CLOSED GEODESICS ON REGULAR POLYHEDRA

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To Askold Khovanskii, a dear friend and an admired mathematician

ABSTRACT. We give a description of closed geodesics, both self-intersecting and non-self-intersecting, on regular tetrahedra, cubes, octahedra and icosahedra.


Key words and phrases. Regular polyhedra, closed geodesics, simple geodesics, prime geodesics.

1. Introduction

Let $S$ be a polyhedral surface in $\mathbb{R}^3$ homeomorphic to a sphere. The Gaussian curvature of $S$ is concentrated in the vertices. The curvature of a vertex $v$ is $2\pi - (\theta_1 + \cdots + \theta_r)$ where $\theta_1, \ldots, \theta_r$ are planar angles at $v$. The Absolute Gauss–Bonnet theorem states that the sum of all the curvatures at the vertices equals $4\pi$.

A geodesic on $S$ is a locally shortest curve not passing through the vertices (the second condition follows from the first one if the surface is convex). Obviously, within any face, any geodesic is straight, and at the point of intersection with an edge, a geodesic forms equal angles with the edge in the two adjacent faces. (Conversely, a polygonal line with these properties is a geodesic.)

If a geodesic $\gamma$ is closed and non-self-intersecting, then the Relative Gauss–Bonnet Theorem states that the sum of the curvatures of vertices within any of two pieces into which $\gamma$ cuts $S$ is equal to $2\pi$. Since for a generic polyhedron no subset of the set of vertices has the total curvature of $2\pi$, generic polyhedra have no closed non-self-intersecting geodesics (this argument has been developed by G. Galperin, [2]). It seems plausible that a polyhedron should be sufficiently symmetric to have a rich family of closed geodesics. This is why we study here the most symmetric case: the case of regular polyhedra.

The case of a regular tetrahedron is very simple, and the result is not new [1]: all closed geodesics are non-self-intersecting, their lengths are unbounded (Section 3). In the case of a cube (Section 4), we give a full description of non-self-intersecting closed geodesics (this description is also contained in [1]) and an almost full description of all closed geodesics (with one natural question remaining open—see Remark 4.5). For regular octahedra (Section 5) we give a full description of both

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self-intersecting and non-self-intersecting geodesics. For regular icosahedra (Section 6) we have only partial result including a description of non-self-intersecting geodesics (the latter was done independently by S. Orevkov [3]). For the case of regular dodecahedra, we have almost nothing to say, so we prefer to keep silence. Section 2 addresses some technical points regarding closed and non-closed geodesics on an arbitrary polyhedron.

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2. Development of a Polyhedron Along a Geodesic

Let a geodesic on a polyhedron $S$ start at a point $X$ of an edge $AB$ and then go into a face $ABC \ldots$ of the polyhedron. Let it arrive at some point $Y$ at another edge of the same face, $DE$, and then pass to a new face, $DEF \ldots$. Draw the faces $ABC \ldots$ and $DEF \ldots$ sharing the edge $DE$ in a plane, and continue doing this along our geodesic (Figure 1).

If the polyhedron is convex, we can visualize this process as a rolling of the polyhedron along the plane in such a way that the geodesic always touches the plane. The development of the polyhedron shown in Figure 1 consists of the traces of the faces of the polyhedron and the geodesic. The geodesic becomes a straight line, and it is closed if this straight line arrives at the initial face $ABC \ldots$, at the point $X'$ at the same position as $X$ on the edge $AB$, and the new face $ABC \ldots$ is the translation image of the old one.

![Figure 1](image-url)

Remark 2.1. It is clear (and also shown in Figure 1) that every closed geodesic on a polyhedron belongs to a family of “parallel” geodesics which travel through the same faces and have the same length.
3. The Case of a Regular Tetrahedron

This case is very simple. Label the vertices of the tetrahedron $A, B, C, D$ (Figure 3, left). Let a geodesic start at a point on $AB$ and proceed into the face $ABC$. The development of a tetrahedron along this geodesic will be contained in the standard triangular tiling of the plane. Moreover, one can label the vertices of the tiling the same letters $A, B, C, D$ in such a way that for any development the labelling of vertices of the tetrahedron will match the labelling of vertices of the tiling (Figure 2). (This is something that will not hold for other types of regular polyhedra.)

