ADAMS OPERATIONS AND POWER STRUCTURES

E. GORSKY

ABSTRACT. We study the relations between Adams operation on a lambda-ring and the power structure on it, introduced by S. Gusein-Zade, I. Luengo and A. Melle-Hernández. We give the explicit equations expressing them by each other. An interpretation of the formula of E. Getzler for the equivariant Euler characteristics of configuration spaces is also given.

2000 Math. Subj. Class. 55S15, 19L20, 05E05, 14H10.

Key words and phrases. \(\lambda\)-rings, Adams operations, plethysms, power structures, moduli of curves.

1. Introduction

We construct a family of additive endomorphisms \(\Psi_k, k = 1, 2, \ldots\), of the Grothendieck ring of quasiprojective varieties and the Grothendieck ring of Chow motives. Each of these maps being applied to a given variety gives a polynomial of its symmetric powers with integer coefficients. For example, \(\Psi_1(X) = X, \Psi_2(X) = 2[S^2X] - [X^2]\), where \(S^2X\) is a symmetric square of \(X\). For a polynomial of the affine line there is a formula

\[\Psi_k(P(L)) = P(L^k).\]

The construction of these maps has a lot in common with the construction of the Adams operations in the K-theory, and for the Grothendieck ring of Chow motives two additional sets of equations

\[\Psi_i(X \cdot Y) = \Psi_i(X) \cdot \Psi_j(Y), \quad \Psi_i \circ \Psi_j = \Psi_{ij},\]

analogous to the ones for the Adams operations, hold. This fact follows from the specialty of the \(\lambda\)-structure over the Grothendieck ring of motives proved by F. Heinloth [12].

These operations are used for study of the so-called power structure over the Grothendieck ring, constructed by S. Gusein-Zade, I. Luengo and A. Melle-Hernández [9]. We prove the inversion formula, which provides a possibility to express explicitly the exponents \(B_i\) via the coefficients \(A_j\) in a formula

\[1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots = (1 - t)^{-B_1}(1 - t^2)^{-B_2}, \ldots,\]

Supported in part by the grants RFBR-007-00593, INTAS-05-7805, NSh-4719:2006.1, and the Moebius Contest fellowship for the young scientists.

\(\copyright 2009\) Independent University of Moscow

305
where the right hand side is considered in a sense of a power structure. This formula looks very similar to the “motivic Moebius inversion” of D. Bourqui [2].

As an example we calculate the class of the variety of irreducible polynomials of a given degree of arbitrary number of variables. This gives a way to compute, for example, all Hodge–Deligne numbers of this variety.

Moreover, we give a clear interpretation in a language of power structures of the E. Getzler’s formulas ([5]) for the characters of the equivariant cohomology corresponding to the natural symmetric group action on the configuration spaces of ordered tuples of points on a given variety.

We also recall some known results concerning power structures like the alternative proof of the L. Goettsche’s formula for the Betti numbers of the Hilbert schemes of points on a surface from [10].

2. Power Structures

The notion of a power structure over a (semi)ring was introduced by S. Gusein-Zade, I. Luengo and A. Melle-Hernandez in [9].

Definition. A power structure over a ring $\mathbb{R}$ is a map

$$(1 + tR[[t]]) \times \mathbb{R} \to 1 + tR[[t]]: (A(t), m) \mapsto (A(t))^m,$$

satisfying the following equations:

1. $(A(t))^0 = 1$,
2. $(A(t))^1 = A(t)$,
3. $((A(t) \cdot B(t))^m = ((A(t))^m \cdot (B(t))^m$,
4. $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,
5. $(A(t))^mn = ((A(t))^n)^m$,
6. $(1 + t)^m = 1 + mt + \text{terms of the higher degree}$,
7. $(A(t^k))^m = ((A(t))^m)|_{t \to t^k}$.

A power structure is said to be finitely determined, if for every $N > 0$ there exists $M > 0$, such that $N$-jet of a series $(A(t))^m$ is uniquely determined by the $M$-jet of the series $A(t)$.

In [9] it is proved that a finitely determined power structure is defined if and only if one has a rule defining $(1 - t)^{-m}$ for every $m \in \mathbb{R}$, and

$$(1 - t)^{-m-n} = (1 - t)^{-m} \cdot (1 - t)^{-n}.$$ 

If we have a series $1 + A_1 t + A_2 t^2 + A_3 t^3 + \cdots$, then, after dividing it by $(1 - t)^{-A_1}$, we will get a series of a form $1 + D_2 t^2 + D_3 t^3 + \cdots$, then we divide it by $(1 - t^2)^{-B_2}$ etc.

Finally we will get a decomposition of a power series into an infinite product

$$1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots = (1 - t)^{-B_1}(1 - t^2)^{-B_2} \cdots$$

(1)

If all series of a form $(1 - t^k)^{-B_k}X$ are known, it is easy to compute $A(t)X$ by setting

$$(1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots)^X = \prod_{k=1}^{\infty} (1 - t^k)^{-B_k} \cdot X.$$
If a ring $R$ is a $\mathbb{Q}$-algebra, one can define the exponential and the logarithmic maps, so one can define a “usual” power structure by the formula

$$(1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots)^X := \exp(X \cdot \ln(1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots)).$$

It is easy to see that the equations 1–7 are satisfied.

It is important to note that this power structure is not unique. Below we will discuss a bunch of examples of useful and important power structures which are far from this one. Most of their definitions do not use division by integers, so they are defined over $\mathbb{Z}$, not only for $\mathbb{Q}$-algebras.

2.1. Power structure over the Grothendieck ring of varieties. By $K_0(\text{Var}_C)$ we denote the Grothendieck ring of quasiprojective algebraic varieties. It is generated by the isomorphism classes of complex quasiprojective algebraic varieties modulo relations of form $[X] = [Y] + [X \setminus Y]$ where $Y$ is a Zariski closed subset of $X$. Multiplication is given by the formula $[X] \cdot [Y] = [X \times Y]$. Let $\mathbb{L} \in K_0(\text{Var}_C)$ denote the class of the affine line.

