NONUNIFORMISABLE FOLIATIONS ON COMPACT COMPLEX SURFACES

MARCO BRUNELLA

Abstract. We give a complete classification of holomorphic foliations on compact complex surfaces which are not uniformisable, i.e., for which universal coverings of the leaves do not glue together in a Hausdorff way. This leads to complex analogs of the Reeb component defined on certain Hopf surfaces and certain Kato surfaces.


Key words and phrases. Holomorphic foliations, Reeb component, uniformisation, nonkahlerian compact complex surfaces.

1. Introduction

A holomorphic (singular) foliation by curves on a compact complex manifold is said to be uniformisable if, roughly speaking, universal coverings of its leaves can be glued together in a coherent way, producing a complex manifold called covering tube. The precise definition will be given below, in Section 2, together with some background material.

When the ambient manifold is Stein instead of compact, Ilyashenko proved in [Ily] that any foliation is uniformisable.1 When the ambient manifold is Kähler compact, we proved in [Br2] (modulo a correction in [Br4]) that any foliation is uniformisable. Here it is essential to take the definition of leaf that we introduced and developed in [Br2], which takes into account the singularities of the foliation. However, there are examples of foliations on compact non-Kähler manifolds which are not uniformisable. Our purpose is to classify them in the two dimensional case.

Theorem 1.1. Let X be a compact connected complex surface, and let F be a nonuniformisable foliation on X. Then F is one of the following:

1 a straight foliation on a (blown up) Hopf surface;
2 a straight foliation on a (blown up) Kato surface.

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1We notice that in some papers by Ilyashenko and Glutsyuk there is a notion of uniformisability that is stronger than ours, which involves not only the existence of covering tubes but also their analytic properties; this notion could be called analytical uniformisability, whereas our notion could be called topological uniformisability.
The straight foliations appearing in this theorem will be introduced and analyzed in Section 2 below. However, the important conclusion in Theorem 1.1 is not really the description of \( F \), but rather the description of the ambient surface \( X \). Indeed, non-Kähler compact complex surfaces are still largely “unknown” [Nak] [Tel]; on the contrary, the theorem above says that the existence of a nonuniformisable foliation \( F \) implies that the surface \( X \) belongs to “well known” classes: Hopf surfaces and Kato surfaces. Once we know that \( X \) is in one of these classes, it is not very difficult to show that \( F \) is a straight foliation.

Theorem 1.1 can be also considered from a different point of view. As we shall see below, nonuniformisability is equivalent to the presence of a vanishing cycle in some (holonomy covering of) leaf. In the classical theory of real codimension one foliations on 3-manifolds [God], vanishing cycles play a prominent rôle, and a classical theorem of Novikov affirms that they are always associated to a Reeb component of the foliation. Theorem 1.1 may be seen as a “complex” counterpart of this result, and indeed foliations of type (1) and (2) may be seen as “complex” Reeb components: see the figure in Section 2.3. Note that a “complex” Reeb component is not a proper subset of the foliation, as in the real case, but it is the full foliation; in some sense, this is due to the fact that, in the complex case, a compact leaf does not disconnect the ambient manifold.

The idea of the proof of Theorem 1.1 is the following. Let us consider an Hartogs figure \( H_\varepsilon = (D \times A_\varepsilon) \cup (\Omega \times D) \), where \( A_\varepsilon \) is the annulus of radii \((1 - \varepsilon, 1)\) and \( \Omega \subset D \) is an open subset containing 0 in its boundary. Suppose that a vanishing cycle is realized by an embedding \( f : H_\varepsilon \rightarrow X \) sending vertical fibers to leaves and \( \{z = 0, |w| = 1 - \frac{1}{2}\varepsilon\} \) to a cycle that is not homotopic to zero in the leaf (the vanishing cycle). By [Iva], this map can be meromorphically extended to a map

\[
\hat{f} : (D \times D) \setminus E \rightarrow X
\]

for some discrete subset \( E \subset D \times D \) of essential singularities; we may assume \( E = \{(0, 0)\} \). Then we take in \( X \) the “global spherical shell” \( \hat{f}(S^3_\delta) \), where \( S^3_\delta \) is the sphere of radius \( \delta \) and \( \delta \) is chosen so that this sphere avoids indeterminacy points of \( \hat{f} \). By [Kai], we conclude that \( X \) is a Hopf or a Kato surface, possibly blown up.

However, there are two substantial difficulties in this sketch.

First of all, a vanishing cycle is not always realized (in principle) by an embedding of the Hartogs figure as above [Cl], but we need to replace \( H_\varepsilon \) with some more general and less trivial figure (as in [Br4], correcting [Br2]). Therefore, we need to replace Ivashkovich’s extension theorem with our unparametrized Levi continuity principle [Br4, Section 2] suitably generalized to the non-Kähler setting. This generalization, which is a combination of [Br4] and arguments from [Iva], is done in the Appendix.

Secondly, and more important, the above maps \( f \) and \( \hat{f} \) are usually not embeddings, but only immersions. Thus \( \hat{f}(S^3_\delta) \) is only an immersed global spherical shell, and Kato’s result does not apply. We need a sort of surgical process which allows to
replace $\widehat{f}(S^3_1)$ with an embedded global pseudoconvex shell (possibly not spherical). Then the proof can be completed by [Ka2]. This is done in Section 3.

Let us conclude this introduction by putting our theorem in a larger perspective.

We would like to classify all foliations on non-Kähler compact complex surfaces. The nonuniformisable case is settled by Theorem 1.1, hence let us consider a uniformisable foliation $F$. By definition, we can construct the corresponding covering tubes. It is conceivable (but probably very difficult to prove...) that these tubes enjoy the same (or a similar) convexity property established in [Br1] in the Kähler case. If it is the case, then we would obtain that the canonical bundle $K_F$ of the foliation is pseudoeffective, at least when $F$ has a hyperbolic leaf (which we assume from now on). On the other hand, this canonical bundle is certainly not big, i.e., its Kodaira dimension is less than 2, because $X$ is not algebraic. Therefore, we could try to apply the Monge–Ampère technique of [Br1, Section 4] to construct a second foliation $G$ on $X$, transverse to $F$ outside some rational or elliptic curves, and for which the leafwise hyperbolic metric on $F$ induces a transverse holonomy invariant hyperbolic metric. Eventually, and with the help of the structural theory of Riemannian foliations [God], this should lead to the conclusion that $F$ is either a turbulent foliation (when $G$ is an elliptic fibration) or a hyperbolic foliation on a Inoue or Inoue–Hirzebruch surface [Nak] (when $G$ is not an elliptic fibration). Note that Inoue and Inoue–Hirzebruch surfaces, and hyperbolic foliations on them, are indeed very closely related to Hilbert Modular surfaces and foliations occurring in [Br1, Section 4].

2. Leaves and Tubes. Vanishing Cycles

2.1. Leaves. Let $X$ be a smooth compact connected complex surface, and let $F$ be a (possibly singular) holomorphic foliation on $X$. Let us firstly recall the definition of leaves, given in [Br2].

Take a point $p \in X^0 = X \setminus \text{Sing}(F)$, and let $L^0_p$ be the (usual) leaf of the nonsingular foliation $F^0 = F|_{X^0}$ through the point $p$. Let $E \subset L^0_p$ be a parabolic end of $L^0_p$, i.e., a closed subset isomorphic to the punctured closed disc $D^* = D \setminus \{0\}$. Assume also that $E$ is adherent to some singular point $q \in \text{Sing}(F)$ (i.e., $E$ is a so-called separatrix of $F$ at $q$), and that its holonomy has finite order $k \in \mathbb{N}$. Then $E$ is called a vanishing end of order $k$ if there exists a meromorphic map

$$f : \mathbb{D} \times \overline{D} \to X$$

such that:

1. $(0, 0)$ is the only indeterminacy point of $f$, and outside that point the map $f$ is an immersion (i.e., a local biholomorphism);
2. $\{0\} \times \overline{D}$ is sent by $f$ to $E$, as a regular cyclic covering of order $k$;
3. for every $t \in \mathbb{D}^*$, the restriction of $f$ to $\{t\} \times \overline{D}$ is an embedding into some leaf of $F^0$.

