LEGENDRIAN DUALITIES AND SPACELIKE HYPERSURFACES IN THE LIGHTCONE

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Abstract. We show four Legendrian dualities between pseudo-spheres in Minkowski space as a basic theorem. We can apply such dualities for constructing extrinsic differential geometry of spacelike hypersurfaces in pseudo-spheres. In this paper we stick to spacelike hypersurfaces in the lightcone and establish an extrinsic differential geometry which we call the lightcone differential geometry.

Key words and phrases. Legendrian duality, spacelike hypersurfaces, the lightcone, conformally flat Riemannian manifolds.

1. Introduction

In this paper we present some results of the project constructing the extrinsic differential geometry on submanifolds of pseudo-spheres in Minkowski space (cf. [15]–[22]). In particular we stick to spacelike hypersurfaces in the lightcone here. It has been known in [2] (cf. Theorem 3.1) that a simply connected Riemannian manifold is conformally flat if and only if it can be embedded as a spacelike hypersurface in the lightcone. According to this result, if we study an extrinsic differential geometry of spacelike hypersurfaces in the lightcone, then we may obtain extrinsic invariants of conformally flat Riemannian manifolds. This is a main motivation for the study of spacelike hypersurfaces in the lightcone. Moreover, the situation in this case is quite different from other submanifold theories because the metric on the lightcone is degenerate (cf. [3], [5]–[8], [10], [11], [15], [30], [35], [36], [38]–[40]). Therefore we cannot apply the ordinary submanifold theory of semi-Riemannian geometry (cf. [33]).

On the other hand, in the classical theory of hypersurfaces in Euclidean space the Gauss map plays a principal role to define geometric invariants. The derivation of the Gauss map (i.e., the Weingarten map) induces the principal curvatures, the Gauss–Kronecker curvature and the mean curvature of the hypersurface. In [5] Bleeker and Wilson studied the singularities of the Gauss map of a surface in...
Euclidean 3-space. In their paper, the main theorem asserts that the generic singularities of Gauss maps are folds or cusps. Later Banchoff et al. [3], Landis [24] and Platonova [35] studied geometric meanings of cusps of the Gauss map of a surface. Bruce [6] and Romero–Fuster [37] have also independently studied the singularities of the Gauss map and the dual of a hypersurface in Euclidean space. The singularity of the dual of a hypersurface is deeply related to the singularity of the Gauss map of the hypersurface. Their main tool for the study is the family of height functions on a hypersurface. It has been classically known that the singular set of the Gauss map is the parabolic set of the surface and it can be interpreted as the criminant set of the family of height functions. This is the reason why they adopted the height function for the study of Gauss maps. They applied the deformation theory of smooth functions to the height function and derived geometric results on Gauss maps. We can interpret that these results on Gauss maps describe the contact of hypersurfaces with hyperplanes. It also has been known that Gauss maps of hypersurfaces are Lagrangian maps. Moreover, the generic singularities of Gauss maps of hypersurfaces and Lagrangian maps are the same [1]. Singularities of projective Gauss maps are also studied by McCrory et al [28], [29]. There are many other articles concerning the singularities of Gauss maps, we only refer here to the book [3].

On the other hand, for a spacelike hypersurface in the lightcone (i.e., the pseudo-sphere with zero radius), we cannot construct “unit normal vector fields” in the tangent space of the lightcone. In this paper we show four Legendrian dualities between pseudo-spheres in Minkowski space as a basic theorem (cf. Theorem 2.2). The case for hypersurfaces in hyperbolic space in [15] can also be interpreted as an application of the basic theorem (cf. Section 2). We can obtain a kind of normal vector fields to a spacelike hypersurface in the lightcone as an application of the basic Legendrian duality theorem. By using this “normal vector field”, we define a mapping to the lightcone which is called the lightcone dual (cf. Section 3). It follows from the properties of the Legendrian duality that we show the derivation of the lightcone dual can be interpreted as a linear transformation on the tangent space. We call it the lightcone Weingarten map. Therefore we can define the lightcone principal curvature $\kappa_\ell$, the lightcone Gauss–Kronecker curvature $K_\ell$ and the lightcone mean curvature $H_\ell$ for a spacelike hypersurface in the lightcone. We study totally umbilic spacelike hypersurfaces under this framework and give a classification in Section 3. Such a spacelike hypersurface is a quadric hypersurface in the lightcone (i.e, the intersection of the lightcone with a hyperplane in Minkowski space). We briefly call it the hyperquadric. There are three kinds of hyperquadrics. The flat one is the parabolic hyperquadric. In Section 4–7 we study local differential geometry from the contact viewpoint of spacelike hypersurfaces with parabolic hyperquadrics as applications of the theory of Legendrian singularities (cf. the appendix). We consider generic properties in Section 7. We study spacelike surfaces in the 3-dimensional lightcone in Section 9. We can show the analogous result of Theorema Egregium of Gauss (cf. Proposition 9.2). However, as a corollary of Proposition 9.2, we show that the lightcone mean curvature is equal to the sectional curvature of the spacelike surface (cf. Theorem 9.3). This is really an “interesting theorem” because the lightcone Gauss–Kronecker curvature is an
extrinsic invariant but the lightcone mean curvature is an intrinsic invariant. In the remaining part of Section 9, we study geometric meanings of generic singularities of the lightcone dual and give a relationship between the Euler number of the global lightcone dual and geometric invariants. We give some examples in Section 10.

The first article of this paper appeared as one of the preprint series in Hokkaido University in 2004 (# 673). In [23] we have considered parallels and evolutes of spacelike hypersurfaces in Minkowski pseudo-spheres as an application of the method in this paper. We assumed some results in this paper (i.e., the assertions in Section 2 and Section 3) without the proofs in [23]. However, the paper [23] has appeared earlier than this paper.

We shall assume throughout the whole paper that all the maps and manifolds are $C^\infty$ unless the contrary is explicitly stated.

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2. Basic Notations and the Duality Theorem

In this section we prepare basic notions on Minkowski space and contact geometry. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n): x_i \in \mathbb{R}, i = 0, 1, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the pseudo scalar product of $x$ and $y$ is defined by $\langle x, y \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$. The space $(\mathbb{R}^{n+1}, \langle \, , \, \rangle)$ is called Minkowski n+1-space and denoted by $\mathbb{R}^{n+1}$.

We say that a vector $x$ in $\mathbb{R}^{n+1}\setminus\{0\}$ is spacelike, lightlike or timelike if $\langle x, x \rangle > 0$, $\langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$ respectively. A submanifold $M \subset \mathbb{R}^{n+1}$ is said to be spacelike if all tangent vectors of $M$ are spacelike. The norm of the vector $x \in \mathbb{R}^{n+1}$ is defined by $|x| = \sqrt{|\langle x, x \rangle|}$. Given a vector $n \neq 0 \in \mathbb{R}^{n+1}$ and a real number $c$, the hyperplane with pseudo normal $n$ is defined by

$$HP(n, c) = \{x \in \mathbb{R}^{n+1}: \langle x, n \rangle = c\}.$$  

We say that $HP(n, c)$ is a spacelike, timelike or lightlike hyperplane if $n$ is timelike, spacelike or lightlike respectively. In this paper we use the following basic facts.

Lemma 2.1. Let $x, y \in \mathbb{R}^{n+1}$ be lightlike vectors. If $\langle x, y \rangle = 0$, then $x, y$ are linearly dependent.

Proof. Suppose that $x, y$ are linearly independent. Let $N$ be the two dimensional subspace of $\mathbb{R}^{n+1}$ generated by $x, y$. Then all vectors in $N$ are lightlike. We consider the subspace $\mathbb{R}^n_0 = \{x = (x_0, x_1, \ldots, x_b) \in \mathbb{R}^{n+1}: x_0 = 0\}$. Then $N \cap \mathbb{R}^n_0$ is a positive dimensional subspace in $\mathbb{R}^{n+1}$. However the vector $x \in N \cap \mathbb{R}^n_0$ is lightlike and spacelike. This is a contradiction. □

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}$: Hyperbolic n-space is defined by

$$H^n(-1) = \{x \in \mathbb{R}^{n+1}: \langle x, x \rangle = -1\},$$

de Sitter n-space by

$$S^n_1 = \{x \in \mathbb{R}^{n+1}: \langle x, x \rangle = 1\}.$$
and the (open) lightcone by
\[ LC^* = \{ x \in \mathbb{R}^{n+1}_1 \setminus \{0\} : \langle x, x \rangle = 0 \}. \]

We also define
\[ LC^*_+ = \{ x \in LC^* : x_0 > 0 \} \]
and call it the future lightcone. If \( x = (x_0, x_1, \ldots, x_n) \) is a non-zero lightlike vector, then \( x_0 \neq 0 \). Therefore we have
\[ \mathbf{x} = \left( 1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \right) \in S^{n-1}_+ = \{ x = (x_0, x_1, \ldots, x_n) : \langle x, x \rangle = 0, \ x_0 = 1 \}. \]

We call \( S^{n-1}_+ \) the lightcone (or, spacelike unit) \((n-1)\)-sphere. In this paper we stick to spacelike hypersurfaces in the lightcone \( LC^* \). Typical such hypersurfaces are given by the intersection of \( LC^* \) with a hyperplane in \( \mathbb{R}^{n+1}_1 \):
\[ HL(n, c) = HP(n, c) \cap LC^*. \]

We say that \( HL(n, c) \) is a quadric hypersurface in the lightcone (or briefly, hyperquadric). We also say that \( HL(n, c) \) is elliptic, hyperbolic or parabolic if \( n \) is timelike, spacelike or lightlike respectively. These hyperquadrics are the candidates of totally umbilic spacelike hypersurfaces in the lightcone (cf. Section 3).

We now review some properties of contact manifolds and Legendrian submanifolds. The detailed properties is described in the appendix. Let \( N \) be a \((2n + 1)\)-dimensional smooth manifold and \( K \) be a tangent hyperplane field on \( N \). Locally such a field is defined as the field of zeros of a 1-form \( \alpha \). The tangent hyperplane field \( K \) is non-degenerate if \( \alpha \wedge (d\alpha)^n \neq 0 \) at any point of \( N \). We say that \((N, K)\) is a contact manifold if \( K \) is a non-degenerate hyperplane filed. In this case \( K \) is called a contact structure and \( \alpha \) is a contact form. Let \( \phi : N \to N' \) be a diffeomorphism between contact manifolds \((N, K)\) and \((N', K')\). We say that \( \phi \) is a contact diffeomorphism if \( d\phi(K) = K' \). Two contact manifolds \((N, K)\) and \((N', K')\) are contact diffeomorphic if there exists a contact diffeomorphism \( \phi : N \to N' \). A submanifold \( i : L \subset N \) of a contact manifold \((N, K)\) is said to be Legendrian if \( \dim L = n \) and \( di_x(T_xL) \subset K_{i(x)} \) at any \( x \in L \). We say that a smooth fiber bundle \( \pi : E \to M \) is called a Legendrian fibration if its total space \( E \) is furnished with a contact structure and its fibers are Legendrian submanifolds. Let \( \pi : E \to M \) be a Legendrian fibration. For a Legendrian submanifold \( i : L \subset E, \pi \circ i : L \to M \) is called a Legendrian map. The image of the Legendrian map \( \pi \circ i \) is called a wavefront set of \( i \) which is denoted by \( W(i) \). For any \( p \in E \), it is known that there is a local coordinate system \((x_1, \ldots, x_m, p_1, \ldots, p_m, z)\) around \( p \) such that \( \pi(x_1, \ldots, x_m, p_1, \ldots, p_m, z) = (x_1, \ldots, x_m, z) \) and the contact structure is given by the 1-form \( \alpha = dz - \sum_{i=1}^m p_i dx_i \) (cf. [1], 20.3).

One of the examples of Legendrian fibrations is given by the unit spherical tangent bundle of a Riemannian manifold. Let \( M \) be a Riemannian manifold and \( TM \) is its tangent bundle. Let \((x_1, \ldots, x_n)\) be local coordinates on a neighbourhood \( U \) of \( M \) and \((v_1, \ldots, v_n)\) coordinates on the fiber over \( U \). Let \( g_{ij} \) be the components of the metric \( \langle \cdot, \cdot \rangle \) with respect to the above coordinates. Then the canonical one-form can be locally defined by \( \theta = \sum_{i,j} g_{ij} v_i dq_j \) where \( q_i = x_i \circ \pi \) for the projection \( \pi : TM \to M \). Let \( \tilde{\pi} : S(TM) \to M \) be the unit spherical tangent bundle with
respect to the metric $(\cdot,\cdot)$. Then the restriction of $\theta$ onto $S(TM)$ gives a contact structure and $\tilde{\pi}: S(TM) \to M$ is a Legendrian fibration (cf. [4]).

We now show the basic theorem in this paper which is the fundamental tool for the study of spacelike hypersurfaces in pseudo-spheres in Minkowski space.

We define one-forms $\langle dv, w \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle v, dw \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and consider the following four double fibrations with one-forms:

1. (a) $H^n(-1) \times S^n \supset \Delta_1 = \{(v, w): \langle v, w \rangle = 0\}$,
   (b) $\pi_{11}: \Delta_1 \to H^n(-1)$, $\pi_{12}: \Delta_1 \to S^n$,
   (c) $\theta_{11} = \langle dv, w \rangle|_{\Delta_1}$, $\theta_{12} = \langle v, dw \rangle|_{\Delta_1}$. 
2. (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(v, w): \langle v, w \rangle = -1\}$,
   (b) $\pi_{21}: \Delta_2 \to H^n(-1)$, $\pi_{22}: \Delta_2 \to LC^*$, 
   (c) $\theta_{21} = \langle dv, w \rangle|_{\Delta_2}$, $\theta_{22} = \langle v, dw \rangle|_{\Delta_2}$. 
3. (a) $LC^* \times S^n \supset \Delta_3 = \{(v, w): \langle v, w \rangle = 1\}$,
   (b) $\pi_{31}: \Delta_3 \to LC^*$, $\pi_{32}: \Delta_3 \to S^n$,
   (c) $\theta_{31} = \langle dv, w \rangle|_{\Delta_3}$, $\theta_{32} = \langle v, dw \rangle|_{\Delta_3}$. 
4. (a) $LC^* \times LC^* \supset \Delta_4 = \{(v, w): \langle v, w \rangle = -2\}$,
   (b) $\pi_{41}: \Delta_4 \to LC^*$, $\pi_{42}: \Delta_4 \to LC^*$,
   (c) $\theta_{41} = \langle dv, w \rangle|_{\Delta_4}$, $\theta_{42} = \langle v, dw \rangle|_{\Delta_4}$. 