Over the Grothendieck ring of quasiprojective varieties one can define a power structure by the formula

$$(1 - t)^{-[X]} = 1 + [S^1 X] t + [S^2 X] t^2 + \cdots,$$

where $S^k X$ denotes the $k$th symmetric power of $X$. Analogously one can define a power structure over the Grothendieck ring of Chow motives. For example, for $j \geq 0$ it is known ([9]) that $[S^k \mathbb{L}]^j = \mathbb{L}^k$, so that

$$(1 - t)^{-\mathbb{L}^j} = \sum_{k=0}^{\infty} t^k \mathbb{L}^k = (1 - t \mathbb{L}^j)^{-1}.$$

This power structure has a nice and clear geometric meaning: if $A_1$, $A_2$, $\ldots$ and $X$ are some quasiprojective varieties and

$$(1 + A_1 \cdot t + A_2 \cdot t^2 + \cdots)^X = 1 + B_1 \cdot t + B_2 \cdot t^2 + \cdots,$$

then the varieties $B_n$ have a following geometric description. Consider a function $I$ on a disjoint union $\bigsqcup_{i=1}^{\infty} A_i$ which equals identically to $i$ on $A_i$. Then $B_n$ is a set of pairs $(K, \varphi)$, where $K$ is a finite subset of $X$ and $\varphi: K \to \bigsqcup_{i=1}^{\infty} A_i$ is a map such that

$$\sum_{x \in K} I(\varphi(x)) = n.$$ 

This set of pairs can be equipped with a structure of a quasiprojective algebraic variety [9]. Furthermore, from this geometric definition it is easy to check that this construction satisfies all properties of a power structure.

Less formally, this construction can be described in a following way: on a variety $X$ there live particles equipped with some natural numbers (multiplicities, masses, charges...). A particle of a given multiplicity $k$ has a complicated space of internal states which are parametrized by points of a quasiprojective variety $A_k$. Now $B_n$ is a configuration space of tuples of particles of total multiplicity $n$. For example,
if all $A_k$ are one-point sets, then $B_n$ consists of all possible tuples of distinct points on $X$ with multiplicities, of total multiplicity $n$, that is, $B_n = S^n X$. Hence

$$1 + [S^1 X] t + [S^2 X] t^2 + \cdots = (1 + t + t^2 + \cdots)^{|X|} = ((1 - t)^{-1})^{|X|} = (1 - t)^{-|X|}.$$

If $A_1$ is a point, and all $A_i$ are empty, then $B_n$ is a set of unordered tuples of distinct points on $X$. A generating function for the classes of these sets has the form $(1 + t)^{|X|}$.

Less trivial examples of power structures are also known. For example, the Jordan normal form of a matrix is a tuple of its eigenvalues with attached partitions (Young diagrams). Therefore the set of Jordan forms of $n \times n$ matrices is the $n$-th coefficient in the series

$$(1 + A_1 \cdot t + A_2 t^2 + \cdots)^X,$$

where $X$ is the set consisting of all possible eigenvalues (whole $\mathbb{C}$, or $\mathbb{C}^*$, if we consider only nondegenerate matrices), and $A_k$ is the set of Young diagrams of weight $k$. This example produce some interesting combinatorial identities which can be found (in a slightly different terminology) in the article [15].

Another application of the technique of power structures is the geometry and the combinatorics of the Hilbert schemes of points on varieties. If $X$ is a smooth projective variety of dimension $d$, then the set of its zero-dimensional subschemes of length $k$ can be equipped with a structure of a projective variety. It is called the Hilbert scheme of $k$ points on $X$ and denoted by $\text{Hilb}^k(X)$. It turns out that the following identity holds [10]:

$$(1 + \text{Hilb}^1(C_d, 0) \cdot t + \text{Hilb}^2(C_d, 0) \cdot t^2 + \cdots)^{|X|} = 1 + [\text{Hilb}^1(X)] \cdot t + [\text{Hilb}^2(X)] \cdot t^2 + \cdots, \quad (2)$$

where $\text{Hilb}^k(C_d, 0)$ is the Hilbert scheme parameterizing subschemes of $C_d$ of length $k$ with the support at the origin. A zero-dimensional subscheme of $X$ can be considered as a pair $(K, \varphi)$, where $K$ is a finite subset of $X$, and $\varphi$ is a map from $K$ to the set of zero-dimensional subsets of $X$ with one-point support; this motivates the formula. This identity does not follow directly from the definition of the power structure, but it can be deduced from it after some technical work ([10]).

Identities like (2) holds in the Grothendieck ring of varieties, so they look quite abstract. Nevertheless, they have concrete and powerful geometric corollaries. For example, the Euler character is an additive invariant of algebraic varieties in the sense that

$$\chi(X) = \chi(Y) + \chi(X \setminus Y),$$

if $Y$ is a Zariski closed subset of $X$. Moreover, $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$, so $\chi: K_0(\text{Var}_{\mathbb{C}}) \to \mathbb{Z}$ is a ring homomorphism. There exist some other invariants of algebraic varieties. For example, the Hodge–Deligne polynomial [3] is the ring homomorphism

$$e: K_0(\text{Var}_{\mathbb{C}}) \to \mathbb{Z}(u, v),$$
which coincides for smooth projective varieties with the generating function for the Hodge numbers:

\[ e(X) = \sum_{i,j} (-1)^{i+j} u^i v^j h_{i,j}(X). \]

Over the polynomial ring there exists a power structure defined by the equation

\[ (1-t)^{-\sum a_k z_k} = \prod (1-t^{\frac{1}{a_k}})^{-a_k}. \]

In [9] it is proved that the Hodge–Deligne polynomial is a morphism of power structures, that is, if \( M, A_1, A_2, \ldots \in K_0(\text{Var}_{\mathbb{C}}) \), then

\[ e((1 + A_1 t + A_2 t^2 + \cdots) M) = (e(1 + A_1 t + A_2 t^2 + \cdots)) e(M). \]

In particular,

\[ \chi((1 + A_1 t + A_2 t^2 + \cdots) M) = (1 + \chi(A_1) t + \chi(A_2) t^2 + \cdots) \chi(M). \]

Therefore we can translate equations in the Grothendieck ring into equations involving Hodge numbers (or Euler characters) of some concrete varieties. For example, in the example with the Hilbert schemes of points, one can use the identity in the Grothendieck ring to obtain the formula of L. Goettsche [8] for the Betti numbers of the Hilbert schemes of points on a surface. In the notation of the power structure over the polynomial ring it has a form:

\[
(1 + e(\text{Hilb}^1(\mathbb{C}^d, 0)) \cdot t + e(\text{Hilb}^2(\mathbb{C}^d, 0)) \cdot t^2 + \cdots)^e(X)
= 1 + e(\text{Hilb}^1(X)) \cdot t + e(\text{Hilb}^2(X)) \cdot t^2 + \cdots.
\]

2.2. Special \( \lambda \)-rings. Let \( R \) be a commutative ring with a unity.

**Definition.** A map \( \lambda_t: R \to 1 + t R[[t]] \) is said to be a \( \lambda \)-structure over \( R \), if for all \( X, Y \in R \)

\[ \lambda_t(X + Y) = \lambda_t(X) \lambda_t(Y) \]

and \( \lambda_t(X) = 1 + tX + \cdots. \)

A ring with a \( \lambda \)-structure is called a \( \lambda \)-ring.

