Brian Conrad kindly pointed out to us that the proof of Proposition 9.8 in the article in question is incomplete. We provide here the missing arguments together with a few other corrections and use the opportunity to indicate some new consequences of our results, and also mention some applications of the results in [S1]. In what follows, the supplementary references, including the original paper itself, are numbered as [S1], [S2], etc., while citations such as [1] refer to those in [S1]. Lemmas, propositions, etc., numbered such as 2.1, 8.4, &c., correspond to those in [S1]. A revised version of [S1] incorporating the corrections in this note is available as arXiv:0808.2169 [math.AG].

**Base Field**

Usually, at the beginning of each section of [S1], the assumptions on the base field \( k \) are specified. In addition, the following modifications are in order.

- In Statements 2.1, 2.4, 2.5 and 2.6, one should mention explicitly that \( X \) and \( Y \) are defined over \( k \).
- In Remark 2.7, one has to assume that there is a proper linear section of codimension \( s + 1 \) of \( X \) defined over \( k \).

If \( k \) is algebraically closed, the Galois group \( g \) is trivial, and these conditions are fulfilled.

Further, the proof of part (i) of Prop. 8.7 uses Corollary 1.4 and it should be modified as follows:

- Take an extension \( k' / k \) in order to get a section \( Y \) defined over \( k' \). This implies that the eigenvalues of the Frobenius of \( k' \) in \( H^{2u-1}(X, \mathbb{Q}_\ell(n)) \) are pure, and the same holds for the eigenvalues of the Frobenius of \( k \), since they are roots of the preceding.
Betti Numbers of Curves

The proof of Lemma 8.4 as given in [S1] is only valid on a finite field. This Lemma and its proof should be stated as follows.

8.4. Lemma. Let $K$ be an algebraically closed field, and $X$ an irreducible projective curve in $\mathbb{P}^N_K$, with arithmetic genus $p_a(X)$. Let $\tilde{X}$ be a nonsingular projective curve birationally equivalent to $X$, with geometric genus $g(\tilde{X})$. Then we have the following.

(i) If $d$ denotes the degree of $X$, then

$$2g(\tilde{X}) \leq b_1(X) \leq 2p_a(X) \leq (d-1)(d-2).$$

(ii) If $K = \bar{k}$, where $k$ is a finite field, and if $X$ is defined over $k$, then

$$b_1^*(X) = 2g(\tilde{X}).$$

During the proof of the Lemma, we shall make use of the following standard construction, when $X$ is a curve. This leads to an inequality between Hilbert polynomials.

8.5. Remark (Comparison of Hilbert polynomials). Let $K$ be an algebraically closed field and $X$ a closed subvariety in $\mathbb{P}^N_K$ disjoint from the whole space, and $r$ an integer such that $\dim X + 1 \leq r \leq N$. Let $\mathcal{C}_r(X)$ be the subvariety of $G_{N-r,N}$ of linear varieties of codimension $r$ meeting $X$. From the properties of the incidence correspondence $\Sigma$ defined by

$$\Sigma = \{(x, E) \in \mathbb{P}^N \times G_{N-r,N} : x \in E\},$$

it is easy to see that $\mathcal{C}_r(X) = \pi_2(\pi_1^{-1}(X))$ is irreducible and that the codimension of $\mathcal{C}_r(X)$ in $G_{N-r,N}$ is equal to $r - \dim X$. Hence, the set of linear subvarieties of codimension $r$ in $\mathbb{P}^N_K$ disjoint from $X$ is a nonempty open subset $D_r(X)$ of $G_{N-r,N}$.