Here, $\pi_{i1}(v, w) = v$, $\pi_{i2}(v, w) = w$ are the canonical projections. Moreover, $\theta_{i1} = \langle dv, w \rangle|_{\Delta_1}$ and $\theta_{i2} = \langle dv, w \rangle|_{\Delta_1}$ are the restrictions of the one-forms $\langle dv, w \rangle$ and $\langle dw, v \rangle$ on $\Delta_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over $\Delta_i$ which is denoted by $K_i$. The basic theorem in this paper is the following theorem:

**Theorem 2.2.** Under the same notations as the previous paragraph, each $(\Delta_i, K_i)$ $(i = 1, 2, 3, 4)$ is a contact manifold and both of $\pi_{ij}$ $(j = 1, 2)$ are Legendrian fibrations. Moreover, those contact manifolds are contact diffeomorphic each other.

**Proof.** By definition we can easily show that each $\Delta_i$ $(i = 1, 2, 3, 4)$ is a smooth submanifold in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and each $\pi_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2)$ is a smooth fibration.

We now show that $(\Delta_1, K_1)$ is a contact manifold. Since $H^n(-1)$ is a spacelike hypersurface in $\mathbb{R}^{n+1}$, $(\cdot,\cdot)_{|_{H^n(-1)}}$ is a Riemannian metric. Let $\pi: S(TH^n(-1)) \to H^n(-1)$ be the unit tangent sphere bundle of $H^n(-1)$. For any $v \in H^n(-1)$, we have local coordinates $(v_1, \ldots, v_n)$ such that $v = (\pm \sqrt{v_1^2 + \cdots + v_n^2 + 1}, v_1, \ldots, v_n)$. We can represent the tangent vector $w \in T_{v}H^n(-1)$ by

$$w = \left(\pm \frac{1}{v_0} \sum_{i=1}^n w_i v_i, w_1, \ldots, w_n\right).$$

It follows that $\langle w, v \rangle = (\pm \frac{1}{v_0} \sum_{i=1}^n w_i v_i)(\mp v_0) + \sum_{i=1}^n w_i v_i = 0$. Therefore $w \in S(T_v H^n(-1))$ if and only if

$$\langle w, w \rangle = 1 \quad \text{and} \quad \langle v, w \rangle = 0.$$
The last conditions are equivalent to the condition that \((v, w) \in \Delta_1\). This means that we can canonically identify \(S(TH^n(-1))\) with \(\Delta_1\). Moreover, the canonical contact structure on \(S(TH^n(-1))\) is given by the one-form \(\theta(V) = \langle d\pi(V), \tau(V) \rangle\) where \(\tau: TS(TH^n(-1)) \to S(TH^n(-1))\) is the tangent bundle of \(S(TH^n(-1))\) (cf. [4], [9]). It can be represented by \(\langle dv, w \rangle|_{\Delta_1}\) through the above identification. Thus \((\Delta_1, \theta^{-1}_1(0))\) is a contact manifold.

We now consider \(\Delta_2 \subset H^n(-1) \times LC^*\). We define a smooth mapping

\[
\Phi_{21}: \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1
\]

by \(\Phi_{21}(v, w) = (v, v - w)\). We can easily check that \(\Phi_{21}(\Delta_2) = \Delta_1\) and \(\Phi_{21}(\Delta_1) = \Delta_2\). Since \(\Phi_{21}\) is an involution, \(\Phi_{21}|_{\Delta_2}\) is a diffeomorphism onto \(\Delta_1\). Moreover, we have

\[
\Phi_{21}^* \theta_{11} = \langle dv, v - w \rangle|_{\Delta_2} = (\langle dv, v \rangle - \langle dv, w \rangle)|_{\Delta_2} = -\langle dv, w \rangle|_{\Delta_2} = -\theta_{21},
\]

so that \(\theta_{21}\) gives a contact structure on \(\Delta_2\) and \(\Phi_{21}|_{\Delta_2}\) is a contact diffeomorphism.

We also consider \(\Delta_3 \subset LC^* \times S^1_1\). We define a smooth mapping

\[
\Phi_{31}: \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1
\]

by \(\Phi_{31}(v, w) = (v - w, w)\). We can also check that \(\Phi_{31}(\Delta_3) = \Delta_1\). The converse mapping \(\Phi_{13}: \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1\) is given by \(\Phi_{13}(v, w) = (v + w, w)\). It can be shown that \(\Phi_{13}(\Delta_1) = \Delta_3\). Therefore \(\Phi_{31}|_{\Delta_2}\) is a diffeomorphism onto \(\Delta_1\). Moreover, we have

\[
\Phi_{31}^* \theta_{11} = \langle d(v - w), w \rangle|_{\Delta_3} = (\langle dv, w \rangle - \langle dw, w \rangle)|_{\Delta_3} = \langle dv, w \rangle|_{\Delta_3} = \theta_{31},
\]

so that \(\theta_{31}\) gives a contact structure on \(\Delta_3\) and \(\Phi_{31}|_{\Delta_3}\) is a contact diffeomorphism. By the same reason as the previous cases, \(\theta_{31}\) and \(\theta_{32}\) give the common contact structure on \(\Delta_3\).

Finally we consider \(\Delta_4 \subset LC^* \times LC^*\). We define a smooth mapping

\[
\Phi_{41}: \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1
\]

by \(\Phi_{41}(v, w) = (v + w, v - w)\). The converse mapping is given by

\[
\Phi_{41}^*(v, w) = \left(\frac{v + w}{2}, \frac{v - w}{2}\right).
\]

We can also check that \(\Phi_{41}(\Delta_1) = \Delta_4\) and \(\Phi_{41}(\Delta_4) = \Delta_1\), so that \(\Phi_{41}|_{\Delta_1}\) and \(\Phi_{41}|_{\Delta_4}\) are diffeomorphisms. Moreover, we have

\[
\Phi_{41}^* \theta_{41} = \langle dv + w, v - w \rangle|_{\Delta_1} = (\langle dv, v \rangle - \langle dw, w \rangle + \langle dw, v \rangle - \langle dw, w \rangle)|_{\Delta_1} = -\langle dv, w \rangle + \langle dw, v \rangle|_{\Delta_1} = 2\theta_{12},
\]

so that \(\theta_{41}\) gives a contact structure on \(\Delta_4\) and \(\Phi_{41}|_{\Delta_1}\) is a contact diffeomorphism. By the same reason as the previous cases, \(\theta_{41}\) and \(\theta_{42}\) give the common contact structure on \(\Delta_4\). Other assertions are trivial by definition.

This completes the proof. \(\square\)
We remark that the previous results on hypersurfaces in hyperbolic space (cf. [15]) can be understood via the duality theorem. For any regular hypersurface $X: U \to H^n(-1)$, we have defined the hyperbolic Gauss indicatrix $L^\pm: U \to LC^*$. By definition, we have a Legendrian embedding $L^+_2: U \to \Delta_2; L^+_2(u) = (X(u), L^+(u))$. Since $\pi_2: \Delta_2 \to LC^*$ is a Legendrian fibration, the hyperbolic Gauss indicatrix $L^\pm = \pi_2 \circ L^+_2$ is the Legendrian map. All results in [15] can be interpreted by using this construction.

3. Geometry of Spacelike Hypersurfaces in the Lightcone

In this section we construct the basic tools for the study of the extrinsic differential geometry on spacelike hypersurfaces in the lightcone $LC^*$. Before we start to develop the theory, we refer the following result [2] why this case is important and interesting.

**Theorem 3.1.** Let $M$ be a simply connected Riemannian manifold with dim $M \geq 3$. Then $M$ is conformally flat if and only if $M$ can be isometrically embedded as a spacelike hypersurface in the lightcone.

By this theorem, if we construct an extrinsic differential geometry on spacelike hypersurfaces in the lightcone, we might obtain some new extrinsic invariants of conformally flat Riemannian manifolds.

Let $X: U \to LC^*$ be a regular spacelike hypersurface. By Theorem 2.2, we can show that the existence and uniqueness of the lightcone normal vector to $M$ follows. We consider the double Legendrian fibration $\pi_{41}: \Delta_4 \to LC^*$ ($i = 1, 2$). The fiber of $\pi_{41}$ is the intersection of $LC^*$ with a lightlike hyperplane (i.e., a parabolic hyperquadric). Therefore the intersection of the fiber with the normal plane (i.e., a time like plane) in $R^{n+1}_1$ of $M$ consists of only one point at each point of $M$. Since $\pi_{41}: \Delta_4 \to LC^*$ is a Legendrian fibration, there is a Legendrian submanifold $L_4: U \to \Delta_4$ such that $\pi_{41} \circ L_4(u) = X(u)$. It follows that we have a smooth map $X^\ell: U \to LC^*$ such that $L_4(u) = (X(u), X^\ell(u))$. Since $L_4$ is a Legendrian embedding, we have $\langle dX(u), X^\ell'(u) \rangle = 0$, so that $X^\ell(u)$ belongs to the normal plane in $R^{n+1}_1$. If we have another Legendrian embedding $L^\ell_3(u) = (X(u), X^\ell_1(u))$, then $X^\ell(u)$ and $X^\ell_1(u)$ are parallel. However, we have a relation $\langle X(u), X^\ell(u) \rangle = (X(u), X^\ell_1(u)) = -2$, so that $X^\ell(u) = X^\ell_1(u)$. This means that $L_4$ is the unique (even in the global sense) Legendrian lift of $X$. We call $X^\ell(u) = \pi_{42} \circ L_4$ the lightcone normal vector to $M = X(U)$ at $X(u)$. Since the equation for $\Delta_4$ is $\langle v, w \rangle = -2$, we define a family of functions

$$H: U \times LC^* \to \mathbb{R}$$

by $H(u, v) = \langle X(u), v \rangle + 2$. We call $H$ a lightcone height function on $X: U \to LC^*$.

Since $X(u), X^\ell(u)$ are linearly independent lightlike vectors and $X$ is a spacelike embedding, we have the following simple lemma.

**Lemma 3.2.** For $p = X(u)$, the $(n+1)$-tuple $X(u), X^\ell(u), X_u(u), \ldots, X_{u_{n-1}}(u)$ is a basis of $T_p R^{n+1}_1$. 
Proposition 3.3. Let $H: U \times LC^* \rightarrow \mathbb{R}$ be a lightcone height function on $X: U \rightarrow LC^*$. Then

1. $H(u, v) = 0$ if and only if $(X(u), v) \in \Delta_4$.
2. $H(u, v) = \frac{\partial H}{\partial u}(u, v) = 0$ (i.e., $n - 1$) if and only if $v = X^\ell(u)$.

Proof. The assertion (1) follows from the definition of $H$ and $\Delta_4$.

Proof. The assertion (2) follows from the definition of $H$ and $\Delta_4$.

(2) There exist real numbers $\lambda, \mu, \xi(i = 1, \ldots, n - 1)$ such that $v = \lambda X^\ell + \mu X + \sum_{i=1}^{n-1} \xi_iX_u$. Since $(X, X) = 0$, we have $(X, X_u) = 0$. Therefore $0 = H(u, v) = (X, \lambda X^\ell) + 2 = -2\lambda + 2$ if and only if $\lambda = 1$. Since $\frac{\partial H}{\partial u}(u, v) = (X_u, v)$, we have

$$0 = (X_u, \lambda X^\ell) + \sum_{j=1}^{n-1} \xi_j g_{ij}(u).$$

The equation $(dX, X^\ell) = 0$ means that $(X_u, X^\ell) = 0$, whence $\sum_{j=1}^{n-1} \xi_j g_{ij}(u) = 0$. Since $g_{ij}$ is positive definite, we have $\xi_j = 0$ ($j = 1, \ldots, n - 1$). We also have $0 = (v, v) = 2\mu (X, X^\ell) = -4\mu$. This completes the proof. \(\square\)

On the other hand, we can directly construct the lightcone normal vector $X^\ell(u)$ as follows: Let $\Pi: \mathbb{R}^{n+1}_1 \rightarrow \mathbb{R}^n_0$ be the canonical projection defined by

$$\Pi(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n).$$

We denote that $r(u) = \Pi \circ X(u)$. Since $\Pi|_{LC^*}: LC^* \rightarrow \mathbb{R}^n_0$ is an embedding, $r: U \rightarrow \mathbb{R}^n_0$ is an embedding (hypersurface). Therefore, we have the ordinary Euclidean unit normal $N(u)$ of $r(U) = \Pi(M)$ and the Euclidean Gauss map $N: U \rightarrow S^{n-1} \subset \mathbb{R}^n$. We now define a transversal vector field $r^\ell(u)$ along $\Pi(M)$ in $\mathbb{R}^n_0$ by

$$r^\ell(u) = \frac{r(u) - 2(r(u) \cdot N(u)N(u))}{(r(u) \cdot N(u))^2},$$

where $a \cdot b$ is the canonical Euclidean scalar product (i.e., $a \cdot b = \langle a, b \rangle_{\mathbb{R}^n}$). As a corollary of Proposition 3.3, we have

$$X^\ell(u) = \left( \frac{\|r(u)\|}{(r(u) \cdot N(u))^2}, r^\ell(u) \right).$$

We call a mapping $X^\ell: U \rightarrow LC^*$ the lightcone dual of $X(U) = M$. We also define the lightcone Gauss map $X^\ell: U \rightarrow S^{n-1}_+$ by $X^\ell(u) = X^\ell(u)$. We can study the extrinsic differential geometry of $X(U) = M$ by using $X^\ell$ like as the Gauss map of a hypersurface in Euclidean space. For the purpose, we have the following fundamental lemma.

