in D(X)$ the condition $F \in D^{\lambda, b}(X)$ can be checked locally. Indeed, since the cohomology functors $H^i$ with respect to our $t$-structures commute with restrictions to open subsets, this follows immediately from the fact that the condition $F = 0$ for $F \in D(X)$ can be checked locally. We derive also that for any vector bundle $V$ on $S$ the functor of tensoring with $f^* V$ is $t$-exact with respect to a $t$-structure, local over $S$. Finally, note that if a $t$-structure on $D(X)$ is nondegenerate and local over $S$ then the same is true for the induced $t$-structures on $D(f^{-1}(U))$ for any open $U \subset S$.

For example, $t$-structures on $D(X)$ considered in [4] (associated with monotone and comonotone perversity functions on the topological space of $X$) are local. Below we will show that for smooth $X$ these are the only nondegenerate local $t$-structures (see Corollary 2.3.6).

The notion of a sheaf of $t$-structures considered in [1], Sec. 2.1, is equivalent to a $t$-structure on $D(X \times S)$ local over $S$. Almost all assertions made in loc. cit. about this notion easily extend to the case of an arbitrary morphism $X \to S$. The following theorem is a slight strengthening of Theorem 2.1.4 of [1].

**Theorem 2.3.2.** Let $f : X \to S$ be a morphism, where $S$ is quasiprojective over an affine scheme. Let $L$ be an ample line bundle on $S$. Then a nondegenerate $t$-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ on $D(X)$ is local over $S$ if and only if tensoring with $f^* L$ is left $t$-exact, i.e., $f^* L \otimes D^{\geq 0}(X) \subset D^{\geq 0}(X)$.

**Proof.** The proof follows the same outline as that of Theorem 2.1.4 of [1]. Let us observe that smoothness assumption used in loc. cit. is not necessary because of Lemma 2.3.1. Another change is in the analogue of Lemma 2.1.6: we claim that it is enough to assume only left $t$-exactness of tensoring with $f^* L$ to deduce that for a closed subset $T \subset S$ an object $F \in D(X)$ is supported on $f^{-1}(T)$ if and only if all cohomology objects $H^i F$ with respect to our $t$-structure are supported on $f^{-1}(T)$. Indeed, one has to check that for an object $F \in D(X)$ and a section $s \in H^0(S, L^d)$ (where $d > 0$) the vanishing of the morphism of multiplication by $s$

$$F \xrightarrow{s} F \otimes f^* L^d$$

induces the vanishing of the similar morphisms for $\tau^{\geq 0} F$ and $\tau^{\leq 0} F$. To this end consider the natural morphism

$$F \otimes f^* L^d \to \tau^{\geq 0} F \otimes f^* L^d.$$
obtained by applying \( \tau^{\geq 0} \) to (2.3.1), coincides with the morphism of multiplication by \( s \) on \( \tau^{\geq 0} F \). Indeed, it is enough to check this equality after composing with the natural morphism \( F \to \tau^{\geq 0} F \), so it follows from the functoriality of the morphism (2.3.1) in \( F \).

\[ \square \]

**Corollary 2.3.3.** If \( S \) is affine then any nondegenerate \( t \)-structure on \( D(X) \) is local over \( S \).

There is a natural gluing procedure for \( t \)-structures, local over the base.

**Lemma 2.3.4.** Let \( f: X \to S \) be a morphism, and let \( S = \bigcup U_i \) be a finite open covering of \( S \). Assume that for every \( i \) we have a nondegenerate \( t \)-structure on \( D(f^{-1}(U_i)) \), local over \( U_i \), and that these \( t \)-structures agree over all pairwise intersection. Then there exists a \( t \)-structure on \( D(X) \), local over \( S \), inducing the given \( t \)-structure on every \( D(f^{-1}(U_i)) \).

**Proof.** Let us set \( X_i = f^{-1}(U_i) \). We want to check that

\[
D^{[a,b]}(X) = \{ F: F|_{X_i} \in D^{[a,b]}(X_i) \text{ for all } i \}
\]

is a \( t \)-structure on \( X \). To show orthogonality of \( F \in D^{\leq 0}(X) \) and \( G \in D^{\geq 1}(X) \) consider the object \( R\text{Hom}_S(F,G) := f_*R\text{Hom}(F,G) \in D_{qc}(S) \). Note that for every open subset \( U \subset S \) we have \( R\text{Hom}(F|_{f^{-1}(U)},G|_{f^{-1}(U)}) \simeq R\text{lim}(U, R\text{Hom}_S(F,G)) \).

Hence, for every \( i \) and every open affine \( U \subset U_i \) we have \( R\text{Hom}_S(F,G)|_{U} \in D^{\geq 1}(U) \) (with respect to the standard \( t \)-structure on \( D(U) \)). It follows that \( R\text{Hom}_S(F,G) \in D^{\geq 1}(S) \) and therefore \( \text{Hom}(F,G) = 0 \). To define the truncation functors it suffices to define \( H^0 \text{F} \) and \( \tau^{\geq 0} \text{F} \) for \( F \in D^{\geq 0}(X) \) (since the \( t \)-structures on \( D(X_i) \) are nondegenerate). Set \( F_i = F|_{X_i} \). Then we have a natural gluing datum for the objects \( H^0 F_i \) in the hearts of \( t \)-structures on \( X_i \). At this point we observe that analogues of Theorem 2.1.9 and of Corollary 2.1.11 of [1] hold in the situation of a general morphism \( X \to S \) (with the same proof). Therefore, we can glue \( (H^0 F_i) \) into an object \( H^0 \text{F} \) equipped with isomorphisms \( H^0 F|_{X_i} \simeq H^0 F_i \). Looking at restrictions to \( X_i \), one easily checks that \( R\text{Hom}_S(H^0 F,F) \in D^{\geq 0}(S) \). By the analogue of Lemma 2.1.10 of [1] this implies that we can glue morphisms \( H^0 F_i \to F_i \) into a global morphism \( H^0 \text{F} \to F \).

\[ \square \]

Using Proposition 2.2.1 we get the following base change construction.

**Theorem 2.3.5.** Let \( f: X \to S \) be a flat morphism, and let \( (D^{\leq 0}(X), D^{\geq 0}(X)) \) be a \( t \)-structure on \( D(X) \), local over \( S \). Then for any finite morphism of finite \( \text{Tor} \) dimension \( g: S' \to S \) there is an induced \( t \)-structure on \( D(X \times_S S') \) given by

\[
D^{[a,b]}(X \times_S S') = \{ F \in D(X \times_S S'): g'_*(F) \in D^{[a,b]}(X) \},
\]

where \( g': X \times_S S' \to X \) is the natural projection. If the original \( t \)-structure on \( D(X) \) is Noetherian then so is the induced \( t \)-structure on \( D(X \times_S S') \).

**Proof.** We claim that in this case the assumptions of Proposition 2.2.1 are satisfied for the morphism \( g': X \times_S S' \to X \). Indeed, it suffices to check that tensoring with \( g'_*\mathcal{O}_{X \times_S S'} \) is right \( t \)-exact. The question is local over \( S \), so we can assume that
$g_*\mathcal{O}_{S'}$ has a finite resolution $V_n \to \ldots \to V_0$ by vector bundles on $S$ (since $g$ has a finite Tor dimension). Then in the derived category $D(S)$ we have

$$g_*\mathcal{O}_{S'} \simeq (V_n \to \ldots \to V_0),$$

where the complex is concentrated in degrees $[-n, 0]$. Using the base change formula we get

$$g_*\mathcal{O}_{X \times_S S'} \simeq f^*g_*\mathcal{O}_{S'} \simeq (f^*V_n \to \ldots \to f^*V_0).$$

Since our $t$-structure on $D(X)$ is local over $S$, the functors of tensoring with $f^*V_i$ are $t$-exact. This implies that tensoring with the above complex is right $t$-exact. It remains to apply Proposition 2.2.1.

For example, assume that $T \subset S$ is a closed subscheme that is a locally complete intersection. Then by the above theorem, a $t$-structure on $D(X)$, local over $S$, induces a $t$-structure on $D(f^{-1}(T))$, local over $T$, such that the push-forward functor $D(f^{-1}(T)) \to D(X)$ is $t$-exact.