If $E$ belongs to $D_{n+2}(X)$, where $n = \dim X$, the projection $\pi$ with center $E$ gives rise to a diagram

\[\begin{array}{ccc}
X & \xrightarrow{i} & \mathbb{P}^N_K - E \\
\pi\downarrow & & \downarrow \pi \\
X' & \xrightarrow{i'} & \mathbb{P}^{n+1}_K \\
\end{array}\]

such that $X'$ is an irreducible hypersurface with $\deg X' = \deg X$, and where the restriction $\pi_X$ is a finite birational morphism: denoting by $S(X)$ the homogeneous coordinate ring of $X$, we have an inclusion $S(X') \subset S(X)$, and $S(X)$ is a finitely generated module over $S(X')$. Hence, if $P_X(T) \in \mathbb{Q}[T]$ is the Hilbert polynomial of $X$ [16, p. 52], we have

$$P_{X'}(t) \leq P_X(t) \quad \text{if} \ t \in \mathbb{N} \ \text{and} \ t \to \infty.$$
Proof of Lemma 8.4. Let $U$ be a regular open subscheme of $X$. Then, there is a commutative diagram

$$
\begin{array}{ccc}
\tilde{U} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
U & \longrightarrow & X
\end{array}
$$

where $\tilde{X}$ is a nonsingular curve, where $\pi$ is a proper morphism which is a birational isomorphism, and an isomorphism when restricted to $\tilde{U}$, and

$$
\text{Sing } X \subset S = X \setminus U, \quad \tilde{S} = \tilde{X} \setminus \tilde{U}.
$$

The excision long exact sequence in compact cohomology [30, Rem. 1.30, p. 94] gives:

$$
0 \longrightarrow H^0_c(X) \longrightarrow H^0_c(S) \longrightarrow H^1_c(U) \longrightarrow H^1_c(S) \longrightarrow 0
$$

and there is a similar exact sequence if we replace $X, U, S$ by $\tilde{X}, \tilde{U}, \tilde{S}$. This implies

$$
b_1(U) = b_1(X) - 1 + |S|, \quad b_1(\tilde{U}) = b_1(\tilde{X}) - 1 + |\tilde{S}|,
$$

and since $U$ and $\tilde{U}$ are isomorphic, we obtain

$$
b_1(X) = b_1(\tilde{X}) + d(X) = 2g(\tilde{X}) + d(X), \quad \text{where } d(X) = |\tilde{S}| - |S|,
$$

since, as is well-known, $b_1(\tilde{X}) = 2g(\tilde{X})$. Let

$$
\delta(X) = p_a(X) - g(\tilde{X}).
$$

Then $0 \leq d(X) \leq \delta(X)$ [59, Prop. 1, p. 68]. Hence

$$
b_1(X) = 2g(\tilde{X}) + d(X) \leq 2g(\tilde{X}) + 2\delta(X) = 2p_a(X).
$$

This proves the first and second inequalities of (i). The Hilbert polynomial of $X$ is

$$
P_X(T) = dT + 1 - p_a(X).
$$

Apply now the construction of Remark 8.5 to $X$, and obtain a morphism $X \longrightarrow X'$, where $X'$ is a plane curve of degree $d$. From the inequality $P_{X'}(t) \leq P_X(t)$ for $t$ large, we get $p_a(X) \leq p_a(X')$. Now, by Example 4.3(ii),

$$
p_a(X') = (d - 1)(d - 2)/2,
$$

since $X'$ is a plane curve of degree $d$, and so

$$
p_a(X) \leq (d - 1)(d - 2)/2,
$$

and this proves the third inequality of (i). Now, under the hypotheses of (ii), we have by [4, Thm. 2.1]:

$$
P_1(X, T) = P_1(\tilde{X}, T) \prod_{j=1}^{d(X)} (1 - \omega_jT),
$$

where the numbers $\omega_j$ are roots of unity, and this implies the inequality in (ii). □
THE PENULTIMATE COHOMOLOGY GROUP

Let $k$ be a perfect field. Assume, as in Sec. 9 of [S1], that all projective varieties over $k$ considered in this section have a $k$-rational nonsingular point. The proof given in [S1] of Prop. 9.8 could be completed as follows.

9.8. Proposition. Let $X$ be a normal projective variety of dimension $n \geq 2$ defined over $k$ which is regular in codimension 2. Then there is a $g$-equivariant isomorphism

$$j_X : V_\ell(\text{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).$$

If $(R_{n,p})$ holds, the same conclusion is true if one only assumes that $X$ is regular in codimension 1.