Lemma 3.4. For any $p = X(u_0) \in M$ and $v \in T_p M$, we have $D_v X^\ell(u_0) \in T_p M$. Here $D_v$ denotes the covariant derivative with respect to the tangent vector $v$.

Proof. We have

$$D_v X^\ell = \lambda X + \eta X^\ell + \mu_1 X_{u_1} + \cdots + \mu_{n-1} X_{u_{n-1}}$$

for some real numbers $\lambda, \eta, \mu_1, \ldots, \mu_{n-1}$. Since $(X^\ell, X) = -2$, we conclude that $v((X^\ell, X)) = 0$. Since $(X^\ell, D_v X) = 0$, we have $(D_v X^\ell, X) + (X^\ell, D_v X) = 0$. Therefore $D_v X^\ell = 0$. Thus $D_v X^\ell(u_0) \in T_p M$. \(\square\)
$D_u((X,X)) = 0$, so that $-2\eta = 0$. Since $\langle X^\ell, X^\ell \rangle = 0$, we have $\langle D_uX^\ell, X^\ell \rangle = 0$. It follows from the fact $\langle X^\ell, X_{u_i} \rangle = 0$ that $-2\lambda = \langle \lambda X + \sum_{i=1}^{n-1} \mu X_{u_i}, X^\ell \rangle = \langle D_uX^\ell, X^\ell \rangle = 0$.

Here we identify $U$ and $M$ through the embedding $X$. Under the identification, the derivatives $dX^\ell(u_0)$ can be considered as linear transformations on the tangent space $T_pM$ where $p = X(u_0)$. We call the linear transformation $S_p^p = -dX^\ell(u_0): T_pM \to T_pM$ the lightcone shape operator. We denote the eigenvalues of $S_p^p$ by $\kappa_i(p)$, which we call the lightcone principal curvature. We may consider that $dX(u_0)$ is the identity mapping on $T_pM$ under the identification between $U$ and $M$ through $X$.

We now define the notion of curvatures of $X(U) = M$ at $p = X(u_0)$ as follows:

$K_\ell(u_0) = \det S_p^p$: The lightcone Gauss–Kronecker curvature,

$H_\ell(u_0) = \frac{1}{n-1} \text{Trace } S_p^p$: The lightcone mean curvature.

We now prove the lightcone Weingarten formula. Since $X_{u_i}$ ($i = 1, \ldots, n-1$) are spacelike vectors, we induce the Riemannian metric (the lightcone first fundamental form) $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$ on $M = X(U)$, where $g_{ij}(u) = \langle X_{u_i}(u), X_{u_j}(u) \rangle$ for any $u \in U$. We also define the lightcone second fundamental invariant by $h_{ij}^\ell(u) = \langle -(X^\ell)_{u_i}(u), X_{u_j}(u) \rangle$ for any $u \in U$.

**Proposition 3.5.** Under the above notations, we have the following lightcone Weingarten formula:

$$\left(\frac{X^\ell}{u}\right)_{u_i} = -\sum_{j=1}^{n-1} \left(\frac{h^\ell}{u}\right)_jX_{u_j},$$

where $\left(\frac{h^\ell}{u}\right)_j = \left(\frac{h_{ij}^\ell}{u}\right)(g^{kj})$ and $\left(\frac{g_{ij}}{u}\right) = \left(\frac{g_{kj}}{u}\right)^{-1}$.

**Proof.** By Lemma 3.4, there exist real numbers $\Gamma^i_j$ such that

$$\left(\frac{X^\ell}{u}\right)_{u_i} = \sum_{j=1}^{n-1} \Gamma^i_jX_{u_j}.$$  

By definition, we have

$$-h_{ij}^\ell = \sum_{a=1}^{n-1} \Gamma^a_i \langle X_{u_a}, X_{u_j} \rangle = \sum_{a=1}^{n-1} \Gamma^a_i g_{a\beta}.$$  

Hence, we have

$$-\left(\frac{h^\ell}{u}\right)_j = -\sum_{j=1}^{n-1} h_{ij}^\ell g^{kj} = \sum_{\beta=1}^{n-1} \sum_{a=1}^{n-1} \Gamma^a_i g_{a\beta}g^{kj} = \Gamma^i_j.$$  

This completes the proof of the lightcone Weingarten formula.

As a corollary of the above proposition, we have an explicit expression of the lightcone Gauss–Kronecker curvature by Riemannian metric and the lightcone second fundamental invariant.
Corollary 3.6. Under the same notations as in the above proposition, the lightcone Gauss–Kronecker curvature is given by

$$K_\ell = \frac{\det(h^\ell_{ij})}{\det(g_{\alpha\beta})}.$$ 

Proof. By the lightcone Weingarten formula, the representation matrix of the lightcone shape operator with respect to the basis \(\{X_{u_1}, \ldots, X_{u_{n-1}}\}\) is \(((h^\ell)^{\ell}_i)^{\ell}_j\) = \((h^\ell_{i\beta})(g^{\beta j})\). It follows from this fact that

$$K_\ell = \det S_\ell^\ell = \det((h^\ell)^{\ell}_i)^{\ell}_j = \frac{\det(h^\ell_{i\beta})}{\det(g_{\alpha\beta})}. \quad \square$$

Since the lightcone dual is written in terms of the projection \(r(u)\) and the Euclidean normal \(N(u)\), the lightcone Weingarten formula can be written by using these information. For the purpose we need the Euclidean Weingarten formula

$$N_{ui} = -\sum_{j=1}^{n-1} (h_0)^j_i (r_{ui}) (r \cdot r_{ui}) \left[ (h_0)^j_i - 2 \sum_{1 \leq s, t \leq n-1} (g_0)^{st} (r \cdot r_{us}) (r \cdot r_{ut}) \right] (h_0)^j_i,$$

where \(\delta_{ij}\) is the Kronecker’s delta. Since \(\Pi|_{LC^*}\) is a diffeomorphism and \((d\Pi) \circ dX^\ell = dr^\ell\), we have

$$\left(\begin{array}{c} r^\ell_{ui} \\ \end{array}\right) = \frac{1}{(r \cdot N)^3} \left\{ (r \cdot N) \delta_{ij} + 2(r \cdot r)(h_0)^j_i \right. - 2 \sum_{1 \leq s, t \leq n-1} (g_0)^{st} (r \cdot r_{us}) (r \cdot r_{ut}) \left( (h_0)^j_i \right) + 2 \sum_{l=1}^{n-1} (h_0)^l_i (r \cdot r_{ui}) \sum_{k=1}^{n-1} (g_0)^{kj} \right\} r_{uj},$$

where \(\delta_{ij}\) is the Kronecker’s delta. Since \(\Pi|_{LC^*}\) is a diffeomorphism and \((d\Pi) \circ dX^\ell = dr^\ell\), we have

$$\left(\begin{array}{c} h^\ell_{ij} \\ \end{array}\right) = \frac{1}{(r \cdot N)^3} \left\{ (r \cdot N) \delta_{ij} + 2(r \cdot r)(h_0)^j_i \right. - 2 \sum_{1 \leq s, t \leq n-1} (g_0)^{st} (r \cdot r_{us}) (r \cdot r_{ut}) \left( (h_0)^j_i \right) + 2 \sum_{l=1}^{n-1} (h_0)^l_i (r \cdot r_{ui}) \sum_{k=1}^{n-1} (g_0)^{kj} \right\}.$$ 

This means that the representation matrix of the lightcone Weingarten map is written by the first and second fundamental invariants of the Euclidean hypersurface \(r\). However, it is rather a complicated relation. Since a parallel translation in \(\mathbb{R}^n_0\) is not induced from a Lorenzian motion of \(\mathbb{R}^{n+1}_1\), the first and the second fundamental invariants of \(r\) are not Lorenzian invariants. Moreover, the lightcone principal directions of \(X\) are different from the Euclidean principal directions of \(r\), so that we
have completely different geometric structures from the induced geometric structure in the Euclidean space $\mathbb{R}^3_0$ by $r$. We say that a point $p = X(u_0)$ (or $u_0$) is a lightcone umbilic point if $S_p^0 = \kappa_\ell(p) 1_{T_p M}$. We also say that $M = X(U)$ is totally lightcone umbilic if all points on $M$ are lightcone umbilic. We also remark that the lightcone umbilicity for $X$ is different from the Euclidean umbilicity for $r$ by the above reason. We have the following classification of totally lightcone umbilic spacelike hypersurfaces in $LC^*$. 

**Proposition 3.7.** Suppose that $M = X(U)$ is totally lightcone umbilic. Then $\kappa_\ell(p)$ is constant $\kappa_\ell$. Under this condition, we have the following classification.

1. If $\kappa_\ell < 0$, then $M$ is a part of hyperbolic hyperquadric $HL(c, 1/\sqrt{-\kappa_\ell})$, where 
   
   $$c = \frac{-1}{2\sqrt{-\kappa_\ell}}(\kappa_\ell X(u) + X^\ell(u)) \in S_1^n$$ 

   is a constant spacelike vector.

2. If $\kappa_\ell = 0$, then $M$ is a part of parabolic hyperquadric $HL(c, -2)$, where $c = X^\ell(u) \in LC^*$ is a constant lightlike vector.

3. If $\kappa_\ell > 0$, then $M$ is a part of elliptic hyperquadric $HL(c, -1/\sqrt{\kappa_\ell})$, where 

   $$c = \frac{1}{2\sqrt{\kappa_\ell}}(\kappa_\ell X(u) + X^\ell(u)) \in H^n(-1)$$ 

   is a constant timelike vector.

**Proof.** By definition, we have $-(X^\ell)_{u_i} = \kappa_\ell(u)X_{u_i}$ for $i = 1, \ldots, n-1$. Therefore, we have

$$-(X^\ell)_{u_i u_j} = (\kappa_\ell)_{u_j}(u)X_{u_i} + \kappa_\ell(u)X_{u_i u_j}.$$ 

The equations $-(X^\ell)_{u_i u_j} = -(X^\ell)_{u_i}u_j$ and $\kappa_\ell(u)X_{u_i u_j} = (\kappa_\ell)_{u_j}(u)X_{u_i}$ imply that $(\kappa_\ell)_{u_j}(u)X_{u_i} = (\kappa_\ell)_{u_i}(u)X_{u_j}$. By definition $\{X_{u_1}, \ldots, X_{u_{n-1}}\}$ is linearly independent, so that $\kappa_\ell$ is constant. Under this condition, we distinguish three cases.

**Case 1.** We assume that $\kappa_\ell < 0$. By definition, we have $-dX^\ell = \kappa_\ell dX$. Since $\kappa_\ell$ is constant, it follows from the above equality that $d(\kappa_\ell X + X^\ell) = 0$. Therefore $c = \frac{-1}{2\sqrt{-\kappa_\ell}}(\kappa_\ell X(u) + X^\ell(u))$ is a constant and we have $\langle c, c \rangle = 1$. On the other hand, we have

$$\langle X(u), c \rangle = \frac{-1}{2\sqrt{-\kappa_\ell}}\langle X(u), \kappa_\ell X(u) + X^\ell(u) \rangle = -\frac{1}{2\sqrt{-\kappa_\ell}} = \frac{1}{\sqrt{-\kappa_\ell}}.$$ 

This means that $M = X(U) \subset HL(c, 1/\sqrt{-\kappa_\ell})$.

**Case 2.** We assume that $\kappa_\ell = 0$. By definition, we have $dX^\ell(u) = 0$, so that $c = X^\ell$ is constant. We also have $\langle X(u), c \rangle = \langle X(u), X^\ell(u) \rangle = -2$. This means that $M = X(U) \subset HL(c, -2)$.

**Case 3.** We assume that $\kappa_\ell > 0$. By the same reasons as the above cases,

$$c = \frac{1}{2\sqrt{\kappa_\ell}}(\kappa_\ell X(u) + X^\ell(u))$$
Proof. By definition, we have an umbilic point if and only if there exists an orthogonal matrix \( v \) and

\[
\langle X(u), c \rangle = -2 \times \frac{1}{2\sqrt{\kappa_t}} = -\frac{1}{\sqrt{\kappa_t}}.
\]

Therefore we have \( M = X(U) \subset HL(c, -1/\sqrt{\kappa_t}) \). This completes the proof. \( \square \)

By the above proposition, we can classify the lightcone umbilic point. Let \( p = X(u_0) \in X(U) = M \) be a lightcone umbilic point, we say that \( p \) is a \emph{timelike lightcone umbilic point} if \( \kappa_t < 0 \), a \emph{lightcone flat point} if \( \kappa_t = 0 \) or a \emph{spacelike lightcone umbilic point} if \( \kappa_t > 0 \).

We say that a point \( p = X(u) \) is a \emph{lightcone parabolic point} if \( K_t(u) = 0 \) and a \emph{lightcone flat point} if it is an umbilic point and \( K_t(u) = 0 \).

We denote the \emph{Hessian matrix} of the lightcone height function \( h_{v_0}(u) = H(u, v_0) \) at \( u_0 \) by \( \text{Hess}(h_{v_0})(u_0) \).