**Corollary 2.3.6.** Assume that $X$ is smooth over a field $k$. Then any nondegenerate local $t$-structure on $D(X)$ is the $t$-structure associated with a monotone and comonotone perversity function on the topological space of $X$ (see [4]).

**Proof.** By Theorem 2.3.5 (applied to $f = \text{id}: X \to X$) for every closed subscheme $i: Z \hookrightarrow X$ we have an induced local $t$-structure on $D(Z)$ such that the functor $i_*: D(X) \to D(Z)$ is $t$-exact. Furthermore, it is easy to see that this $t$-structure on $D(Z)$ is nondegenerate. Now assume that $Z$ is irreducible and reduced. By locality the $t$-structure on $D(Z)$ induces a nondegenerate $t$-structure on $D(\eta_Z)$, where $\eta_Z \in Z$ is the generic point, such that the restriction functor $D(Z) \to D(\eta_Z)$ is $t$-exact. There is a unique integer $p = p(\eta_Z)$ such that this $t$-structure on $D(\eta_Z)$ coincides with $(D^\leq(\eta_Z), D^\geq(\eta_Z))$, where $(D^\leq, D^\geq)$ denotes the standard $t$-structure. Indeed, since $D(\eta_Z)$ is semisimple (i.e., every exact triangle splits) and the object $k(\eta_Z) \in D(\eta_Z)$ is indecomposable, it has a unique nonzero cohomology with respect to any nondegenerate $t$-structure.

Next, let us show that if we have an embedding of irreducible closed subsets $Z \subset Y$ then $p(\eta_Z) \geq p(\eta_Y)$, i.e., the function $p$ is monotone. By locality it suffices to study the situation in a neighborhood of $\eta_Z$ in $Y$. Let $A$ be the local ring of $Y$ at $\eta_Z$, $C$ the heart of the induced $t$-structure on $D(\text{Spec} A)$, and let $k(\eta_Y), k(\eta_Z)$ denote the residue fields at $\eta_Y$ and $\eta_Z$. Since $k(\eta_Y)[-p(\eta_Y)]$ belongs to the heart of the $t$-structure on $D(\eta_Y)$, there exists $F \in C$ such that $F|_{\eta_Y} \simeq k(\eta_Y)[-p(\eta_Y)]$. On the other hand, viewing $k(\eta_Z)$ as an $A$-module we have $k(\eta_Z)[-p(\eta_Z)] \in C$. Assume that $M = H^0_{\text{st}}F \neq 0$ and $H^0_{\text{st}}F = 0$, where $H^0_{\text{st}}$ denote the cohomology functors with respect to the standard $t$-structure on $D(\text{Spec} A)$. Note that $n \geq p(\eta_Y)$. By Nakayama lemma $M$ has a nonzero morphism to $k(\eta_Z)$. Thus, we get a nonzero morphism $F \to k(\eta_Z)[-n]$. Since $k(\eta_Z)[-n] \in C[p(\eta_Z) - n]$, this implies that $p(\eta_Z) - n \geq 0$. Hence, $p(\eta_Z) \geq n \geq p(\eta_Y)$.

Let $\mathbb{D}: D(X) \to D(X), F \mapsto \mathbb{R}\text{Hom}(F, \omega_X[\dim X])$ be the duality functor. Then $(\mathbb{D}(D^\geq(\mathcal{X})), \mathbb{D}(D^\leq(\mathcal{X})))$ is also a nondegenerate local $t$-structure on $D(X)$, so we can apply the above argument to it as well. We claim that the corresponding function on points is $\overline{p}(x) = -\dim x - p(x)$. Indeed, for every irreducible closed
subset $Z \subset X$ there exists an object $F$ in the heart of the original t-structure on $D(Z)$ such that $F|_{\eta_Z} \cong k(\eta_Z)[-p(\eta_Z)]$. Then $D(F)|_{\eta_Z} \cong k(\eta_Z)[\dim Z + p(\eta_Z)]$. But $D(F)$ is in the heart of the new t-structure, so we obtain the above formula for $\mathfrak{p}$. Since $\mathfrak{p}$ is monotone, we get that $p$ is comonotone. Finally, using the fact that for every closed subset $i: Z \hookrightarrow X$ the functor $i^*$ (resp., $i^!$) is right t-exact (resp., left t-exact) we easily see that $D^{\leq 0} \subset D^{p,\leq 0}$ (resp., $D^{\geq 0} \subset D^{p,\geq 0}$), where $(D^{p,\leq 0}, D^{p,\geq 0})$ is the t-structure associated with $p$ (see [4]). Therefore, we have equalities $D^{\leq 0} = D^{p,\leq 0}$, $D^{\geq 0} = D^{p,\geq 0}$.

Exactly the same argument as in Proposition 3.3.2 of [1] proves the following open heart property of a Noetherian t-structure on $D(X)$, local over $S$.

**Proposition 2.3.7.** Let $f: X \to S$ be a flat morphism, $(D^{\leq 0}(X), D^{\geq 0}(X))$ a Noetherian t-structure on $D(X)$, local over $S$. For every open subset $U \subset S$ we denote by $C_U \subset D(f^{-1}(U))$ the heart of the corresponding t-structure. Let also $T \subset S$ be a closed subscheme that is a locally complete intersection, and let $C_T \subset D(f^{-1}(T))$ be the heart of the induced t-structure. Then for every $F \in D(X)$ such that $F|_{f^{-1}(T)} \in C_T$ there exists an open neighborhood $T \subset U \subset S$ such that $F|_{f^{-1}(U)} \in C_U$.

The Noetherian hypothesis is used in the proof to guarantee the existence for any object in $C_S$ of a maximal subobject supported on $f^{-1}(T)$. Without this hypothesis the result is false (see the counterexample in [1], Sec. 6.3). In Section 3.5 we will describe one nice application of the open heart property to the invariance of a t-structure with respect to a continuous group of autoequivalences.

For a future reference we record the following technical result about the base change with respect to a closed embedding.

**Lemma 2.3.8.** Let $f: X \to S$ be a flat morphism, and let $(D^{\leq 0}(X), D^{\geq 0}(X))$ be a t-structure on $D(X)$, local over $S$. Then for any closed embedding of finite Tor dimension $i: T \to S$ we have

$$H^0k^*k_*F \cong F,$$

where $k: X_T := f^{-1}(T) \to X$ is the natural embedding, $H^0$ is taken with respect to the induced t-structure on $D(X_T)$ and $F$ is an object in the heart of this t-structure.

**Proof.** Let us define $G \in D(X_T)$ from the exact triangle

$$G \to k^*k_*F \xrightarrow{\alpha} F \to \ldots,$$

where $\alpha$ is the natural adjunction morphism. It suffices to show that $G \in D^{\leq -1}(X_T)$. By the definition of the t-structure on $D(X_T)$ this is equivalent to $k_*G \in D^{\leq -1}(X)$. But in the exact triangle

$$k_*G \to k_*k^*k_*F \xrightarrow{k_*\alpha} k_*F \to \ldots$$

the morphism $k_*\alpha$ is the projection onto the direct summand. Indeed, if $\beta: k_*F \to k_*k^*(k_*F)$ is the natural adjunction morphism then $k_*\alpha \circ \beta = \text{id}_{k_*F}$. Hence, we have $k_*k^*k_*F \cong k_*F \oplus k_*G$. On the other hand, by the projection formula $k_*k^*k_*F \cong$
Moreover, the morphism $\beta$ is induced by the natural map $\mathcal{O}_X \to k_*\mathcal{O}_{X_T}$ that has the cone $f^*J_T[1]$, where $J_T \subset \mathcal{O}_S$ is the ideal sheaf of $T$. Therefore,

$$k_*G \simeq f^*J_T \otimes k_*F[1].$$

Using locality of the $t$-structure and local finite resolutions of $J_T$ over $\mathcal{O}_S$ we derive that $k_*G \in D^{\leq -1}(X)$. \hfill \Box

3. Constant Families of $t$-Structures

3.1. Gluing of $t$-structures. Let us recall some definitions and constructions involving admissible subcategories and semiorthogonal decompositions (see [6], [7], [14]).