Proof. Step 1. Assume that $X$ is a subvariety in $\mathbb{P}_k^N$. Since $X$ is regular in codimension 2, we deduce from Proposition 1.3 and Corollary 1.4 that $U_{n-2}(X)$ contains a nonempty Zariski open set $U_0$ in the Grassmannian $G_{N-n+2,N}$. On the other hand, any open set defined over $k$ is defined over a field $k'/k$. Let $U_1 \subset U_0$ be an open set defined over $k$. If $E \subset U_1$, then $Y = X \cap E$ is a typical surface on $X$ over $k$, i.e., a nonsingular proper linear section of dimension 2 in $X$. For such a typical surface $Y$, the closed immersion $\iota : Y \to X$ induces a homomorphism $\iota_* : \text{Alb}_w Y \to \text{Alb}_w X$. By Proposition 9.4(i), the set of linear varieties $E \subset U_1$ such that $\iota_*$ is a purely inseparable isogeny contains as well as none a nonempty open subset $U \subset G_{N-n+2,N}$. On the other hand, any open set defined over $k$ is defined over a field $k'/k$. Let $U_1 \subset U_0$ be an open set defined over $k$. If $E \subset U_1$, then $Y = X \cap E$ is a typical surface on $X$ over $k$, i.e., a nonsingular proper linear section of dimension 2 in $X$.

Step 2. Assume that $U(k)$ is nonempty. If $E \subset U(k)$, we get a $g$-equivariant nondegenerate pairing $H^1(\bar{Y}, \mathbb{Q}_\ell) \times H^3(\bar{Y}, \mathbb{Q}_\ell(2)) \to \mathbb{Q}_\ell$, from which we deduce a $g$-equivariant isomorphism

$$\psi : \text{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \to H^3(\bar{Y}, \mathbb{Q}_\ell(2)).$$

Since $(X, Y)$ is a semi-regular pair with $Y$ nonsingular, from Corollary 2.1 we know that the Gysin map

$$\iota_* : H^3(\bar{Y}, \mathbb{Q}_\ell(2-n)) \to H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$$

is an isomorphism. Now a $g$-equivariant isomorphism of vector spaces over $\mathbb{Q}_\ell$:

$$j_X : V_\ell(\text{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$$

is defined as the isomorphism making the following diagram commutative:

$$\begin{array}{ccc}
\text{Hom}(V_\ell(\text{Pic}_w Y)(-1), \mathbb{Q}_\ell) & \xrightarrow{\xi} & V_\ell(\text{Alb}_w Y) \\
\downarrow \iota_{\text{by}} & & \downarrow V_\ell(\text{Alb}_w X) \\
\text{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) & \xrightarrow{\psi} & H^3(\bar{Y}, \mathbb{Q}_\ell)(2) \\
\downarrow j_{\Pi} & & \downarrow \iota_* \circ j_X \\
\text{Hom}(H^1(\bar{X}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) & \xrightarrow{j_{\Pi}} & H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).
\end{array}$$
Here $\varpi$ is defined by the Weil pairing, and $^{t}h_Y$ is the transpose of the map $h_Y$ defined in Proposition 9.6. Hence, the conclusion holds if $U(k) \neq \emptyset$.

**Step 3.** Assume that $k$ is an infinite field. One checks successively that if $U$ is an open subset in an affine line, an affine space, or a Grassmannian, then $U(k) \neq \emptyset$ and the conclusion follows from Step 2.

**Step 4.** Assume that $k$ is a finite field. Then the following elementary result holds (as a consequence of Proposition 12.1, for instance).

**Claim.** Let $U$ be a nonempty Zariski open set in $G_{r,N}$, defined over $k$, and $k_s = \mathbb{F}_q^s$, the extension of degree $s$ of $k = \mathbb{F}_q$. Then there is an integer $s_0(U)$ such that $U(k_s) \neq \emptyset$ for every $s \geq s_0(U)$.