A homomorphism of \( \lambda \)-rings is a ring homomorphism \( f: R_1 \to R_2 \) such that for all \( X \in R_1 \)

\[ f(\lambda_t(X)) = \lambda_t(f(X)), \]

where \( f \) is naturally extended to \( R_1[[t]] \) and \( R_2[[t]] \).

If a \( \lambda \)-structure over a ring is given, one can define a power structure over it by setting \( (1 - t)^{-X} = \lambda_t(X) \). On the other hand, a power structure over a ring induces a bunch of different power structures of the form

\[ \lambda^A_t(X) = (A(t))^X, \]

where \( A(t) \) is an arbitrary series of the form \( 1 + t + \cdots. \) Nevertheless, below, if the contrary is not said, a \( \lambda \)-structure over a ring with a power structure will be considered equal to \( (1 - t)^{-X} \).
Let \( \sigma_i \) be the \( i \)th elementary symmetric polynomial of variables \( \xi_1, \ldots, \xi_N \), and let \( s_i \) be the \( i \)th elementary symmetric polynomial of variables \( x_1, \ldots, x_N \). Let \( P_n(\sigma_1, \ldots, \sigma_n; s_1, \ldots, s_n) \) be the coefficient at \( t^n \) in the series

\[
\prod_{1 \leq i,j \leq N} (1 + \xi_i x_j t),
\]

and let \( P_{n,r}(\sigma_1, \ldots, \sigma_{nr}) \) be the coefficient at \( t^n \) in the series

\[
\prod_{1 \leq i_1 < \cdots < i_r \leq N} (1 + \xi_{i_1} \cdots \xi_{i_r} t).
\]

Consider the following \( \lambda \)-structure over a ring \( 1 + tR[[t]] \). Addition is given by multiplication, multiplication \( \circ \) is given by the formula

\[
(1 + a_1 t + a_2 t^2 + \cdots) \circ (1 + b_1 t + b_2 t^2 + \cdots) = 1 + \sum_{n=1}^{\infty} P_n(a_1, \ldots, a_n; b_1, \ldots, b_n) t^n,
\]

and \( \lambda \)-structure is given by the formula

\[
\Lambda_r(1 + a_1 t + a_2 t^2 + \cdots) = 1 + \sum_{n=1}^{\infty} P_{n,r}(a_1, \ldots, a_{nr}) t^n.
\]

**Definition.** A \( \lambda \)-structure over a ring \( R \) is said to be special if \( \lambda_t : R \to 1 + tR[[t]] \) is a ring homomorphism preserving the \( \lambda \)-structure.

F. Heinloth [12] proved that the \( \lambda \)-structure over the Grothendieck ring of Chow motives is special. It is not known whether the analogous statement holds for the Grothendieck ring of varieties. Since natural additive maps like the Hodge–Deligne polynomial factorize through the Chow motives, it is not a big problem.

To use the theorem of Heinloth for calculations, it is useful to reformulate, following [13], the definition of a special \( \lambda \)-structure.

Let

\[
\lambda_t(X)^{-1} \cdot \frac{d}{dt} \lambda_t(X) = \sum_{n=1}^{\infty} \Psi_n(X) t^{n-1}.
\]

The definition of a \( \lambda \)-structure is equivalent to the identity

\[
\Psi_i(X + Y) = \Psi_i(X) + \Psi_i(Y),
\]

\( \lambda \)-structure is special, if and only if [13]

\[
\Psi_i(XY) = \Psi_i(X) \Psi_i(Y) \quad \text{and} \quad \Psi_i \circ \Psi_j = \Psi_{ij}
\]

for all \( i \) and \( j \).

**Definition.** These homomorphisms \( \Psi_k \) are called the Adams operations on the \( \lambda \)-ring \( R \).

**Example.** Let \( X = L^k \). Then \( \lambda_t(X) = (1 - L^k t)^{-1} \), so

\[
\lambda_t(X)^{-1} \cdot \frac{d}{dt} \lambda_t(X) = \frac{L^k}{1 - L^k t},
\]

hence

\[
\Psi_n(L^k) = L^{kn}.
\]
Since $\Psi_n$ are additive operations, $\Psi_n(P(L)) = P(L^n)$. Therefore over the subring of polynomials of $L$, the $\lambda$-structure is special.

In fact, the Adams operation on the polynomial ring of arbitrary number of variables have a similar form. If $q(x) = \sum a_k x^k$, then

$$(1 - t)^{-q(x)} = (1 - t)^{-\sum a_k x^k} = \prod (1 - t a_k)^{-a_k},$$

hence

$$(1 - t)^{q(x)} \frac{d}{dt} (1 - t)^{-q(x)} = \sum \frac{a_k x^k}{1 - t a_k} = \sum_{n=1}^{\infty} q(x_1^n, x_2^n, \ldots) t^{n-1},$$

so that

$$\Psi_n(q(x_1, x_2, \ldots)) = q(x_1^n, x_2^n, \ldots).$$

For example, this means that the $\lambda$-structure over the polynomial ring is special.

**Example.** Another important example of a ring with a natural special $\lambda$-structure is the Grothendieck ring of representations of a given finite group $G$. Representations $\rho: G \to GL(V)$ are in 1-to-1 correspondence with their characters $\chi(g) = \text{tr} \rho(g)$, and the Grothendieck ring of representations is isomorphic to the ring of functions on $G$ invariant under the conjugation. The character of the sum (tensor product) of two representations is equal to the sum (product) of the characters of these representations, so that the character map is a ring homomorphism.