In other words, we require that the (singular) disc $E \cup \{q\} \subset L^0_p \cup \{q\}$ can be “meromorphically deformed” to discs embedded into nearby leaves, up to a ramification at $q$ due to the holonomy. These discs, however, are not close to $E \cup \{q\}$, but rather to $E \cup R$, where $R$ is a union of rational curves, invariant by the foliation, image
by $f$ of its indeterminacy point (note that $(0, 0)$ is necessarily an indeterminacy point, otherwise $q$ would be a regular point of the foliation). Equivalently, the indeterminacy point of $f$ can be resolved by a sequence of blow-ups over $(0, 0)$, getting a holomorphic map $\tilde{f}: (D \times \overline{D})^\sim \to X$, and then $R$ is the image by $\tilde{f}$ of the exceptional divisors of the resolution.

The leaf of $\mathcal{F}$ through $p$, denoted by $L_p$, is then obtained by compactifying all the vanishing ends of $L^0_p$, in the orbifold’s sense: a vanishing end of order $k$ is compactified by adding one point of multiplicity $k$. Such a leaf has a natural map $i_p: L_p \to X$ which sends the discrete set $L_p \setminus L^0_p$ to $\text{Sing}(\mathcal{F})$. In other words, the leaf of $\mathcal{F}$ is obtained from the leaf of $\mathcal{F}^0$ by adding some special singular points of the foliation, possibly with multiplicities.

We shall denote by $\hat{L}_p$ the holonomy covering of $L_p$, that is the covering associated to the Kernel of the holonomy representation [God, II.2], and by $\tilde{L}_p$ its universal covering, both with basepoint $p$ (and we shall identify $p$ also with the distinguished point of $\hat{L}_p$ or $\tilde{L}_p$ corresponding to the constant path).

By construction, $\hat{L}_p$ naturally contains $\hat{L}^0_p$, the holonomy covering of $L^0_p$. More precisely, $\hat{L}_p$ can be seen as the result of compactifying (now without multiplicity) all the parabolic ends of $\hat{L}^0_p$ which project to vanishing ends of $L^0_p$. Indeed, if $E \subset L^0_p$ is a vanishing end of order $k$, then its preimage in $\hat{L}^0_p$ is a collection (finite or infinite) of parabolic ends $\hat{E}_j \subset \hat{L}^0_p$, and every map $\hat{E}_j \to E$ is a regular cyclic covering of order $k$. Such a map extends to a still regular (in orbifold’s sense) cyclic covering $\hat{E}_j \cup \{0\} \to E \cup \{0\}$, provided that $0 \in E \cup \{0\}$ has multiplicity $k$ and $0 \in \hat{E}_j \cup \{0\}$ has multiplicity 1. This extension gives the local structure of the covering $\hat{L}_p \to L_p$ over points in $L_p \setminus L^0_p$.

Remark 2.1. We could also take the slightly different, but equivalent, approach of [Br1] (see also [Br3]). Let us assume (to simplify, and without a serious loss of generality) that $\mathcal{F}$ has reduced singularities, in Seidenberg’s sense. By contracting certain $\mathcal{F}$-invariant rational curves [Br3, p. 57], we can obtain a new surface $X'$ and a new foliation $\mathcal{F}'$ such that:

(a) $X'$ is possibly not smooth, but it has at most cyclic quotient singularities;

(b) $\mathcal{F}'$ is nonsingular around these singularities, in the sense that it is locally the cyclic quotient of a nonsingular foliation;
(b) the canonical bundle $K_{F'}$ of $F'$ has nonnegative degree on every rational
curve of negative selfintersection.

(Indeed, if $C \subset X$ is a rational curve with $C \cdot C < 0$ and $K_{F'} \cdot C < 0$, then [Br3, p. 57] $C$
can be contracted to a cyclic quotient singularity around which the contracted
foliation is nonsingular; then we continue like this, until we get property (b)).

We shall say that $F'$ is a relatively minimal model of $F$. For such a foliation,
we can define the leaves as in [Br1] and [Br3]: that is, leaves are just the leaves of
$F'$ outside $\text{Sing}(F')$, but with multiple points arising from the (possible) passages
through $\text{Sing}(X')$. With this definition, and thanks to property (b), leaves of $F'$
have no vanishing ends, see [Br1, Lemma 1]. Then it is not difficult to see that,
if $p \in X'$ is a point outside the exceptional divisor of $X \to X'$, then the leaf
of $F$ through $p$ in the sense of [Br2]) projects isomorphically to the leaf of $F'$
through $p' = \pi(p)$ (in the sense of [Br1]). More precisely, every $q \in \text{Sing}(F)$ that
compactifies a vanishing end (of order $k$) projects to a nonsingular point of $F'$ (of
multiplicity $k$).

**Remark 2.2.** There is an equivalent definition of vanishing end which will be useful
later. Let $E \subset L^0_p$ be a parabolic end, adherent to $q \in \text{Sing}(F)$, with holonomy
of order $k$. This holonomy is therefore generated, in a suitable coordinate, by the
germs $h_{k, \ell}(x) = e^{2\pi i \ell/k}x$, for some uniquely determined $\ell \in \{1, \ldots, k\}$ prime to $k$
(here we take as generator of $\pi_1(E)$ the positively oriented boundary $\partial E$). Set

$$\Omega_{k, \ell} = \mathbb{D} \times \mathbb{D}^*/\{(z, w) \sim (e^{2\pi i \ell/k}z, e^{2\pi i \ell/k}w)\}.$$ 

It is a complex surface with a cyclic quotient singularity $q_{k, \ell}$. The projection

$$\text{pr}_{k, \ell}: \Omega_{k, \ell} \to \mathbb{D}$$

induced by the first coordinate defines a foliation on $\Omega_{k, \ell} \setminus \{q_{k, \ell}\}$, with a central
leaf isomorphic to $\mathbb{D}^*$ and whose holonomy is generated by $h_{k, \ell}$; all the other leaves
are isomorphic to $\mathbb{D}$.

Then, in the definition of vanishing end given above, conditions (1), (2) and (3)
can be equivalently replaced by: There exists a meromorphic map

$$f: \Omega_{k, \ell} \to X$$

such that

1. $q_{k, \ell}$ is the only indeterminacy point of $f$, and outside that point the map $f$
is an immersion;
2. $\text{pr}_{k, \ell}^{-1}(0) \setminus \{q_{k, \ell}\}$ is sent by $f$ to $E$, isomorphically;
3. for every $t \in \mathbb{D}^*$, the restriction of $f$ to $\text{pr}_{k, \ell}^{-1}(t)$ is an embedding into some
leaf of $F'$.

This sheds more light on the previous Remark 2.1: when we pass to the relatively
minimal model $(X', F')$, the map $f' = \pi \circ f: \Omega_{k, \ell} \to X'$ becomes a foliated chart
around a nonsingular point of $F'$.
2.2. Tubes. Now we can introduce holonomy tubes and covering tubes, and arrive to the definition of uniformisability.

Take an embedded disc $T \subset X^0$ transverse to $\mathcal{F}$, and consider the union of the holonomy coverings of the leaves through $T$:

$$V_T = \bigsqcup_{t \in T} \hat{L}_t$$

(the holonomy tube). As in [Br2, p. 145], [Br1, p. 123] (see also [Suz]) we can give to $V_T$ a natural structure of complex surface (extending, of course, the complex structure already present on the fibers $\hat{L}_t$) such that the natural projection $Q_T: V_T \to T$ is holomorphic, and the maps $\hat{L}_t \to L_t \to X$, $t \in T$, glue together to a meromorphic map

$$\pi_T: V_T \to X.$$ 

The indeterminacy set of $\pi_T$ is equal to $\bigsqcup_{t \in T}(\hat{L}_t \setminus \hat{L}_t^0)$, and outside this set $\pi_T$ is an immersion into $X^0$.

We shall say that the foliation is uniformisable if a similar construction can be done with the universal coverings $\tilde{L}_t$ replacing the holonomy coverings $\hat{L}_t$:

**Definition 2.1.** A foliation $\mathcal{F}$ on a compact connected surface $X$ is uniformisable if for every embedded disc $T \subset X^0$ transverse to $\mathcal{F}$ there exists a (necessarily unique) structure of complex surface on

$$U_T = \bigsqcup_{t \in T} \tilde{L}_t$$

(the covering tube) such that:

1. the projection $P_T: U_T \to T$ is a holomorphic submersion;
2. the basepoint section $p_T: T \to U_T$ is holomorphic;
3. there exists a meromorphic map

$$\Pi_T: U_T \to X$$

whose restriction to $\tilde{L}_t$ coincides (after removal of indeterminacies) with $\tilde{L}_t \to L_t \to X$, for every $t \in T$.

