THE ANTI-SYMMETRIC GUE MINOR PROCESS

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Abstract. Our study is initiated by a multi-component particle system underlying the tiling of a half hexagon by three species of rhombi. In this particle system species $j$ consists of $\lfloor j/2 \rfloor$ particles which are interlaced with neighbouring species. The joint probability density function (PDF) for this particle system is obtained, and is shown in a suitable scaling limit to coincide with the joint eigenvalue PDF for the process formed by the successive minors of anti-symmetric GUE matrices, which in turn we compute from first principles. The correlations for this process are determinantal and we give an explicit formula for the corresponding correlation kernel in terms of Hermite polynomials. Scaling limits of the latter are computed, giving rise to the Airy kernel, extended Airy kernel and bead kernel at the soft edge and in the bulk, as well as a new kernel at the hard edge.


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1. Introduction

A classical problem in random matrix theory is to compute the statistical properties of the eigenvalues, given the distribution of the entries of the matrix. For the well known Gaussian orthogonal, unitary and symplectic ensembles (GOE, GUE and GSE; see, e.g., [F]) this problem can be solved exactly, with $k \times k$ determinant, and $2k \times 2k$ Pfaffian formulas available for the $k$-point correlations in the case of the GUE, and GOE, GSE respectively. Of these the GUE is special because determinantal expressions can also be derived for the correlations of the process specified by the eigenvalues of all the successive minors [JN, OR], whereas for the GOE and GSE the correlations for the minors are not known.

While interest in the eigenvalues of GUE matrices stems from its use in modelling the highly excited energy levels of classically chaotic quantum systems with no time reversal symmetry, the interest in the GUE minor process stems from a work of Baryshnikov [B1] relating to queues. Subsequently the GUE minor process has been identified by Johansson and Nordenstam within a number of random tiling problems [JN]. In the language of stepped surfaces (which are equivalent to certain
tilings) similar applications were noticed in [OR], while in [BFS] the GUE minor process is encountered in the asymmetric exclusion process.

One of the random tiling problems identified in [JN] as relating to the GUE minor process is the tiling of a hexagon by three species of rhombi. This tiling problem is equivalent to non-intersecting lattice paths which at each step move up one unit or down one unit in a so called watermelon formation (see [KGV]). A very natural variation of the non-intersecting paths model is to impose the boundary condition that the paths cannot go below the x-axis [F1, KGV, G], which in fact corresponds to the tiling of a half hexagon by the same three species of rhombi as referred to above (see Figure 1 below).

This variation initiates the study of the present work, where we begin in Section 2 by analyzing such non-intersecting paths from the same viewpoint which in the study [JN] gave rise to the GUE minor process. We obtain in Corollary 2.11 a joint probability density function (PDF) which can be interpreted as a multi-component particle system in which the number of particles in species $j$ consists of $\lfloor j/2 \rfloor$ particles ($j = 1, \ldots, 2n + 1$). The marginal distribution of species $j$ can also be computed exactly (see Theorem 2.8), and is precisely that of a $j \times j$ anti-symmetric GUE matrix (see, e.g., [F, Ch. 1]). By analogy with the findings from [JN], this motivates us to seek the joint eigenvalue PDF of the anti-symmetric GUE minor process, and to compare this against the joint PDF obtained from analyzing the non-intersecting lattice paths model. We do this in Section 3, and find the two joint PDFs are identical.

It should be noted that the discrete processes in this article and in [JN] give rise to random matrix distributions not in the positions of the random walkers, but the holes between them, as detailed in Section 2. It is also possible to obtain random matrix distributions by looking at the positions of random walkers directly. The works [G, KMFW] contain examples of such models. For tilings of a hexagon, it is well known that the limit shape of the disordered region is a circle, [CLP]. We believe that the limit shape in our half-hexagon model is a semicircle, but do not address this question.

As already mentioned, the GUE minor process is determinantal, with the correlations between eigenvalues from prescribed minors being given by an $r \times r$ determinant. Explicitly, with the coordinate $(s, y)$ denoting an eigenvalue from the $s$-th principal minor having value $y$, it was found that [JN]

$$
\rho_{(r)}(\{(s_j, y_j)\}_{j=1,\ldots,r}) = \det[K_{\text{GUE}m}(\{(s_j, y_j), (s_k, y_k)\})]_{j,k=1,\ldots,r} \quad (1.1)
$$