To construct a closed geodesic, it is sufficient to choose in the tiling two identically oriented edges, say, $AB$, not on the same line, and to join two generic points $X, X'$ of these edges at the same distance from $A$ by a straight segment (the points are generic in the sense that the segment $XX'$ does not pass through any vertex of the tiling); this will produce a closed geodesic on the tetrahedron, and every closed geodesic can be obtained in this way.

![Figure 2.](image)

If the coordinates (in the plane) of the vertices $A, B, C$ of the first face visited by our geodesic are $(0, 0), (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, then the coordinates of the points $X$ and $X'$ are $(\alpha, 0), (\alpha + q + 2p, q\sqrt{3})$ where $0 < \alpha < 1$. (In Figure 2, $p = 2, q = 3$.) If $(p, q) = 1$, then the geodesic is prime (does not repeat itself several times).

**Proposition 3.1.** Any geodesic on a regular tetrahedron is non-self-intersecting.

**Proof.** Two identically labelled triangles of the tiling of Figure 2 are either parallel or obtained from each other by rotation by $180^\circ$. The latter does not change
directions of the lines. Thus, segments of any geodesic (closed or not) within each face are parallel to each other.\end{proof}

Thus, the regular tetrahedron contains simple non-self-intersecting geodesics of an arbitrarily large length. A geodesic corresponding to the segment $XX'$ of Figure 2 is shown in Figure 3. It cuts the tetrahedron into two pieces (also shown in Figure 3), each contains two vertices. All closed geodesics on a regular tetrahedron have similar appearance.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{tetrahedron.png}
\caption{Figure 3.}
\end{figure}

Remark 3.2. A regular tetrahedron is double covered, with 4 branching points $A, B, C, D$, by a flat torus obtained by the usual identification from the parallelogram shown in Figure 4. This explains why the behavior of geodesics on a tetrahedron is similar to behavior of geodesics on a flat torus not passing through 4 marked points $A, B, C, D$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{torus.png}
\caption{Figure 4.}
\end{figure}

Remark 3.3. Besides tetrahedra with equal faces and equal opposite edges there are no polyhedra with arbitrarily long prime non-self-intersecting geodesics.
4. The Case of a Cube

Label the vertices of a cube $A, B, C, D, A', B', C', D'$ (see Figure 5). Consider a geodesic starting at a point $X$ of the edge $AB$ and proceeding into the face $ABCD$. The development of a cube along this geodesic is a part of the standard tiling of the plane by squares, and the vertices of squares along the geodesic get labels (see Figure 6).

It is impossible, however, to label of all the vertices of the tiling by letters $A, B, \ldots, C', D'$ in such a way that the labels along the geodesic match this labelling. For example, Figure 7 shows labellings corresponding to two parallel geodesics starting at two different points of the edge $AB$.

Let the coordinates of the vertices $A, B, D$ of the initial square of the tiling be $(0, 0), (1, 0), (0, 1)$ (we will use these coordinates throughout the section). The geodesic segments shown in Figure 7 join the points $(\alpha, 0)$ and $(1 + \alpha, 2)$. If $\alpha < \frac{1}{2}$ (Figure 7, left), then we first roll the cube about the edge $CD$; the points $(0, 2)$ and $(1, 2)$ become $D'$ and $C'$. Then we roll the cube about $CC'$, and the points $(2, 1)$ and $(2, 2)$ become $B$ and $B'$. If $\alpha > \frac{1}{2}$ (Figure 7, right), the we first roll the cube about $BC$ (the points $(2, 0), (2, 1)$ become $B, C'$) and then about $CC'$, and the points $(1, 2), (2, 2)$ become $D, D'$. We see that the final result depends on $\alpha$. 