**Definition.** Let $\mathcal{D}$ be a triangulated category. A full triangulated subcategory $\mathcal{A} \subset \mathcal{D}$ is called right admissible (resp., left admissible) if there exist right (resp., left) adjoint functors to the inclusion $\mathcal{A} \to \mathcal{D}$.

**Definition.** A (weak) semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \quad (3.1.1)$$

is given by a collection of full triangulated subcategories such that $\text{Hom}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $i < j$ and for every object $X \in \mathcal{D}$ there exists a sequence of exact triangles

$$A_i \to X_i \to X_{i-1} \to A_i[1], \quad i = 1, \ldots, n,$$

with $A_i \in \mathcal{A}_i$, where $X_n = X$ and $X_0 = 0$.

We will not use the stronger notion of semiorthogonal decomposition that requires all subcategories $\mathcal{A}_i$ to be right and left admissible, so we will omit the attribute “weak”.

In the case $n = 2$ the semiorthogonal decomposition is determined by one of the subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$. Namely, $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ if and only if $\mathcal{A}_1$ is left admissible and $\mathcal{A}_2 = \perp \mathcal{A}_1$. Equivalently, $\mathcal{A}_2$ should be right admissible and $\mathcal{A}_1 = \mathcal{A}_2 \perp$. In general if (3.1.1) is a semiorthogonal decomposition then $\mathcal{A}_1$ is left admissible and there is a semiorthogonal decomposition

$$\perp \mathcal{A}_1 = \langle \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle. \quad (3.1.2)$$

Also, $\mathcal{A}_n$ is right admissible and

$$\mathcal{A}_n^\perp = \langle \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} \rangle.$$

This leads to an alternative definition of a semiorthogonal decomposition as a sequence of left admissible subcategories $\mathcal{D}_1 = \mathcal{A}_1 \subset \mathcal{D}_2 \subset \ldots \subset \mathcal{D}_n = \mathcal{D}$ such that $\mathcal{A}_1$ is the left orthogonal of $\mathcal{D}_{i-1}$ in $\mathcal{D}_i$.

In the next lemma we define the gluing of $t$-structures in the situation when one has a semiorthogonal decomposition. This is a particular case of the formalism of [3], sec. 1.4, rewritten in slightly different terms. It will be convenient to use the following notion analogous to that of a pre-aisle. We say that a full subcategory $Q \subset \mathcal{D}$ is an anti-pre-aisle if it is closed under extensions and under the functor $X \to X[-1]$. For a collection of subcategories $S_1, \ldots, S_n \subset \mathcal{D}$ we denote by
Lemma 3.1.1. Assume we are given a semiorthogonal decomposition (3.1.1) and t-structures \((A_i^<, A_i^>)\) on each \(A_i\). If all the subcategories \(A_i \subset D\) are right admissible then the following formulas define a t-structure on \(D\):

\[
D^<_\rho = \text{p-a}[A^<_1, \ldots, A^<_n], \\
D^>_\rho = \{ X \in D : \rho_1(X) \in A^>_1, \ldots, \rho_n(X) \in A^>_n \},
\]

where for each \(i\) the functor \(\rho_i : D \to A_i\) is the right adjoint to the inclusion \(A_i \to D\). Similarly, if all the subcategories \(A_i \subset D\) are left admissible then we have a t-structure on \(D\) defined by

\[
D^<_\lambda = \{ X \in D : \lambda_1(X) \in A^<_1, \ldots, \lambda_n(X) \in A^<_n \}, \\
D^>_\lambda = \text{a-p-a}[A^>_1, \ldots, A^>_n],
\]

where \(\lambda_i : D \to A_i\) are the left adjoint functors to the inclusions.

Proof. Let us prove that \((D^<\rho, D^>_\rho)\) is a t-structure on \(D\) provided all \(A_i\)'s are right admissible (for the second t-structure the argument is similar). First, let us consider the case \(n = 2\). To show orthogonality of \(D^<\rho\) and \(D^>_\rho\) it is enough to check that for \(X \in D^>_\rho\) one has \(\text{Hom}(A^<_i, X) = 0\) for \(i = 1, 2\). But this follows immediately from the assumption \(\rho_i X \in A^>_i\) and from the orthogonality of \(A^<_i\) and \(A^>_i\). Next, for every \(X \in D\) we have to find an exact triangle \(Y \to X \to Z \to Y[1]\) with \(Y \in D^<\rho\) and \(Z \in D^>_\rho\). By the definition of \(\rho_2\) we have an exact triangle

\[
\rho_2(X) \to X \to W \to \rho_2(X)[1],
\]

where \(\rho_2(W) = 0\). Also, we have an exact triangle

\[
\tau^<\rho_2(X) \to \rho_2(X) \to \tau^>_1\rho_2(X) \to \tau^<\rho_2(X)[1],
\]

where we use the truncation functors on \(A_2\). We can embed the composed map \(\tau^<\rho_2(X) \to \rho_2(X) \to X\) into an exact triangle

\[
\tau^<\rho_2(X) \to X \to X' \to \tau^<\rho_2(X)[1]
\]

By the octahedron axiom we also have an exact triangle

\[
\tau^>_1\rho_2(X) \to X' \to W \to \tau^>_1\rho_2(X)[1]
\]

that implies that \(\rho_2(X') \simeq \tau^>_1\rho_2(X)\). Similarly, we can embed the composed map \(\tau^<\rho_1(X') \to \rho_1(X') \to X'\) (the truncation is taken in \(A_1\)) into an exact triangle

\[
\tau^<\rho_1(X') \to X' \to Z \to \tau^<\rho_1(X')[1],
\]

where \(\rho_1(Z) \simeq \tau^>_1\rho_1(X')\). Also, since \(\rho_2(A_1) = 0\), it follows that \(\rho_2(Z) \simeq \rho_2(X') \simeq \tau^>_1\rho_2(X)\). Therefore, \(Z \in D^>_\rho\). Finally, let us embed the composed map \(X \to X' \to Z\) into an exact triangle \(Y \to X \to Z \to Y[1]\). By the octahedron axiom we have an exact triangle

\[
\tau^<\rho_2(X) \to Y \to \tau^<\rho_1(X') \to \tau^<\rho_2(X)[1],
\]

hence, \(Y \in D^<_\rho\).
The case of general \( n \) is deduced by induction: one has to use the semiorthogonal decompositions (3.1.2) and \( \mathcal{D} = \langle A_1, A_1^\perp \rangle \) (note that \( A_1^\perp \) is automatically right admissible).

Under additional assumptions one can rewrite the above definition of the glued \( t \)-structure in a more symmetric way. We keep the notation \( \rho_i \) (resp., \( \lambda_i \)) for the right (resp., left) adjoint functor to the inclusion \( A_i \to \mathcal{D} \).

**Lemma 3.1.2.** Assume we are given a semiorthogonal decomposition (3.1.1) and \( t \)-structures \( (A_i^{\leq 0}, A_i^{\geq 0}) \) on each \( A_i \). Assume in addition that all the subcategories \( A_i \) are right admissible, and for every \( i < j \) the functor \( \rho_i|_{A_i^j} : A_j \to A_i \) is right \( t \)-exact (with respect to the \( t \)-structures on \( A_i \) and \( A_j \)). Then one has

\[
\mathcal{D}^\rho_{a,b} = \{ X \in \mathcal{D} : \rho_1(X) \in A_1^{[a,b]}, \ldots, \rho_n(X) \in A_n^{[a,b]} \}. \tag{3.1.6}
\]

Similarly, if we assume that \( A_i \)'s are left admissible, and for every \( i < j \) the functor \( \lambda_j|_{A_i} : A_i \to A_j \) is left \( t \)-exact, then

\[
\mathcal{D}^\lambda_{a,b} = \{ X \in \mathcal{D} : \lambda_1(X) \in A_1^{[a,b]}, \ldots, \lambda_n(X) \in A_n^{[a,b]} \}. \tag{3.1.7}
\]

**Proof.** First, consider the case \( n = 2 \). Note that for every \( X \in \mathcal{D} \) the exact triangle (3.1.5) can be rewritten as

\[
\rho_2(X) \to X \to \lambda_1(X) \to \rho_2(X)[1]. \tag{3.1.8}
\]