If such a complex surface exists, then the natural map $F_T: U_T \to V_T$ is a local diffeomorphism which restricts to the universal covering over each fiber, so that $U_T$ is a sort of “fiberwise” universal covering of $V_T$. As firstly discovered by Ilyashenko [Ily], it is quite evident how to put a topology on $U_T$ (by seeing $U_T$ as a quotient of a space of paths, see below) and even a complex atlas (using for instance $F_T$ and pulling-back the complex atlas of $V_T$). The only nontrivial property to check is the Hausdorff property of the topological space $U_T$. This is equivalent to the absence of vanishing cycles in $V_T$ [Ily], [Br1], [Br2].

**Definition 2.2.** Let $\mathcal{F}$ be a foliation on a compact connected surface $X$, and let $V_T$ be a holonomy tube (associated to some transverse disc $T$). A vanishing cycle in $V_T$ is a loop $\gamma: [0, 1] \to \hat{L}_t$, $\gamma(0) = \gamma(1) = t$ (for some $t \in T$), such that:

1. $\gamma$ is not homotopic to zero in $\hat{L}_t$;
(2) \( \gamma \) can be uniformly approximated by loops \( \gamma_n : [0, 1] \to \hat{L}_{t_n}, \ \gamma_n(0) = \gamma_n(1) = t_n \) such that \( \gamma_n \) is homotopic to zero in \( \hat{L}_{t_n} \), for every \( n \).

For sake of completeness, let us sketch the proof of the following basic fact \([\text{Ily}], [\text{Br1}], [\text{Br2}]\).

**Proposition 2.1.** A foliation \( \mathcal{F} \) on a compact connected surface \( X \) is uniformisable if and only if for every transverse disc \( T \) the holonomy tube \( V_T \) has no vanishing cycle.

**Proof.** The set \( U_T \) can be thought as the quotient \( \Omega_T / \sim \), where \( \Omega_T \) is the space of paths \( \gamma : [0, 1] \to V_T \) starting on \( T \) (\( \sim q_T(T) \subset V_T \)) and contained in fibers of \( V_T \), and \( \sim \) is the equivalence relation “\( \gamma_1 \sim \gamma_2 \) if they are in the same fiber, they have the same endpoint, and \( \gamma_1 \ast \gamma_2^{-1} \) is zero-homotopic in the fiber”. We put on \( \Omega_T \) the uniform convergence topology, and on \( U_T \) the quotient topology. The endpoint map \( U_T \xrightarrow{p} V_T \) is then a local homeomorphism, and the complex atlas of \( V_T \) can be lifted to \( U_T \).

It remains the Hausdorff property. If \( U_T \) is not a Hausdorff topological space, then there exist two nonseparated points \( [\gamma_1], [\gamma_2] \in U_T \) which are both limit of the same sequence in \( U_T \), i.e., there are two sequences of paths \( \{\gamma_{1,n}\}, \{\gamma_{2,n}\} \) such that \( \gamma_{1,n} \xrightarrow{\text{unif}} \gamma_1, \ \gamma_{2,n} \xrightarrow{\text{unif}} \gamma_2 \), and \([\gamma_{1,n}] = [\gamma_{2,n}] \) for every \( n \). Clearly, \([\gamma_1]\) and \([\gamma_2]\) must project to the same point of \( V_T \), that is, \( \gamma_1 \) and \( \gamma_2 \) are contained in the same fiber of \( V_T \) and have the same endpoint. The loop \( \gamma_1 \ast \gamma_2^{-1} \) is then a vanishing cycle, uniformly approximated by the zero-homotopic cycles \( \gamma_{1,n} \ast \gamma_{2,n}^{-1} \).

Thus, if \( U_T \) is not Hausdorff then \( V_T \) contains a vanishing cycle. The converse is totally similar. \( \square \)

Note that the structure of \( V_T \) around a fibre \( \hat{L}_t \) does not really depend on the chosen transverse disc \( T \). Hence, by abuse of terminology, we shall sometime speak about vanishing cycle “in a leaf”, meaning the projection of a vanishing cycle in the corresponding holonomy covering.

Remark that vanishing cycles are not totally unrelated with vanishing ends. Indeed, if \( E \subset L^0_p \) is a vanishing end, then obviously \( L^0_p \) contains a vanishing cycle (\( \partial E \)) for the foliation \( \mathcal{F}^0 \) (here the ambient manifold \( X^0 \) is not compact, so we take the definition above in a larger sense). Our definition of leaf is concocted in such a way that this special vanishing cycle disappears when we pass from \( L^0_p \) to \( L^0_p \). Of course, there is no reason to exclude, a priori, some “transcendental” vanishing cycle which has nothing to do with vanishing ends, and which cannot be absorbed in a definition.

**2.3. Examples of vanishing cycles.** When \( X \) is Kähler, it is shown in \([\text{Br1}], [\text{Br2}]\) (modulo a correction in \([\text{Br4}]\), concerning the unparametrized Levi continuity principle) that vanishing cycles do not exist, i.e., every foliation on \( X \) is uniformisable. If \( X \) is not Kähler, however, the uniformisability may fail. Let us see some examples.

**Example 2.1 (Straight foliations on blown up Hopf surfaces).** Let \( X \) be a Hopf surface \([\text{BPV}, \text{V.18}]\), i.e., a compact complex surface whose universal covering is
$(\mathbb{C}^2)^* = \mathbb{C}^2 \setminus (0,0)$. Its fundamental group $G$ is an extension of $\mathbb{Z}$ by a finite subgroup of $U(2)$; normal forms for the action of $G$ on $(\mathbb{C}^2)^*$ can be found in [MN].

Any foliation $\mathcal{F}$ on $X$ can be lifted to a foliation $\mathcal{G}^*$ on $(\mathbb{C}^2)^*$ which extends to a foliation $\mathcal{G}$ on $\mathbb{C}^2$. This extension may be singular or not at $(0,0)$, and so let us distinguish two cases:

1. $\mathcal{G}$ is singular at $(0,0)$. As a general fact, note that any vanishing cycle is homotopic to zero in the ambient manifold, and so it can be lifted in the universal covering, giving a vanishing cycle for the lifted foliation; and, conversely, any vanishing cycle in the universal covering projects to a vanishing cycle. Thus, $\mathcal{F}$ has a vanishing cycle if and only if $\mathcal{G}^*$ has such one. The extended foliation $\mathcal{G}$ is singular only at $(0,0)$ (otherwise, use the action of $G$ to get a contradiction), hence $\mathcal{G}^*$ coincides with the restriction of $\mathcal{G}$ to the complement of its singularities. Now, for any foliation in $\mathbb{C}^2$ Ilyashenko proved that its restriction to the complement of its singularities has no vanishing cycle [Ily]. We conclude that $\mathcal{G}^*$ has no vanishing cycle, as well as $\mathcal{F}$. Of course, instead of referring to [Ily] we can also put $\mathcal{G}$ in some simple normal form (using its $G$-invariance and the normal forms for $G$ of [MN]), and then do some explicit analysis of the foliation.

2. $\mathcal{G}$ is not singular at $(0,0)$. The group $G$ always contains an element which acts on $\mathbb{C}^2$ as a global contraction towards $(0,0)$, and which can be put in some simple (global) normal form of Poincaré–Dulac type [MN]: $(z, w) \mapsto (\alpha z + cw^m, \beta w)$, with $(\beta^m - \alpha)e = 0$ and $\alpha, \beta$ of modulus less than 1. The invariance of $\mathcal{G}$ by this contraction, and the fact that $\mathcal{G}$ is not singular at the origin, readily show that $\mathcal{G}$ is necessarily a constant slope foliation, given by the equation $dw = 0$ up to a further (linear) change of coordinates. The foliation $\mathcal{G}^*$ has a vanishing cycle: the circle $\gamma_0 = \{z = 1, w = 0\}$ is not homotopic to zero in the leaf $\{z \neq 0, w = 0\}$ of $\mathcal{G}^*$, whereas the circle $\gamma_\varepsilon = \{z = 1, w = \varepsilon\}, \varepsilon \neq 0$, is homotopic to zero in the leaf $\{w = \varepsilon\}$. This vanishing cycle projects to $X$ to a vanishing cycle for $\mathcal{F}$, contained in the elliptic leaf arising from $\{w = 0\}$. All the other leaves are simply connected, isomorphic to $\mathbb{C}$, and they spiral around this elliptic leaf. Remark that $X$ may be a “secondary” Hopf surface, i.e., its fundamental group may be strictly larger than $\mathbb{Z}$; however, using the fact that $\mathcal{G}$ is invariant by the action of $G$, it is easy to see that such a fundamental group must be abelian, that is $\mathbb{Z} \oplus \mathbb{Z}_k$ for some $k$.