![Figure 5](image5.png)  
![Figure 6](image6.png)  
![Figure 7](image7.png)
To get a closed geodesic on the cube, we must join the points $X_1(\alpha, 0)$ and $X_2(p + \alpha, q)$ (with $0 < \alpha < 1$) by a straight interval not passing through points with integral coordinates (for this, it is sufficient to assume that $d\alpha \notin \mathbb{Z}$ where $d = \frac{q}{(p,q)}$); then we roll the cube along $X_1X_2$ and label the vertices of the squares hit by this interval accordingly. The interval $X_1X_2$ represents a closed geodesic on the cube, if and only if the points $(p, q), (p + 1, q)$ (surrounding the point $X_2$) acquire the labels $A, B$. Our goal is to find $(\alpha, p, q)$ with this property. From now on, we will always assume that $0 \leq p \leq q, 0 < q$ (the general case can be reduced to this case by reflections in horizontal, vertical, and diagonal lines).

Our previous considerations (Figure 7) show that the triple $(\alpha, 1, 2)$ does not satisfy our condition for any $\alpha$; moreover, we observed that the final results (labels at $(p, q), (p + 1, q)$) may depend on $\alpha$. (More precisely, these labels are the same for $\frac{1}{2} < \alpha < \frac{1 + \sqrt{5}}{2}$, but may be different for different $i$’s.) The next statement shows that the latter cannot happen in the most important case.

**Proposition 4.1.** If for given $p, q$, the equalities $(p, q) = A, (p + 1, q) = B$ hold for some $\alpha$, then they hold for every $\alpha$ ($0 < \alpha < 1$, $d\alpha \notin \mathbb{Z}$, $d = \frac{q}{(p,q)}$).

**Proof.** Let $X_1X_2$, where $X_1 = (\alpha, 0), X_2 = (p + \alpha, q)$, correspond to a closed geodesic. Let $Y_1, Y_2$ be the intersection points of the line $X_1X_2$ with the horizontal lines $y = m, y = m + q (0 \leq m < q)$ (see Figure 8, where $p = 2, q = 3, m = 1$). Then $Y_1Y_2$ also corresponds to a closed geodesic, with the same $p$ and $q$ but a different $\alpha$: the new $\alpha$ is the old $\alpha$ plus the fractional part of $\frac{mp}{q}$. It is clear that when $m$ varies from 0 to $q$, this fractional part assumes values $\frac{r}{d}$ with all possible $r$ mod $d$. Thus, if $X_1X_2$ is a closed geodesic for some $\alpha$, it is a closed geodesic for all $\alpha$. \qed

![Figure 8.](image-url)
Thus, the question is, which pairs \((p, q)\) (where, as before, \(0 \leq p \leq q\), \(0 < q\) correspond to prime (that is, maybe, self-intersecting, but not repeating itself more than one time) closed geodesics. Before addressing this question we will state two simpler propositions.

**Proposition 4.2.** Within every face, a geodesic may have at most two directions, and these directions are perpendicular to each other.

*Proof.* In the development of a geodesic, every face may appear in 4 positions obtained from each other by rotation by a multiple of \(90^\circ\). These rotations either preserve the direction of a line or turn it into the perpendicular direction. \(\square\)

**Proposition 4.3.** If \((p, q)\) corresponds to a closed geodesic and \(p + q \geq 7\), then the geodesic is self-intersecting.