Also, we have

\[
\mathcal{D}^\rho_{\leq 0} = \{ X \in \mathcal{D} : \lambda_1(X) \in A_1^{\leq 0}, \rho_2(X) \in A_2^{\leq 0} \}.
\]

Applying \( \rho_1 \) to (3.1.8) we get the exact triangle

\[\rho_1\rho_2(X) \to \rho_1(X) \to \lambda_1(X) \to \rho_1\rho_2(X)[1].\]

Thus, if \( \rho_2(X) \in A_2^{\leq 0} \) then by our assumption \( \rho_1\rho_2(X) \in A_1^{\leq 0} \). Thus, if \( \rho_2(X) \in A_2^{\leq 0} \) then the conditions \( \rho_1(X) \in A_1^{\leq 0} \) and \( \lambda_1(X) \in A_1^{\leq 0} \) are equivalent. This proves that

\[
\mathcal{D}^\rho_{\leq 0} = \{ X \in \mathcal{D} : \rho_1(X) \in A_1^{\leq 0}, \rho_2(X) \in A_2^{\leq 0} \}.
\]

The case \( n > 2 \) follows by induction. Namely, we apply the above argument to the semiorthogonal decompositions (3.1.2) and \( \mathcal{D} = \langle A_1, A_1^\perp \rangle \). We only have to check that the restriction of the functor \( \rho_1 \) to \( A_1^\perp = \langle A_2, \ldots, A_n \rangle \) is right \( t \)-exact (with respect to the glued \( t \)-structure on \( \langle A_2, \ldots, A_n \rangle \)). But this immediately follows from the fact that \( \langle A_2, \ldots, A_n \rangle^{\leq 0} = p-a[A_2^{\leq 0}, \ldots, A_n^{\leq 0}] \).

**3.2. Constant \( t \)-structure over \( \mathbb{P}^r \) following [1].** Starting from this point we always assume our schemes to be of finite type over a fixed field \( k \). The product of such schemes is taken over \( k \).

Let \( X \) be a scheme.

**Theorem 3.2.1** (Theorem 2.3.6 of [1]). For every Noetherian nondegenerate \( t \)-structure \( (D^{\leq 0}(X), D^{\geq 0}(X)) \) on \( D(X) \) there is a Noetherian nondegenerate \( t \)-structure \( (D^{\leq 0}(X \times \mathbb{P}^r), D^{\geq 0}(X \times \mathbb{P}^r)) \), local over \( \mathbb{P}^r \), characterized by the property

\[
D^{[a,b]}(X \times \mathbb{P}^r) = \{ F \in D(X \times \mathbb{P}^r) : p_*(F(n)) \in D^{[a,b]}(X) \text{ for all } n \gg 0 \}, \tag{3.2.1}
\]

where \( p : X \times \mathbb{P}^r \to X \) is the natural projection.
We will refer to the above $t$-structure on $D(X \times \mathbb{P}^r)$ as the constant $t$-structure. We show in [1] that it is obtained as a certain limit of the sequence of glued $t$-structures on $D(X \times \mathbb{P}^r)$. More precisely, one has
\[ D^{\leq 0}(X \times \mathbb{P}^r) = \bigcup_n D_n^{\leq 0}(X \times \mathbb{P}^r), \tag{3.2.2} \]
where the $n$-th $t$-structure $(D_n^{\leq 0}, D_n^{\geq 0})$ is glued from standard $t$-structures with respect to the semiorthogonal decomposition
\[ D(X \times \mathbb{P}^r) = \langle p^* D(X)(-r-n), \ldots, p^* D(X)(-1-n), p^* D(X)(-n) \rangle. \]

In the notation of Lemma 3.1.1 we have $D_{n,b}^{[a,b]} = D_n^{[a,b]}$. The conditions of Lemma 3.1.2 are also satisfied in this case, so the $n$-th $t$-structure has a nice description in terms of the functors $F \mapsto p_*(F(m))$ which leads to (3.2.1).

It is important to observe that the assumption that $X$ is smooth made in [1] is not needed for Theorem 2.3.6 (nor for Theorem 2.1.4) of loc. cit. Indeed, it is used there only to guarantee the essential surjectivity of the restriction functor $j^* : D(X \times S) \to D(X \times U)$ for open subsets $U \subset S$. However, this is true without this assumption (see Lemma 2.3.1). The smoothness is used seriously in Section 2.4 of loc. cit. to characterize the essential image of the push-forward under a closed embedding. This characterization is then used in loc. cit. to construct the constant $t$-structure on $D(X \times S)$ for arbitrary smooth quasiprojective base $S$. Using Theorem 2.1.2 we will give an alternative construction of such a $t$-structure on $D(X \times S)$ assuming only that $X$ and $S$ are of finite type of $k$. We will also extend the construction of constant $t$-structures to the case of close to Noetherian $t$-structures (see Section 1.2).

### 3.3. Constant $t$-structures

We start with the case when the base is $\mathbb{P}^r$. Using Lemma 3.1.1 we can make the following observation.

**Lemma 3.3.1.** Let $(D^{\leq 0}(X), D^{\geq 0}(X))$ be a Noetherian nondegenerate $t$-structure on $D(X)$. Then the corresponding constant $t$-structure on $D(X \times \mathbb{P}^r)$ satisfies
\[ D^{\leq 0}(X \times \mathbb{P}^r) = p_{\ast}[D^{\leq 0}(X) \otimes \mathcal{O}_{\mathbb{P}^r}(n) : n \in \mathbb{Z}]. \]

**Proof.** This immediately follows from (3.2.2) and the formula for $D^{\leq 0}_n(X \times \mathbb{P}^r)$ obtained by Lemma 3.1.1. \qed

Now we can consider the case of a close to Noetherian $t$-structure on $D(X)$.

**Lemma 3.3.2.** Let $(D^{\leq 0}(X), D^{\geq 0}(X))$ be a close to Noetherian nondegenerate $t$-structure on $D(X)$. Then we can define a $t$-structure on $D(X \times \mathbb{P}^r)$, local over $\mathbb{P}^r$, by the formula
\[ D^{[a,b]}(X \times \mathbb{P}^r) = \{ F \in D(X \times \mathbb{P}^r) : p_*(F \otimes \mathcal{O}(n)) \in D^{[a,b]}(X) \text{ for all } n \gg 0 \}. \tag{3.3.1} \]

We also have
\[ D^{\leq 0}(X \times \mathbb{P}^r) = p_{\ast}[D^{\leq 0}(X) \otimes \mathcal{O}(n) : n \in \mathbb{Z}]. \tag{3.3.2} \]

**Proof.** Suppose that $(D^{\leq 0}_0(X), D^{\geq 0}(X))$ is a Noetherian $t$-structure such that
\[ D^{\leq -1}_0(X) \subset D^{\leq 0}(X) \subset D^{\leq 0}_0(X). \]
Then we have the corresponding Noetherian constant $t$-structure ($D_0^{<0}(X \times \mathbb{P}^r)$, $D_0^{\leq 0}(X \times \mathbb{P}^r)$) on $D(X \times \mathbb{P}^r)$. Now let us use formula (3.3.2) to define the pre-aisle $D^{\leq 0}(X \times \mathbb{P}^r)$. It follows from Lemma 3.3.1 that

$$D_0^{< -1}(X \times \mathbb{P}^r) \subset D_0^{< 0}(X \times \mathbb{P}^r) \subset D_0^{\leq 0}(X \times \mathbb{P}^r).$$

By Theorem 1.2.1 this implies that $D^{< 0}(X \times \mathbb{P}^r)$ extends to a $t$-structure. Computing the right orthogonal to $D^{< -1}(X \times \mathbb{P}^r)$ gives

$$D^{\geq 0}(X \times \mathbb{P}^r) = \{ F \in D(X \times \mathbb{P}^r) : p_*(F \otimes \mathcal{O}(n)) \in D^{\geq 0}(X) \text{ for all } n \in \mathbb{Z} \}.$$ (3.3.3)

It is easy to see that in this formula it is enough to consider $n \gg 0$: one has to use the exactness of the Koszul complex

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^r}(n-r) \rightarrow \bigwedge^r V \otimes \mathcal{O}_{\mathbb{P}^r}(n-r+1) \rightarrow \ldots \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^r}(n) \rightarrow \mathcal{O}_{\mathbb{P}^r}(n+1) \rightarrow 0,$$ (3.3.4)

where $V = H^0(\mathbb{P}^r, \mathcal{O}(1))$. On the other hand, (3.3.2) implies that for every $F \in D^{< 0}(X \times \mathbb{P}^r)$ one has $p_*(F(n)) \in D^{< 0}(X)$ for $n \gg 0$. Hence, the left-hand side of (3.3.1) is contained in its right-hand side. Now the equality in (3.3.1) follows from Lemma 1.1.1(ii).