with

$$
K_{\text{GUE}m}(\{(s, x), (t, y)\}) = \begin{cases}
\frac{e^{-(x^2+y^2)/2}}{\sqrt{\pi}} \sum_{k=1}^{t} \frac{H_{s-k}(x)H_{t-k}(y)}{2^{t-k}(t-k)!}, & s \geq t, \\
\frac{e^{-(x^2+y^2)/2}}{\sqrt{\pi}} \sum_{k=-\infty}^{0} \frac{H_{s-k}(x)H_{t-k}(y)}{2^{t-k}(t-k)!}, & s < t,
\end{cases} \quad (1.2)
$$
here $H_j(x)$ denotes the Hermite polynomials of degree $j$, satisfying
\[
\int_{\mathbb{R}} H_i(x)H_j(x)e^{-x^2} \, dx = \delta_{ij} i! \sqrt{\pi}.
\]
In Section 4, Proposition 4.3, we show that similarly for the anti-symmetric GUE minor process,
\[
\rho(r)\{(s_j, y_j)\}_{j=1,...,r} = \det[K^{aGUEm}\{(s_j, y_j), (s_k, y_k)\}]_{j,k=1,...,r}
\]
with
\[
K^{aGUEm}(s, x, t, y) = \begin{cases}
2e^{-((x^2+y^2)/2)}/\sqrt{\pi} \sum_{l=1}^{[t/2]} H_{s-2l}(x)H_{t-2l}(y)/2^{t-2l}(t-2l)!, & s \geq t, \\
-2e^{-((x^2+y^2)/2)}/\sqrt{\pi} \sum_{l=-\infty}^{0} H_{s-2l}(x)H_{t-2l}(y)/2^{t-2l}(t-2l)!, & s < t.
\end{cases}
\]

In Section 5 we turn our attention to various scaled limits of (1.3). In particular, two distinct soft edge scaling limits (neighbourhood of the largest eigenvalues of the respective minors), as well as a bulk and hard edge scaling (the latter referring to the eigenvalues in the neighbourhood of the origin) are computed. At the soft edge, with the species (i.e. minors) labels chosen to differ by a fixed amount, the limiting correlation kernel is found to be the well known Airy kernel (see e.g. [F, Ch. 7])
\[
K^{soft}(s, x, t, y) = \int_{0}^{\infty} \text{Ai}(x + u) \, \text{Ai}(y + u) \, du
\]
independent of the species. The other soft edge scaling limit analyzed is when the species differ by $O(n^{2/3})$. In the case of the GUE minor process, study of this limit is motivated by a mapping to the interface of a certain droplet polynuclear growth model [PS]. Fluctuations of the latter are governed by the dynamical extension of the Airy kernel
\[
K^{soft}(s, x, t, y) = \begin{cases}
\int_{0}^{\infty} e^{-(\tau_y-\tau_x)u} \, \text{Ai}(x + u) \, \text{Ai}(y + u) \, du, & \tau_y \geq \tau_x \\
-\int_{-\infty}^{0} e^{-(\tau_y-\tau_x)u} \, \text{Ai}(x + u) \, \text{Ai}(y + u) \, du, & \tau_y < \tau_x.
\end{cases}
\]
It is (1.6) which again appears in this second soft edge scaling. In the bulk scaling of the GUE minor process, the limiting correlation kernel was found to be [FN]
\[
K^{bead}(s, x, t, y) = \begin{cases}
\frac{1}{2} \int_{1}^{(is)\tau_y-\tau_x} e^{is\pi(s-y)} \, ds, & \tau_y > \tau_x \\
-\frac{1}{2} \int_{R\setminus[-1,1]} (is)\tau_y-\tau_x e^{is\pi(s-y)} \, ds, & \tau_y < \tau_x.
\end{cases}
\]
This corresponds to the $\gamma = 0$ (isotropic) case of the correlation kernel for the bead process [B]. It too is reclaimed in the appropriate bulk scaling limit of the
anti-symmetric GUE minor process. The hard edge scaling limit has no analogue in the GUE minor process. The limiting correlations are given in Proposition 5.5.

Note: While this work was being prepared, a research announcement by M. Defosseaux was posted on the arXiv, [D], stating results equivalent to our Theorem 3.3 and Proposition 4.3.

2. The Half Hexagon

Consider $p$ simple symmetric random walks started at $(0, 2i - 2)$ conditioned to end up at $(2N, 2i - 2)$, for $i = 1, \ldots, p$. They are conditioned never to intersect and never to go below the $x$ axis. Such configurations are in bijection to tilings of a half-hexagon, as seen in the Figure 1. To count such configurations we need the following result of Krattenthaler et al. [KGV].