**Proposition 3.8.** Let \( X: U \to LC^* \) be a spacelike hypersurface in the lightcone and \( v_0 = X^\ell(u_0) \). Then

1. \( p = X(u_0) \) is a lightcone parabolic point if and only if \( \det \text{Hess}(h_{v_0})(u_0) = 0 \).
2. \( p = X(u_0) \) is a lightcone flat point if and only if \( \text{rank} \text{Hess}(h_{v_0})(u_0) = 0 \).

**Proof.** By definition, we have

\[
\text{Hess}(h_{v_0})(u_0) = \langle (X_{u_i}^\ell(u_0), X_{u_j}^\ell(u_0)), (-X_{u_i}^\ell(u_0), X_{u_j}^\ell(u_0)) \rangle.
\]

By definition, we have \(-\langle X_{u_i}, X_{u_j}^\ell \rangle = h_{ij}^\ell \), so that we have

\[
K_t(u_0) = \frac{\det \text{Hess}(h_{v_0})(u_0)}{\det(g_{\alpha_\beta}(u_0))}.
\]

The first assertion follows from this formula.

For the second assertion, by the lightcone Weingarten formula, \( p = X(u_0) \) is an umbilic point if and only if there exists an orthogonal matrix \( A \) such that \( A^t((h^\ell)^\alpha^\beta)A = \kappa_t I \). Therefore, we have \((h^\ell)^\alpha^\beta = \kappa_t I \) and \( \kappa_t = \kappa_t(g_{ij}) \). Thus, \( p \) is a lightcone flat point (i.e., \( \kappa_t(u_0) = 0 \)) if and only if \( \text{rank} \text{Hess}(h_{v_0})(u_0) = 0 \). \( \square \)

**4. Lightcone Duals as Wave Fronts**

In this section we naturally interpret the lightcone dual of a spacelike hypersurface in the lightcone as a wave front set in the framework of contact geometry. For the purpose, we now give a quick review on the Legendrian singularity theory mainly due to Arnol’d–Zakalyukin [1], [45]. Almost all results have been known at least implicitly. Let \( \pi: PT^*(M) \to M \) be the projective cotangent bundle over an \( n \)-dimensional manifold \( M \). This fibration can be considered as a Legendrian fibration with the canonical contact structure \( K \) on \( PT^*(M) \). We now review geometric properties of this space. Consider the tangent bundle \( \tau: TPT^*(M) \to PT^*(M) \) and the differential map \( d\pi: TPT^*(M) \to TM \) of \( \pi \). For any \( X \in TPT^*(M) \), there exists an element \( \alpha \in T^*(M) \) such that \( \tau(X) = [\alpha] \). For an element \( V \in T_x(M) \),
the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(M)$ by

$$K = \{ X \in TPT^*(M) : \tau(X)(d\pi(X)) = 0 \}.$$ 

For a local coordinate neighbourhood $(U, (x_1, \ldots, x_n))$ on $M$, we have a trivialisation $PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$ and we call

$$((x_1, \ldots, x_n), [\xi_1 : \cdots : \xi_n])$$

homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$.

It is easy to show that $X \in K(x, [\xi])$ if and only if $\sum_{i=1}^{n} \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial \xi_i}$. An immersion $i : L \rightarrow PT^*(M)$ is said to be a Legendrian immersion if $\dim L = n$ and $d\pi(T_q L) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the Legendrian map and the set $W(i) = \text{image} \pi \circ i$ the wave front of $i$. Moreover, $i$ (or, the image of $i$) is called the Legendrian lift of $W(i)$.

The main tool of the theory of Legendrian singularities is the notion of generating families. Here we only consider local properites, we may assume that $M = \mathbb{R}^n$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ. We say that $F$ is a Morse family of hypersurfaces $\{ f_x^{-1}(0) \}_{x \in (\mathbb{R}, 0)}$ if the map germ

$$\Delta^*F = \left( F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case we have a smooth $(n - 1)$-dimensional submanifold germ

$$\Sigma^*_F(0) = \left\{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) : F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\} = (\Delta^*F)^{-1}(0)$$

and a map germ $L_F : (\Sigma^*_F, 0) \rightarrow PT^*\mathbb{R}^n$ defined by

$$L_F(q, x) = \left( x, \left[ \frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol’d–Zakalyukin [1], [45].

**Proposition 4.1.** All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.

We call $F$ a generating family of $L_F(\Sigma^*_F)$. Therefore the wave front is

$$W(L_F) = \left\{ x \in \mathbb{R}^n : \exists q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$ 

We sometime denote $D_F = W(L_F)$ and call it the discriminant set of $F$.

We consider a point $v = (v_0, v_1, \ldots, v_n) \in LC^*$, then we have the relation $v_0 = \pm \sqrt{v_1^2 + \cdots + v_n^2}$. We have two components $LC^* = LC^*_+ \cup LC^*_*$, where $LC^*_+ = \{ v = (v_0, v_1, \ldots, v_n) \in LC^* : v_0 > 0 \}$ which is called a future component and $LC^*_* = \{ v = (v_0, v_1, \ldots, v_n) \in LC^* : v_0 < 0 \}$ which is called a past component.
So we adopt the coordinate systems \((v_1, \ldots, v_n)\) on both \(LC_+^\ast\) and \(LC_-^\ast\). We consider the projective cotangent bundle \(\pi: PT^\ast(LC^\ast) \rightarrow LC^\ast\) with the canonical contact structure. We claim here that we have a trivialization:

\[
\Phi: PT^\ast(LC^\ast) \equiv LC^\ast \times P(\mathbb{R}^{n-1})^\ast;
\]

\[
\Phi \left( \sum_{i=1}^{n} \xi_i dv_i \right) = \left( (v_0, v_1, \ldots, v_n), [\xi_1 : \cdots : \xi_n] \right).
\]

by using the above coordinate systems.

On the other hand, we define the following mapping:

\[
\Psi: \Delta_4 \rightarrow LC^\ast \times P(\mathbb{R}^{n-1})^\ast; \quad \Psi(v, w) = (v_0 w_1 - v_1 w_0 : \cdots : v_0 w_n - v_n w_0).
\]

For the canonical contact form \(\theta = \sum_{i=1}^{n} \xi_i dv_i\) on \(PT^\ast(LC^\ast)\), we have

\[
\Psi^\ast \theta = (v_0 w_1 - v_1 w_0) dw_1 + \cdots + (v_n w_0 - v_0 w_n) dw_n |\Delta_4,
\]

\[
= -w_0 (-v_0 dw_0 + v_1 dw_1 + \cdots + v_n dw_n) |\Delta_4 = -w_0(v, dw)|\Delta_4 = -w_0 \theta_{12}.
\]

Thus \(\Psi\) is a contact morphism.

**Proposition 4.2.** The lightcone height function \(H: U \times LC^\ast \rightarrow \mathbb{R}\) is a Morse family of hypersurfaces \(\{(h_u)^{-1}(0)\}_{u \in LC^\ast}\). Here, we denote that \(h_u(v) = H(u, v)\) for \(v \in LC^\ast\).

**Proof.** Without the loss of the generality, we consider on the future component \(LC^\ast_+\). For any \(v = (v_0, v_1, \ldots, v_n) \in LC^\ast_+\), we have \(v_0 = \sqrt{v_1^2 + \cdots + v_n^2}\), so that

\[
H(u, v) = -x_0(u) \sqrt{v_0^2 + v_1^2 + \cdots + v_n^2} + x_4(u)v_1 + \cdots + x_n(u)v_n + 2,
\]

where \(X(u) = (x_0(u), \ldots, x_n(u))\). We define a mapping

\[
\Delta^\ast H: U \times LC^\ast \rightarrow \mathbb{R} \times \mathbb{R}^{n-1}
\]

by \(\Delta^\ast H = (H, \frac{\partial H}{\partial u_0}, \ldots, \frac{\partial H}{\partial u_{n-1}})\). We have to prove that \(\Delta^\ast H\) is non-singular at any point on \(\Sigma_{\ast}(H) = (\Delta^\ast H)^{-1}(0)\). If \((u, v) \in \Sigma_{\ast}(H)\), then \(v = X^\prime(u)\) by Proposition 3.3. The Jacobian matrix of \(\Delta^\ast H\) is given as follows:

\[
\begin{pmatrix}
X_{u_1}, v & X_{u_{n-1}}, v \\
X_{u_1 u_1}, v & X_{u_{n-1} u_{n-1}}, v \\
X_{u_{n-1} u_1}, v & X_{u_{n-1} u_{n-1}}, v
\end{pmatrix} = \begin{pmatrix}
-x_0 \frac{v_1}{v_0} + x_1 & -x_0 \frac{u_1}{v_0} + x_n \\
-x_0u_1 \frac{v_1}{v_0} + x_1u_1 & -x_0u_1 \frac{u_1}{v_0} + x_{nu_1} \\
-x_0u_{n-1} \frac{v_1}{v_0} + x_{1u_{n-1}} & -x_0u_{n-1} \frac{u_1}{v_0} + x_{nu_{n-1}}
\end{pmatrix},
\]

We now show that the determinant of the matrix

\[
A = \begin{pmatrix}
-x_0 \frac{v_1}{v_0} + x_1 & -x_0 \frac{u_1}{v_0} + x_n \\
-x_0u_1 \frac{v_1}{v_0} + x_1u_1 & -x_0u_1 \frac{u_1}{v_0} + x_{nu_1} \\
-x_0u_{n-1} \frac{v_1}{v_0} + x_{1u_{n-1}} & -x_0u_{n-1} \frac{u_1}{v_0} + x_{nu_{n-1}}
\end{pmatrix}
\]

is non-zero.
does not vanish at \((u, v) \in \Sigma_4(H)\). We denote that

\[
a = \begin{pmatrix}
x_0 \\
x_{0u_1} \\
\vdots \\
x_{0u_{n-1}}
\end{pmatrix}, \quad b_1 = \begin{pmatrix}
x_1 \\
x_{1u_1} \\
\vdots \\
x_{1u_{n-1}}
\end{pmatrix}, \quad \ldots, \quad b_n = \begin{pmatrix}
x_n \\
x_{nu_1} \\
\vdots \\
x_{nu_{n-1}}
\end{pmatrix}.
\]

Then we have

\[
\det A = \frac{v_0}{v_0} \det(b_1 \ldots b_n) - \frac{v_1}{v_0} \det(a \ b_2 \ldots b_n) - \cdots - \frac{v_n}{v_0} \det(b_1 \ldots b_{n-1} \ a).
\]

On the other hand, we have

\[
X \wedge X_{u_1} \wedge \cdots \wedge X_{u_{n-1}} = (-\det(b_1 \ldots b_n), -\det(a \ b_2 \ldots b_n), \ldots, -\det(b_1 \ldots b_{n-1} \ a)).
\]

We now consider a hyperplane \(HP(c, 0)\), where \(c = X \wedge X_{u_1} \wedge \cdots \wedge X_{u_{n-1}}\). By definition, the basis of the vector subspace \(HP(c, 0)\) is \(\{X, X_{u_1}, \ldots, X_{u_{n-1}}\}\). Since \(X, X_{u_i}\) \((i = 1, \ldots, n - 1)\) are tangent to the lightcone \(LC^*\), the hyperplane \(HP(c, 0)\) is a lightlike hyperplane. By Lemma 2.1, \(c\) and \(X\) are linearly dependent, so that there exists a non-zero real number \(\lambda\) such that \(\lambda X = X \wedge X_{u_1} \wedge \cdots \wedge X_{u_{n-1}}\).

Therefore we have

\[
\det A = \left\langle \frac{v_0}{v_0} (X^\ell, X \wedge X_{u_1} \wedge \cdots \wedge X_{u_{n-1}}) \right\rangle = \frac{1}{v_0} \left\langle X^\ell, \lambda X \right\rangle = \frac{2\lambda}{v_0} \neq 0. \quad \Box
\]

We now show that \(H\) is a generating family of \(\mathcal{L}_4(U) \subset \Delta_4\), where we fix the Legendrian fibration \(\pi_{42}: \Delta_4 \to LC^*\).

**Theorem 4.3.** For any spacelike hypersurface \(X: U \to LC^*\), the lightcone height function \(H: U \times LC^* \to \mathbb{R}\) of \(X\) is a generating family of the Legendrian immersion \(\mathcal{L}_4\).

**Proof.** We remember the contact morphism

\[
\Psi: \Delta_4 \to LC^* \times P(\mathbb{R}^{n-1})^*.
\]

Since the lightcone height function \(H: U \times LC^* \to \mathbb{R}\) is a Morse family of hypersurfaces, we have a Legendrian immersion

\[
\mathcal{L}_H: \Sigma_4(H) \to LC^* \times P(\mathbb{R}^{n-1})^*
\]

defined by

\[
\mathcal{L}_H(u, v) = \left(v, \left[\frac{\partial H}{\partial v_1}, \ldots, \frac{\partial H}{\partial v_n}\right]\right),
\]

where \(v = (v_0, \ldots, v_n)\) and \(v_0 = \pm \sqrt{v_1^2 + \cdots + v_n^2}\). By Proposition 3.3, we have

\[
\Sigma_4(H) = \{(u, X^\ell(u)) \in U \times LC^* : u \in U\}.
\]
Since \( v = X^\ell(u) \) and \( v_0 = \pm \sqrt{v^2_1 + \cdots + v^2_n} \), we have
\[
\frac{\partial H}{\partial v_i}(u, X^\ell(u)) = -x_0(u)\frac{x^\ell_i(u)}{x^\ell_0(u)} + x_i(u),
\]
where \( X(u) = (x_0(u), \ldots, x_n(u)) \) and \( X^\ell(u) = (x^\ell_0(u), \ldots, x^\ell_n(u)) \). It follows that
\[
\mathcal{L}_H(u, X^\ell(u)) = (X^\ell(u), [x^\ell_0(u)x_1(u) - x^\ell_1(u)x_0(u)] + \cdots : x^\ell_0(u)x_n(u) - x^\ell_n(u)x_0(u)).
\]
Therefore we have \( \Psi \circ \mathcal{L}_1(u) = \mathcal{L}_H(u) \). This means that \( H \) is a generating family of \( \mathcal{L}_4(U) \subset \Delta_4 \).
\( \blacksquare \)

5. The Lightcone Dual and the Lightcone Gauss Map of a Spacelike Hypersurface in the Lightcone

In this section we consider the relationship between the lightcone dual and the lightcone Gauss map of a spacelike hypersurface in the lightcone. For any spacelike hypersurface \( X: U \rightarrow LC^* \), we define a function \( \mathfrak{f}: U \times S^{n-1}_+ \rightarrow \mathbb{R} \) by
\[
\mathfrak{f}(u, v) = -\frac{\langle X(u), v \rangle}{2}.
\]
We call \( \mathfrak{f} \) the lightlike height function of \( X(U) = M \). We also define a function \( \mathfrak{f}^*: U \times S^{n-1}_+ \times \mathbb{R}^* \rightarrow \mathbb{R} \) by
\[
\mathfrak{f}^*(u, v, y) = \mathfrak{f}(u, v) - y = -\frac{\langle X(u), v \rangle}{2} - y,
\]
where \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). We call \( \mathfrak{f}^* \) the extended lightlike height function on \( X(U) = M \).