*Proof.* The segment \(X_1X_2\) (not counting the points \(X_1\) and \(X_2\)) crosses \(p - 1\) horizontal lines and \(q\) vertical lines. Hence, the geodesic crosses edges at \((p - 1) + q + 1\) points (the last 1 stands for \(X_1\) and \(X_2\)) and hence it visits \(p + q\) faces. If \(p + q \geq 7\), it must visit at least one face twice. If the two segments on a face are perpendicular to each other, they either cross within the face, or have ends on the same edge, say, \(IJ\). Then they meet within the other face with the edge \(IJ\) (see Figure 9, left). If they are parallel, we follow them until they have a vertex, say, \(L\), between them (they must have a vertex between them by the Gauss–Bonnet Theorem, see Section 1). Then they meet within one of two other faces attached to the vertex \(L\) (see Figure 9, right). \(\square\)

![Figure 9](image)

Now we can state the main result of this section. For brevity, we call a pair \((p, q)\), \(0 \leq p \leq q\), \(q > 0\) good, if for some \(\alpha\) (and then for any \(\alpha\) such that \(\frac{\alpha q}{(p, q)} \notin \mathbb{Z}\)) the segment \([\alpha, 0), (p + \alpha, q)]\) corresponds to a prime closed geodesic on the cube.

**Theorem 4.4.** (1) For any pair of integers \((p, q)\) such that \(0 \leq p \leq q\), \(q > 0\), \((p, q) = 1\), there exists a unique positive integer \(k\) such that \((kp, kq)\) is a good pair.

(2) If \(p, q\) are both odd, then \(k = 3\).

(3) If one of \(p, q\) is even, then \(k = 2\) or 4.
Remark 4.5. The statement (3) looks inconclusive. It will be seen from the proof that the cases \( k = 2 \) and \( k = 4 \) can be distinguished by a relatively simple condition formulated in terms of the group \( S(4) \) of symmetries of the cube, but one can expect that there is a more direct condition formulated in terms of arithmetic properties of \( p \) and \( q \). A simple computer program yields, in no time, a list of pairs with \( k = 2 \) and \( k = 4 \), as long as one could wish. To get the reader interested, we provide here a beginning of this list. Of 79 pairs \( (p, q) \) with \( 0 \leq p \leq q \), \( 0 \leq q < 20 \), \( (p, q) = 1 \), \( pq \equiv 0 \mod 2 \), 25 pairs have \( k = 2 \) (and the remaining 54 have \( k = 4 \)). Here are these 25 pairs:

\[
(1, 2), \quad (1, 6), \quad (1, 10), \quad (1, 14), \quad (1, 18), \quad (2, 7), \quad (2, 9), \\
(2, 15), \quad (2, 17), \quad (4, 5), \quad (4, 11), \quad (5, 16), \quad (7, 8), \quad (7, 10), \\
(7, 18), \quad (8, 11), \quad (8, 13), \quad (8, 19), \quad (9, 14), \quad (10, 11), \quad (10, 17), \\
(12, 19), \quad (13, 14), \quad (16, 17), \quad (16, 19).
\]

Proof of Theorem 4.4. Let \( T_1, T_2 \) be the automorphisms of the cube shown in Figure 10 (rolling over the edges \( DC \) and \( BC \)). The canonical identification of the group of symmetries of the cube with the group \( S(4) \) (of permutations of the diagonals \( AC', BD', CA', DB' \)) takes \( T_1 \) and \( T_2 \) to the permutations \( \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \) which we also denote by \( T_1 \) and \( T_2 \). For given relatively prime numbers \( p, q \) consider the fractions \( \frac{1}{p}, \ldots, \frac{p-1}{p} \) and \( \frac{1}{q}, \ldots, \frac{q-1}{q} \) and arrange them in the increasing order

\[ 0 < r_1 < r_2 < \cdots < r_{p+q-2} < 1. \]

Figure 10.

For \( i = 1, 2, \ldots, p + q - 2 \) set

\[ c(i) = \begin{cases} 
1, & \text{if } pr_i \in \mathbb{Z}, \\
2, & \text{if } qr_i \in \mathbb{Z}.
\end{cases} \]

Then define

\[ T_{p,q} = T_{c(1)}T_{c(2)} \cdots T_{c(p+q-2)}T_2T_1. \]

(Our definition will not work for \( p = 0, q = 1 \); we set \( T_{0,1} = T_1 \).) This is the total rolling of the cube along the geodesic \( [(\alpha, 0), (p + \alpha, q)] \) for very small \( \alpha \) (\( T_1 \)'s and \( T_2 \)'s correspond to the intersections of this interval with the horizontal and vertical
lines of the tiling; the last two correspond to the last two intersections near the point \((p, q)\).