A nonuniformisable foliation as in (2) will be called a straight foliation on the Hopf surface $X$. More generally, let $X$ be a (multiple) blow-up of a Hopf surface $X_0$. Every foliation $\mathcal{F}$ on $X$ is the blow-up of a foliation $\mathcal{F}_0$ on $X_0$, and the exceptional divisor of $X \to X_0$ is necessarily invariant by $\mathcal{F}$ (recall that $\mathcal{F}_0$ has no singularity). We then have a bijective correspondence between vanishing cycles of $\mathcal{F}$ and vanishing cycles of $\mathcal{F}_0$, and $\mathcal{F}$ is nonuniformisable if and only if $\mathcal{F}_0$ is straight. Again, we shall say that $\mathcal{F}$ is a straight foliation on $X$.

Example 2.2 (Straight foliations on blown up Kato surfaces). Let $X$ be a Kato surface [Kal], [Dlo]. Similarly to Hopf surfaces, a Kato surface is associated to a (germ of) contracting mapping $F$: $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, however, contrary to Hopf surfaces, such a $F$ is not invertible. A foliation $\mathcal{F}$ on $X$ corresponds to a (germ of) foliation $\mathcal{G}$ on $(\mathbb{C}^2, 0)$, invariant by $F$. A thorough study of foliations on Kato surfaces has been carried out in [DK] and [DO]. We have only two possibilities:
(1) $\mathcal{G}$ is singular at $(0,0)$. In suitable coordinates $(z,w)$, it is given by the equation $wdz + \lambda z dw = 0$, where $\lambda$ is a quadratic irrational. Note that the two separatrices of this singularity have infinite order holonomy. The foliation $\mathcal{F}$ admits one or two cycles of invariant rational curves, each rational curve contains a leaf isomorphic to $\mathbb{C}^*$, and this leaf has infinite holonomy, so that its holonomy covering is simply connected. All the other leaves are simply connected. It follows that $\mathcal{F}$ does not have any vanishing cycle, for the trivial reason that every holonomy covering is simply connected. Thus $\mathcal{F}$ is uniformisable. Remark that this case occurs if and only if $X$ belongs to a special subclass of Kato surfaces, called Inoue–Hirzebruch surfaces.

(2) $\mathcal{G}$ is not singular at $(0,0)$. Then, as in the case of Hopf surfaces, $\mathcal{F}$ is not uniformisable, and we shall call it a straight foliation on the Kato surface $X$. The foliation $\mathcal{F}$ admits an invariant compact curve which is composed by a cycle of rational curves $C$ to which some trees of rational curves $T_j$ are attached (if we contract to a point each one of these trees, we obtain a relatively minimal model of the foliation, in the sense of Remark 2.1). The vanishing cycle is here contained in a leaf contained in $C$.

Remark 2.3. In all these examples, the vanishing cycle is contained in an algebraic leaf which is either an elliptic curve, or a part of a cycle of rational curves. This algebraicity property is not at all evident a priori: by definition, a vanishing cycle could well belong to some transcendental leaf. A natural approach to prove, a priori, such an algebraicity could be the following, close to Sullivan’s approach to

(This picture does not reflect the true complexity of the foliation. In the non-generic case “$\sigma_n > 2n$” [DO], the piece of leaf between the circle $\gamma_\varepsilon$ and its first return close to $\gamma_\varepsilon$ is not an annulus, but a disc with $k$ holes, $k \geq 2$. The disc $\Gamma_\varepsilon$ turns $\ell$ times along the rational curves, with $\ell$ large when $\varepsilon$ is small, and then it goes away and caps. Thus the part of $\Gamma_\varepsilon$ “close” to the rational curves is is a disc with $k^\ell$ holes, and its part “far” from the rational curves is a collection of $k^\ell$ small discs. In the generic case “$\sigma_n = 2n$” [DO], instead, $k = 1$ and the situation is closer to the one of straight foliations on Hopf surfaces, with the elliptic curve replaced by a cycle of rational curves).

A similar dichotomy holds for foliations on blown up Kato surfaces, and blow-ups of (2) will be called again straight foliations.
Novikov’s theorem [God, V.3]. By taking a limit of normalized integration currents over the discs \( \{\Gamma_n\} \) bounded by \( \{\gamma_n\} \), we can construct a closed positive current \( T \), and then we could try to prove that such a current has an algebraic nature (for instance, by evaluating its Lelong numbers...). However, already in the example of straight foliations on Kato surfaces this approach does not work. Indeed, if the Kato surface is not of generic type [DO], then the current \( T \) that we obtain (which is not only closed but even exact) is a diffuse one, which gives no mass to the rational curves (this follows from the fact that on a nongeneric Kato surface there is no positive divisor cohomologous to zero). This is also related to the fact that, in that case, the leaves of the foliation have a large limit set which is not reduced only to the set of rational curves [DO].

3. Nonuniformisable Foliations

3.1. Pseudoconvex shells. Given a compact connected surface \( X \), by a pseudoconvex shell in \( X \) we mean a compact connected real analytic hypersurface \( M \subset X \) such that:

1. \( M \) is global, i.e., \( X \setminus M \) is connected;
2. \( M \) is strictly pseudoconvex;
3. \( M \) bounds a (possibly singular) Stein surface \( Y \).

The condition (3) must be understood in the “abstract” sense, i.e., \( Y \) is not a subset in \( X \) (which is anyway prohibited by condition (1)). Pseudoconvex shells were introduced by Kato in [Ka2], where he obtains a classification of surfaces that admit such a shell.

In this section we shall firstly prove the following result, which is the key step toward the classification of nonuniformisable foliations.

**Theorem 3.1.** Let \( X \) be a compact connected complex surface which is not elliptic, and let \( \mathcal{F} \) be a nonuniformisable foliation on \( X \). Then \( X \) contains a pseudoconvex shell.

In order to start the proof of this theorem, let us consider a vanishing cycle \( \gamma : [0, 1] \to \hat{L}_t \) contained in some fiber of some holonomy tube \( V_T \). As observed in [Br1, p. 125], we may assume that \( \gamma \) is a simple loop: by elementary surface topology, the existence of a vanishing cycle implies the existence of a simple one. Hence we may assume that the image of \( \gamma \) (still denoted by \( \gamma \), by abuse of notation) is an embedded real analytic circle in \( \hat{L}_t \).

Set \( A_r = \{ r < |w| \leq 1 \} \) and \( \partial A_r = \{|w| = 1\} \). Then, for some \( r \in (0, 1) \), we may find an embedding

\[
f_0 : \mathbb{D} \times A_r \to V_T
\]

which sends fibers \( \{z\} \times A_r \) to fibers of \( V_T \) and \( \{0\} \times \partial A_r \) to \( \gamma \). By definition of vanishing cycle, for some sequence \( \{z_n\} \subset \mathbb{D}, z_n \to 0 \), the image by \( f_0 \) of \( \{z_n\} \times \partial A_r \) is a circle \( \gamma_n \) which is homotopic to zero in the fiber \( \hat{L}_{t_n} \) containing it. Of course, this remains true not only for \( z_n \) but also for every \( z \) sufficiently close to \( z_n \), so that we find an open subset \( U \subset \mathbb{D} \) with the property that: for every \( z \in U \) the circle

\[
\gamma_z = f_0(\{z\} \times \partial A_r) = f_0(z, \partial A_r)
\]
is homotopic to zero in the corresponding fiber $\hat{L}_{t(z)}$, and hence it bounds a disc
\[ \Gamma_z \subset \hat{L}_{t(z)} \]
The subset $U$ contains 0 in its closure.