Using calculations similar to the proof of Proposition 3.3, we have
\[
\mathcal{D}_\mathfrak{f} = \left\{ \left( \tilde{X}^\ell(u), -\frac{\langle X(u), \tilde{X}^\ell(u) \rangle}{2} \right) \in S^{n-1}_+ \times \mathbb{R}^*: u \in U \right\}.
\]
Let \( \pi_1: S^{n-1}_+ \times \mathbb{R}^* \rightarrow S^{n-1}_+ \) be the canonical projection, then \( \pi_1|_{\mathcal{D}_\mathfrak{f}} \) can be identified with the lightcone Gauss map of \( X(U) = M \).

We define a diffeomorphism \( \psi: S^{n-1}_+ \times \mathbb{R}^* \rightarrow LC^* \) by \( \psi(v, y) = (1/y)v \). Since \( X^\ell(u) = -\frac{2}{\langle X(u), \tilde{X}^\ell(u) \rangle} \tilde{X}^\ell(u) \), we have
\[
\psi(\mathcal{D}_\mathfrak{f}) = \{ X^\ell(u): u \in U \} = \mathcal{D}_H.
\]
By these arguments, we say that the lightcone dual is the lift of the lightcone Gauss map. In fact, we also have
\[
\Sigma_*\left( \mathfrak{f}^* \right) = \left\{ \left( u, \tilde{X}^\ell(u), -\frac{\langle X(u), \tilde{X}^\ell(u) \rangle}{2} \right): u \in U \right\}.
\]
We now consider a local coordinate neighbourhood of \( S^{n-1}_+ \). Without the loss of generality, we choose
\[
U_1 = \{ v = (1, v_1, \ldots, v_n) \in S^{n-1}_+: v_1 > 0 \},
\]
so that
$$\overline{H}(u, v, y) = \frac{1}{2} \left( x_0(u) + \sqrt{1 - (v_2^2 + \cdots + v_n^2)} x_1(u) + v_2 x_2(u) + \cdots + v_n x_n(u) \right) + y.$$ 

We can calculate that
$$\frac{\partial \overline{H}}{\partial v_i} = \frac{1}{2} \left( -\frac{v_i}{v_1} x_1(u) + x_i(u) \right),$$
$$\frac{\partial \overline{H}}{\partial y} = 1,$$
where $i = 2, \ldots, n$ and $v_1 = \sqrt{1 - (v_2^2 + \cdots + v_n^2)}.$ Therefor we have a Legendrian embedding
$$L_{\overline{H}}: (\tilde{X}^{\ell})^{-1}(U_1) \subset U \to T^*U_1 \times R^*$$
defined by
$$L_{\overline{H}}(u) = \left( \tilde{X}^{\ell}(u), \frac{1}{2} \left( -\frac{v_2}{v_1} x_1(u) + x_2(u) \right), \ldots, \frac{1}{2} \left( -\frac{v_n}{v_1} x_1(u) + x_n(u) \right), \langle X(u), \tilde{X}^{\ell}(u) \rangle \right).$$

Here we consider the canonical contact structure on $T^*U_1 \times R^*$ given by the contact form $\theta = dy - \alpha,$ where $\alpha$ is the Liouville one-form on $T^*U_1.$ We can also consider a Lagrangian embedding (for basic properties of Lagrangian singularities, see [1]):
$$L_{\tilde{H}}: (\tilde{X}^{\ell})^{-1}(U_1) \subset U \to T^*U_1$$
defined by
$$L_{\tilde{H}}(u) = \left( \tilde{X}^{\ell}(u), \frac{1}{2} \left( -\frac{v_2}{v_1} x_1(u) + x_2(u) \right), \ldots, \frac{1}{2} \left( -\frac{v_n}{v_1} x_1(u) + x_n(u) \right) \right),$$
whose generating family is the lightlike height function $\tilde{H}.$ Here we also consider the canonical symplectic structure $\omega = d\alpha$ on $T^*U_1.$ We now consider the canonical projection $\Pi_1: T^*S^{n-1}_+ \times R^* \to T^*S^{n-1}_+,$ then
$$\Pi_1 \circ L_{\tilde{H}} = L_{\tilde{H}}.$$ 

We remark that if we adopt other local coordinates on $S^{n-1}_+,$ exactly the same results hold. Therefore we have the following proposition.

**Proposition 5.1.** Under the same assumptions as in the previous paragraph, we have the following:

1. The lightcone Gauss map is a Lagrangian map. The corresponding Lagrangian embedding is called the Lagrangian lift of the lightcone Gauss map.
2. The Legendrian lift of the lightcone dual (i.e., $L_4$) is a covering of the Lagrangian lift of the lightcone Gauss map.
6. Contact with Parabolic Hyperquadrics

Before we start to consider the contact between spacelike hypersurfaces and parabolic hyperquadrics, we briefly review the theory of contact due to Montaldi [31]. Let $X_i, Y_i$ ($i = 1, 2$) be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the contact of $X_1$ and $Y_1$ at $y_1$ is the same type as the contact of $X_2$ and $Y_2$ at $y_2$ if there is a diffeomorphism germ $\Phi: (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1; X_2; Y_1; Y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [31], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory. Let $f, g: (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be map germs. We say that $f, g$ are $K$-equivalent if there exists a diffeomorphism germ $\phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $I(f \circ \phi) = I(g)$, where $I(f) = (f_1, \ldots, f_p)_{\mathcal{E}_n}$ is the ideal generated by the component function germs $f_1, \ldots, f_p$ of $f$ (i.e., $f = (f_1, \ldots, f_p)$) in the local ring $\mathcal{E}_n = \{h: h: (\mathbb{R}^n, 0) \to \mathbb{R}\}$ of function germs at 0.

**Theorem 6.1.** Let $X_i, Y_i$ ($i = 1, 2$) be submanifolds of $\mathbb{R}^n$; suppose that $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i: (X_i, x_i) \to (\mathbb{R}^n, y_i)$ be immersion germs and $f_i: (Y_i, y_i) \to (\mathbb{R}^p, 0)$ be submersion germs with $(Y_1, y_1) = (f_1^{-1}(0), y_i)$. Then $K(X_1, Y_1; x_i) = K(X_2, Y_2; y_i)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $K$-equivalent.

We now consider a function $\mathcal{H}: \mathcal{L}^n \times \mathcal{L}^n \to \mathbb{R}$ defined by $\mathcal{H}(u, v) = (u, v) + 2$. For any $v_0 \in \mathcal{L}^n$, we denote that $\mathcal{h}_{v_0}(u) = \mathcal{H}(u, v_0)$ and we have a parabolic hyperquadric $\mathcal{h}_{v_0}^{-1}(0) = H\mathcal{P}(v_0, -2) \cap \mathcal{L}^n = H\mathcal{L}(v_0, -2)$. For any $v_0 \in U$, we consider the lightlike vector $v_0 = X^f(u_0)$, then we have

$$\mathcal{h}_{v_0} \circ X(u_0) = \mathcal{H} \circ (X \times 1_{\mathcal{L}^n})(u_0, v_0) = H(u_0, X^f(u_0)) = 0.$$  

By Proposition 3.3, we also have relations

$$\frac{\partial \mathcal{h}_{v_0} \circ X}{\partial u_i}(u_0) = \frac{\partial \mathcal{H}}{\partial u_i}(u_0, X^f(u_0)) = 0 \quad \text{for} \quad i = 1, \ldots, n - 1.$$  

This means that the parabolic hyperquadric $\mathcal{h}_{v_0}^{-1}(0) = H\mathcal{L}(v_0, -2)$ is tangent to $M = X(U)$ at $p = X(u_0)$. In this case, we call $H\mathcal{L}(v_0, -2)$ the tangent parabolic hyperquadric of $M = X(U)$ at $p = X(u_0)$ (or at $u_0$), which we write $TP\mathcal{H}(X, u_0)$. Let $v_1, v_2$ be lightlike vectors. If $v_1, v_2$ are linearly dependent, then corresponding lightlike hyperplanes $H\mathcal{P}(v_1, -2), H\mathcal{P}(v_2, -2)$ are parallel. Therefore, we say that parabolic hyperquadrics $H\mathcal{L}(v_1, -2), H\mathcal{L}(v_2, -2)$ are parallel if $v_1, v_2$ are linearly dependent. Then we have the following simple lemma.

**Lemma 6.2.** Let $X: U \to \mathcal{L}^n$ be a spacelike hypersurface. Consider two points $u_1, u_2 \in U$. Then

1. $X^f(u_1) = X^f(u_2)$ if and only if $TP\mathcal{H}(X, u_1) = TP\mathcal{H}(X, u_2)$.
2. $\tilde{X}^f(u_1) = \tilde{X}^f(u_2)$ if and only if $TP\mathcal{H}(X, u_1) = TP\mathcal{H}(X, u_2)$ are parallel.

Eventually, we have tools for the study of the contact between spacelike hypersurfaces and parabolic hyperquadrics.

Let $X^f_1: (U, u_1) \to (\mathcal{L}^n, v_1)$ ($i = 1, 2$) be lightcone dual germs of spacelike hypersurface germs $X_i: (U, u_i) \to (\mathcal{L}^n, u_i)$. We say that $X^f_1$ and $X^f_2$
are $A$-equivalent if there exist diffeomorphism germs $\phi: (U, u_1) \to (U, u_2)$ and $\Phi: (LC^*, v_1) \to (LC^*, v_2)$ such that $\Phi \circ X_1^t = X_2^t \circ \phi$. In order to understand the meanings of the $A$-equivalence among the lightcone dual germs, we need the theory of Legendrian equivalence [1], [45], [46]. Let $i: (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i': (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that $i$ and $i'$ are Legendrian equivalent if there exists a contact diffeomorphism germ $H: (PT^*\mathbb{R}^n, p) \to (PT^*\mathbb{R}^n, p')$ such that $H$ preserves fibers of $\pi$ and that $H(L) = L'$. A Legendrian immersion germ $i: (L, p) \subset PT^*\mathbb{R}^n$ (or, a Legendrian map $\pi \circ i$) at a point is said to be Legendrian stable if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney $C^\infty$ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i: (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

**Proposition 6.3.** Let $i: (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i': (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representative of both of germs are proper mappings and the regular sets of the projections $\pi \circ i, \pi \circ i'$ are dense. Then $i$, $i'$ are Legendrian equivalent if and only if wave front sets $W(i)$, $W(i')$ are diffeomorphic as set germs.

This result has been firstly pointed out by Zakalyukin [46]. The assumption in the above proposition is a generic condition for $i$, $i'$. Specially, if $i$, $i'$ are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We consider the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n : h(0) = 0\}$ of the local ring $\mathcal{E}_n$. Let $F, G: (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that $F$ and $G$ are $P$-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Psi: (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$ such that $\Psi^*(F)|_{\mathbb{R}^k+n} = (G)|_{\mathbb{R}^k+n}$. Here $\Psi^*: \mathbb{R}^k+n \to \mathbb{R}^k+n$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F: (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that $F$ is a $\mathcal{K}$-versal deformation of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_k = T_{e}(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \bigg|_{\mathbb{R}^k \times \{0\}}, \ldots, \frac{\partial F}{\partial x_n} \bigg|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}},$$

where

$$T_{e}(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k} \bigg|_{f} \right\rangle_{\mathcal{E}_k}.$$

(See [25].)

The main result in Arnol’d–Zakalyukin’s theory [1], [45] is the following:

**Theorem 6.4.** Let $F, G: (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then...
(1) $\Phi_F$ and $\Phi_G$ are Legendrian equivalent if and only if $F$, $G$ are $P$-$K$-equivalent.

(2) $\Phi_F$ is Legendrian stable if and only if $F$ is a $K$-versal deformation of $F_{|R^k \times \{0\}}$.

Since $F$, $G$ are function germs on the common space germ $(R^k \times R^n, 0)$, we do no need the notion of stably $P$-$K$-equivalences under this situation (cf. [1]). By the uniqueness result of the $K$-versal deformation of a function germ, Proposition 6.3 and Theorem 6.4, we have the following classification result of Legendrian stable germs (cf. [15]). For any map germ $f:(R^n, 0) \rightarrow (R^p, 0)$, we define the local ring of $f$ by $Q_r(f) = E_n/f^*(M^*_p)E_n + M^*_n+1$.

Proposition 6.5. Let $F, G: (R^k \times R^n, 0) \rightarrow (R, 0)$ be Morse families of hypersurfaces. Suppose that $\Phi_F, \Phi_G$ are Legendrian stable. The following conditions are equivalent.