**Proposition 2.1** (Theorem 6 in [KGV]). Let $e_1 < e_2 < \cdots < e_p$ with $e_i \equiv m \pmod{2}$, $i = 1, 2, \ldots, p$. The number of stars with $p$ branches, the $i$-th branch running from $A_i = (0, 2i - 2)$ to $E_i = (m, e_i)$, $i = 1, 2, \ldots, p$, and never going below the $x$ axis equals

$$\prod_{i=1}^{p} \frac{(e_i+1)(m+2i-2)!}{(m+e_i+2)!} \prod_{1 \leq i < j \leq p} \left( \frac{e_j+1}{2} - \frac{e_j+1}{2} \right)^2 \right). \quad (2.1)$$

Here, a star means a configuration of $p$ simple symmetric random walks, or branches. Given that the red particles on line $x = n$ occur at positions $e_1 < e_2 < \cdots < e_p$.

![Figure 1. Tiling with rhombuses of “half a hexagon” and the corresponding non-intersecting random walks. The positions of blue particles are marked by triangles and the red particles by squares. The picture was generated with the method of [P].](image-url)
\[ \prod_{i=1}^{p} \frac{(e_i + 1)(n + 2i - 2)!}{(\frac{n + e_i}{2} + p - 1)!} \prod_{1 \leq i < j \leq p} \left( \left( \frac{e_i + 1}{2} \right)^2 - \left( \frac{e_j + 1}{2} \right)^2 \right). \] (2.2)

The number of ways to tile the area to the right of line \( n \) is

\[ \prod_{i=1}^{p} \frac{(e_i + 1)(2N - n + 2i - 2)!}{(\frac{2N - n - e_i}{2} + p - 1)!} \prod_{1 \leq i < j \leq p} \left( \left( \frac{e_i + 1}{2} \right)^2 - \left( \frac{e_j + 1}{2} \right)^2 \right). \] (2.3)

Thus the probability of a certain configuration of red particles on line \( n \) is the product of the above two expressions divided by the total number of tilings of the area.

Lemma 2.2. The probability distribution of red dots along line \( x = n \) is

\[ P_{\text{red}}^n[0 < e_1 < \cdots < e_p] = Z^{-1} \prod_{1 \leq i < j \leq p} \left( \frac{(e_i + 1)^2 - (e_j + 1)^2}{2} \right)^2 \times \prod_{i=1}^{p} \frac{(e_i + 1)(n + 2i - 2)!}{(\frac{n + e_i}{2} + p - 1)!} \prod_{1 \leq i < j \leq p} \left( \left( \frac{e_i + 1}{2} \right)^2 - \left( \frac{e_j + 1}{2} \right)^2 \right). \] (2.4)

for \( 0 < e_1 < \cdots < e_p \leq 2p + n - 2 \) and \( e_i = n \mod 2 \) for \( i = 1, \ldots, p \).

We are more interested in the distribution of the holes, i.e., the blue dots. This can be computed using a result from \([B2]\). To set this up, let \( X = \{x_1, \ldots, x_M\} \) be a set of \( M \) real numbers. Consider measures on the following form. For \( w: X \to \mathbb{R} \),

\[ \mathcal{P}_w^{(m)}[x_{i_1}, \ldots, x_{i_m}] = \text{const} \prod_{1 \leq k < \ell \leq m} (x_{i_k} - x_{i_\ell})^2 \prod_{k=1}^{m} w(x_{i_k}) \] (2.5)

is a probability measure which is supported on sets of \( m \) points. Notice that the measure above for the red particles is on this form. Denote the measure on the holes by

\[ \mathcal{P}_w^{M-m} (A) = \mathcal{P}_w^m (X \setminus A), \] (2.6)

this is of course a measure supported on sets of \( M - m \) elements. Borodin proves the following formula.

Proposition 2.3 (Proposition 2 in \([B2]\)). Let \( u, v: X \to \mathbb{R} \) be functions such that for all \( x_k \in X \),

\[ u(x_k)v(x_k) = \frac{1}{\prod_{i \neq k}(x_k - x_i)^2}. \] (2.7)

Then

\[ \mathcal{P}_u^{M-m} = \mathcal{P}_v^{M-m}. \] (2.8)