Consider now the composite map
\[ f: \mathbb{D} \times A_r \xrightarrow{f_0} V_T \xrightarrow{\pi_T} X. \]
Up to a small deformation of the initial $\gamma$, we may assume that the image of $f_0$ does not meet the indeterminacy points of $\pi_T$, so that $f$ is a holomorphic immersion into $X$. Moreover, by a standard argument in foliation theory [God, p. 96], a generic leaf of $F$ has trivial holonomy, and this means that the map $\pi_T$ is injective when restricted to generic fibers of $V_T$. It follows from this that, up to restricting the $\mathbb{D}$-factor, the map $f$ is an almost embedding (in the sense of the Appendix below: it is an embedding outside a finite set of fibers).

We now apply the unparametrized Levi continuity principle stated and proved in the Appendix. For every $z \in U$, the annulus $f(z, A_r)$ clearly extends to a disc, image of $\Gamma_z$ by $\pi_T$. Hence, by Proposition 4.1, $f(\mathbb{D} \times A_r)$ extends to a punctured meromorphic family of discs
\[ g: W \setminus E \rightarrow X. \]
Here $W$ is a $\mathbb{D}$-fibration over $\mathbb{D}$, $E \subset X$ is the discrete set of essential singularities, and $g$ sends the boundary of the fiber $W_z \simeq \mathbb{D}$ to the circle $f(z, \partial A_r)$, for every $z \in \mathbb{D}$. For generic $z$, $g(W_z)$ is a disc embedded in a leaf $L_{t(z)}$, and indeed in $L^0_{t(z)}$ (note that for a generic leaf we have $L_{t(z)} = L^0_{t(z)}$, because a generic fiber of $V_T$ is free of indeterminacy points of $\pi_T$).

By the very definition of leaf, the map $g$ can be lifted to an embedding
\[ g_0: W \setminus E \rightarrow V_T. \]
Thus, every circle $\gamma_z \subset \hat{L}_{t(z)} \subset V_T$, $z \in \mathbb{D}$, bounds in the fiber a pluripunctured disc $\Gamma^*_z$, image of $W^*_z = W_z \setminus E_z$ by $g_0$. Because $\gamma_0 = \gamma$ is not zero-homotopic, by assumption, we have $E_0 \neq \emptyset$. Up to restricting to a neighbourhood of some point of $E_0$, we may assume that $E_0 = \{1 \text{ point}\}$ and $E_z = \emptyset$ for every $z \neq 0$ (note that this may change the vanishing cycle, by replacing the initial cycle by a “primitive” one). Also, we may assume that $W = \mathbb{D} \times \mathbb{D}$ and $E_0 = (0, 0)$.

Let us resume this first part of the proof:

**Lemma 3.1.** If $V_T$ has a vanishing cycle, then there exists an embedding
\[ g_0: (\mathbb{D} \times \mathbb{D}) \setminus (0, 0) \rightarrow V_T \]
such that:

1. $g_0$ sends $\{0\} \times \mathbb{D}^*$ to a parabolic end of some fiber $\hat{L}_{t(0)}$;
2. $g_0$ sends $\{z\} \times \mathbb{D}$ to a disc in some fiber $\hat{L}_{t(z)}$, for every $z \in \mathbb{D}^*$.

From now on we shall identify $(\mathbb{D} \times \mathbb{D}) \setminus (0, 0)$ with its image in $V_T$, so that $\gamma_z = \{z\} \times \partial \mathbb{D}$ for every $z$ and $\Gamma_z = \{z\} \times \mathbb{D}$ if $z \neq 0$. The restriction of $\pi_T$ to this subset will be denoted by $g$. 


The idea for constructing a pseudoconvex shell is now the following one. We take a 3-sphere around the origin \( S^3 \subset (\mathbb{D} \times \mathbb{D}) \setminus (0, 0) \) (with generic radius, so that it does not meet the indeterminacy points of \( \pi_T \)), and then we take its image in \( X \) by \( g \). The problem, however, is that we get in this way only an immersed hypersurface, not an embedded one. A similar problem occurs in [Iva], concerning the difference between spherical shells in Kato’s sense and spherical shells in the Ivashkovich’s sense.

3.2. A first case. Let us firstly consider the case in which the meromorphic immersion

\[
g: (\mathbb{D} \times \mathbb{D}) \setminus (0, 0) \rightarrow X
\]

is injective (and free of indeterminacies) on \( \gamma_0 \). Up to restricting the first factor, we may assume that \( g \) is injective (and free of indeterminacies) on the solid torus \( R = \mathbb{D} \times \partial \mathbb{D} \). Note that indeterminacy points of \( g \) could accumulate to the essential singularity \((0, 0)\), and so they cannot totally avoided in the following lines.

For every \( z \in \mathbb{D}^\ast \) with \( \Gamma_z \cap \text{Indet}(g) = \emptyset \), the immersion \( g_z = g|_{\Gamma_z}: \Gamma_z \rightarrow X \) sends \( \Gamma_z \) to some leaf \( L_0^{(z)} \), and it is an embedding around \( \partial \Gamma_z \). The leaf \( L_0^{(z)} \) is certainly not a rational curve, otherwise it would be a rational curve of zero selfintersection [Br3, Section 2] and therefore \( X \) would be a rational or ruled surface [BPV, V.4], hence Kählerian. It is a general fact that, given a holomorphic map of \( \mathbb{D} \) into a non-rational curve \( C \), the injectivity on \( \partial \mathbb{D} \) implies the injectivity on the full \( \mathbb{D} \): this can be checked by lifting the map to the universal covering of \( \mathbb{D} \) (or \( \mathbb{C} \)), and then by applying the maximum principle. Therefore, our map \( g_z \) is actually an embedding over the full \( \Gamma_z \). The image

\[
D_z = g_z(\Gamma_z)
\]

is therefore a closed disc embedded into \( X^0 \) and with boundary on the embedded solid torus \( S = g(R) \).

However, in principle, such a disc \( D_z \) could “return” and intersect \( S \) not only along \( \partial D_z \), but also along some other circles embedded in \( D_z \setminus \partial D_z \), see the figure below. These circles correspond to boundaries of other closed discs \( D_{z'} \), contained in \( D_z \). Thus, even if every \( g_z \) is injective, the full map \( g \) may fail to be injective, because \( D_{z'} \subset D_z \) for some \( z' \neq z \).
Lemma 3.2. There exists a domain $B \subset \mathbb{D}$ such that:

1. $B$ is relatively compact in $\mathbb{D}$, it contains the origin, and $\partial B$ is piecewise real analytic;
2. for every $z \in \partial B$, $\Gamma_z$ is disjoint from the indeterminacies of $g$, and $D_z = g_z(\Gamma_z)$ cuts $S_B = g(\overline{B} \times \partial \mathbb{D})$ only along $\partial D_z$.

Proof. We start by trying with $B_0 = \mathbb{D}(r)$, with $r \in (0, 1)$ such that $\Gamma_z \cap \text{Indet}(g) = \emptyset$ for every $z \in \partial \mathbb{D}(r)$ (this is just a generic condition, and we will no more worry about it in the following). Consider the subset of $\overline{B}_0$ defined by

$$C = \{z' \in \overline{B}_0 : \exists z \in \partial B_0 \text{ s.t. } D_{z'} \subset D_z\}$$

(it is implicitly assumed in this definition that $z' \neq 0$ and $\Gamma_{z'} \cap \text{Indet}(g) = \emptyset$, so that $D_{z'}$ is well defined). In other words, $\bigcup_{z \in \partial B_0} D_z = g(\partial B_0 \times \mathbb{D})$ cuts $S_{B_0} = g(\overline{B}_0 \times \partial \mathbb{D})$ along the compact set $g(C \times \partial \mathbb{D})$. Note that in the definition we don’t exclude $z = z'$, so that $C$ contains the full $\partial B_0$. The curve $C$ is a (singular) real analytic curve in $B_0$, and we now set

$$B_1 = \text{connected component of } B_0 \setminus C \text{ containing } 0.$$  

For this new choice, we have that for every $z \in \partial B_1$ the disc $D_z$ can still contain some other disc $D_{z'}$, with $z' \in B_1$, but if this occurs then $z' \in \partial B_1$. This follows from the fact that $z \in C$ and $D_{z'} \subset D_z$ imply $z' \in C$.