1. $(W(\Phi_F), 0)$ and $(W(\Phi_G), 0)$ are diffeomorphic as germs.
2. $\Phi_F$ and $\Phi_G$ are Legendrian equivalent.
3. $Q_{n+1}(f)$ and $Q_{n+1}(g)$ are isomorphic as $R$-algebras, where $f = F_{|R^k \times \{0\}}, g = G_{|R^k \times \{0\}}$.

If both the regular sets of $X_i^1$ and $X_i^2$ are dense in $(U, u_i)$, it follows from Proposition 6.3 that $X_i^1$ and $X_i^2$ are $A$-equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}_i^1: (U, u_i) \rightarrow (\Delta_1, z_1)$ and $\mathcal{L}_i^2: (U, u_2) \rightarrow (\Delta_2, z_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families $H_1$ and $H_2$ are $P$-$K$-equivalent by Theorem 6.4. Here, $H_i: (U \times LC^*, (u_i, v_i)) \rightarrow R$ is the lightcone height function germ of $X_i$.

On the other hand, we denote that $h_{i, v_i}(u) = H_i(u, v_i)$, then we have $h_{i, v_i}(u) = h_{v_i} \circ X_i(u)$. By Theorem 6.1,

\[
K(X_1(U), TPH(X_1, u_1), v_1) = K(X_2(U), TPH(X_2, u_2), v_2)
\]

if and only if $h_{1, v_1}$ and $h_{1, v_2}$ are $K$-equivalent. Therefore, we can apply the arguments in the appendix to our situation. We denote $Q(X, u_0)$ the local ring of the function germ $h_{v_0}: (U, u_0) \rightarrow R$, where $v_0 = X'(u_0)$. We remark that we can explicitly write the local ring as follows:

\[
Q_{n+1}(X, u_0) = \frac{C_{u_0}^\infty(U)}{\langle (X(u), X'(u_0)) + 2C_{u_0}^\infty(U) + M_{u_0}^{n+2}(U) \rangle},
\]

where $C_{u_0}^\infty(U)$ is the local ring of function germs at $u_0$ with the unique maximal ideal $M_{u_0}(U)$.

Theorem 6.6. Let $X_i: (U, u_i) \rightarrow (LC^*, u_i) (i = 1, 2)$ be spacelike hypersurface germs such that the corresponding Legendrian map germs $\pi_{1,2} \circ \mathcal{L}_i^1: (U, u_i) \rightarrow (LC^*, v_i)$ are Legendrian stable. Then the following conditions are equivalent:

1. Lightcone dual germs $X_i^1$ and $X_i^2$ are $A$-equivalent.
2. $H_1$ and $H_2$ are $P$-$K$-equivalent.
3. $h_{1, v_1}$ and $h_{1, v_2}$ are $K$-equivalent.
4. $K(X_1(U), TPH(X_1, u_1), v_1) = K(X_2(U), TPH(X_2, u_2), v_2)$.
(5) \( Q_{n+1}(X_1, u_1) \) and \( Q_{n+1}(X_2, u_2) \) are isomorphic as \( \mathbb{R} \)-algebras.

Proof. By the previous arguments (mainly from Theorem 6.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from Theorem 5.2 and Proposition 6.5. \( \square \)

In the next section, we will prove that the assumption of the theorem is generic in the case when \( n \leq 6 \). For general dimensions, we need the topological theory (cf. Proposition A.3).

**Theorem 6.7.** Let \( X_i : (U, u_i) \to (LC^*, X_i(u_i)) \) \( (i = 1, 2) \) be spacelike hypersurface germs such that the map germ given by \( \pi_H : (H^{-1}_i(v_1), (u_i, v_1)) \to (LC^*, v_i) \) at any point \( u_i \in U \) is an \( MT^* \)-stable map germ, where \( H_i \) is the lightcone height function of \( X_i \) and \( v_i = X_i^t(u_i) \). If \( Q(X_1, u_1) \) and \( Q(X_2, u_2) \) are isomorphic as \( \mathbb{R} \)-algebras, then \( (X_1^t(U), u_1) \) and \( (X_2^t(U), u_2) \) are stratified equivalent as set germs.

In general we have the following proposition.

**Proposition 6.8.** Let \( X_i : (U, u_i) \to (LC^*, X_i(u_i)) \) \( (i = 1, 2) \) be spacelike hypersurface germs such that their lightcone parabolic sets have no interior points as subspaces of \( U \). If lightcone dual germs \( X_1^t, X_2^t \) are \( A \)-equivalent, then

\[
K(X_1(U), TPH(X_1, u_1), v_1) = K(X_2(U), TPH(X_2, u_2), v_2).
\]

In this case, \( (X_1^{-1}(TPH(X_1, u_1)), u_1) \) and \( (X_2^{-1}(TPH(X_2, u_2)), u_2) \) are diffeomorphic as set germs.

Proof. The lightcone parabolic set is the set of singular points of the lightcone dual. So the corresponding Legendrian lifts \( L^t_i \) satisfy the hypothesis of Proposition A.2. If lightcone dual germs \( X_1^t, X_2^t \) are \( A \)-equivalent, then \( L^t_1, L^t_2 \) are Legendrian equivalent, so that \( H_1, H_2 \) are \( P^\sim \)-equivalent. Therefore, \( h_{1,v_1}, h_{1,v_2} \) are \( K \)-equivalent. By Theorem 6.1, this condition is equivalent to the condition that \( K(X_1(U), TPH(X_1, u_1), v_1) = K(X_2(U), TPH(X_2, u_2), v_2) \).

On the other hand, we have \( (X_1^{-1}(TPH(X_1, u_1)), u_1) = (h_{1,v_1}^{-1}(0), u_1) \). It follows from this fact that \( (X_1^{-1}(TPH(X_1, u_1)), u_1) \) and \( (X_2^{-1}(TPH(X_2, u_2)), u_2) \) are diffeomorphic as set germs because the \( K \)-equivalence preserves the zero level sets. \( \square \)

For a spacelike hypersurface germ \( X : (U, u_0) \to (LC^*, X(u_0)) \), we say that \( (X^{-1}(TPH(X, u_0)), u_0) \) is the tangent parabolic indicatrix germ of \( X \). By Proposition 6.5, the diffeomorphism type of the tangent parabolic indicatrix germ is an invariant of the \( A \)-classification of the lightcone dual germ of \( X \). Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need \( K \)-invariants for function germ. The local ring of a function germ is a complete \( K \)-invariant for generic function germs. It is, however, not a numerical invariant. The \( K \)-codimension (or, Tyurina number) of a function germ is a numerical \( K \)-invariant of function germs [25]. We denote that

\[
P-\text{ord}(X, u_0) = \dim \frac{C^\infty_{u_0}(U)}{\langle \langle X(u), X^t(u_0) \rangle \rangle + 2, (X_{u_0}(u), X^t(u_0)) \rangle C^\infty_{u_0}}.
\]
Usually $P\text{-ord}(X, u_0)$ is called the $K$-codimension of $h_{u_0}$. However, we call it the order of contact with the tangent parabolic hyperquadric at $X(u_0)$. We also have the notion of corank of function germs

$$P\text{-corank}(X, u_0) = (n - 1) - \text{rank} \text{Hess}(h_{u_0}(u_0)),$$

where $v_0 = X^t(u_0)$.

By Proposition 3.8, $X(u_0)$ is a lightcone parabolic point if and only if $P\text{-corank}(X, u_0)$ is $\geq 1$. Moreover $X(u_0)$ is a lightcone flat point if and only if $P\text{-corank}(X, u_0)$ equals $n - 1$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, a) \to \mathbb{R}$ has the $A_k$-type singularity if $f$ is $K$-equivalent to the germ $\pm u_1^2 + \cdots + \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $P\text{-corank}(X, u_0) = n - 2$, the lightcone height function $h_{u_0}$ has the $A_k$-type singularity at $u_0$ in generic. In this case we have $P\text{-ord}(X, u_0) = k$. This number is equal to the order of contact in the classical sense (cf. [7]). This is the reason why we call $P\text{-ord}(X, u_0)$ the order of contact with the tangent parabolic hyperquadric at $X(u_0)$.

### 7. Generic Properties

In this section we consider generic properties of spacelike hypersurfaces in $LC^*$. The main tool is a kind of transversality theorems. We consider the space of space-like embeddings $\text{Emb}_{pU}(U, LC^*)$ with Whitney $C^\infty$-topology. We also consider the function $\mathcal{H} : LC^* \times LC^* \to \mathbb{R}$ which is given in Section 6. We claim that $\mathcal{H}_u$ is a submersion for any $u \in LC^*$, where $\mathcal{H}_u(v) = \mathcal{H}(u, v)$. For any $X \in \text{Emb}_{pU}(U, LC^*)$, we have $H = \mathcal{H} \circ (X \times 1_{LC^*})$. We also have the $k$-jet extension

$$j^k_H : U \times LC^* \to J^k(U, \mathbb{R})$$

defined by $j^k_H(u, v) = j^k h_u(v)$. We consider the trivialisation $J^k(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^k(n-1, 1)$. For any submanifold $Q \subset J^k(n-1, 1)$, we denote that $\tilde{Q} = U \times \{0\} \times Q$.

Then we have the following proposition as a corollary of Lemma 6 in Wassermann [43]. (See also Montaldi [32]).

**Proposition 7.1.** Let $Q$ be a submanifold of $J^k(n-1, 1)$. Then the set

$$T_Q = \{ X \in \text{Emb}_{pU}(U, LC^*) : j^k_H \text{ is transversal to } \tilde{Q} \}$$

is a residual subset of $\text{Emb}_{pU}(U, LC^*)$. If $Q$ is a closed subset, then $T_Q$ is open.

On the other hand, we already have the canonical stratification $A_k^0(U, \mathbb{R})$ of $J^k(\mathbb{R}^{n-1}, \mathbb{R}) \setminus W^k(\mathbb{R}^{n-1}, \mathbb{R})$ (cf. the appendix). By the above proposition and arguments in the appendix, we have the following theorem.

**Theorem 7.2.** There exists an open dense subset $\mathcal{O} \subset \text{Emb}_{pU}(U, LC^*)$ such that for any $X \in \mathcal{O}$, the germ of the corresponding lightcone dual $X^t$ at each point is the critical part of an $MT$-stable map germ.

In the case when $n \leq 6$, for any $X \in \mathcal{O}$, the germ of the Legendrian map $X^t = \pi_{4,2} \circ L_4$ at each point is Legendrian stable.

We remark that we can also prove the multi-jet version of Proposition 7.1. As an application of such a multi-jet transversality theorem, we can show that the lightcone dual is the critical part of an (global) $MT$-stable map for a generic spacelike...
hypersurface \( X: U \to LC^* \) (cf. the appendix). However, the arguments are rather tedious, so that we omit it.

8. OTHER CURVATURES OF SPACELIKE HYPERSURFACES IN THE LIGHTcone

In this section we introduce the other curvatures of spacelike hypersurfaces in the lightcone induced by the basic duality theorem (Theorem 2.2) which will be studied in the next section. Let \( X: U \to LC^* \) be a spacelike embedding. We have a diffeomorphism

\[ \Phi_{41}: \Delta_4 \to \Delta_1 \]

defined by

\[ \Phi_{41}(v, w) = \left( \frac{v + w}{2}, \frac{v - w}{2} \right). \]

We can calculate that

\[ \Phi_{41}^* \theta_{12} = \left\langle \frac{d v + w}{2}, \frac{v - w}{2} \right\rangle \bigg|_{\Delta_4} = -\frac{1}{4} \langle dv, w \rangle + \frac{1}{4} \langle dw, v \rangle \bigg|_{\Delta_4} = \frac{1}{2} \theta_{42}, \]

so that \( \Phi_{41} \) is a contact diffeomorphism. Therefore, we have a Legendrian submanifold \( L_1: U \to \Delta_1 \) defined by \( L_1(u) = \Phi_{41} \circ L_4(u) \). If we denote that \( L_1(u) = (X^h(u), X^d(u)) \), then we have

\[ X^h(u) = \frac{X(u) + X^f(u)}{2}, \quad X^d(u) = \frac{X(u) - X^f(u)}{2}. \]

We call \( X^h(u) \) the hyperbolic normal vector to \( M = X(U) \) at \( X(u) \) and \( X^d(u) \) the de Sitter normal vector to \( M = X(U) \) at \( X(u) \). As a consequence of Lemma 3.4, we can define the hyperbolic shape operator \( S^h_p = -dX^h(u_0): T_p M \to T_p M \) and the de Sitter shape operator \( S^d_p = -dX^d(u_0): T_p M \to T_p M \). Moreover, we can define curvatures of \( X(U) = M \) at \( p = X(u_0) \) as follows:

\[ K_h(u_0) = \text{det} \, S^h_p; \quad K_d(u_0) = \text{det} \, S^d_p; \quad H_h(u_0) = \frac{1}{n-1} \text{Trace} \, S^h_p; \quad H_d(u_0) = \frac{1}{n-1} \text{Trace} \, S^d_p. \]

We also have the following expressions on the hyperbolic Gauss–Kronecker curvature and the de Sitter Gauss–Kronecker curvature as a corollary of Proposition 3.5.

**Proposition 8.1.** Under the same notations in Corollary 3.6, we have the following formulae:

\[ (1) \quad K_h = \frac{1}{2^{n-1}} \frac{\text{det}(h^f_{ij} - g_{ij})}{\text{det}(g_{a\beta})}, \]

\[ (2) \quad K_d = \frac{1}{2^{n-1}} \frac{\text{det}(-h^f_{ij} - g_{ij})}{\text{det}(g_{a\beta})}. \]
Proof. (1) Since $X^h = (X + X^\ell)/2$, we have

$$(X^h)_u = \sum_{j=1}^{n-1} \frac{(\delta^j - (h^\ell)_{ij})}{2} X_{u_j},$$

It follows from the similar calculation as the proof of the above corollary that we have the desired formula. The second formula also follows from the equation that $X^d = (X - X^\ell)/2$.