With this tool we can now compute the distribution of blue particles.
Theorem 2.4. On line \( x = n \leq N \), for \( n \) even, the distribution of the blue particles is

\[
\mathcal{P}_{\text{blue}}^n \{ e_1, e_2, \ldots, e_{p'} \} = Z^{-1} \prod_{1 \leq i < j \leq p'} ((e_j + 1)^2 - (e_i + 1)^2)^2 \\
\times \prod_{i=1}^{p'} \frac{(2N-n-e_i+p)!(\frac{2N-n-e_i}{2}+p-1)!}{(n+2i-2)!(2N-n+2i-2)!(\frac{2n+2i}{2}+p)!} \\
(2.9)
\]

for \( p' = n/2 \) and \( 0 \leq e_1 < e_2 < \cdots < e_{p'} \leq 2p + n - 2 \) and \( e_i \) even for all \( i \). For odd \( n \) it is

\[
\mathcal{P}_{\text{blue}}^n \{ e_1, e_2, \ldots, e_{p'} \} = Z^{-1} \prod_{1 \leq i < j \leq p'} ((e_j + 1)^2 - (e_i + 1)^2)^2 \\
\times \prod_{i=1}^{p'} \frac{(2N-n+e_i+p)!(\frac{2N-n-e_i}{2}+p-1)!}{(n+2i-2)!(2N-n+2i-2)!(\frac{2n+2i}{2}+p)!} \\
(2.10)
\]

for \( p' = (n-1)/2 \) and again \( 1 \leq e_1 < e_2 < \cdots < e_{p'} \leq 2p + n - 2 \) and \( e_i \) odd for all \( i \).

Proof. For \( n \) even, let \( X = \{0, 2, \ldots, 2M-2\} \). By Proposition 2.3, we need to compute

\[
\prod_{y \in X \setminus \{x\}} ((x+1)^2 - (y+1)^2)^{-2} = 2^{-4M} \left( \frac{(M-\frac{1}{2})!(M+\frac{1}{2})!}{x+1} \right)^{-2} \\
(2.11)
\]

For \( n \) odd, \( X = \{1, 3, \ldots, 2M-1\} \) and we need to compute

\[
\prod_{y \in X \setminus \{x\}} ((x+1)^2 - (y+1)^2)^{-2} = 2^{-4M} \left( \frac{(M-\frac{1+x}{2})!(M+\frac{1+x}{2})!}{x+1} \right)^{-2} \\
(2.12)
\]

Combining these with the formula in Lemma 2.2 gives the result. \( \square \)

Let us fix some notation. For any \( k \leq n \), let the positions of the \( \lfloor k/2 \rfloor \) blue particles on line \( k \) be denoted \( x_i^{(k)} > \cdots > x_{\lfloor k/2 \rfloor}^{(k)} \). Let \( x^{(k)} = (x_1^{(k)}, \ldots, x_{\lfloor k/2 \rfloor}^{(k)}) \).

For simplicity of notation later, let \( x_i^{(2l+1)} = 0 \) for \( l = 0, 1, \ldots, \lfloor n/2 \rfloor \). Obviously, for \( k \) odd, \( x_i^{(k)} \) is always even and for \( k \) even, \( x_i^{(k)} \) is always odd.

It so happens, as is easily seen from the picture, that the blue particles must fulfill certain interlacing requirements,

\[
x_i^{(k+1)} < x_i^{(k-1)} < x_i^{(k)} \\
(2.13)
\]

for all \( k = 2, \ldots, n \) and \( i = 1, \ldots, \lfloor k/2 \rfloor \). Henceforth we will write \( x^{(k+1)} \succ x^{(k)} \) when \( x^{(k+1)} \) and \( x^{(k)} \) satisfy the above interlacing. Let \( K(x^n) = \{ (x^{(1)}, \ldots, x^{(n)}) : x^{(n)} \succ x^{(n-1)} \succ \cdots \succ x^{(1)} \} \).

The next observation to make is that the distribution \( x^{(1)}, \ldots, x^{(n)} \), given \( x^n \) is uniform in the cone given by the above inequalities. This multiplied by \( \mathcal{P}_{\text{blue}}^n \{ x^{(n)} \} \) gives the distribution of \( \{ x^{(1)}, \ldots, x^{(n)} \} \).
Proposition 2.5. The joint probability of all blue particles on the first \( n \) lines is
\[
P_{\text{blue}}^{(1,n)} \{x^{(1)}, \ldots, x^{(n)}\} = \frac{\chi_1(x^{(1)}, x^{(2)}) \ldots \chi_{n-1}(x^{(n-1)}, x^{(n)})}{\text{card} \, K(x^{(n)})} P_{\text{blue}} \{x^{(n)}\}, \quad (2.14)
\]
where \( \chi_k(x^{(k)}, x^{(k+1)}) = 1\{x^{(k+1)} > x^{(k)}\} \) and \( \text{card} \, K(x^{(n)}) \) is the cardinality of \( K(x^{(n)}) \).

Seeing this, one would immediately want to compute \( \text{card} \, K(x^{(n)}) \).