Since the constructed $t$-structure is invariant under tensoring with $\mathcal{O}_{\mathbb{P}^r}(1)$, by Theorem 2.3.2 it is local over $\mathbb{P}^r$. □

In the situation of the above lemma let $\mathcal{C} \subset D(X)$ (resp., $\mathcal{C}_{\mathbb{P}^r} \subset D(X \times \mathbb{P}^r)$) be the heart of the $t$-structure on $D(X)$ (resp., $D(X \times \mathbb{P}^r)$). We have the following property (similar to Lemma 2.3.7 of [1]).

**Lemma 3.3.3.** *In the situation of Lemma 3.3.2 for every $F \in \mathcal{C}_{\mathbb{P}^r}$ there exists a surjection $p^*G(n) \rightarrow F$ in $\mathcal{C}_{\mathbb{P}^r}$ with $G \in \mathcal{C}$ and $n \in \mathbb{Z}$.***

**Proof.** Set $V = H^0(\mathbb{P}^r, \mathcal{O}(1))$. First, we observe that for every $F \in \mathcal{C}_{\mathbb{P}^r}$ the natural map $V \otimes p_*(F(n)) \rightarrow p_*(F(n+1))$ is surjective in $\mathcal{C}$ for $n \gg 0$. Indeed, this follows immediately from the exactness of the Koszul complex (3.3.4). Therefore, if we fix a sufficiently large $n$ then for $N > n$ the natural map

$$S^{N-n}V \otimes p_*(F(n)) \rightarrow p_*(F(N))$$ (3.3.5)

is surjective. We claim that this implies surjectivity of the map

$$f : p^*p_*(F(n))(-n) \rightarrow F$$

in $\mathcal{C}_{\mathbb{P}^r}$. Indeed, if $C = \operatorname{coker}(f)$ (the cokernel taken in $\mathcal{C}_{\mathbb{P}^r}$) then for $N \gg 0$ the object $p_*(C(N)) \in \mathcal{C}$ can be identified with the cokernel of the map (3.3.5) in $\mathcal{C}$. Hence, we get $p_*(C(N)) = 0$ for $N \gg 0$. Therefore, $C = 0$, i.e., $f$ is surjective. □

In the following lemma we compute the restriction of the above constant $t$-structure to $X \times \mathcal{A}^r$. We denote by $(D_0^{< 0}(X), D_0^{\geq 0}(X))$ the extension of our $t$-structure from $D(X)$ to $D_{qc}(X)$, where $D_0^{< 0}(X) = p_{\mathbb{A}^r,*}(\mathcal{D}^{\leq 0}(X))$ and $D_0^{\geq 0}(X)$ is the right orthogonal to $D^{<-1}(X)$ in $D_{qc}(X)$ (see Lemma 2.1.1).
Lemma 3.3.4. Let \((D^{\leq 0}(X), D^{\geq 0}(X))\) be a close to Noetherian nondegenerate t-structure on a scheme \(X\). Then there exists a t-structure on \(D(X \times \mathbb{A}^r)\), local over \(\mathbb{A}^r\), given by
\[
D^{[a,b]}(X \times \mathbb{A}^r) = \{ F \in D(X \times \mathbb{A}^r) : p_*(F) \in D^{[a,b]}_q(X) \},
\]
where \(p : X \times \mathbb{A}^r \to X\) is the projection. In addition we have
\[
D^{\leq 0}(X \times \mathbb{A}^r) = p_*(p^*D^{\leq 0}(X)).
\]
If the original t-structure on \(D(X)\) is Noetherian then so is the constructed t-structure on \(D(X \times \mathbb{A}^r)\).

Proof. Since the t-structure on \(D(X \times \mathbb{P}^r)\) constructed in Lemma 3.3.2 is local over \(\mathbb{P}^r\), it induces a t-structure on \(D(X \times U)\) such that for an open subset \(U \subset \mathbb{P}^r\) the subcategory \(D^{[a,b]}(X \times U) \subset D(X \times U)\) consists of restrictions of objects of \(D^{[a,b]}(X \times \mathbb{P}^r)\). This immediately gives (3.3.7). Computing the orthogonal to \(p^*D^{\leq -1}(X)\) we get (3.3.6) for \([a, b] = [0, +\infty)\). Also, using (3.3.7) one easily checks that \(p_*D^{\leq 0}(X \times \mathbb{A}^r) \subset D^{\leq 0}_q(X)\). Therefore, the functor \(p_* : D(X \times \mathbb{A}^r) \to D_q(X)\) is t-exact with respect to our t-structures. By Lemma 1.1.1(i) this implies (3.3.6). The assertion about Noetherian t-structures was proved in Theorem 2.3.6 of [1]. □

Below we will also use the following technical observation.

Lemma 3.3.5. Let \((D^{\leq 0}(X), D^{\geq 0}(X))\) be a close to Noetherian nondegenerate t-structure on a scheme \(X\) of finite type over \(k\). Extend it to a t-structure \(D_q(X)\) by setting \(D^{\leq 0}_q(X) = p_*(D_q(X))\). Then for every \(k\)-vector space \(V\) with a countable basis one has
\[
V \otimes_k D^{[a,b]}(X) \subset D^{[a,b]}_q(X).
\]
Proof. It suffices to prove this for one vector space with an infinite countable basis. Let us equip the categories \(D(X \times \mathbb{P}^1)\) and \(D(X \times \mathbb{A}^1)\) with constant t-structures from Lemmas 3.3.2 and 3.3.4. It is clear from the definition and the projection formula that the functor of pull-back \(D(X) \to D(X \times \mathbb{P}^1)\) is t-exact. The restriction functor \(D(X \times \mathbb{P}^1) \to D(X \times \mathbb{A}^1)\) is also t-exact by locality of the constant t-structure over \(\mathbb{P}^1\). It follows that the pull-back functor \(p^* : D(X) \to D(X \times \mathbb{A}^1)\) is t-exact, i.e., \(p^*D^{[a,b]}(X) \subset D^{[a,b]}(X \times \mathbb{A}^1)\). Combining this with (3.3.6) we deduce that \(H^0(\mathbb{A}^1, \mathcal{O}) \otimes D^{[a,b]}(X) \subset D^{[a,b]}_q(X)\) as required. □

Now we are ready to prove our main result about constant t-structures.

Theorem 3.3.6. Let \((D^{\leq 0}(X), D^{\geq 0}(X))\) be a close to Noetherian nondegenerate t-structure on a scheme \(X\) of finite type over \(k\).

(i) For every scheme \(S\) of a finite type over \(k\) we have a close to Noetherian nondegenerate t-structure on \(D(X \times S)\), local over \(S\), such that
\[
D^{[a,b]}(X \times S) = \{ F \in D(X \times S) : p_*(F|_{X \times U}) \in D^{[a,b]}_q(X) \text{ for every open affine } U \subset S \},
\]
where \(p : X \times U \to X\) is the natural projection. If \(S = \bigcup_i U_i\) is an open affine covering of \(S\) then \(F \in D^{[a,b]}(X \times S)\) if and only if \(p_*(F|_{X \times U_i}) \in D^{[a,b]}_q(X)\) for
every i. If the original $t$-structure is Noetherian then so is the obtained $t$-structure on $D(X \times S)$. The functor $p^*: D(X) \rightarrow D(X \times S)$ is $t$-exact with respect to the original $t$-structure on $D(X)$ and the above $t$-structure on $D(X \times S)$.