We thus have a sort of “dynamical system” on $\partial B_1$, which associates to every $z$ the (finite, and possibly empty) “orbit”

$$\mathcal{O}_z = \{z' \in \partial B_1 : z' \neq z, D_{z'} \subset D_z\}.$$  

Note that each such orbit $\mathcal{O}_z$ has a natural partial order, given by inclusion; a minimal element in $\mathcal{O}_z$ is one of the “last” intersections of $D_z$ with $S_{B_1}$. We would like that this dynamical system on $\partial B_1$ be trivial, i.e., $\mathcal{O}_z = \emptyset$ for every $z$. Indeed, this is exactly the condition (2) that we are looking for. Set

$$\ell = \max_{z \in \partial B_1} \{\#\mathcal{O}_z\}.$$  

We next modify $B_1$ in such a way that $\ell$ decreases, so that the proof will be concluded by induction.

Consider the subset of $\partial B_1$ defined by $E = \{z \in \partial B_1 : \mathcal{O}_z = \emptyset\}$. It is an open subset, and in fact a finite union of intervals (where interval may mean also an entire component of the boundary). Note that $E$ is not empty: for any $z \in \partial B_1$, any minimal element in $\mathcal{O}_z$ belongs to $E$. We modify $B_1$ by pushing a little any such interval $I$ inside the domain:
Call $B_2$ this new domain. The deformed interval $I_{\text{def}} \subset \partial B_2$ is chosen so close to $I \subset \partial B_1$ that for every $z \in I_{\text{def}}$ we still have $D_z \cap S_{B_1} = \partial D_z$, and a fortiori $D_z \cap S_{B_2} = \partial D_z$. Denote by $E_{\text{def}} \subset \partial B_2$ the deformation of $E$ constructed in this way, and by $F$ its complement in $\partial B_2$, or equivalently in $\partial B_1$.

If $z \in F$ and $D_z \subset D_z$ for some $z' \in \overline{B}_2$, then actually $z' \in \partial B_2$, and more precisely $z' \in F$: indeed, $z$ belongs also to $\partial B_1$, hence $z'$ belongs also to $\partial B_1$, and $\partial B_1 \cap \overline{B}_2 = F$. Therefore, even for $B_2$ we have a dynamical system as for $B_1$, associating an orbit $O_z \subset \partial B_2$ to every $z \in \partial B_2$. Setting $\ell^* = \ell$ equal to the maximal orbit cardinality, we are showing that $\ell^* < \ell$, concluding the proof.

If $z \in E_{\text{def}}$, then $O_z^* = \emptyset$. If $z \in F$, then $O_z$ (sic) contains at least one minimal element $z''$, which belongs to $E$. This point has been removed from $O_z^*$ by deforming $E$ to $E_{\text{def}}$. More precisely, we have $O_z^* = O_z \setminus (O_z \cap E)$. Hence $\# O_z^* < \# O_z$, and $\ell^* < \ell$.

With this lemma it is now easy to construct a pseudoconvex shell. From property (2) which implies that $D_z$ and $D_z'$ are disjoint for every $z, z' \in \partial B, z \neq z'$, we see that $g$ is injective on

$$M_0 = (\overline{B} \times \partial \mathbb{D}) \cup (\partial B \times \overline{\mathbb{D}}) = \partial (B \times \mathbb{D}) \subset \mathbb{D} \times \overline{\mathbb{D}}.$$  

Because $B \times \mathbb{D}$ is Stein, we can approximate $M_0$ with a real analytic and strictly pseudoconvex hypersurface $M \subset \mathbb{D} \times \mathbb{D}$ over which $g$ is still free of indeterminacies and injective. Note that such an $M$ bounds a Stein domain $Y$, containing the origin. Its image $g(M) \subset X$ is then a pseudoconvex shell. The “global” requirement follows from [Iva, Lemma 2.5]: $g(M)$ is not homologous to zero in $X$, i.e., $X \setminus g(M)$ is connected, because $g$ has an essential (and unique) singularity in $Y$.

Remark that here we have not yet used the hypothesis that $X$ is not elliptic. Remark also that in Lemma 3.2 it is not asserted that $B$ is simply connected, hence the pseudoconvex shell so far obtained could have a topology different from a 3-sphere.

### 3.3. A second case

Let us now remove the assumption that the meromorphic immersion

$$g: (\mathbb{D} \times \overline{\mathbb{D}}) \setminus \{0, 0\} \to X$$

is injective on the vanishing cycle $\gamma_0 = \{0\} \times \partial \mathbb{D}$. We need to understand better this failure of injectivity.

As a preliminary fact, let us recall a theorem of Ohnaka [Roy]: if $C$ is a hyperbolic curve and $h: \mathbb{D} \to C$ is a holomorphic map, then either $h$ extends to the origin, or there exists a parabolic end $e$ of $C$ such that $h$ extends to a holomorphic map $h: \mathbb{D} \to C \cup \{e\}$, sending $0$ to $e$. This theorem holds also when $C$ is a hyperbolic orbifold, even if the underlying space (forgetting multiplicities) is parabolic. Indeed, by using the contracting property of holomorphic maps with respect to hyperbolic metrics, we see that if $h$ does not extend to $0$ then $h(z)$, $z \to 0$, is divergent in $C$ (note that the hyperbolic length of $\partial \mathbb{D}(\varepsilon) \subset \mathbb{D}^*$ tends to zero as $\varepsilon \to 0$). Hence, up to restricting the domain of definition, we may assume that the image of $h$ is contained in some proper subregion of $C$. This subregion is
hyperbolic, even forgetting multiplicities, and so we may apply Ohtsuka’s theorem to get the parabolic end \( e \) towards which \( h \) converges.

Returning to our proof, let us firstly consider the case in which the leaf \( L_{t(0)} \) is hyperbolic, i.e., \( \hat{L}_{t(0)} = \mathbb{D} \). In the holonomy covering \( \hat{L}_{t(0)} \), the cycle \( \gamma_0 \) bounds a parabolic end \( \Gamma_0^* \). By Ohtsuka’s theorem, under the covering \( \hat{L}_{t(0)} \to L_{t(0)} \) the parabolic end \( \Gamma_0^* \) is sent to a parabolic end of \( L_{t(0)} \): there exists a parabolic end \( \Sigma_0^* \) of \( L_{t(0)} \) such that the map \( \hat{L}_{t(0)} \to L_{t(0)} \) extends to a map between \( \hat{L}_{t(0)} \cup \{0\} \) and \( L_{t(0)} \cup \{0\} \), sending \( 0 \in \Gamma_0^* \) to \( 0 \in \Sigma_0^* \). Up to deforming (restricting) \( \Gamma_0^* \), we may assume that \( \Gamma_0^* \) is sent exactly to \( \Sigma_0^* \), as a cyclic covering of order \( k \). The (noninjective) image of \( \gamma_0 \) is then an embedded circle \( \sigma_0 = \partial \Sigma_0 \subset L_{t(0)} \), whose holonomy has order \( k \), and which is covered \( k \) times by \( \gamma_0 \).

Take now the (singular) domain \( \Omega_{k,\ell} = (\mathbb{D} \times \mathbb{D})/\mathbb{Z}_k \) introduced in Remark 2.2, with \( \ell \) given by the type of the holonomy along \( \sigma_0 \). The map \( g \) factorizes through a map \( g': \Omega_{k,\ell} \setminus \{q_{k,\ell}\} \to X \), and now \( g' \) is injective on the cycle \( \gamma_0/\mathbb{Z}_k \subset \Omega_{k,\ell} \). We can repeat all the arguments of the previous section, with \( g \) replaced by \( g' \), and we get a pseudoconvex shell in \( X \). Note that in this case the shell bounds a singular Stein domain, contained in \( \hat{\Omega}_{k,\ell} \).

Let us now consider the case in which \( L_{t(0)} \) is not hyperbolic, i.e., \( \hat{L}_{t(0)} = \mathbb{C} \). We have three possibilities for such a leaf:

1. \( L_{t(0)} \) is noncompact: \( \mathbb{C} \) or \( \mathbb{C}^* \) or \( \mathbb{C} \) with one multiple point or \( \mathbb{C} \) with two multiple points, both of multiplicity two;
2. \( L_{t(0)} \) is an elliptic curve;
3. \( L_{t(0)} \) is a rational curve with three or four multiple points, whose multiplicities \( m_j \) satisfy the relation \( \sum_j (1 - \frac{1}{m_j}) = 2 \).