Over the ring of representations, we have a natural $\lambda$-structure:

$$\lambda_t(V) = 1 + \sum_{k=1}^{\infty} S^k V \cdot t^k.$$ 

Let $\xi_1, \ldots, \xi_n$ are eigenvalues of $\rho(g)$ for some $g \in G$. The eigenvalues of the operator $S^k \rho(g)$ acting on the space $S^k V$ are all products of the form $\xi_{i_1} \cdots \xi_{i_k}, i_1 \leq \cdots \leq i_k$. Therefore

$$1 + \sum_{k=1}^{\infty} t^k \text{tr} S^k \rho(g) = \prod_{j=1}^{n} (1 - \xi_j t)^{-1},$$

so the value of the character of $\Psi_k(\rho)$ at the element $g$ is equal to $\xi_1^k + \cdots + \xi_n^k = \chi(g^k)$. Therefore

$$\Psi_k \chi(g) = \chi(g^k),$$

and in particular, the power structure is special.

**Example.** First appearance of the Adams operations was in K-theory [4]. Let $X$ be an arbitrary topological space, and let $K_0(X)$ be the Grothendieck group of vector bundles over it. Over $K_0(X)$, there is a natural $\lambda$-structure: if $E$ is a (virtual) bundle, then

$$(1 - t)^{-E} = 1 + \sum_{k=1}^{\infty} S^k E \cdot t^k.$$
For the calculation of the Adams operations, one can use the splitting principle: if $E = \bigoplus_{i=1}^{n} E_i$, $\text{rk} E_i = 1$, then
\[ \Psi_k(E) = \bigoplus_{i=1}^{n} E_i^k. \]

The proof of this fact is completely analogous to the previous example, and the power structure is special.

**Example.** Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, $R_i \cdot R_j \subset R_{i+j}$. For every $x \in R_j$, let
\[ \Psi_k(x) = k^j \cdot x. \]
It is easy to see that these operations are ring homomorphisms, and
\[ \Psi_k(\Psi_m(x)) = \Psi_{km}(x). \]

Therefore $\Psi_k$ are Adams operations for some special power structure. It is clear how to reconstruct this structure: if $x \in R_j$, then
\[
(1 - t)^{-x} = \exp\left(\sum_{k=1}^{\infty} \Psi_k(x) \frac{t^k}{k}\right) = \exp\left(\sum_{k=1}^{\infty} k^{j-1} t^k\right) = 1 + t + \left(2^{j-1} + \frac{1}{2}\right) t^2 + \left(3^{j-1} + 2^{j-1} + \frac{1}{6}\right) t^3 + \cdots.
\]

This structure is strange at a first glance, but it appears naturally, for example, on the even-dimensional cohomology of an arbitrary topological space $X$. The Chern character $\text{ch}$ is a homomorphism from $K_0(X)$ to $H^{2\ast}(X)$. If $E$ is a line bundle over $X$, then
\[ c_1(\Psi_k(E)) = c_1(E^k) = kc_1(E) = \Psi_k(c_1(E)). \]

Since $\Psi_k$ are ring homomorphisms,
\[ \text{ch}(\Psi_k(E)) = e^{c_1(\Psi_k(E))} = \Psi_k(e^{c_1(E)}) = \Psi_k(\text{ch}(E)) \]
From the splitting principle and the properties of the Adams operations, it follows that for every bundle $E$
\[ \text{ch}(\Psi_k(E)) = \Psi_k(\text{ch}(E)), \quad \text{ch}((1 - t)^{-E}) = (1 - t)^{-\text{ch}(E)}. \]

3. An Inversion Formula

The proof of the formula (1) taken from [9] is clear, but it does not give any explicit formulas expressing $B_k$ through the coefficients of $A(t)$. It turns out that such formulas can be written in terms of the Adams operations.

**Theorem 1.** Let $A(t)^{-1} \frac{d}{dt} A(t) = \sum_{n=1}^{\infty} C_n t^{n-1}$. Then
\[
nB_n = \sum_{i_1, \ldots, i_{s-1} > 1} (-1)^{s-1} \Psi_{i_1} \cdots \Psi_{i_{s-1}}(C_{i_s}),
\]
and if the corresponding $\lambda$-structure over a ring is special, then

$$nB_n = \sum_{d \mid n} \mu(d)\Psi_d(C_{\frac{n}{d}}),$$

where $\mu$ is the Moebius function.

Proof. Remark that

$$(1 - t^k)B_k \frac{d}{dt}(1 - t^k)^{-B_k} = kt^{k-1} \sum_{n=1}^{\infty} \Psi_n(B)t^{kn-1} = \sum_{n=1}^{\infty} \Psi_n(kB_k)t^{kn-1},$$

so that the equation (1) is equivalent to the equation

$$\sum_{n=1}^{\infty} C_n t^{n-1} = \sum_{k,m=1}^{\infty} \Psi_m(kB_k)t^{km-1},$$

that is,

$$\sum_{km=n} \Psi_m(kB_k) = C_n.$$

We have $B_1 = C_1$, $2B_2 + \Psi_2(B_1) = C_2$ etc, so that the solution for this system of equations is unique. On the other hand, it is easy to see that the expressions for $B_k$ in the statement of the lemma satisfy the last equation. □

Example 1. Let $A(t) = \exp(at)$, then $C(t) = a$, so that $C_1 = a$, $C_i = 0$, $i > 1$. Suppose that the $\lambda$-structure is special. Then

$$nB_n = \sum_{d \mid n} \mu(d)\Psi_d(C_{\frac{n}{d}}) = \mu(n)\Psi_n(a), \quad B_n = \frac{\mu(n)}{n} \Psi_n(a).$$

Hence,

$$\exp(at) = \prod_{n=1}^{\infty} (1 - t^n)^{-\frac{\mu(n)}{n} \Psi_n(a)},$$

where right hand side is considered in a sense of the power structure. For example, for $a = 1$ we get the equality of formal power series

$$e^t = \prod_{n=1}^{\infty} (1 - t^n)^{-\frac{\mu(n)}{n}}.$$

Example 1a. Analogously to the previous example, one can also prove a couple of interesting identities:

$$\prod_{k=1}^{\infty} (1 - t^k)^{-\varphi(k)} = e^{\frac{t^2}{2}},$$

where $\varphi(n)$ is the Euler function, that is, the number of positive integers less than $n$ and coprime with $n$, and

$$\prod_{\text{g.c.d}(k,m)=1} (1 - x^k y^m)^{\frac{1}{x^k y^m}} = (1 - x)^{\frac{1}{x}},$$

\[\text{where g.c.d}(k,m) = \gcd(k,m)\].
Example 2. Let $A(t) = 1 + at$, then $C(t) = \frac{a}{t+a}$, so that $C_j = -a^j$. Suppose that the $\lambda$-structure is special, then

$$nB_n = - \sum_{d|n} \mu(d)\Psi_d((-a)^{\frac{n}{d}}).$$

Therefore

$$\frac{d}{dt} \ln((1 + at)^x) = \sum_{n=1}^{\infty} D_n t^{n-1},$$

where

$$D_n = \sum_{km=n} \Psi_m(kB_k \cdot x) = - \sum_{km=n} \Psi_m \left( \sum_{d|k} \mu(d)\Psi_d((-a)^{\frac{n}{d}}) \cdot x \right) = - \sum_{dsm=n} \mu(d)\Psi_{md}((-a)^{s})\Psi_m(x).$$