Lemma 2.6. Let \( Z_2 = 1, \ Z_n = \prod_{j=1}^{n-2} j!, \ n \geq 3 \). For \( n \) even,
\[
\text{card} \, K(x^{(n)}) = Z_n^{-1} \prod_{1 \leq i < j \leq n/2} \left( \frac{x_i^{(n/2)} - x_j^{(n/2)}}{4} \right) \quad (2.15)
\]
and for \( n \) odd \( (n \geq 3) \),
\[
\text{card} \, K(x^{(n)}) = Z_n^{-1} \prod_{1 \leq i < j \leq (n-1)/2} \left( \frac{x_i^{(n)} - x_j^{(n)}}{4} \right) \prod_{i=1}^{(n-1)/2} \frac{x_i^{(n)}}{2} \quad (2.16)
\]

Proof. The proof is an inductive one. With the products over \( i < j \) interpreted as unity in the cases \( n = 2 \) and \( 3 \), we see by inspection that both (2.15) and (2.16) are correct in these base cases.

Suppose (2.15) has been established for \( n = 2N \). Then we see that
\[
\text{card} \, K(x^{(2N+1)}) = \sum_{x^{(2N)}; x^{(2N+1)} \succ x^{(2N)}} \text{card} \, K(x^{(2N)}). \quad (2.17)
\]

Use of the Vandermonde determinant identity in (2.15) shows
\[
\text{card} \, K(x^{(2N)}) = Z_{2N}^{-1} \det \left[ \left( \frac{x_{N+1-i}^{(2N)} - 1}{4} \right)^{j-1} \right]_{i,j=1,\ldots,\ N}. \quad (2.18)
\]
The important feature is that each row in the determinant depends on a distinct coordinate, and allows the sum in (2.17) to be carried out by summing each \( x_{N+1-i}^{(2N)} \) from 1 to \( x_{N+1-i}^{(2N)} - 1 \) over odd values, or equivalently summing \( (x_{N+1-i}^{(2N)})^{j-1} \) from zero to \( x_{N+1-i}^{(2N)} - 2 - 1 \) over integer values. Thus
\[
\text{card} \, K(x^{(2N+1)}) = Z_{2N}^{-1} \det \left[ \frac{x_{N+1-i}^{(2N+1)} - 1}{2} \right]_{i,j=1,\ldots,\ N} \quad (2.19)
\]
The sum in column \( j \) is an odd polynomial in \( x_{N+1-i}^{(2N+1)} \) of degree \( 2j - 1 \) with leading term \( (1/(2j-1)) (x_{N+1-i}^{(2N+1)} - 2)^{j-1} \) in row \( i \). Adding appropriate multiples of columns 1, \ldots, \( j - 1 \) to column \( j \) shows that this term can replace the sum, and so
\[
\text{card} \, K(x^{(2N+1)}) = Z_{2N}^{-1} \prod_{j=1}^{N} \frac{1}{2j - 1} \det [(x_{N+1-i}^{(2N+1)} - 2)^{j-1}]_{i,j=1,\ldots,\ N} \quad (2.20)
\]
Removing a common factor of \( x_{N+1-i}^{(2N+1)} - 2 \) from each row and further use of the Vandermonde determinant identity gives (2.16).
2.7. Asymptotics

**Theorem 2.8.** Under the rescaling \( x_i^{(n)} = \sqrt{2N(1 - 1/\sqrt{3})} \lambda_i^{(n)} \) the measure \( \mathcal{P}^{(n)}_{\text{blue}} \) converges weakly to

\[
\mathcal{P}^{(n)}_{\text{aGUE}}(\lambda_1^{(n)}, \ldots, \lambda_{n/2}^{(n)}) = W_n^{-1} \prod_{1 \leq i < j \leq n/2} ((\lambda_j^{(n)})^2 - (\lambda_i^{(n)})^2)^2 \prod_{i=1}^{n/2} e^{-(\lambda_i^{(n)})^2} \quad (2.18)
\]

if \( n \) is even and

\[
\mathcal{P}^{(n)}_{\text{aGUE}}(\lambda_1^{(n)}, \ldots, \lambda_{[n/2]}^{(n)}) = W_n^{-1} \prod_{1 \leq i < j \leq [n/2]} ((\lambda_j^{(n)})^2 - (\lambda_i^{(n)})^2)^2 \prod_{i=1}^{[n/2]} (\lambda_i^{(n)})^2 e^{-(\lambda_i^{(n)})^2} \quad (2.19)
\]

if \( n \) is odd. \( W_n \) is the normalisation constant \( W_n = \prod_{i=1}^{[n/2]} N_{n-2l} \), where \( N_{j} \) is specified by (4.37) below.