(ii) Assume that $S$ is projective. Then the above $t$-structure satisfies

$$D^{[a,b]}(X \times S) = \{ F \in D(X \times S) : p_*(F \otimes p_S^n L^n) \in D^{[a,b]}(X) \text{ for all } n \gg 0 \}, \quad (3.3.9)$$

where $p_S: X \times S \rightarrow S$ is the projection, $L$ is an ample line bundle on $S$.

**Proof.** (i) Assume first that $S$ is affine. Let us choose a closed embedding $S \hookrightarrow A^r$. Note that this embedding has finite Tor dimension since $A^r$ is smooth. Applying Theorem 2.3.5 to the constant $t$-structure on $D(X \times A^r)$ constructed in Lemma 3.3.4 we get an induced $t$-structure on $D(X \times S)$ with

$$D^{[a,b]}(X \times S) = \{ F \in D(X \times S) : p_*(F) \in D^{[a,b]}_q(X) \}. \quad (3.3.10)$$

First, we can check that the left-hand side is contained in the right-hand side. Indeed, for $[a, b] = [-\infty, 0]$ the right-hand-side of (3.3.10) is a cocomplete pre-aisle containing $D^{\leq 0}(X \times S)$. Hence, it also contains $D^{\leq 0}_q(X \times S)$. On the other hand, we have the inclusion $p^* D^{\leq -1}(X) \subset D^{\leq -1}_q(X \times S)$ (since $p_* p^* F \simeq F \otimes H^0(S, \mathcal{O}_S)$ and $D^{\leq -1}_q(X)$ is closed under coproducts). Passing to right orthogonals in $D_q(X \times S)$ we derive that $p_* D^{\geq 0}_q(X \times S) \subset D^{\geq 0}_q(X)$. Now the equality in (3.3.10) follows from Lemma 1.1.1(i) applied to the functor $p_* : D_q(X \times S) \rightarrow D_q(X)$.

Now let $j: U \hookrightarrow S$ be an open affine subset. Then we have a similar constant $t$-structure on $D(X \times U)$ and its extension to $D_q(X \times U)$. From the above formulas it is clear that the functor $(\text{id}_X \times j)_*: D_q(X \times U) \rightarrow D_q(X \times S)$ is $t$-exact. By adjunction we derive that $(\text{id}_X \times j)^* D^{\leq 0}_q(X \times S) \subset D^{\leq 0}_q(X \times U)$. On the other hand,

$$D^{\leq 0}_q(X \times U) = (\text{id}_X \times j)^* (\text{id}_X \times j)_* D^{\leq 0}_q(X \times S) \subset (\text{id}_X \times j)^* D^{\leq 0}_q(X \times S),$$

hence $D^{\leq 0}_q(X \times U) = (\text{id}_X \times j)^* D^{\leq 0}_q(X \times S)$. It follows that

$$D^{\leq 0}_q(X \times U) = p^* D^{\leq 0}_q(X \times S). \quad (3.3.8)$$

Since $(\text{id}_X \times j)^* D^{\leq 0}_q(X \times S)$ is an aisle in $D(X \times S)$ (by locality of the constant $t$-structure over $S$), using Lemma 2.1.1 we get the following equality of the aisles in $D(X \times U)$:

$$D^{\leq 0}_q(X \times U) = (\text{id}_X \times j)^* D^{\leq 0}_q(X \times S).$$

Therefore, the corresponding $t$-structures coincide, so the functor

$$(\text{id} \times j)^*: D(X \times S) \rightarrow D(X \times U)$$

is $t$-exact with respect to the constant $t$-structures. This implies formula (3.3.8) for affine $S$. 

Note that if the original t-structure is Noetherian then so is the constant t-structure on \( D(X \times \mathbb{A}^n) \). By Theorem 2.3.5 this also implies that the induced t-structure on \( D(X \times S) \) is Noetherian for affine \( S \).

Now let us consider the case of arbitrary \( S \). Let \( S = \bigcup_i U_i \) be a finite open affine covering of \( S \). By the preceding part of the proof, for every \( i \) we have a t-structure on \( D(X \times U_i) \) local over \( U_i \). We claim that these t-structures agree on intersections \( U_i \cap U_j \). Indeed, both the t-structures on \( D(X \times (U_i \cap U_j)) \), restricted from \( X \times U_i \) and from \( X \times U_j \), are local over \( U_i \cap U_j \). Since their restrictions to \( X \times U \) agree for every open affine subset \( U \subseteq U_i \cap U_j \), our claim follows. By Lemma 2.3.4 the above t-structures on \( D(X \times U_i) \) can be glued into a t-structure on \( D(X \times S) \). It is easy to see that it is still given by (3.3.8).

If the original t-structure is Noetherian then the constructed t-structure on \( D(X \times S) \) will also be Noetherian (this immediately reduces to the affine case considered above). Since our construction preserves inclusions between pre-aisles, we derive that the constant t-structure on \( D(X \times S) \) is close to Noetherian.

Finally, let us check that the functor \( p^* : D(X) \to D(X \times S) \) is t-exact. If \( S \) is affine then we have \( p_* p^* F \simeq H^n(S, \mathcal{O}_S) \otimes F \), so the assertion follows from Lemma 3.3.5. The general case is deduced easily by covering \( S \) with open affine subsets and using locality of our t-structure on \( D(X \times S) \) over \( S \).

(ii) Since the constant t-structure on \( D(X \times S) \) is local over \( S \), it is invariant under tensoring with the pull-back of any line bundle on \( S \). Hence, for any integer \( d > 0 \) and \( F \in D(X \times S) \) we have \( F \in D^{[a,b]}(X \times S) \) if and only if \( F \otimes p_2^* L^i \in D^{[a,b]}(X \times S) \) for \( i = 0, \ldots, d - 1 \). Therefore, it is enough to prove (3.3.9) for \( L^d \) instead of \( L \), so we can assume that \( L \) is very ample. Let \( i : S \to \mathbb{P}^r \) be a closed embedding such that \( L = i^* \mathcal{O}(1) \). Applying Theorem 2.3.5 to the constant t-structure on \( D(X \times \mathbb{P}^n) \) we derive that the right-hand side of (3.3.9) gives a t-structure on \( D(X \times S) \) (automatically local over \( S \) by Theorem 2.3.2). Considering the standard open covering of \( \mathbb{P}^n \) by the affine pieces, it is easy to see that the restrictions of this t-structure on \( D(X \times S) \) to the induced open affine pieces of \( S \) agree with the t-structures constructed in (i). Hence, it coincides with the t-structure given by (3.3.8).

In the case of an affine base \( S = \text{Spec}(A) \) the heart of the constant t-structure on \( D(X \times S) \) has a natural description in terms of \( A \)-modules in the heart of the t-structure on \( D_{qc}(X) \). More precisely, let \( \mathcal{C} \subset D(X) \) be the heart of the original t-structure on \( D(X) \), and let \( \mathcal{C}_{qc} \subset D_{qc}(X) \) be the heart of the corresponding t-structure on \( D_{qc}(X) \) (such that \( D_{qc}^{\leq 0}(X) = p_{a,D_{qc}(X)}[D_{qc}^{\leq 0}(X)] \)). Recall that an \( A \)-module in \( \mathcal{C}_{qc} \) is an object \( F \in \mathcal{C}_{qc} \) equipped with a homomorphism of algebras \( A \to \text{End}(F) \) (see [15]). They form an abelian category that we will denote by \( A\text{-mod-}\mathcal{C}_{qc} \). Note that since \( A \) has a countable basis as a \( k \)-vector space, by Lemma 3.3.5 for every \( F \in \mathcal{C} \) we have \( F \otimes_k A \in \mathcal{C}_{qc} \). Let us say that an \( A \)-module \( M \in \mathcal{C}_{qc} \) is finitely presented if there exists a pair of objects \( F_0, F_1 \in \mathcal{C} \) and a morphism of free \( A \)-modules \( f : F_1 \otimes_k A \to F_0 \otimes_k A \) in \( A\text{-mod-}\mathcal{C}_{qc} \) such that \( M \simeq \text{coker}(f) \).