Hence,

$$\ln((1 + at)^x) = - \sum_{d,s,m=1}^{\infty} \mu(d)\Psi_{md}((-a)^{s})\Psi_m(x) \frac{\mu_{dsm}}{dsm} \ln(1 + \Psi_{md}(a)t^{md})\Psi_m(x) = \sum_{n=1}^{\infty} \ln(1 + \Psi_{n}(a)t^{n}) \frac{1}{n} \sum_{m|n} \mu\left( \frac{n}{m} \right) \Psi_m(x).$$

We get a formula

$$(1 + at)^x = \prod_{n=1}^{\infty} \left( 1 + \Psi_{n}(a)t^{n} \right)^{\frac{1}{n} \sum_{m|n} \mu\left( \frac{n}{m} \right) \Psi_m(x)},$$

where powers in the left hand side are considered in the sense of the power structure, and in the right hand side — in the “usual” sense of the “exponent of the logarithm”.

Let us consider, for example, the ring $\Lambda$ of polynomial of infinite number of variables $x_1, x_2, \ldots$ with integer coefficients. Let $p_k = x_1^k + x_2^k + \cdots$ denote the Newton symmetric polynomials. Then $\Psi_k(p_1) = p_k$.

Let $X$ be a quasiprojective variety, let $F(X, n)$ denote the set of ordered $n$-tuples of points on $X$, and let $e^{S_n}_{F(X,n)}(u, v)$ be the equivariant Hodge-Deligne polynomial ([5]) for the natural $S_n$-action on $F(X, n)$. Let $B(X, n)$ be the set of unordered tuples of distinct points on $X$. E. Getzler ([5]) proved the following identity

$$1 + \sum_{n=1}^{\infty} t^n e^{S_n}_{F(X,n)}(u, v) = \prod_{n=1}^{\infty} (1 + \Psi_{n}(a)t^{n})^{\frac{1}{n} \sum_{m|n} \mu\left( \frac{n}{m} \right) \Psi_m(e^X(u,v))}.$$
Example 3a. One can prove [14] that after replacing all \( p_i \) with 1 in the character of a representation, one gets a (virtual) multiplicity of the trivial representation in a given one. Thus if we change all \( p_i \) to 1 in the Getzler’s formula, we will get a generating function for the dimensions of the \( S_n \)-invariant subspaces in the cohomology of \( F(X, n) \), that is, a generating function for the Poincare polynomials (with compact support) of the quotients \( B(X, n) = F(X, n)/S_n \). Therefore

\[
1 + \sum_{n=1}^{\infty} t^n P_{B(X, n)}(q) = \prod_{n=1}^{\infty} \left(1 + t^n\right)^{\frac{1}{n} \sum_{m|n} \mu\left(\frac{n}{m}\right) \Psi_m(P(X))},
\]

which coincides with \((1 + t)^{P(X)}\) in the sense of the power structure. This coincidence is not accidental, since from the geometric interpretation of the power structure over the Grothendieck ring of varieties, we have

\[
1 + \sum_{n=1}^{\infty} t^n [B(X, n)] = (1 + t)^{|X|}.
\]

If \( b_k \) is the \( k \)th Betti number of the variety \( X \), then

\[
(1 + t)^{P(X)} = \frac{(1 - t^2)^{P(X)}}{(1 - t)^{P(X)}} = \prod_{k=0}^{\infty} \frac{(1 - t^2 q^k)^{(-1)^k b_k}}{(1 - t q^k)^{(-1)^k b_k}}.
\]

Example 3b. A loop on the \( B(X, n) \) corresponds to an automorphism of the covering \( F(X, n) \rightarrow B(X, n) \), that is, an element of the symmetric group \( S_n \). Then in particular, the sign representation of \( S_n \) corresponds to some one-dimensional representation of \( \pi_1(B(X, n)) \).

It is known ([14]) that the change of \( p_i \) to \((-1)^i-1\) in the character of a representation gives a (virtual) multiplicity of the sign representation in a given one. Hence if we change all \( p_i \) to \((-1)^i-1\) in the Getzler’s formula, we will get a generating function for the Poincare polynomials (with compact support) of \( B(X, n) \) with the coefficients in the sign representation. Therefore

\[
1 + \sum_{n=1}^{\infty} t^n \sum_{k=0}^{\infty} (-1)^k q^k \dim H^k_*(B(X, n), \pm \mathbb{C})
\]

\[
= \prod_{n=1}^{\infty} \left(1 + (-1)^{n-1} t^n\right)^{\frac{1}{n} \sum_{m|n} \mu(n/m) \Psi_m(P(X))},
\]

which coincides with \((1 - u)^{P(X)}|_{u=-t}\) in the sense of the power structure. If \( b_k \) is the \( k \)th Betti number of \( X \), then

\[
(1 - u)^{P(X)}|_{u=-t} = \prod_{k=0}^{\infty} (1 - u^k q^k)^{(-1)^k b_k}|_{u=-t} = \prod_{k=0}^{\infty} (1 + t q^k)^{(-1)^k b_k}.
\]

Example 3c. One can prove [14] that the coefficient at \( p_i^n \) in the character of an \( S_n \)-representation is equal to the (virtual) dimension of this representation multiplied by \( n! \). Therefore if we change all \( p_i \) to 0 for \( i > 1 \) in the Getzler’s formula, we obtain the exponential generating function for the Poincare polynomials (with the
compact support) of $F(X, n)$. Therefore

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} P_{F(X,n)}(q) = (1 + t)^{P(X)},$$

where the power is considered in the “usual” sense.