Obviously, the pair \((kp, kq)\) is good if and only if \(k\) is the order of \(T_{p,q}\) in \(S(4)\). This proves Part (1) of the Theorem.

All elements of \(S(4)\) have orders 1, 2, 3, 4. If one of \(p, q\) is even, then \(p + q\) is odd, and, since the permutations \(T_1\), \(T_2\) are odd, the permutation \(T_{p,q}\) is also odd. Hence, its order is even, that is, 2 or 4. This proves Part (3).

Consider now the homomorphism of \(S(4)\) onto \(S(3)\) (in terms of cube automorphisms, it assigns to an automorphism of a cube the corresponding permutation of \(x, y,\) and \(z\)-directions). This homomorphism takes \(T_1\) and \(T_2\) into \(t_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\) and \(t_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\).

Now notice that the sequence \(r_1, \ldots, r_{p+q-2}\) is symmetric (that is, \(1 - r_i = r_{p+q-1-i}\)), and hence the sequence \(c(i)\) is symmetric \((c(i) = c(p+q-1-i)\). If \(p\) and \(q\) are both odd, then \(p + q - 2\) is even, and hence

\[t_{c(1)} \cdots t_{c(p+q-2)} = \left(t_{c(1)} \cdots t_{c(p+q-2)}\right)\left(t_{c(p+q-2)} \cdots t_{c(1)}\right)\]

and the two products in the parentheses are inverse to each other (since one is a mirror reflection of the other, and \(t_i = t_i^{-1}\)). Thus the homomorphism \(S(4) \rightarrow S(3)\) annihilates \(T_{c(1)} \cdots T_{c(p+q-1)}\) and therefore takes \(T_{p,q}\) into \(t_2 t_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\). The latter has order 3 in \(S(3)\), hence it the order of \(T_{p,q}\) is divisible by 3, hence it is equal to 3. This proves Part (2).

**Corollary 4.6** (cf. [1]). Up to parallelism and cube symmetries, there are precisely three non-self-intersecting closed geodesics on a cube. They correspond to the good pairs \((0, 4), (3, 3),\) and \((2, 4)\).

These three geodesics are shown in Figure 11. Two of these closed geodesics are planar: these are a square section of the cube by a plane parallel to a pair of parallel faces and a hexagonal section of the cube by a plane perpendicular to the long diagonal. The squares of their lengths are 16 and 18 (if the length of the edge of the cube is taken for 1). The third one is not planar (and by this reason less obvious). The square of its length is 20.

![Figure 11](image-url)

The next four shortest geodesics correspond to the good pairs \((3, 9), (2, 12), (4, 14), (8, 12)\) (so, the squares of the lengths are 90, 148, 208, 212). They are shown in Figure 12.
Two longer geodesics, corresponding to the pairs (21, 39) and (33, 63) are shown in Figure 13.

5. The Case of an Octahedron

The word octahedron refers below to a fixed regular octahedron.

The faces of the octahedron can be painted black and white in such a way that adjacent faces always have opposite colors. We label the vertices of the octahedron by (unordered) pairs 12, 13, 14, 23, 24, 34; two vertices are joined by an edge if they share a digit, and three vertices belong to one face if either they all share a digit (a white face) or they all avoid some digit (a black face). (See Figure 14.)