**Proposition 3.3.7.** Keep the assumptions of Theorem 3.3.6. Assume in addition that \( S = \text{Spec}(A) \) for a finitely generated \( k \)-algebra \( A \). Then the heart \( \mathcal{C}_S \) of the constant t-structure on \( D(X \times S) \) is equivalent to the category of finitely presented \( A \)-modules in \( \mathcal{C}_{qc} \).
Proof. We have the exact functor $p_*: \mathcal{C}_S \to \mathcal{C}_{qc}$. Furthermore, for every $F \in \mathcal{C}_S$ the object $p_* F \in \mathcal{C}_{qc}$ has a natural $A$-module structure given by the homomorphism

$$A \to H^0(X \times S, \mathcal{O}) \to \text{End}(F) \to \text{End}(p_* F).$$

Thus, we obtain an exact functor $p_*: \mathcal{C}_S \to A\text{-mod-}\mathcal{C}_{qc}$. It sends $p^* G$, where $G \in \mathcal{C}$, to the free $A$-module $p_* p^* G \simeq G \otimes_k A$. We claim that for every $F \in \mathcal{C}_S$ there exists a surjection $p^* G \to F$ in $\mathcal{C}_S$ with $G \in \mathcal{C}$. Indeed, assume first that $S = \mathbb{A}^r$. Then we can extend every $F \in \mathcal{C}_{\mathbb{A}^r}$ to an object $\overline{F} \in \mathcal{C}_S$ such that $\overline{F}|_{X \times \mathbb{A}^r} \simeq F$. Applying Lemma 3.3.3 to $\overline{F}$ and restricting the obtained surjection to $X \times \mathbb{A}^r$ we deduce our claim for $S = \mathbb{A}^r$. For general $S$ let us consider a closed embedding $i: S \to \mathbb{A}^r$. Then there exists a surjection of the required type for $(\text{id}_X \times i)_* F$ in $\mathcal{C}_{\mathbb{A}^r}$. It remains to restrict it to $S$ and use Lemma 2.3.8 together with the fact that $(\text{id}_X \times i)^*$ is right $t$-exact (as the left adjoint of the $t$-exact functor $(\text{id}_X \times i)_*$).

This proves our claim. It follows that every object in $\mathcal{C}_S$ can be represented as the cokernel of a morphism $p^* G_1 \to p^* G_0$, where $G_0, G_1 \in \mathcal{C}$. Hence, the $A$-module $p_* F$ is finitely presented for every $F \in \mathcal{C}$. Next, we have a natural isomorphism

$$\text{Hom}_{\mathcal{C}_S}(p^* G, F) \simeq \text{Hom}_{\mathcal{C}_{qc}}(G, p_* F) \simeq \text{Hom}_{A\text{-mod-}\mathcal{C}_{qc}}(G \otimes_k A, p_* F)$$

for $G \in \mathcal{C}$, $F \in \mathcal{C}_S$. Representing arbitrary $F \in \mathcal{C}_S$ as the cokernel of a morphism $p^* G_1 \to p^* G_0$ with $G_0, G_1 \in \mathcal{C}$ we deduce that

$$\text{Hom}_{\mathcal{C}_S}(F, F') \simeq \text{Hom}_{A\text{-mod-}\mathcal{C}_{qc}}(p_* F, p_* F')$$

for $F, F' \in \mathcal{C}_S$. It is also clear that every finitely presented $A$-module in $\mathcal{C}_{qc}$ is in the essential image of the functor $p_*$, so our assertion follows.

3.4. Localization. Let $f: X \to S$ be a morphism, where $S$ is quasiprojective. As an application of our techniques, we show that some $t$-structures on $D(X)$ naturally give rise to new $t$-structures on $D(X)$, local over $S$.

Theorem 3.4.1. Let $L$ be an ample line bundle on $S$, and let $(D^{\leq 0}(X), D^{\geq 0}(X))$ be a nondegenerate close to Noetherian $t$-structure on $D(X)$ such that tensoring with $f^* L$ is right $t$-exact, i.e., $f^* L \otimes D^{\leq 0}(X) \subset D^{\leq 0}(X)$. Then there exists a $t$-structure on $D(X)$, local over $S$, given by

$$D_{f,a}^{[a,b]}(X) = \{ F \in D(X) : F \otimes f^* L^n \in D_{qc}^{[a,b]}(X) \text{ for all } n \gg 0 \}. \quad (3.4.1)$$

We also have

$$D_{f,a}^{[a,b]}(X) = \{ F : j_* j^* F \in D_{qc}^{[a,b]}(X) \text{ for every open } U \subset S, \text{ where } j : f^{-1}(U) \hookrightarrow X \}. \quad (3.4.2)$$

If the original structure is Noetherian then so is the new one.

Proof. First, let us check that (3.4.1) is a $t$-structure on $D(X)$, local over $S$. Let us denote the right-hand side of (3.4.1) by $D_{f,a}^{[a,b]}(X)$. We claim that it is enough to prove that $D_{f,a}^{[a,b]}(X)$ is a $t$-structure for some $d > 0$. Indeed, by Theorem 2.3.2 this $t$-structure is local over $S$. Hence, it is stable under tensoring with $f^* L$, so we have $F \in D_{f,L,a}^{[a,b]}(X)$ if and only if $F \otimes f^* L^i \in D_{f,L,a}^{[a,b]}(X)$ for $i = 0, \ldots, d-1$. Therefore,

$$D_{f,a}^{[a,b]}(X) = D_{f,L,a}^{[a,b]}(X),$$

and our claim follows. Thus, we can assume that $L$ is very
ample. Let \( i : S \to \mathbb{P}^r \) be the locally closed embedding such that \( i^* \mathcal{O}_{\mathbb{P}^r}(1) = L \).

Since the right-hand-side of (3.4.1) depends only on \( f^* L \), it is enough to prove that (3.4.1) gives a \( t \)-structure after replacing the data \((S, f, L)\) with \((\mathbb{P}^r, i \circ f, \mathcal{O}_{\mathbb{P}^r}(1))\).

Thus, we can assume from the beginning that \( S = \mathbb{P}^r \) and \( L = \mathcal{O}_{\mathbb{P}^r}(1) \). Consider the closed embedding \( i = (\text{id}, f) : X \to X \times \mathbb{P}^r \). We claim that under our assumptions the functor of tensoring with \( i_* \mathcal{O}_X \) on \( D(X \times \mathbb{P}^r) \) is right \( t \)-exact with respect to the constant \( t \)-structure. Indeed, by (3.3.2) it suffices to prove the inclusions

\[ i_* \mathcal{O}_X \otimes (D^{\leq 0}(X) \boxtimes \mathcal{O}_{\mathbb{P}^r}(n_0)) \subset D^{\leq 0}(X \times \mathbb{P}^r) \]

for all \( n_0 \in \mathbb{Z} \). For \( F \in D^{\leq 0}(X) \) we have

\[ i_* \mathcal{O}_X \otimes (F \boxtimes \mathcal{O}_{\mathbb{P}^r}(n_0)) \simeq i_*(F \otimes f^* \mathcal{O}_{\mathbb{P}^r}(n_0)). \]

Tensoring this with \( \mathcal{O}_{\mathbb{P}^r}(n) \) and pushing forward to \( X \) gives \( F \otimes f^* \mathcal{O}_{\mathbb{P}^r}(n_0 + n) \) which belongs to \( D^{\leq 0}(X) \) for \( n \geq 0 \) by the assumption. Note also that \( i \) has finite Tor dimension since \( \mathbb{P}^r \) is smooth. Thus, we can apply Proposition 2.2.1. The induced \( t \)-structure on \( D(X) \) will be given by (3.4.1).

From (3.4.1) we deduce that the push-forward with respect to the closed embedding \((\text{id}, f) : X \to X \times S\) is \( t \)-exact with respect to our new \( t \)-structure on \( D(X) \) and the constant \( t \)-structure \((D^{\leq 0}(X \times S), D^{\geq 0}(X \times S))\) on \( D(X \times S) \) induced by the old \( t \)-structure on \( D(X) \). Hence,

\[ D^{[a,b]}_f(X) = \{ F \in D(X) : (\text{id}, f)_* F \in D^{[a,b]}(X \times S) \}. \]

Now (3.4.2) follows from (3.3.8).