We also get in this context the lightcone Gauss equations as we shall see next and it will be used in the next section. Since $X(U) = M$ is a Riemannian manifold, it makes sense to consider the Christoffel symbols:

$$\{ \begin{array}{c} k \\ i \end{array} \} = \frac{1}{2} \sum_m g^{km} \left\{ \frac{\partial g_{jm}}{\partial u_i} + \frac{\partial g_{im}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_m} \right\}.$$ 

Proposition 8.2. Let $X: U \rightarrow LC^*$ be a spacelike hypersurface. Then we have the following lightcone Gauss equations:

$$X_{u_iu_j} = \sum_k \{ \begin{array}{c} k \\ i \end{array} \} X_{u_k} + \frac{1}{2} (g_{ij} X^\ell - h_{ij}^\ell X).$$

Proof. Since $\{X, X_{u_1}, \ldots, X_{u_{n-1}}, X^\ell\}$ is a basis of $\mathbb{R}^{n+1}$, we can write $X_{u_iu_j} = \sum_k \Gamma^k_{ij} X_{u_k} + \Gamma_{ij} X^\ell + \Gamma^i_{j} X$. We now have

$$\langle X_{u_iu_j}, X_{u_m} \rangle = \sum_k \Gamma^k_{ij} \langle X_{u_k}, X_{u_m} \rangle = \sum_k \Gamma^k_{ij} g_{km}.$$ 

Since $\frac{\partial g_{ij}}{\partial u_m} = \langle X_{u_iu_j}, X_{u_k} \rangle + \langle X_{u_k}, X_{u_iu_j} \rangle$ and $X_{u_iu_j} = X_{u_ju_i}$, we get $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\Gamma_{ij} = \Gamma_{ji}$, $\Gamma^i = \Gamma_{ji}$. By exactly the same calculations as those of the case for hypersurfaces in Euclidean space, $\Gamma^k_{ij} = \{ i, j \}$.

On the other hand, we have $\langle X_{u_i}, X^\ell \rangle = \langle X^\ell, X^\ell \rangle = 0$ and $\langle X, X^\ell \rangle = -2$, so that $-2\Gamma_{ij} = \langle X_{u_iu_j}, X^\ell \rangle = h_{ij}^\ell$. Moreover $\langle X_{u_iu_j}, X \rangle = -2\Gamma_{ij}$ and since $\langle X_{u_i}, X \rangle = 0$, we have $\langle X_{u_iu_j}, X \rangle = -\langle X_{u_i}, X_{u_j} \rangle = -g_{ij}$. which implies that $2\Gamma_{ij} = g_{ij}$. 

Since $X^h = (X + X^\ell)/2$ and $X^d = (X - X^\ell)/2$, we have the following corollary.

Corollary 8.3. Under the same assumption as the above proposition, we have

$$X_{u_iu_j} = \sum_k \{ \begin{array}{c} k \\ i \end{array} \} X_{u_k} + \frac{1}{2} (g_{ij} - h_{ij}^\ell) X^h - \frac{1}{2} (g_{ij} + h_{ij}^\ell) X^d.$$ 

9. Spacelike Surfaces in the 3-Dimensional Lightcone

In this section we stick to the case $n = 3$. First of all we need to make some local calculations. Let $X: U \rightarrow LC^*$ be a spacelike surface, where $U \subset \mathbb{R}^2$ is an open region, and consider the Riemannian curvature tensor

$$R^\alpha_{\beta\gamma} = \frac{\partial}{\partial u_\gamma} \{ \begin{array}{c} \delta \\ \alpha \end{array} \} - \frac{\partial}{\partial u_\beta} \{ \begin{array}{c} \delta \\ \alpha \end{array} \} + \sum_{\epsilon} \{ \begin{array}{c} \epsilon \\ \alpha \end{array} \} \left\{ \begin{array}{c} \delta \\ \gamma \end{array} \right\} - \sum_{\epsilon} \{ \begin{array}{c} \epsilon \\ \alpha \end{array} \} \left\{ \begin{array}{c} \epsilon \\ \beta \end{array} \right\}.$$
We also consider the tensor $R_{\alpha\beta\gamma\delta} = \sum_\epsilon g_{\alpha\epsilon} R^{\epsilon}_{\beta\gamma\delta}$. Standard calculations, analogous to those used in the study of the classical differential geometry on surfaces in Euclidean space (cf. [41]), lead to the following formula.

**Proposition 9.1.** Under the above notations, we have

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} \{g_{\beta\gamma} h^\epsilon_{\alpha\delta} - g_{\beta\gamma} h^\epsilon_{\alpha\gamma} + h^\epsilon_{\beta\gamma} g_{\alpha\delta} - h^\epsilon_{\beta\delta} g_{\alpha\gamma}\}.$$  

We denote that

$$h^h_{\alpha\beta} = -\frac{1}{2} (g_{\alpha\beta} - h^\epsilon_{\alpha\beta}) \quad \text{and} \quad h^d_{\alpha\beta} = -\frac{1}{2} (g_{\alpha\beta} + h^\epsilon_{\alpha\beta}).$$

It follows from Proposition 8.1 that we have

$$K_h = \frac{h^h_{11} h^h_{22} - h^h_{21} h^h_{12}}{g_{11} g_{22} - g_{12} g_{21}}, \quad K_d = \frac{h^d_{11} h^d_{22} - h^d_{21} h^d_{12}}{g_{11} g_{22} - g_{12} g_{21}}.$$  

Therefore we obtain the analogous result of Theorema Egregium of Gauss for the lightcone case:

**Proposition 9.2.** Under the above notations, we have

$$K_d - K_h = -\frac{R_{1212}}{g},$$

where $g = g_{11} g_{22} - g_{12} g_{21}$.

We remark that $-\frac{R_{1212}}{g}$ is the sectional curvature of the surface, so we denote it by $K_s$.

On the other hand, let $\kappa_i^\ell$ (i = 1, 2) be eigenvalues of $(h^\ell)_{ij}$ (i.e., lightcone principal curvatures of the spacelike surface). We remind that $\kappa_i^\ell = \kappa_h^i - \kappa_d^i$, where $\kappa_h^i$ (respectively, $\kappa_d^i$) is a hyperbolic (respectively, de Sitter) principal curvature. By direct calculations, we have the following “Theorema Egregium” as a corollary of the above proposition.

**Theorem 9.3.** The following relation holds:

$$K_s = K_d - K_h = H_\ell = H_h - H_d.$$  

We study in the remaining of this section some generic properties of spacelike surfaces in $LC^*$. By Theorem 7.2 and the classification result on wave fronts (cf. [1]), we have the following local classification of singularities for the lightcone Gauss image of a generic spacelike surface in $LC^*$.

**Theorem 9.4.** Let $Emb_{sp}(U, LC^*)$ be the space of spacelike embeddings from an open region $U \subset \mathbb{R}^2$ into $LC^*$ equipped with the Whitney $C^\infty$-topology. There exists an open dense subset $\mathcal{O} \subset Emb_{sp}(U, LC^*)$ such that for any $X \in \mathcal{O}$, the following conditions hold:

1. The lightcone parabolic set $K^{-1}_h(0)$ is a regular curve. We call such a curve the lightcone parabolic curve.

2. The lightcone dual $X^\ell$ along the lightcone parabolic curve is locally diffeomorphic to the cuspidal edge except at isolated points. At such isolated points, $X^\ell$ is locally diffeomorphic to the swallowtail.
Here, the cuspidal edge is \( C = \{ (x_1, x_2, x_3) : x_1^2 = x_2^3 \} \) and the swallowtail is \( SW = \{(x_1, x_2, x_3) : x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v \} \) (cf. Fig. 1).

Following the terminology of Whitney [44], we say that a spacelike surface \( X : U \to LC^* \) has the excellent lightcone dual \( \hat{X} \) if \( L_4 \) is a stable Legendrian embedding at each point. In this case, the lightcone dual \( \hat{X} \) has only cuspidal edges and swallowtails as singularities. Theorem 7.2 asserts that a spacelike surface with the excellent lightcone dual is generic in the space of all spacelike surfaces in \( LC^* \).

We now consider the geometric meanings of cuspidal edges and swallowtails of the lightcone dual. We have the following results analogous to the results in Banchoff et al. [3].

**Theorem 9.5.** Let \( \hat{X} : (U, u_0) \to (LC^*, v_0) \) be the excellent lightcone Gauss image germ of a spacelike surface \( X \) and \( h_{v_0} : (U, u_0) \to \mathbb{R} \) be the lightcone height function germ at \( v_0 = \hat{X}(u_0) \). Then we have the following:

1. \( u_0 \) is a lightcone parabolic point of \( X \) if and only if \( \text{P-corank}(X, u_0) = 1 \) (i.e., \( u_0 \) is not a lightcone flat point of \( X \)).

2. If \( u_0 \) is a lightcone parabolic point of \( X \), then \( h_{v_0} \) has the \( A_k \)-type singularity for \( k = 2, 3 \).

3. Suppose that \( u_0 \) is a lightcone parabolic point of \( X \). Then the following conditions are equivalent:
   
   (a) \( \hat{X} \) has a cuspidal edge at \( u_0 \).
   
   (b) \( h_{v_0} \) has the \( A_2 \)-type singularity.
   
   (c) \( \text{P-ord}(X, u_0) = 2 \).
   
   (d) The tangent parabolic indicatrix germ is a ordinary cusp, where a curve \( C \subset \mathbb{R}^2 \) is called a ordinary cusp if it is diffeomorphic to the curve given by \( \{(u_1, u_2) : u_1^2 - u_2^3 = 0 \} \).
   
   (e) For each \( \varepsilon > 0 \), there exist two distinct points \( u_1, u_2 \in U \) such that \( |u_0 - u_i| < \varepsilon \) for \( i = 1, 2 \), both of \( u_1, u_2 \) are not lightcone parabolic points and the tangent parabolic quadrics to \( \hat{M} = \hat{X}(U) \) at \( u_1, u_2 \) are parallel.
(4) Suppose that \( u_0 \) is a lightcone parabolic point of \( X \). Then the following conditions are equivalent:

(a) \( X^\ell \) has a swallowtail at \( u_0 \).
(b) \( h_{u_0} \) has the \( A_3 \)-type singularity.
(c) \( P\text{-ord}(X, u_0) = 3 \).
(d) The tangent parabolic indicatrix germ is a point or a tachnodal, where a curve \( C \subset \mathbb{R}^2 \) is called a tachnodal if it is diffeomorphic to the curve given by \( \{(u_1, u_2): u_1^2 - u_2^2 = 0\} \).
(e) For each \( \varepsilon > 0 \), there exist three distinct points \( u_1, u_2, u_3 \in U \) such that \( |u_0 - u_i| < \varepsilon \) for \( i = 1, 2, 3 \) and the tangent parabolic quadrics to \( M = X(U) \) at \( u_1, u_2, u_3 \) are parallel.
(f) For each \( \varepsilon > 0 \), there exist two distinct points \( u_1, u_2 \in U \) such that \( |u_0 - u_i| < \varepsilon \) for \( i = 1, 2 \) and the tangent parabolic quadrics to \( M = X(U) \) at \( u_1, u_2 \) are equal.

Proof. We have shown that \( u_0 \) is a lightcone parabolic point if and only if \( P\text{-corank} \) of \( (X, u_0) \) is \( \geq 1 \). Since \( n = 3 \), we have \( P\text{-corank}(X, u_0) \leq 2 \). Since the lightcone height function germ \( H: (U \times \text{LC}^*, (u_0, v_0)) \to \mathbb{R} \) can be considered as a generating family of the Legendrian immersion germ \( L_4, h_{v_0} \) has only the \( A_k \)-type singularities \( (k = 1, 2, 3) \). This means that the corank of the Hessian matrix of \( h_{v_0} \) at a lightcone parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3a)–(3c) (respectively, (4a)–(4c)) are equivalent. If the height function germ \( h_{v_0} \) has the \( A_2 \)-type singularity, it is \( K \)-equivalent to the germ \( \pm u_1^2 + u_2^2 \). Since the \( K \)-equivalence send the zero level sets, the tangent parabolic indicatrix germ is diffeomorphic to the curve given by \( \pm u_1^2 + u_2^2 = 0 \). This is the ordinary cusp. The normal form for the \( A_3 \)-type singularity is given by \( \pm u_1^2 + u_2^2 \), so that the tangent parabolic indicatrix germ is diffeomorphic to the curve \( \pm u_1^2 + u_2^2 = 0 \). This means that the condition (3d) (respectively, (4d)) is also equivalent to the other conditions.

Suppose that \( u_0 \) is a lightcone parabolic point, by Proposition 7.1, the lightcone Gauss map has only folds or cusps. If the point \( u_0 \) is a fold point, there is a neighbourhood of \( u_0 \) on which the lightcone Gauss map is 2 to 1 except the lightcone parabolic curve (i.e., fold curve). By Lemma 6.2, the condition (3e) is satisfied. If the point \( u_0 \) is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the lightcone Gauss map is 3 to 1 inside region of the critical value. Moreover, the point \( u_0 \) is in the closure of the region. This means that the condition (3e) holds. We can also observe that near by a cusp point, there are 2 to 1 points which approach to \( u_0 \). However, one of those points are always lightcone parabolic points. Since other singularities do not appear for in this case, so that the condition (3e) (respectively, (4e)) characterizes a fold (respectively, a cusp).