**Example 4.** O. Tommasi [16], [1] proved that the moduli space of smooth hyperelliptic curves of genus $g \geq 2$ has always the cohomology of a point. Using the power structure, it is easy to check that the Hodge–Deligne polynomial of the moduli space of hyperelliptic curves of genus $g$ is equal to $(uv)^{4g-2}$.

Hyperelliptic curves is in 1-to-1 correspondence with unordered $(2g + 2)$-tuples of distinct points on $\mathbb{P}^1$ considered up to the action of the group $\text{PGL}(2, \mathbb{C})$. Projectivization of the space of $2 \times 2$-matrices is isomorphic to $\mathbb{C}\mathbb{P}^3$, its class in the Grothendieck ring is equal to $L^3 + L^2 + L + 1$. Degenerate matrices lay on the Segre quadric whose class is equal to $(1 + L)^2 = 1 + 2L + L^2$. Hence

$$[\text{PGL}(2, \mathbb{C})] = (L^3 + L^2 + L + 1) - (1 + 2L + L^2) = (L^3 - L).$$

On the other hand, the generating function for the classes of unordered tuples of points on $\mathbb{P}^1$ has the following form:

$$(1 + t)^{1+L} = (1 + t) \cdot \frac{1 - t^{-L}}{1 - t} = \frac{(1 - L^2)(1 + t)}{(1 - L)}.$$ 

The coefficient at $t^k$ in the decomposition of this function is equal to $L^k - L^{k-2}$ for $k \geq 4$.

Therefore the class of the moduli space of hyperelliptic curves in the Grothendieck ring is equal to

$$\frac{L^{2g+2} - L^{2g}}{L^3 - L} = L^{2g-1},$$

so we get the desired statement about the Hodge–Deligne polynomial.

It turns out that the equation from the example 2 can be generalized to the arbitrary series.

**Theorem 2.**

$$\left(1 + A_1 t + A_2 t^2 + \cdots \right)^X = \prod_{n=1}^{\infty} \left(1 + \Psi_n(A_1)t^n + \Psi_n(A_2)t^{2n} + \cdots \right)^{\frac{1}{n} \sum_{m|n} \mu(\frac{n}{m})\Psi_m(X)},$$

where powers in the left hand side are considered in the sense of a power structure, and in the right hand side in the sense of “exponent of the logarithm”.

**Proof.** Let us prove first that the right hand side of the equation (4) defines a power structure. The properties 1, 3, 4, 6, 7 are obvious. The property 2 follows from the equality $\sum_{m|n} \mu(n/m) = \delta_{n,1}$. Let us prove the property 5. From the viewpoint of
this (conjectural) structure

\[
((A(t))^X)^Y = \left[ \prod_{n=1}^{\infty} (1 + \Psi_n(A_1)t^n + \Psi_n(A_2)t^{2n} + \ldots)^{\frac{1}{n}} \sum_{m|n} \mu(n/m)\Psi_m(X) \right]^Y
\]

\[
= \prod_{k=1}^{\infty} \prod_{n=1}^{\infty} \Psi_k \left[ \left( \Psi_n(A)(t^{kn}) \right)^{\frac{1}{n}} \sum_{m|n} \mu(n/m)\Psi_m(X) \right]^{\frac{1}{kn}} \sum_{i|k} \mu(k/l)\Psi_i(Y)
\]

\[
= \prod_{k=1}^{\infty} \prod_{n=1}^{\infty} \left( \Psi_{kn}(A)(t^{kn}) \right)^\frac{1}{kn} \sum_{m|n} \sum_{i|k} \mu(n/m)\mu(k/l)\Psi_m(X)\Psi_i(Y).
\]

Let \( a = kn \), \( b = km \). Then \( n/m = a/b \). Let us note that

\[
\sum_{k: k|b,l} \mu(k/l) = \sum_{f|(b/l)} \mu(f) = \delta_{b,l,1}.
\]

Therefore,

\[
\sum_{kn=a} \sum_{m|n} \sum_{l|k} \mu(n/m)\mu(k/l)\Psi_{kn}(X)\Psi_l(Y) = \sum_{b|a} \mu(a/b)\Psi_b(X)\Psi_b(Y)
\]

\[
= \sum_{b|a} \mu(a/b)\Psi_b(XY).
\]

Hence

\[
\prod_{k=1}^{\infty} \prod_{n=1}^{\infty} \left( \Psi_{kn}(A)(t^{kn}) \right)^\frac{1}{kn} \sum_{m|n} \sum_{i|k} \mu(n/m)\mu(k/l)\Psi_m(X)\Psi_i(Y)
\]

\[
= \prod_{a=1}^{\infty} \left( \Psi_{a}(A)(t^{a}) \right)^{\frac{1}{a}} \sum_{b|a} \mu(a/b)\Psi_b(XY).
\]

Therefore the property 5 is also satisfied, and the right hand side of the equality (4) defines a power structure. What remains to be proved is that this structure agrees with the original one at \( A(t) = (1 - t)^{-X} \). Taking the logarithm of the expression

\[
\prod_{n=1}^{\infty} (1 - t^n)^{\frac{1}{n}} \sum_{m|n} \mu(n/m)\Psi_m(X),
\]

we get

\[
- \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m|n} \mu(n/m)\Psi_m(X) \ln(1 - t^n) = \sum_{n=1}^{\infty} \sum_{m|n} \sum_{k=1}^{\infty} \frac{1}{k} \mu(n/m) t^{kn} \Psi_m(X)
\]

\[
= \sum_{s,m=1}^{\infty} \sum_{l|m} \mu(l) \frac{1}{s} t^s \Psi_m(X) = \sum_{s=1}^{\infty} \frac{1}{s} t^s \Psi_m(X).
\]

Therefore the logarithmic derivative of (5) equals

\[
\sum_{s=1}^{\infty} t^{s-1} \Psi_m(X),
\]

which finishes the proof. \qed
Example 5. Let $P_N$ denote the projectivization of the set of polynomials of degree $N$ of $n$ variables, and let $\text{Irr}_N$ be the projectivization of the set of the irreducible polynomials of degree $N$ of $n$ variables.