A development of the octahedron along any geodesic is a part of the standard triangular tiling (Figure 2) but we can furnish this tiling by a black and white coloring (Figure 15, left), and the colors of the faces will match the colors of the tiles for any development. We can transform this tiling into a square tiling as shown in Figure 15, right. A closed geodesic on the octahedron, as in the case of a cube, corresponds to a segment \([\alpha, 0], (\alpha + p, q)\] where we can assume that \(0 \leq p \leq q, 0 < q\). (We still have to find out which pairs \((p, q)\) correspond to closed geodesics.) Notice that the coordinates of the second end of the segment above with respect to the triangular tiling are \((\alpha + \frac{p}{2} + p, \frac{\alpha + p}{2})\); this shows that the length of a geodesic corresponding to the pair \((p, q)\) is \(\sqrt{p^2 + pq + q^2}\).
Propositions 4.1–4.3 hold in the case of the octahedron with minor changes. A proposition similar to Proposition 4.1 states that the fact that a segment $[(\alpha, 0), (\alpha + p, q)]$ corresponds to a closed geodesic depends on $p$ and $q$, but not on $\alpha$; the proof is the same as in the case of the cube. An octahedral version of Proposition 4.2 states that a geodesic can have, within any face, at most three directions, forming the angles $60^\circ$ and $120^\circ$. And the last of the three propositions states that if $p + q \geq 5$, then a prime closed geodesic must be self-intersecting (the geodesic corresponding to the segment above crosses edges $2(p + q)$ times, if $p + q > 4$, it should visit at least one face at least twice, and the rest of the proof is similar to that of Proposition 4.4).
As in the case of the cube, we call a pair \((p, q)\) good if it corresponds to a prime geodesic (that is, maybe, self-intersecting but not repeating itself).

**Theorem 5.1.** (1) For every pair \((p, q)\) of integers such that \(0 \leq p \leq q, 0 < q, \) and \((p, q) = 1,\) there exists a unique positive integer \(k\) such that the pair \((kp, kq)\) is good.

(2) If \(p \equiv q \mod 3,\) then \(k = 2.\)

(3) If \(p \not\equiv q \mod 3,\) then \(k = 3.\)

**Proof.** Transition from a square to a square along the segment \([(\alpha, 0), (\alpha + p, q)]\) provides a combination of two rollings of the octahedron, and this combination, followed by a parallel translation, is an automorphism of the octahedron preserving the orientation and also the color of the faces. There are 12 such automorphisms, and they comprise a group isomorphic to \(A(4):\) a permutation \(\sigma \in A(4)\) takes the vertex \((ij)\) into the vertex \((\sigma(i)\sigma(j)).\) For \(p \leq q,\) there are only two types of these permutations, \(U_1\) and \(U_2,\) they are shown in Figure 16.

\[
\begin{align*}
12 & \rightarrow 14 \\
13 & \rightarrow 34 \\
14 & \rightarrow 24
\end{align*}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 3 & 2
\end{pmatrix}
\]

\(U_1\)

\[
\begin{align*}
12 & \rightarrow 13 \\
13 & \rightarrow 23 \\
14 & \rightarrow 34
\end{align*}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & 1 & 2 & 4
\end{pmatrix}
\]

\(U_2\)

**Figure 16.**

If \((p, q) = 1,\) then the total rolling of the octahedron along our segment amounts to

\[U_{p,q} = U_{c(1)} \cdots U_{c(p+q)} U_2 U_1,\]

where \(c(1), \ldots, c(p+q-2)\) are defined precisely as in Proof of Theorem 4.4. Obviously, \((kp, kq)\) is a good pair, if and only if \(k\) is the order of \(U_{p,q}\) in \(A(4).\) This proves Part (1) and establishes that \(k = 1, 2, \) or \(3.\)

The projections of \(U_1\) and \(U_2\) into \(A(3) = \mathbb{Z}_3\) are \(u_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \) and \(u_2 = u_1^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.\) The image of \(U_{p,q}\) in \(A(3)\) is \(u_{p,q} = u_1^{q-p}.\) If \(p \not\equiv q \mod 3,\) then \(u_{p,q}\) has order 3. In this case \(U_{p,q}\) must also have order 3, which proves part (3).