If we consider the lightcone dual instead of the lightcone Gauss map, the only singularities are cuspidal edges or swallowtails. For the swallowtail point \( w_0 \), there are self intersection curve (cf. Fig. 1) approaching to \( w_0 \). On this curve, there are two distinct point \( u_1, u_2 \) such that \( X^\ell(u_1) = X^\ell(u_2) \). By Lemma 6.2, this means that tangent parabolic hyperquadric to \( M = X(U) \) at \( u_1, u_2 \) are equal. Since there
are no other singularities in this case, the condition (4f) characterize a swallowtail point of $X^\ell$. This completes the proof.

Let $M$ be a compact 2-manifold without boundary. When considering a global embedding $f: M \to LC^*$ induced by a Legendrian embedding $L_4: M \to \Delta_4$, one must also pay attention to the multilocal phenomena. So we must add the double point locus, the intersection of a regular surface and the cuspidal edge and the triple point to the list of local normal forms of the singular image of lightcone duals of generic embeddings. These follow from the multi-jet version of Proposition 7.1. Given a point $p_0 \in M$ and the lightlike vector $v_0 = L(p_0)$, we have the tangent parabolic quadric $TPH(f, p_0)$ of $f(M)$ at $f(p_0)$ (cf. Section 6). By Lemma 6.2, $L(p_1) = L(p_2)$ if and only if $TPH(f, p_1) = TPH(f, p_2)$. Analogously, a triple point of the lightcone dual of $f: M \to LC^*$ corresponds to a tritangent parabolic quadric.

On the other hand, we have a geometric characterizations of the swallowtail point in Theorem 9.5. Remember that a point $p \in M$ is called the lightcone parabolic point provided $K_\ell(p) = 0$. Denote by $T(f)$ the number of tritangent parabolic quadrics and by $SW(f)$ the number of swallowtail points of a generic embedding $f: M \to LC^*$. By definition, the lightcone dual of a hypersurface can be interpreted as a wave front in the theory of Legendrian singularities (cf. the appendix). Therefore, we have the following formula as a particular case of the relation obtained in [14] for wave fronts:

$$
\chi(\mathbb{L}(M)) = \chi(M) + \frac{1}{2}SW(f) + T(f).
$$

Since $f: M \to LC^*$ is a spacelike embedding, so that it covers $S^2_+ \subset S^2$, and, hence is diffeomorphic to $S^2$. Therefore we have the following formula:

$$
\chi(\mathbb{L}(M)) = 2 + \frac{1}{2}SW(f) + T(f).
$$

This formula and Theorem 9.5 tells us that the Euler number of the lightcone dual of a generic spacelike embedding in to $LC^*$ can be obtained in terms of the invariants of the lightcone differential geometry.

10. Examples

In this section we give some examples. Consider a function germ $f(u_1, \ldots, u_{n-1})$ around the origin with $f(0) = 1$ and $f_{u_i}(0) = 0$ ($i = 1, \ldots, n-1$). Then we have a spacelike hypersurface in $LC^*_+$ defined by

$$
X_f(u) = (g(u), u_1, \ldots, u_{n-1}, f(u)),
$$

where

$$
g(u) = \sqrt{u_1^2 + \cdots + u_{n-1}^2 + f^2(u_1, \ldots, u_{n-1})}
$$

and $u = (u_1, \ldots, u_{n-1})$. We have $X_f(0) = (1, 0, \ldots, 0, 1)$. We can easily calculate that $X_{f_{u_i}}(0) = e_i$ ($i = 1, \ldots, n-1$), where $e_i$ is the canonical unit spacelike vector of $\mathbb{R}^{n+1}_1$. It follows that we have $X_f^\ell(0) = (1, 0, \ldots, 0, -1)$. In this case, the tangent
parabolic hyperquadric of $X_f$ at $X_f(0)$ is
\[
TP_f(u) = \left( \frac{u_1^2 + \cdots + u_{n-1}^2}{4} + 1, u_1, \ldots, u_{n-1}, 1 - \frac{u_1^2 + \cdots + u_{n-1}^2}{4} \right).
\]
Therefore the tangent parabolic indicatrix germ of $X_f$ at the origin is
\[
\{ u \in (\mathbb{R}^{n-1}, 0) : 4f(u) + (u_1^2 + \cdots + u_{n-1}^2) - 4 = 0 \}.
\]

We remark that a parallel translation in $\mathbb{R}^n_0$ is not induced from a Lorenzian motion, we cannot assert that any spacelike hypersurface can be obtained as the above construction even locally. We now give two examples in the case when $n = 3$.

**Example 10.1.** Consider the function given by
\[
f(u_1, u_2) = 1 + \left( \frac{1}{3}u_1^3 - \frac{1}{4}u_1^2 - \frac{1}{2}u_2 \right).
\]
Then
\[
4f(u_1, u_2) + (u_1^2 + u_2^2) - 4 = 2 \left( \frac{1}{3}u_1^3 - \frac{1}{4}u_1^2 - \frac{1}{2}u_2 \right),
\]
so that the tangent parabolic indicatrix germ at the origin is the ordinary cusp. By Theorem 9.5, $X_f(0)$ is a parabolic point and $X^j_{\ell}f(0)$ might be the cuspidaledge.

**Example 10.2.** Consider the function given by
\[
f(u_1, u_2) = 1 + \left( \frac{1}{4}u_1^4 - \frac{1}{4}u_1^2 - \frac{1}{2}u_2 \right).
\]
Then
\[
4f(u_1, u_2) + (u_1^2 + u_2^2) - 4 = u_1^4 - u_2^2,
\]
so that the tangent parabolic indicatrix germ at the origin is the tachnode. Therefore, $X_f(0)$ is a parabolic point and $X^j_{\ell}f(0)$ might be the swallowtail.

**Appendix. Generic Properties of Wave Fronts**

In which we consider the following question: How does a wave front look like generically?

In Section 5 and Section 6 we have given the quick reviews on the theory of Legendrian singularities due to [1], [45], [46]. The stable Legendrian map is characterized by the $K$-versality of its generating family. We now have another characterization of $K$-versal deformations of function germs. Let $J^\ell(\mathbb{R}^k, \mathbb{R})$ be the $\ell$-jet bundle of $n$-variable functions which has the canonical decomposition: $J^\ell(\mathbb{R}^k, \mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times J^\ell(k, 1)$. For any Morse family of hypersurfaces $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we define a map germ
\[
j^\ell_F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to J^\ell(\mathbb{R}^k, \mathbb{R})
\]
by $j^\ell_F(q, x) = j^\ell_Fx(q)$, where $Fx(q) = F(q, x)$. We denote $K^\ell(z)$ the $K$-orbit through $z = j^\ell_f(0) \in J^\ell(k, 1)$. (cf. [25]). If $f(q) = F(q, 0)$ is $\ell$-determined relative
to $\mathcal{K}$, then $F$ is a $K$-versal deformation of $f$ if and only if $j_1^F$ is transversal to $\mathbb{R}^k \times \{0\} \times \mathcal{K}'(z)$ (cf. [25]).

We now consider the stratification of the $\ell$-jet space $J^\ell(\mathbb{R}^k, \mathbb{R})$ such that $K$-versal deformations are transversal to the stratification and the pull back stratification in the parameter space corresponds to the canonical stratification of the discriminant set. By Theorem 6.4, such a stratification should be $K$-invariant, where we have the $K$-action on $J^\ell(k, 1)$ (cf. [25], [26]). By this reason, we use Mather’s canonical stratification here [13], [27]. Let $\mathcal{A}^\ell(k, 1)$ be the canonical stratification of $J^\ell(k, 1) \setminus W^\ell(k, 1)$, where

$$W^\ell(k, 1) = \{ j^\ell f(0) : \dim \mathbb{R}^k / ((T_x \mathcal{K})(f) + \mathfrak{M}_f^\ell) \geq \ell \}.$$  

We now define the stratification $\mathcal{A}_0^\ell(\mathbb{R}^k, \mathbb{R})$ of $J^\ell(\mathbb{R}^k, \mathbb{R}) \setminus W^\ell(\mathbb{R}^k, \mathbb{R})$ by

$$\mathbb{R}^k \times (\mathbb{R} \setminus \{0\}) \times (J^\ell(k, 1) \setminus W^\ell(k, 1)), \quad \mathbb{R}^k \times \{0\} \times \mathcal{A}^\ell(k, 1),$$

where $W^\ell(\mathbb{R}^k, \mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times W^\ell(k, 1)$. In [42], Y.-H. Wan has shown that if $j^\ell F(0) \not\in W^\ell(k, 1)$ and $j^\ell F$ is transversal to $\mathcal{A}_0^\ell(\mathbb{R}^k, \mathbb{R})$ then $\pi_F : (F^{-1}(0), \mathbb{O}) \to (\mathbb{R}^n, \mathbb{O})$ is a MT-stable map germ. (See also [16]). Here, we call a map germ $MT$-stable if it is transversal to the canonical stratification of a jet space which is introduced in [13], [27]. The main assertion of Mather’s topological stability theorem is that an MT-stable map germ is a topological stable map germ. Moreover, the critical value set of an MT-stable map germ is canonically stratified. For the classification, we refer to the following theorem of Fukuda–Fukuda [12].

**Theorem A.1.** Let $f, g : (\mathbb{R}^n, \mathbb{O}) \to (\mathbb{R}^p, \mathbb{O})$ be MT-stable map germs. If $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, then these map germs are topological equivalent.

If we carefully read their proof, we can conclude that critical value sets (discriminant sets) of $f$, $g$ are stratified equivalent. Here we say that two stratified sets are stratified equivalent if there exists a homeomorphism between stratified sets such that the homeomorphism maps a strata onto a strata and the restriction on each strata is smooth.

In order to apply Theorem A.1 to our situation, we need to review the theory of unfoldings of map germs. The definition of an $r$-dimensional unfolding of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ (originally due to Thom) is a germ $\tilde{F} : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0)$ given by $\tilde{F}(x, u) = (F(x, u), u)$, where $F(x, u)$ is a germ of an $r$-dimensional parameterized family of germs with $F(x, 0) = f(x)$. This definition depends on the coordinates of both of spaces $(\mathbb{R}^n \times \mathbb{R}^r, 0)$ and $(\mathbb{R}^p \times \mathbb{R}^r, 0)$. For our purpose, we need the coordinate free definition of unfoldings [13]. Let $f : (N, x_0) \to (P, y_0)$ be a map germ between manifolds. An unfolding of $f$ is a triple $(\tilde{F}, i, j)$ of map germs, where $i : (N, x_0) \to (N', x_0'), j : (P, y_0) \to (P', y_0')$ are immersions and $j$ is transverse to $\tilde{F}$, such that $\tilde{F} \circ i = j \circ f$ and $(i, f) : N \to \{ (x', y) \in N' \times P : \tilde{F}(x') = j(y) \}$ is a diffeomorphism germ. The dimension of $(\tilde{F}, i, j)$ as an unfolding is $\dim N' - \dim N$. We can easily prove that the above two definitions are equivalent. We can show that the local ring of a map germ does not depend on the choice of the local coordinates at the points. Therefore we can define the local ring $Q(\tilde{F})$ for a Morse family of hypersurfaces $F$. We can easily show that $Q(f)$ and $Q(\tilde{F})$ are canonically isomorphic as $\mathbb{R}$-algebras.
We now apply the above arguments to our case. The idea used here comes from Martinet’s study of stable map germs [25]. Let \( F: (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be a Morse family of hypersurfaces. Corresponding to \( F \), we have an unfolding of \( f = F|_{\{0\} \times \mathbb{R}^n} \)

\[ \tilde{F}: (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R} \times \mathbb{R}^n, 0) \]

given by \( \tilde{F}(q, x) = (F(q, x), x) \). Then we can easily show the following lemma.

**Lemma A.2.** We consider inclusions

\[ i: (F^{-1}(0), 0) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, 0) \quad \text{and} \quad j: (\{0\} \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R} \times \mathbb{R}^n, 0). \]

Then \( (\tilde{F}, i, j) \) is an unfolding of \( \pi_F: (F^{-1}(0), 0) \rightarrow (\mathbb{R}^n, 0) \).

By the previous arguments, \( Q(\pi_F), Q(\tilde{F}) \) and \( Q(f) \) are isomorphic to each other as \( \mathbb{R} \)-algebras. By Theorem A.1, we have the following proposition:

**Proposition A.3.** Let \( F, G: (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be Morse families of hypersurfaces such that \( \pi_F \) and \( \pi_G \) are MT-stable map germs. If \( Q(f) \) and \( Q(g) \) are isomorphic as \( \mathbb{R} \)-algebras, then \( \pi_F \) and \( \pi_G \) are topological equivalent. Moreover, in this case, \( D_F \) and \( D_G \) are stratified equivalent.

Let \( F: (\mathbb{R}^n \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be a Morse family of hypersurfaces. Suppose that \( j^1_F(0) \notin W^F(1, 1) \) and \( j^1_F(0) \) is transversal to \( A^j(\mathbb{R}^k, \mathbb{R}) \) for sufficient large \( \ell \) (i.e., \( \text{codim} W^k(k, 1) > k + n \)). By the transversality assumption, we cannot avoid strata \( X_j \) of codimension \( \leq k + n \). For \( n \leq 6 \) and \( \ell \geq 8 \), by the classification of \( K \)-simple function germs [1], \( \text{codim} W^k(k, 1) > k + 6 \) and each strata of \( A^k(k, 1) \) is a \( K \)-orbit in \( J^k(k, 1) \). In this case, we can say that \( F \) is a \( K \)-versal deformation of \( f = F|_{\mathbb{R}^n \times \{0\}} \) by the characterization of \( K \)-versal deformations. Therefore \( L_F \) is Legendrian stable. For general \( n \geq 7 \), by the previous arguments, the wave front \( W(L_F) \) is the critical value set of the MT-stable map germ \( \pi_F: (F^{-1}(0), 0) \rightarrow (\mathbb{R}^n, 0) \).

**References**


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