**Remark.** The assumption \( f^* L \otimes D^{\leq 0}(X) \subset D^{\leq 0}(X) \) in the above theorem is equivalent (by passing to right orthogonals) to the condition \( f^* L^{-1} \otimes D^{\geq 0}(X) \subset D^{\geq 0}(X) \). Therefore, the formula for \( D^{[a,b]}_f(X) \) can be rewritten as

\[ D_f^{[a,b]}(X) = \{ F \in D(X) : F \otimes f^* L^n \in D^{\geq 0}(X) \text{ for all } n \in \mathbb{Z} \}, \]

so that \( D_f^{[a,b]}(X) = \bigcap_{n \in \mathbb{Z}} D_n^{\geq 0}(X) \), where \( D_n^{\geq 0}(X) = f^* L^{-n} \otimes D^{\geq 0}(X) \). Note that we have a chain of inclusions

\[ \ldots \supset D_n^{\geq 0} \supset D_{n+1}^{\geq 0} \supset \ldots \]

Thus, Theorem 3.4.1 can be viewed as an example of a “limiting” \( t \)-structure, like Corollary 1.2.2. Note also that if we apply this theorem to the glued \( t \)-structures \((D_n^{\leq 0}(X \times \mathbb{P}^r), D_n^{\geq 0}(X \times \mathbb{P}^r))\) on \( D(X \times \mathbb{P}^r) \) (associated with a \( t \)-structure on \( D(X) \), see Section 3.3) and take \( L = \mathcal{O}_{\mathbb{P}^r}(1) \) then it will produce the constant \( t \)-structure on \( D(X \times \mathbb{P}^r) \).

In the case when \( X \) is quasiprojective and \( f \) is the identity morphism the above construction produces a local \( t \)-structure on \( D(X) \).

**Corollary 3.4.2.** Assume that \( X \) is quasiprojective. Let \( L \) be an ample line bundle on \( X \). Let also \((D^{\leq 0}(X), D^{\geq 0}(X))\) be a nondegenerate close to Noetherian \( t \)-structure on \( D(X) \) such that \( L \otimes D^{\leq 0}(X) \subset D^{\leq 0}(X) \). Then for every smooth point \( x \in X \) the structure sheaf \( \mathcal{O}_x \) has only one nonzero cohomology object with respect to \((D^{\leq 0}(X), D^{\geq 0}(X))\).
Proof. By Theorem 2.3.5 the corresponding local t-structure \((D_{\text{id}}^{\leq 0}, D_{\text{id}}^{\geq 0})\) is compatible with some t-structure on \(D(x)\) for every smooth point \(x \in X\) (so that the push-forward functor with respect to the embedding \(x \hookrightarrow X\) is t-exact). Hence, \(\mathcal{O}_x\) has only one nonzero cohomology with respect to \((D_{\text{id}}^{\leq 0}, D_{\text{id}}^{\geq 0})\). It remains to use the fact that \(L^n \otimes \mathcal{O}_x \simeq \mathcal{O}_x\).

\[\square\]

Remark. The condition that \(L \otimes D^{\leq 0}(X) \subset D^{\leq 0}(X)\) is crucial in the above corollary. Without it the assertion may be wrong even for a nondegenerate Noetherian t-structure. For example, let \(X\) be a K3 surface, \(C \subset X\) a \((-2)\)-curve. Take \((D^{\leq 0}(X), D^{\geq 0}(X))\) to be the image of the standard t-structure under the reflection functor \(T_{\mathcal{D}_C}^{-1}\) (see [16]). Then for \(x \in C\) the structure sheaf \(\mathcal{O}_x\) will have two nontrivial cohomology objects.

3.5. Invariance under a connected group of autoequivalences. In this section we assume that our ground field \(k\) is algebraically closed.

Recall that if \(K \in D(X \times Y)\) is an object of finite Tor-dimension, such that its support is proper over \(Y\), then it induces an exact functor

\[\Phi_K : D(X) \to D(Y),\quad F \mapsto p_{2*}(p_1^*F \otimes K),\]

where \(p_i\) are the projections from \(X \times Y\) to its factors. We say that \(K\) is the kernel giving the functor \(\Phi_K\). It follows from the theorem of Orlov in [13] that if \(X\) is a smooth projective variety then every exact autoequivalence of \(D(X)\) is given by some kernel.

Let us denote by \(\text{Autoeq}\ D(X)\) the group of (isomorphism classes of) exact autoequivalences of \(D(X)\). By an action of a group \(G\) on \(D(X)\) we mean a homomorphism \(G \to \text{Autoeq}\ D(X), g \mapsto \Phi_g\). In the case when \(G\) is an algebraic group there is a natural way to strengthen this definition by requiring the existence of a family of kernels.

Definition. We say that an algebraic group \(G\) acts on \(D(X)\) by kernel autoequivalences if we are given a homomorphism \(G \to \text{Autoeq}\ D(X), g \mapsto \Phi_g\), and an object \(K \in D(G \times X \times X)\) of finite Tor dimension with the support proper over \(G \times X\) (with respect to the projection \(p_{13}\)), such that for every \(g \in G(k)\) we have \(\Phi_g = \Phi_{K_g}\), where \(K_g = K|_{g \times X \times X}\).

For example, the Poincaré line bundle \(\mathcal{P}\) on \(\text{Pic}^0(X) \times X\) gives rise to an action of \(\text{Pic}^0(X)\) on \(D(X)\) by kernel autoequivalences. Namely, we should take \(K = (\text{id} \times \Delta) \circ \mathcal{P} \in D(\text{Pic}^0(X) \times X \times X)\), where \(\Delta : X \to X \times X\) is the diagonal.

Theorem 3.5.1. Let \((D^{\leq 0}(X), D^{\geq 0}(X))\) be a nondegenerate Noetherian t-structure on \(D(X)\). Assume that a connected smooth algebraic group \(G\) acts on \(D(X)\) by kernel autoequivalences. Then \((D^{\leq 0}(X), D^{\geq 0}(X))\) is invariant under this action.

Proof. Let \(\mathcal{C} = D^{\leq 0}(X) \cap D^{\geq 0}(X)\). For \(F \in \mathcal{C}\) consider the object \(K \ast F \in D(G \times X)\) defined by

\[K \ast F = p_{13*}(K \otimes p_2^*F),\]

where \(p_{ij}\) and \(p_i\) are projections from \(G \times X \times X\). \(K\) is the kernel defining the action of \(G\). Then \((K \ast F)|_{e \times X} \simeq \Phi_E(F)\). In particular, \((K \ast F)|_{e \times X} \simeq F \in \mathcal{C}\), where \(e \in G\) is the neutral element. By the open heart property (see Proposition
2.3.7) this implies that there exists an open neighborhood $U$ of $e$ in $G$ such that $(K^*F)_{U \times X}$ belongs to the heart of the constant $t$-structure on $D(U \times X)$. Since for any $g \in U$ the restriction functor $D(U \times X) \to D(\{g\} \times X)$ is right $t$-exact (as the left adjoint to the $t$-exact push-forward functor), this implies that $\Phi_g(F) \in D^{\leq 0}(X)$ for all $g \in U$. Thus, the functors $\Phi_g$ are right $t$-exact for all $g \in U$. It follows that for $g \in U \cap U^{-1}$ the functors $\Phi_g$ are $t$-exact. Hence, the set of $g$ such that $\Phi_g$ is $t$-exact is an open subgroup in $G$, so it is equal to $G$.

\textbf{Corollary 3.5.2.} Assume $X$ is smooth and projective. Let $\Sigma$ be a connected component in the space of numerical stability conditions on $D(X)$ such that the corresponding subspace $V(\Sigma) \subset (N(X) \otimes \mathbb{C})^*$ is defined over $\mathbb{Q}$ (see [8]). Then any stability in $\Sigma$ is invariant under the action of a connected group of kernel autoequivalences.

\textbf{Proof.} Indeed, for a stability with Noetherian $\mathcal{P}(0, 1]$ this follows from Theorem 3.5.1. Since the set of such stabilities is dense in $\Sigma$ and autoequivalences act by isometries, the general case follows.

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