**Proof.** Apply Stirling’s approximation to the formulas from Theorem 2.4. \( \square \)

As indicated by the use of the subscript aGUE, \( \mathcal{P}^{(n)}_{\text{aGUE}} \) happens to be the eigenvalue measure for the anti-symmetric GUE ensemble, as we shall see in the next section. We are interested in the full measure \( \mathcal{P}^{(1,n)}_{\text{blue}} \), and how its limit relates to aGUE matrices.

Let \( \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_{[k/2]}^{(k)}) \in (\mathbb{R}^+)^n \), for \( k = 1, 2, \ldots \). Also let \( \lambda^{(2l+1)}_l = 0 \) for \( l = 1, 2, \ldots \). Consider the cone \( K(\lambda^{(n)}) = \{ (\lambda^{(1)}, \ldots, \lambda^{(n)}) : \lambda^{(n)} \succ \lambda^{(n-1)} \succ \cdots \succ \lambda^{(1)} \} \), where \( \lambda^{(k+1)} \succ \lambda^{(k)} \) means that \( \lambda^{(k+1)}_l > \lambda^{(k)}_l \) for \( l = 1, 2, \ldots \).

**Theorem 2.9.** Under the rescaling \( x_i^{(n)} = \sqrt{2N(1 - 1/\sqrt{3})} \lambda_i^{(n)} \), as \( N = p \to \infty \), the measure \( \mathcal{P}^{(1,n)}_{\text{blue}} \) converges weakly to

\[
\mathcal{P}^{(1,n)}_{\text{aGUEm}}(\lambda^{(1)}, \ldots, \lambda^{(n)}) = \frac{\chi_k(\lambda^{(1)}, \lambda^{(2)}) \cdots \chi_{n-1}(\lambda^{(n-1)}, \lambda^{(n)})}{\text{vol} K(\lambda^{(n)})} \mathcal{P}^{(n)}_{\text{aGUE}}(\lambda^{(n)}),
\]

where \( \chi_k(\lambda^{(k)}, \lambda^{(k+1)}) = 1 \{ \lambda^{(k+1)} \succ \lambda^{(k)} \} \).

**Proof.** The convergence of a Riemann sum to an integral. The number of integer point in a cone \( K(x^n) \) suitably normalised converges to the volume of the cone \( K(\lambda^{(n)}) \). \( \square \)

**Lemma 2.10.** Let \( Z_n \) be as in Lemma 2.6. One has

\[
\text{vol} K(\lambda^{(n)}) = Z_n^{-1} \prod_{i<j} ((\lambda_i^{(n)})^2 - (\lambda_j^{(n)})^2) \quad (2.21)
\]
for $n$ even and
\[
\text{vol } K(\lambda^{(n)}) = Z_n^{-1} \prod_{i<j} ((\lambda_i^{(n)})^2 - (\lambda_j^{(n)})^2) \prod_i (\lambda_i^{(n)})
\] (2.22)
for $n$ odd.

**Proof.** The proof of Lemma 2.6, with summation replaced by integration. □

The interlacing conditions $\chi_k$ can be written in terms of a determinant,
\[
\chi_k(\lambda^{(k)}, \lambda^{(k+1)}) = \det[1\{\lambda_i^{(k)} < \lambda_j^{(k+1)}\}]_{1 \leq i, j \leq M_k}
\] (2.23)
(see, e.g., [FR1, Lemma 1]), where $M_k = \lfloor (k+1)/2 \rfloor$ and $\lambda_i^{(2l+1)} = 0$ for $l = 0, 1, \ldots$. Also, introduce the standard notation for the Vandermonde determinant,
\[
\Delta(a^2) = \prod_{1 \leq i < j \leq n} (a_i^2 - a_j^2)
\] (2.24)
for $a \in \mathbb{R}^n$. This enables us to write the limiting measure in a very nice form.

**Corollary 2.11.**
\[
P_{\text{aGUEm}}^{(1,n)}\{\lambda^{(1)}, \ldots, \lambda^{(n)}\} = \frac{Z_n}{W_n} \Delta((\lambda^{(n)})^2) \prod_{k=1}^{n-1} \det[1\{\lambda_i^{(k)} < \lambda_j^{(k+1)}\}] \prod_{i=1}^{n/2} e^{-\lambda_i^{(n)}^2}
\] (2.25)
for $n$ even and
\[
P_{\text{aGUEm}}^{(1,n)}\{\lambda^{(1)}, \ldots, \lambda^{(n)}\} = \frac{Z_n}{W_n} \Delta((\lambda^{(n)})^2) \prod_{k=1}^{n-1} \det[1\{\lambda_i^{(k)} < \lambda_j^{(k+1)}\}] \prod_{i=1}^{(n-1)/2} (\lambda_i^{(n)}) e^{-\lambda_i^{(n)}^2}
\] (2.26)
for $n$ odd.