The class of $P_N$ in the Grothendieck ring of varieties equals

$$[P_N] = \frac{\mathbb{L}_{(n+N)} - \mathbb{L}_{(n+N-1)}}{ \mathbb{L} - 1}.$$ 

Let

$$P(\mathbb{L}, t) = 1 + \sum_{N=1}^\infty [P_N]t^N.$$ 

Let

$$P(\mathbb{L}, t)^{-1} \frac{\partial}{\partial t} P(\mathbb{L}, t) = \sum_{n=1}^\infty C_n(L)t^{n-1}.$$ 

Theorem 3.

$$n[Irr_n] = \sum_{d|n} \mu(d)C_{\frac{n}{d}}(L^d).$$

Proof. Since in a ring of polynomials, the decomposition into irreducible factors is unique up to multiplication by a constant, one can define the set $P_{i_1i_2\ldots}$ of polynomials which are products of $i_1$ irreducible factors of degree 1, $i_2$ of degree 2 ($i_k = 0$ for $k$ large enough).

Note that

$$P_{i_1i_2\ldots} = (S^{i_1}\text{Irr}_1) \times (S^{i_2}\text{Irr}_2) \times \cdots$$

and

$$P_N = \bigsqcup_{i_1 + 2i_2 + \cdots = N} P_{i_1i_2\ldots},$$

so that

$$P(\mathbb{L}, t) = \sum_{N=0}^\infty [P_N]t^N = \prod_{k=1}^\infty (1 - t^k)^{-[\text{Irr}_k]}.$$ 

Now the statement of the theorem follows from Lemma 1 and the example preceding it. □

Corollary 1. The Hodge–Deligne polynomial of the projectivization of the set of irreducible polynomials is given by a formula

$$n \cdot e_{\text{Irr}_n}(u, v) = \sum_{d|n} \mu(d)C_{\frac{n}{d}}(u^dv^d).$$

One can calculate the Euler character of $\text{Irr}_1$. We have

$$\prod_{i=1}^\infty (1 - t^i)^{-\chi(\text{Irr}_i)} = \frac{1}{(1 - t)^n}.$$ 

Taking the logarithms, we obtain

$$\sum_i \chi(\text{Irr}_i) \ln(1 - t^i) = n \ln(1 - t),$$
and
\[
\sum_n \chi(\text{Irr}_i) \sum_l t^l/l = n \sum_k t^k/k.
\]
Comparing the coefficients, we get
\[
\sum_{i \mid k} i \chi(\text{Irr}_i) = n,
\]
so by the Mobius inversion formula,
\[
\chi(\text{Irr}_i) = 2 \sum_{k \mid i} \mu(i),
\]
which is equal to \(n\) for \(i = 1\) and 0 for \(i > 1\).

4. Plethysms and Representations of the Symmetric Groups

Let \(\Lambda\) be the ring of symmetric polynomials of infinite number of variables. Over \(\Lambda\), as over any polynomial ring, there is a natural \(\lambda\)-structure. It is easy to check that this structure is special.

For any \(\lambda\)-ring \(R\), one can construct a natural map sending a pair of elements \(f \in \Lambda\) and \(X \in R\) to an element \(f \circ X \in R\) such that the following properties are satisfied:

1. \((f_1 + f_2) \circ X = f_1 \circ X + f_2 \circ X\);
2. \((f_1 f_2) \circ X = (f_1 \circ X)(f_2 \circ X)\);
3. \(p_k \circ X = \Psi_k(X)\).

An easy check shows that
\[
(1 - t)^{-X} = \sum_{k=0}^{\infty} h_k \circ X t^k.
\]

One can prove that a \(\lambda\)-ring is special if and only if for any \(f, g, X\) we have \((f \circ g) \circ X = f \circ (g \circ X)\).

Consider the direct sum of representation rings of groups \(S_k\) over all \(k\). For each representation \(V\) of the group \(S_k\) one can construct its character – it is the homogeneous polynomial in \(\Lambda\) of degree \(k\), defined by the formula
\[
\text{ch}(V) = \sum_{\sigma \in S_k} \text{Tr}(\sigma|_V) \cdot p_1^{i_1(\sigma)} \cdots p_k^{i_k(\sigma)},
\]
where \(i_s(\sigma)\) is the number of cycles of length \(s\) in the permutation \(\sigma\).

It is proved in [14] that the “natural” operation on representations corresponds to natural operations on their characters: the character of the direct sum of representations is the sum of their characters, and if \(V\) is a representation of \(S_k\), and \(W\) is a representation of \(S_m\), then the character of the representation \(\text{Ind}_{S_k \times S_m}^{S_{k+m}}(V \otimes W)\) is equal to a product of the characters of \(V\) and \(W\). Moreover, there is a so-called plethysm operation on representations: if \(V\) is a representation of \(S_k\), and \(W\) is a representation of \(S_m\), then there is a natural action of the semidirect product of the
groups $S_k$ and $(S_m)^k$ on the product $V \otimes W^{\otimes k}$. The plethysm is the representation $V \circ W = \text{Ind}_{S_k \times (S_m)^k}^{S_k \times (S_m)^k} (V \otimes W^{\otimes k})$. It turns out ([14]) that

$$ch(V \circ W) = ch(V) \circ ch(W).$$

In particular, if $V_k$ is the trivial one-dimensional representation of the $S_k$, then $ch(V_k) = h_k$ and $ch(V_k \circ W) = h_k \circ ch(W)$. The generating function for characters of such representations is equal to

$$\sum_{k=0}^{\infty} h_k \circ ch(W) = (1 - t)^{-ch(W)}.$$

Consider the direct sum of the Grothendieck rings of the representations of all symmetric groups:

$$R = \bigoplus_{n=1}^{\infty} R(S_n).$$

The multiplication in the ring $R$ is given by the formula

$$V \otimes W = \text{Ind}_{S_{k+m}}^{S_{k \times (S_m)^k}} (V \otimes W),$$

where $V$ is a representation of $S_k$ and $W$ is a representation of $S_m$ (the product is a representation of $S_{k+m}$).

It is proved in [14], [13] that the character map is an isomorphism between this ring and the ring $\Lambda$. It follows from the above discussion that the power structure over the polynomial ring $\Lambda$ corresponds to a structure over the ring $R$ such that

$$(1 - t)^{-W} = \sum_{k=0}^{\infty} t^k \cdot \text{Ind}_{S_k \times (S_m)^k}^{S_{k+m}} W^{\otimes k}$$

for a representation $W$ of the group $S_m$.

5. Moduli Spaces of Curves

Since every complex curve of genus 2 is hyperelliptic, one can try to compute the $S_n$-equivariant Euler character of the moduli space of genus 2 curves with $n$ marked points.