Now take for $U$ the open set in $G_{N-n+2,N}$ introduced in Step 1. Choose any $s \geq s_0(U)$, and let $g_s = \text{Gal}(k/k_s)$. Since $U(k_s) \neq \emptyset$, upon replacing $k$ by $k_s$, we deduce from Step 2 a $g_s$-equivariant isomorphism of $\mathbb{Q}_r$-vector spaces:

$$j_{X,s} : \nu_{\ell}(\text{Alb}_w X) \sim H^{2n-1}(\tilde{X}, \mathbb{Q}_\ell(n)).$$

This implies in particular that if $m = 2\dim \text{Alb}_w X$, then

$$\dim H^{2n-1}(\tilde{X}, \mathbb{Q}_\ell(n)) = \dim \nu_{\ell}(\text{Alb}_w X) = m.$$

In each of these spaces, there is an action of $g = g_1$. By choosing bases, we identify both of them with $\mathbb{Q}_r^n$. Denote by $g_1 \in \text{GL}_m(\mathbb{Q}_\ell)$ the matrix of the endomorphism $\nu_{\ell}(\varphi)$, where $\varphi \in g$ is the geometric Frobenius operator in $H^{2n-1}(\tilde{X}, \mathbb{Q}_\ell(n))$. The existence of the $\mathbb{g}_s$-equivariant isomorphism $j_{X,s}$ implies that $g_1^n$ and $g_2^n$ are conjugate. In order to finish the proof when $k$ is finite, we must show that $g_1$ and $g_2$ are conjugate. This follows from the Conjugation Lemma below, since $g_1$ is semi-simple by [31, p. 203].

**Step 5.** Assume now that $(\mathbb{R}_{m,p})$ holds and that $X$ is regular in codimension 1. Take $\tilde{X}$ to be a nonsingular projective variety birationally equivalent to $X$ over $k$. Then $\text{Alb}_w \tilde{X} = \text{Alb}_w X$ since the Albanese-Weil variety is a birational invariant, and

$$H^{2n-1}_x(\tilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)) = H^{2n-1}(\tilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)),$$

by Proposition 8.1(ii). Now it is well known that $H^{2n-1}(\tilde{X} \otimes \bar{k}, \mathbb{Q}_\ell)$ is pure, and the same holds for $X$, by Prop. 8.7(i). Hence,

$$H^{2n-1}(\tilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)) = H^{2n-1}(X \otimes \bar{k}, \mathbb{Q}_\ell(n)).$$

Since the conclusion is true for a nonsingular variety, we obtain a $g$-equivariant map

$$j_{\tilde{X}} : \nu_{\ell}(\text{Alb}_w \tilde{X}) \sim H^{2n-1}(\tilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)),$$

and this gives the required $g$-equivariant isomorphism. \qed
Conjugation Lemma. Let $K$ be a field of characteristic zero, and let $g_1$ and $g_2$ be two matrices in $\text{GL}_n(K)$, with $g_1$ semi-simple. If $g_2^s$ is conjugate to $g_1^s$ for infinitely many prime numbers $s$, then $g_2$ is conjugate to $g_1$.

Proof. Let $g_2 = su$ be the multiplicative Jordan decomposition of $g_2$ into its semi-simple and unipotent part. Take $a$ and $b$ prime with $g_2^a$ conjugate to $g_1^a$. Then $s^au^a$ is conjugate to $g_1^a$, and hence, $u^a = I$, by the uniqueness of the Jordan decomposition. Similarly, we find $u^b = I$. Hence $u = I$ with the help of Bézout’s equation, and $g_2$ is semisimple.