In the next section the measures $P_{\text{aGUEm}}^{(1,n)}$ will be identified with the joint eigenvalue PDF for the minors of anti-symmetric GUE matrices.

**3. Eigenvalue PDF for the Anti-Symmetric GUE Ensemble**

Consider the anti-symmetric GUE ensemble, which is the probability measure on purely imaginary Hermitian matrices with density $Z^{-1} e^{-\text{Tr} H^2/2}$. Here, $Z$ is a normalisation constant. Equivalently, form a real Gaussian matrix with entries chosen independently from $N(0, 1/\sqrt{2})$ and set $H = \frac{i}{2}(X - X^T)$. We seek to find the PDF of the eigenvalues of $H$ and its principal minors. For this we adapt workings from [B1, FR2] in which this task is carried out for $H$ a GUE matrix. First, a technical lemma is required.

**Lemma 3.1.** Let $0 < a_1 < \cdots < a_n$ be fixed real numbers. Let $q_1, \ldots, q_n$ be independent exp(1) random variables, specified by the PDF $e^{-x} (x > 0)$. Consider the random rational function
\[
p(\lambda) = \lambda - \sum_{i=1}^n \frac{\lambda q_i}{\lambda^2 - a_i^2}.
\] (3.1)
It has \( n \) positive zeros denoted \( 0 < b_1 < \cdots < b_n \), and their PDF is

\[
2^n \frac{\Delta(b^2)}{\Delta(a^2)} \prod_{i=1}^n b_i e^{-b_i^2 + a_i^2}
\]  

supported on \( a_1 < b_1 < a_2 < \cdots < a_n < b_n \).

**Proof.**  
Claim 1: \( p \) has at least \( n \) positive roots. Since the \( q_i \) are all non-negative, \( p(\lambda) \to -\infty \) as \( \lambda \to a_i^- \) and \( p(\lambda) \to \infty \) as \( \lambda \to a_i^+ \). So there must be a root on each of the intervals \((a_i, a_{i+1})\), for \( i = 1, \ldots, n - 1 \). This also accounts for the inequalities \( a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n \). Further, since \( p(\lambda) \to \infty \) as \( \lambda \to \infty \), there must be a root \( b_n > a_n \).

Claim 2: \( p \) has at most \( n \) positive roots. The rational function \( p \) has simple poles at \( \pm a_i \) for \( i = 1, \ldots, n \). Let

\[
q(\lambda) = \prod_{i=1}^n (\lambda^2 - a_i^2),
\]

then \( pq \) is a polynomial of degree \( 2n + 1 \) that has the same zeros as \( p \). The zeros already accounted for by the above argument are \( \pm b_1, \ldots, \pm b_n \) and 0, so there can be no more.

It follows from the above that it is possible to write

\[
p(\lambda) = \frac{\lambda \prod_{i=1}^n (\lambda^2 - b_i^2)}{\prod_{i=1}^n (\lambda^2 - a_i^2)}
\]  

Comparing the residue at \( a_i \) of \( p \) in (3.1) and (3.4), an elementary computation gives that

\[
- q_i = \frac{\prod_{j=1}^n (a_i^2 - b_j^2)}{\prod_{j=1, j \neq i}^n (a_i^2 - a_j^2)}.
\]

The PDF for the variables \( \{q_i\}_{i=1}^n \) is

\[
\exp\left(- \sum q_i \right)
\]

and we want to change variables to \( \{b_i\}_{i=1}^n \). The Jacobian \( J \) for that transformation is, up to a sign,

\[
J = \det \left[ \frac{-2b_j q_i}{a_i^2 - b_j^2} \right] = \prod_{1 \leq i < j \leq n} (a_i^2 - a_j^2)(b_i^2 - b_j^2) \prod_i (2b_i q_i),
\]

where the determinant is evaluated with the Cauchy double alternating identity. Inserting the expression for \( q_i \) from (3.5) simplifies this to

\[
J = 2^n \prod_{1 \leq i < j \leq n} (b_i^2 - b_j^2) \prod_{i=1}^n b_i.
\]

By expanding (3.1) and (3.4) at infinity and comparing the \( 1/\lambda \) coefficient it follows that

\[
- \sum_{i=1}^n q_i = \sum_{i=1}^n a_i^2 - b_i^2.
\]
Inserting this in (3.6) and multiplying by the Jacobian (3.8) gives the sought form (3.2).