Consider the forgetful map $\mathcal{M}_{2,n} \to \mathcal{M}_2$. It is not a locally trivial fibration since a curve can have a nontrivial automorphism group (for example, every hyperelliptic curve has a nontrivial automorphism – a hyperelliptic involution), and the true fiber of the forgetful map is the quotient of the space of distinct unordered points on a curve by the action of the automorphism group of this curve.

Consider the following situation: a finite group $G$ acts on a variety $X$. Let $X_k(g)$ denote the set of points with orbit of length $k$ for the action of an element $g \in G$.

**Theorem 4.**

$$\sum_{n=0}^{\infty} t^n \chi^{S_n} (F(X, n)/G) = \frac{1}{|G|} \sum_{g \in G} \prod_{k} (1 + pt^k)^{\chi(X_k(g))}.$$
Proof. Let $R \in \text{Rep}(G)$ be the alternating sum of the cohomology groups of $X$ considered as representations of $G$ ($R$ belongs to the Grothendieck ring of representations of the group $G$). By Getzler’s formula, the analogous sum for the $S_n$-equivariant cohomology groups of $F(X, n)$ is equal to

$$\prod_{k=1}^{\infty} (1 + p_k t^k)^\frac{1}{k} \sum_{d|k} \mu(k/d) \Psi_d(R),$$

and its character at an element $g$ is equal to

$$\xi(g) = \prod_{k=1}^{\infty} (1 + p_k t^k)^\frac{1}{k} \sum_{d|k} \mu(k/d) \chi(g^d),$$

where $\chi$ is the character of the representation $R$. Note that it follows from the Lefschetz theorem that $\chi(e) = \chi(X)$, and for all other $g$, $\chi(g)$ is equal to the Euler character of the fixed point set of $g$. The dimension of a $G$-invariant part in the cohomology of $F(x, n)$ is equal to

$$\frac{1}{|G|} \sum_{g \in G} \xi(g).$$

The number $\chi(g^d)$ is equal to the Euler character of the $g^d$-fixed point set, that is, $\sum_{l|d} \chi(X(l))$. Therefore

$$\frac{1}{k} \sum_{d|k} \mu(k/d) \chi(g^d) = \frac{1}{k} \sum_{d|k} \mu(k/d) \sum_{l|d} \chi(X_l(g)) = \frac{1}{k} \sum_{l|k} \chi(X_l(g)) \sum_{d|k} \mu(d)$$

$$= \frac{1}{k} \chi(X_k(g)).$$

Combining these results, we obtain

$$\xi(g) = \prod_k (1 + p_k t^k)^\frac{\chi(X_k(g))}{k},$$

which finishes the proof. \qed

Corollary 2.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \chi(F(X, n)/G) = \frac{1}{|G|} \left( (1 + t)^\chi(X) + \sum_{g \neq e} (1 + t)^L(g) \right),$$

where $L(g) = \chi(X_1(g))$ is the Euler character of the fixed point set of $g$.

To calculate the equivariant Euler character of $\mathcal{M}_{2,n}$, one has to describe hyperelliptic curves with additional symmetries. For this one has to describe all 6-tuples of points on $\mathbb{CP}^1$ having nontrivial symmetry groups, and to compute the Euler characters of the moduli spaces of corresponding curves.

The sum of these Euler characters is equal to 1, since a theorem of O. Tommasi [16], [1] states that the moduli space of hyperelliptic curves has cohomology of a point.
The sum of the Euler characters of strata divided by the orders of corresponding symmetry groups (the orbifold Euler character) is equal to $\frac{-1}{240}$, in agreement with the Harer–Zagier formula [11]:

$$\chi_{\text{orb}}(M_{g,n}) = (-1)^n \frac{(2g-3+n)!}{(2g)!} B_{2g},$$

where $B_{2g}$ are Bernoulli numbers. For $g = 2$, $B_{2g} = \frac{1}{30}$, so that

$$\chi_{\text{orb}}(M_{2,0}) = \frac{1 \cdot 3 \cdot -1}{4!} \cdot \frac{-1}{30} = \frac{-1}{240}.$$ 

Summing up the answers for different strata, we get (see [7]) the following Proposition.

$$\sum_{n=0}^{\infty} t^n \chi_{S_n}(M_{2,n}) = \frac{-1}{240}(1 + p_1 t)^{-2} - \frac{1}{240}(1 + p_1 t)^6(1 + p_2 t^2)^{-4} + \frac{2}{5}(1 + p_1 t)^3(1 + p_3 t^5)^{-1} + \frac{2}{5}(1 + p_1 t)(1 + p_2 t^2)(1 + p_3 t^5)(1 + p_4 t^{10})^{-1} + \frac{1}{6}(1 + p_1 t)^2(1 + p_2 t^2)(1 + p_4 t^6)^{-1} - \frac{1}{12}(1 + p_1 t)^4(1 + p_2 t^3)^{-2} - \frac{1}{12}(1 + p_2 t^2)(1 + p_3 t^3)^2(1 + p_6 t^6)^{-2} + \frac{1}{12}(1 + p_1 t)^2(1 + p_2 t^2)^2(1 + p_4 t^4)^{-2} + \frac{1}{4}(1 + p_1 t)^2(1 + p_4 t^4)(1 + p_8 t^8)^{-1} - \frac{1}{8}(1 + p_1 t)^2(1 + p_2 t^2)^2(1 + p_4 t^4)^{-2}.$$ 

Up to the case of 4 points, we get

$$\sum_{n=0}^{\infty} t^n \chi_{S_n}(M_{2,n}) = 1 + 2p_1 \cdot t + p_1^2 \cdot t^2 + \left(\frac{1}{7}p_4 + \frac{2}{3}p_1 p_3 - \frac{1}{6}p_1^4\right) \cdot t^4 + \cdots = 1 + 2s_1 \cdot t^4 + (s_1 + s_2) \cdot t^2 + (s_4 - s_{3,1} - s_{2,2}) \cdot t + \cdots,$$

which coincides with the values obtained by Getzler [6], and Tommasi and Bergstrom [1].

**Acknowledgments.** Author is grateful to S. Gusein-Zade, M. Kazaryan, and S. Lando for many useful discussions.

**References**


Moscow State University, Faculty of Mathematics and Mechanics, Department of Higher Geometry and Topology, Russia, 119991, Moscow, Leninskie Gory, 1

E-mail address: gorsky@ccme.ru