To prove Part (2), let us observe the following relations between \(U_1\) and \(U_2:\)

\[U_1^3 = 1, \quad U_2^3 = 1, \quad U_1 U_2 U_1 = U_2^{-1}, \quad U_2 U_1 U_2 = U_1^{-1},\]
and hence
\[(U_1U_2)^2 = (U_2U_1)^2 = 1.\]
The left relations show that Part (2) of Theorem follows from the following

**Lemma 5.2.** If \( p \equiv q \mod 3 \), then \( U_{p,q} = U_1U_2 \) or \( U_2U_1 \).

**Proof of Lemma.** \( U_{p,q} = V_{p,q}U_1 \), where \( V_{p,q} = U_{(i)} \ldots U_{c(p+q-2)} \) is a symmetric word (palindrome) in \( U_1 \) and \( U_2 \). We want to prove that \( (\text{if } p \equiv q \mod 3) \) \( V_{p,q} \)
is equal to 1, or \( (U_1U_2^{-1}U_1)^{\pm 1} \), or \( (U_2U_1^{-1}U_2)^{\pm 1} \). First, using the relations \( U_1^3 = 1 \), \( U_2^2 = U_1^{-1} \) into a symmetric word of the form
\[
\ldots U_2^{\pm 1}U_1^{\pm 1}U_2^{\pm 1}U_1^{\pm 1} \ldots
\]
(and the total degrees in \( U_1 \) and \( U_2 \) modulo 3 stay unchanged). The number of letters after this transformation will be 0 or even. If the total number of letters is more than 1, then the middle 3-letter section will be one of \( (U_1U_2U_1)^{\pm 1} \), \( (U_2U_1U_2)^{\pm 1} \);
\( (U_1^{-1}U_2U_1^{-1})^{\pm 1} \), \( (U_2^{-1}U_1U_2^{-1})^{\pm 1} \). In the first two cases, we apply the relations \( U_1U_2U_1 = U_2^{-1} \), \( U_2U_1U_2 = U_1^{-1} \) and reduce the number of letters. In the second two cases, if the total number of letters is more than 3, we also can reduce the number of letters:
\[
U_1(U_1U_2^{-1}U_1)^{-1}U_1 = (U_1U_2U_1)^{-1} = U_2, \quad U_1^{-1}(U_1U_2^{-1}U_1)^{-1} = U_2^{-1},
\]
\[
U_2(U_2U_1^{-1}U_1)^{-1}U_2 = (U_2U_1U_2)(U_2U_1U_2) = U_1^{-1}U_1^{-1} = U_1,
\]
\[
U_2(U_2U_1^{-1}U_1)^{-1}U_2 = U_2(U_2U_1U_2)(U_2U_1U_2)U_2 = U_2U_1^{-1}U_1^{-1}U_2 = U_1^{-1},
\]
and similarly in all other cases. It is important that all these transformations keep \( (\deg U_1 - \deg U_2) \mod 3 \) unchanged.

Thus, our palindrome can be reduced to one of
\[
1, \ (U_1)^{\pm 1}, \ (U_2)^{\pm 1}, \ (U_1U_2^{-1}U_1)^{\pm 1}, \ (U_2U_1^{-1}U_2)^{\pm 1},
\]
and the options \( (U_1)^{\pm 1}, \ (U_2)^{\pm 1} \) are actually excluded, since the degrees in \( U_1 \) and \( U_2 \) should be the same \( \mod 3 \). It remains to observe that
\[
1 \cdot U_2U_1 = U_2U_1,
\]
\[
U_1U_2^{-1}U_1 \cdot U_2U_1 = U_1U_2^{-1}U_2^{-1} = U_1U_2,
\]
\[
U_1^{-1}U_2U_1^{-1} \cdot U_2U_2 = U_1(U_1U_2U_1)(U_1U_2U_1) = U_1U_2^{-1}U_2^{-1} = U_1U_2,
\]
\[
U_1^{-1}U_2U_1^{-1} \cdot U_2U_2 = U_1U_2U_1^{-1}(U_1U_2U_1) = U_1U_2^{-1}U_2^{-1} = U_1U_2,
\]
\[
U_2^{-1}U_1U_2^{-1} \cdot U_2U_1 = U_2^{-1}U_1^{-1} = (U_1U_2)^{-1} = U_1U_2.
\]

This completes the proof of Theorem 5.1.