In case (1) it is easy to see that the parabolic end \( \Gamma_0^* \) of \( \hat{L}_{t(0)} \) covers a parabolic end of \( L_{t(0)} \), with some order \( k \), and so we may conclude as in the hyperbolic case. In case (2) we use the following obvious fact: any cycle on a 2-torus is homotopic to a multiple of an embedded cycle. This means that, up to deforming \( \gamma_0 \), we can assume that its projection to \( L_{t(0)} \) is an embedded cycle \( \sigma_0 \), and \( \gamma_0 \to \sigma_0 \) is a covering of order \( k \). Again, this is sufficient to proceed as in the hyperbolic case.

In case (3) this strategy of reducing to a \( k \)-fold covering fails. We shall prove, instead, that \( X \) is elliptic, against the hypotheses. Note that, a posteriori, this case does not exist.

**Lemma 3.3.** In case (3) the surface \( X \) admits an elliptic fibration, and \( L_{t(0)} \) is an irreducible component of a fiber.

*Proof.* It is convenient here to work with the relatively minimal model \( (X', \mathcal{F}') \) of \( (X, \mathcal{F}) \) mentioned in Remark 2.1, by resolving singularities of \( \mathcal{F} \) and then contracting (iteratively) those rational curves of negative selfintersection over which the canonical bundle of the foliation has negative degree. Following the procedure described in [Bu3, p. 57], it is clear that the leaf \( L_{t(0)} \) of \( \mathcal{F} \) becomes a leaf \( L'_{t(0)} \) of \( \mathcal{F}' \) of the same type. However, for \( \mathcal{F}' \) the leaves do not pass through \( \text{Sing}(\mathcal{F}') \), only through \( \text{Sing}(X') \). Thus \( L'_{t(0)} \) is a rational curve in \( X' \) which is disjoint from
Sing($\mathcal{F}'$), and which passes three or four times through Sing($X'$), with appropriate multiplicities.

We can compute the selfintersection of this rational curve, using Camacho-Sad formula [Br3, Section 2]: it is zero, because the curve is free of singularities of the foliation. In other words, denoting by $\pi: X \to X'$ the contraction map, our original surface $X$ contains a tree of rational curves $\pi^{-1}(L_{t(0)}') = \bigcup R_j = R$ which supports a positive divisor with zero selfintersection $\pi^*(L_{t(0)}') = \sum m_j R_j = D$.

Because the surface is not Kähler, its algebraic dimension $a(X)$ is 0 or 1. If $a(X) = 1$ then [BPV, VI.4] the surface is an elliptic one, and moreover the above divisor $D$ is a (singular) fiber (of which $L_{t(0)}'$ is an irreducible component), because on a nonalgebraic elliptic surface there are no compact curves besides the fibers. If $a(X) = 0$ then [BPV, VI.5] [Nak] the surface belongs to the class VII$_0$ (possibly blown up). There are, in this case, finitely many compact curves on $X$, whose possible configurations are described in [Nak, Section 6–7], and one checks that in no case there is a divisor $D$ with $D^2 = 0$ and with support equal to a tree of rational curves. 

This lemma completes the proof of Theorem 3.1

### 3.4. Classification

With Theorem 3.1 and a Kato’s theorem [Ka2], we can now conclude the proof of Theorem 1.1. Let $\mathcal{F}$ be a nonuniformisable foliation on the compact connected surface $X$.

Let us firstly consider the nonelliptic case. Then, by Theorem 3.1, $X$ contains a pseudoconvex shell, and by [Ka2] $X$ is either a (blown up) Hopf surface or a (blown up) Kato surface. We have already analyzed nonuniformisable foliations on these classes of surfaces, in Examples 2.1 and 2.2: they are all straight.

Let us suppose now that $X$ is an elliptic surface. Because $X$ is not Kähler, and therefore not algebraic, there are only two possible types of foliations: the elliptic fibration $\mathcal{G}$ itself, and the turbulent foliations, i.e., foliations which are transverse to a generic elliptic fiber [Br3, Section 7]. This dichotomy follows from the fact that if $\mathcal{F} \neq \mathcal{G}$ then the tangency divisor between $\mathcal{F}$ and $\mathcal{G}$ must be supported on some (components of) fibers (there are no other curves on a nonalgebraic elliptic surface [BPV, VI.4]), and this means precisely that $\mathcal{F}$ is turbulent. The elliptic fibration
is uniformisable (note that around any fiber we can always find a Kähler metric), hence let us consider the case of a turbulent foliation $\mathcal{F}$.

Up to resolving singularities and contracting certain curves, we may suppose that $\mathcal{F}$ is in the relatively minimal form described in [Br3, p. 67], i.e., around each fiber of $X$ the structure of $\mathcal{F}$ is given by one of the six models described in [Br3, Fig. 5]:

Note that $X$ may be singular, but $\mathcal{F}$ is not. Note also that the elliptic fibration is isotrivial, i.e., all the smooth fibers are isomorphic to the same elliptic curve, the isomorphism being defined by the holonomy of $\mathcal{F}$ between transverse fibers. Denote by $B$ the base of the elliptic fibration, with projection $\pi: X \to B$, and by $B_0$ the set of points over which the fiber is not invariant by $\mathcal{F}$, with its natural orbifold structure given by multiplicities of fibers, so that every leaf of $\mathcal{F}$ outside the invariant fibers projects to $B_0$ as a regular orbifold covering [Br3, Section 7].

Outside the invariant fibers, the foliation is the suspension of its monodromy representation, therefore it is the quotient of a trivial foliation and it is obviously uniformisable. Thus a vanishing cycle $\gamma$ must be contained in some invariant fiber $F$, over some point $q \in B \setminus B_0$. Let $\gamma'$ be a cycle close to $\gamma$, contained in some leaf $L$ and homotopic to zero there. Its projection $\gamma''$ to $B$ is then a cycle close to $q$, which turns around $q$ a certain number of times. Because $L$ projects to $B_0$, as a regular covering, this cycle $\gamma''$ must be homotopic to zero in $B_0$, that is, the “boundary component” of $B_0$ corresponding to $q$ must be a torsion element in $\pi_{\text{orb}}^1(B_0)$ (the orbifold fundamental group of $B_0$). It follows that $B_0$ must be isomorphic to $\mathbb{C}$ with at most one multiple point. Therefore $B$ is rational, there is only one invariant fiber $F$ (possibly multiple), and at most one fiber of type (a)-with-multiplicity, (b) or (c) (type (a)-without-multiplicity is the type of a generic fiber; types (b) and (c) always have a multiplicity in $\{3, 4, 6\}$). In particular, there are at most two multiple fibers $F_1$ and $F_2$ of the elliptic fibration, over $q_1$ and $q_2$, with multiplicities $m_1$ and $m_2$.

By the canonical bundle formula [BPV, p. 161], we have

$$K_X = \pi^*(K_{\mathbb{C}P^1}) \otimes \mathcal{O}_X \left( \sum_j (m_j - 1)F_j \right) = \pi^* \left( K_{\mathbb{C}P^1} \otimes \mathcal{O}_{\mathbb{C}P^1} \left( \sum_j \frac{m_j - 1}{m_j} \mathcal{O}_{\mathbb{C}P^1} \right) \right),$$

where we have used $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{C}P^1}$ because the fibration is isotrivial [BPV, p. 162]. Therefore $K_X = \pi^*(\text{negative } \mathbb{Q}\text{-divisor})$, for $\deg K_{\mathbb{C}P^1} = -2$ and $\sum_j \frac{m_j - 1}{m_j} < 2$,
and consequently\[ \text{kod}(X) = -\infty. \]

The same negativity of Kodaira dimension holds for our initial \( X \), before passing to the relatively minimal model. By a theorem of Kodaira [BPV, V. 18], such a \( X \) is a (blown up) Hopf surface, and by our discussion in Example 2.1 \( \mathcal{F} \) is straight.