Take now two diagonal matrices $d_1$ and $d_2$ in $\text{GL}_n(\bar{K})$ such that $g_i$ is conjugate to $d_i$ in $\text{GL}_n(\bar{K})$. Two conjugate diagonal matrices are conjugate by an element of the group $W$ of permutation matrices: if $d_1^s$ and $d_2^s$ are conjugate, then $d_2^s = (w_s d_1 w_s^{-1})^s$ with $w_s \in W$. Since $W$ is finite, one of the sets

$$T(w) = \{ s \in \mathbb{N} : d_2^s = (w d_1 w^{-1})^s \}$$

contains infinitely many prime numbers. Take two prime numbers $a$ and $b$ in that set, then

$$d_2^a = h_1^a, \quad d_2^b = h_1^b, \quad h_1 = w d_1 w^{-1},$$

from which we deduce $d_2 = h_1$ by Bézout’s equation. This implies that $d_1$ and $d_2$ are conjugate in $\text{GL}_n(\bar{K})$, and the same holds for $g_1$ and $g_2$. But two elements of $\text{GL}_n(K)$ which are conjugate in $\text{GL}_n(\bar{K})$ are conjugate in $\text{GL}_n(K)$.

□

ADDENDA

One can improve some results in the paper, assuming that $(R_{n,p})$ holds. This may provide indications on the range of validity of the statements. For instance, the following proposition shows that the conclusion of Cor. 9.10 of [S1] is true without assuming that $X$ is regular in codimension 2.

In what follows $k$ is a perfect field of characteristic $p \geq 0$. Recall that a projective variety, regular in codimension one, which is a local complete intersection, is normal.

A1. Proposition. Assume that $(R_{n,p})$ holds. Let $X$ be a projective variety of dimension $n \geq 2$ defined over $k$.

(i) If $X$ is normal, there is a $g$-equivariant injective linear map

$$H^1(\bar{X}, \mathbb{Q}_\ell) \rightarrow \text{Hom}(H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)), \mathbb{Q}_\ell).$$

(ii) If $X$ is regular in codimension one and is a local complete intersection, this linear map is bijective.

Proof. The proof of (i) follows the lines of the proof of [S1, Cor. 9.10], taking in account the last statement in Prop. 9.8 of the present note. In order to see that (ii) holds, remark that

$$\dim H^1(\bar{X}, \mathbb{Q}_\ell) = \dim H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$$

by Poincaré duality [S1, Rem. 2.7].

□
A2. Proposition. Assume that $(R_{n,p})$ holds. Let $X$ be a projective variety of dimension $n \geq 2$ defined over $k$, regular in codimension one, which is a local complete intersection. Then the canonical map

$$\nu: \text{Alb}_w X \longrightarrow \text{Alb}_s X$$

is an isomorphism.

Proof. From Prop. A1(ii) and the proof of [S1, Cor. 9.10], we deduce that the homomorphism

$$V(t\nu): V(Pic_s X) \longrightarrow V(Pic_w X)$$

is bijective, hence, $\nu$ is an isogeny by Tate's Theorem [31, Appendix I]. Since the kernel of $\nu$ is connected by Prop. 9.1(ii), it is trivial. $\square$

Now Prop. A2 leads to an improvement of Prop. 10.10:

A3. Corollary. Assume that $(R_{n,p})$ holds. Let $X$ be a projective variety of dimension $n$, defined over the finite field $k = F_q$, regular in codimension one, which is a local complete intersection. If $g = \dim \text{Alb}_w X$, then

$$q^{-g}P_1^w(X, q^n T) = P_{2n-1}(X, T) .$$

$\square$

Applications and complements

It may be interesting to note that some of the results of [S1] have found application in such diverse fields as group theory by T. Bandman & al. [S2], [S3], the study of Boolean functions by F. Rodier [S7], and Waring’s problem in function fields by Y.-R. Liu and T. Wooley [S6]. None of these applications are based on the results whose proof needed modifications or corrections, as outlined here. Improvements of some of the estimates in [S1] have also been obtained by A. Cafure and G. Matera [S4], [S5]. Finally, since Section 1 of [S1] includes a version of Bertini Theorem, it is worthwhile to notice that deep results on Bertini Theorems over finite fields have been recently obtained by B. Poonen [S7].

Supplementary References


S.R.G.: DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY, POWAI, MUMBAI 400076, INDIA

E-mail address: srg@math.iitb.ac.in

G.L.: INSTITUT DE MATHÉMATIQUES DE LUMINY, CNRS, LUMINY CASE 907, 13288 MARSEILLE CEDEX 9, FRANCE

E-mail address: lachaud@univmed.fr