**Corollary 5.3 (cf. [1]).** Up to octahedral symmetries and parallelism, there are two non-self-intersecting closed geodesics on the octahedron.

They correspond to the pairs \((0, 3)\) and \((2, 2)\); their lengths (if the length of the edge is 1) are 3 and \(2\sqrt{3}\).

The closed geodesics are shown in Figure 17, left and middle. The first of these geodesics is the hexagonal section of the octahedron by a plane parallel to two opposite faces. The second is not planar. Figure 17, right, shows one of the self-intersecting geodesics (this one corresponds to the pair \((4, 10)\)).
6. The Case of Icosahedron

We consider a regular icosahedron (see Figure 18). The group of orientation preserving automorphisms of an icosahedron is $A(5)$. (There exists a five-color coloring of the icosahedron such that no two of the four faces of the same color share a vertex; this coloring can be found in many popular books on geometry. An orientation-preserving automorphism of the icosahedron yields an even permutation of the colors and is fully determined by this permutation.) A development of the icosahedron along a geodesic is a part of the standard triangular tiling. It looks like (any of the two diagrams in Figure 15, only without a black and white (or any other) coloring. Thus there appear again “good pairs” $(p, q)$ corresponding to prime (maybe, self-intersecting but not repeating themselves) closed geodesics; the length of the corresponding closed geodesic is again $\sqrt{p^2 + pq + q^2}$. Similarly to Proposition 4.3 (and its octahedral version), a geodesic must be self-intersecting, if $p + q > 10$.

The classification of good pairs is described, partially, by the following result.

**Theorem 6.1.** For every pair $(p, q)$ of integers such that $0 \leq p \leq q$, $0 < q$, and $(p, q) = 1$, there exists a unique $k = 2, 3, 5$ such that the pair $(kp, kq)$ is good.

The proof follows the patterns of the proofs of Part (1) of Theorems 4.4 and 5.1. The permutations $T_1, T_2$ of Theorem 4.4 and $U_1, U_2$ of Theorem 5.1 should be replaced by permutations $W_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \ 3 & 5 & 2 & 1 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 1 & 2 & 4 & 5 \ 4 & 1 & 5 & 3 \end{pmatrix}$ which represent in the group $A(5)$ the combinations of rollings of the icosahedron described by Figure 16. For a pair $(p, q)$ as in the statement of the Theorem, $(kp, kq)$ is a good pair if and only if $k$ is the order of $W_{p,q} = W_{c(1)} \cdots W_{c(p+q-2)} W_2 W_1$ in $A(5)$. This shows that $k$ is unique and is one of the numbers $1, 2, 3, 5$. The easiest way to see that $1$ is actually impossible is to notice that $W_{p,q}$ is a product of a palindrome and $W_2 W_1$, and $1 \in A(5)$ does not have this form.

Here are the values of $k$ for several small $p$ and $q$:

- $k(0, 1) = 5, k(1, 1) = 3, k(1, 2) = 2, k(1, 3) = 3,$
- $k(1, 4) = 5, k(1, 5) = 5, k(1, 6) = 3, k(2, 3) = 5,$
- $k(2, 5) = 5, k(3, 4) = 5, k(3, 5) = 3, k(4, 5) = 2.$
Thus, the three shortest closed geodesics correspond to the pairs (0, 5), (3, 3), and (2, 4); the squares of their lengths are 25, 27, and 28. These geodesics are non-self-intersecting; they are presented in Figure 18. The shortest of them is a decagonal plane section perpendicular to the longest diagonal, the other two are non-planar. The next shortest geodesic corresponds to the pair (3, 9). It is interesting that it has just one self-intersection. Other closed geodesics are substantially longer and have multiple self-intersections.

REFERENCES


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