**Remark 3.1.** There is indeed another, less direct, argument to show that, in this last part of the proof, \( \text{kod}(X) \) must be negative. When we take \( X \) in the relatively minimal form as above, it is not difficult to compute its universal covering \( \tilde{X} \) (in orbifold’s sense, as usual). Using again the canonical bundle formula and the isotriviality of the fibration, one finds that for \( \text{kod}(X) \geq 0 \) this universal covering is \( \mathbb{D} \times \mathbb{C} \) or \( \mathbb{C} \times \mathbb{C} \) (depending on the universal covering of the orbifold base). In particular, \( \tilde{X} \) is Stein, and by [Ily] \( \tilde{\mathcal{F}} \), and therefore \( \mathcal{F} \), has no vanishing cycle.

### 4. Appendix: the Unparametrized Levi Continuity Principle

Let us recall some terminology from [Br4, Section 2], adapted to our context. For every \( r \in (0, 1) \), set \( A_r = \{ r < |w| \leq 1 \} \) and \( \partial A_r = \{ |w| = 1 \} \subset A_r \). Let \( X \) be a compact complex surface.\(^2\)

Given an immersion \( f : A_r \to X \), we shall say that \( f(A_r) \) extends to a disc if there exists a holomorphic map \( g : \overline{\mathbb{D}} \to X \) and an embedding \( j : A_r \to \overline{\mathbb{D}} \), such that \( f = g \circ j \). Given an immersion \( f : \mathbb{D} \times A_r \to X \), we shall say that \( f(\mathbb{D} \times A_r) \) extends to a meromorphic family of discs if there exists a complex surface \( W \) with boundary, a meromorphic map \( g : W \to X \), and an embedding \( j : \mathbb{D} \times A_r \to W \), such that: (1) \( W \) has a fibration (submersion) over \( \mathbb{D} \), all of whose fibers \( W_z \), \( z \in \mathbb{D} \), are isomorphic to the closed disc; (2) for every \( z \in \mathbb{D} \), \( j \) embeds \( \{ z \} \times A_r \) into \( W_z \), sending \( \{ z \} \times \partial A_r \) to \( \partial W_z \); (3) \( f = g \circ j \). In particular, for every \( z, f(z, A_r) \) extends to a disc, provided by \( g(z, \cdot) : W_z \to X \) after removal of indeterminacies.

If the meromorphic map \( g \) is not defined on the full \( W \) but only on \( W \setminus E \), with \( E \subset W \) a discrete subset of essential singularities, then we shall say that \( f(\mathbb{D} \times A_r) \) extends to a punctured meromorphic family of discs.

We shall say that an immersion \( f : \mathbb{D} \times A_r \to X \) is an almost embedding if, for some finite subset \( I \subset \mathbb{D} \), the restriction of \( f \) to \( (\mathbb{D} \setminus I) \times A_r \) is an embedding.

The following result is a combination of the unparametrized Levi continuity principle of [Br4] and arguments from [Iva] to bypass the absence of a Kähler metric on \( X \). It is the unparametrized version of [Iva, Corollary 1]. As in [Iva], we shall use a Gauduchon metric [Gau] on \( X \), i.e., a \( \partial \bar{\partial} \)-closed positive \((1, 1)\)-form, which exists on any compact complex surface.

**Proposition 4.1.** Let \( X \) be a compact complex surface and let \( f : \mathbb{D} \times A_r \to X \) be an almost embedding. Suppose that \( f(z, A_r) \) extends to a disc for every \( z \) in some uncountable subset \( U \subset \mathbb{D} \). Then \( f(\mathbb{D} \times A_r) \) extends to a punctured meromorphic family of discs.

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\(^2\)Mutatis mutandis, the result below holds also in a higherdimensional context.
Proof. We follow very closely [Br4, Prop. 2.1]. Consider the subset \( Z \subset \mathbb{D} \setminus I \) of those points \( z \) such that \( f(z, A_r) \) extends to a disc. Exactly as in [Br4], we can decompose \( Z \) as a countable collection of disjoint components, each component being either an open subset of \( \mathbb{D} \setminus I \) or a single point. By hypothesis, there exists an open-type component \( V \). Moreover, over this component we have a “tautological” family of closed discs \( W_0 \), equipped with a “tautological” map \( h_0 : W_0 \to X \) which extends \( f(V \times A_r) \). We can glue \( D \times A_r \) to \( W_0 \), using \( h_0^{-1} \circ f \), obtaining a new surface \( W_1 \) and a new map \( h_1 : W_1 \to X \) which extend \( W_0 \) and \( h_0 \). The surface \( W_1 \) fibers over \( D \), the fibers over \( \mathbb{D} \) are closed discs, the ones over \( \mathbb{D} \setminus V \) are annuli \( A_r \).

Let us look at the boundary of \( V \) in \( \mathbb{D} \). Using a Gauduchon metric \( \omega \), as in [Iva], we will show that this boundary, and even \( \mathbb{D} \setminus V \), is a polar subset of the disc, i.e., it is contained in the set of poles of a subharmonic function [Ran].

Over \( V \) we have the area function

\[
a : V \to \mathbb{R}^+, \quad a(z) = \int_{(W_0)_z} h_0^*(\omega),
\]

that is, \( a(z) \) is the area of the disc extending \( f(z, A_r) \). As in [Iva, p. 815], using the \( \partial \partial \)-closedness of \( \omega \) we see that \( a \) can be written as

\[
a_0 + a_1,
\]

where \( a_0 \) is bounded (when restricted to \( V \cap K, K \subset \mathbb{D} \) compact) and \( a_1 \) is harmonic. Indeed, if we compute the laplacian of \( a_0 \), at some point \( z \in V \), we obtain an expression which involves only some boundary integrals on \( \partial(W_0)_z \), and which can be smoothly extended to \( \mathbb{D} \). We then set \( a_0 \) equal to the logarithmic potential of such expression, so that \( a_0 \) is bounded and \( a_1 = a - a_0 \) is harmonic.

Take \( z_\infty \in \partial V \setminus I \). If for some sequence \( \{z_j\} \subset V \) converging to \( z_\infty \), we have that \( \{a(z_j)\} \) stays bounded, then (as in [Br4], using Bishop’s theorem) we obtain \( z_\infty \in Z \), and of course such a \( z_\infty \) belongs to a point-type component of \( Z \). There is therefore only a countable subset \( B \) of \( \partial V \) with such a boundedness property. For any other point \( z_\infty \in \partial V \setminus B \) we have \( \lim \inf_{z \to z_\infty} a(z) = +\infty \), and therefore also \( \lim \inf_{z \to z_\infty} a_1(z) = +\infty \). We can find on \( \mathbb{D} \) a subharmonic function \( b_0 \) such that \( \{b_0 = -\infty\} \supset B \cup I \) [Ran, Section 3.5]. Take, on \( \mathbb{D} \), the function \( b \) equal to \( b_0 - a_1 \) on \( V \) and \( -\infty \) on \( \mathbb{D} \setminus V \): it is upper semicontinuous, and subharmonic on the full disc. Hence \( \mathbb{D} \setminus V \subset \{b = -\infty\} \) is a polar subset of \( \mathbb{D} \), as claimed above.

Now the proof can be completed by the same embedding trick of [Br4], which reduces the unparametrized continuity principle to the parametrized one of [Iva]. Indeed, the proof of [Br4, Lemma 2.1] works even if \( \mathbb{D} \setminus V \) is polar, instead of thin: the only thing that we need is that any bounded holomorphic function on \( V \) extends holomorphically to \( \mathbb{D} \), which is true if the complement of \( V \) is polar [Ran, Section 3.6]. Hence, by [Br4, Lemma 2.1], we can embed \( W_1 \) into \( \mathbb{D} \times \mathbb{C}P^1 \), so that \( W_1 \) can be completed to a family of closed discs \( W \) over \( \mathbb{D} \), still contained in \( \mathbb{D} \times \mathbb{C}P^1 \), by “filling the holes”. At this point, we can apply [Iva, Corollary 1] and deduce that the map \( h_1 : W_1 \to X \) has a meromorphic extension \( g : W \setminus E \to X \), with \( E \) discrete. This is the punctured meromorphic family of discs that we was looking for. \( \square \)
References


Marco Brunella, Institut de Mathématiques de Bourgogne – UMR 5584 – 9 Avenue Savary, 21078 Dijon, France
E-mail address: Marco.Brunella@u-bourgogne.fr