Lemma 3.2. Let $0 < a_1 < \cdots < a_n$ be fixed real numbers. Let $q_1, \ldots, q_n$ be i.i.d. exp(1) distributed random variables and let $q_0$ be $\Gamma(1/2, 1)$ distributed, having PDF $(\pi x)^{-1/2}e^{-x}$ for $x > 0$. Consider the random rational function

$$p(\lambda) = \lambda - \frac{q_0}{\lambda} - \sum_{i=1}^{n} \frac{\lambda q_i}{\lambda^2 - a_i^2}.$$  

(3.10)

It has $n + 1$ positive zeros denoted $0 < b_0 < \cdots < b_n$ and their PDF is

$$\frac{2n+1}{\sqrt{\pi}} \frac{\Delta(b^2)}{\Delta(a^2)} \prod_{i=0}^{n} e^{-b_i^2} \prod_{i=1}^{n} \frac{a_i^2}{a_i^2}$$

(3.11)

supported on $0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n$.

Proof. It is convenient to introduce $a_0 = 0$. Instead of (3.3), choose

$$q(\lambda) = \lambda \prod_{i=1}^{n} (\lambda^2 - a_i^2).$$  

(3.12)

The proof of Lemma 3.1 goes through virtually unchanged with indices starting from zero instead of from one. The Jacobian expression from (3.8) can then be simplified as

$$J = 2^{n+1} \prod_{0 \leq i < j \leq n} (b_i^2 - b_j^2) \prod_{i=0}^{n} b_i = 2^{n+1} (-1)^n \prod_{0 \leq i < j \leq n} (a_i^2 - a_j^2) \prod_{i=0}^{n} b_i$$

(3.13)

Computing the residue at the origin of $p$ in (3.10) and (3.4) gives

$$\prod_{i=0}^{n} b_i^2 = q_0 \prod_{i=1}^{n} a_i^2.$$  

(3.14)

The expression corresponding to (3.9) is

$$-\sum_{i=0}^{n} q_i = \sum_{i=1}^{n} a_i - \sum_{i=0}^{n} b_i.$$  

(3.15)

The PDF for the variables $\{q_i\}_{i=0}^{n}$ is

$$\exp \left( -\sum_{i=0}^{n} q_i \right) \sqrt{\pi q_0}.$$  

(3.16)

Multiplying this with the Jacobian (3.8), inserting (3.14) and (3.15) gives the sought form (3.10).

□

Theorem 3.3. Let $H$ be an $n \times n$ matrix from the anti-symmetric GUE ensemble. Let $H_k$ be the $k \times k$ principal minor of $H$. Let $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_{\lfloor k/2 \rfloor}^{(k)})$ be the positive eigenvalues of $H_k$, ordered so that $\lambda_k^{(k)} > \lambda_{k+1}^{(k)}$. Then the joint PDF of $\lambda^{(1)}, \ldots, \lambda^{(n)}$ is precisely that given in Corollary 2.11.
Proof. Such a matrix $H$ has the property that if $\lambda$ is an eigenvalue of $H$, then so is $-\lambda$. Also, if the size of $H$ is odd, this implies that one eigenvalue will be zero.

The proof is an inductive one. A $2\times 2$ matrix from this ensemble is of the form
\[
\begin{pmatrix}
0 & a \\
-a & 0
\end{pmatrix},
\]
where $a \in N(0, 1/\sqrt{2})$. Its eigenvalues are $\pm a$, confirming the theorem in the case $n = 2$.

First, let $n$ be even. Consider an $n \times n$ matrix $A$ from this ensemble. The induction assumption is that its eigenvalue PDF is known. Consider the $(n + 1) \times (n + 1)$ matrix given by bordering $A$,
\[
\begin{pmatrix}
A & w \\
w^* & 0
\end{pmatrix}
\]
(3.17)

Here, $w$ is a column vector of $n$ purely imaginary numbers, all $N(0, 1/\sqrt{2})$. The star means transpose and complex conjugate.

The eigenvectors of $A$ can be paired up in the following way. If $v$ is an eigenvector corresponding to eigenvalue $\lambda$ then $\bar{v}$ is an eigenvector corresponding to eigenvalue $-\lambda$. Consider a normalised eigenvector, $|v| = 1$. Since $v$ and $\bar{v}$ must be orthogonal,
\[
|\text{Re} v|^2 = \frac{1}{2} (v + \bar{v}, v + \bar{v}) = \frac{1}{2},
\]
where $(\cdot, \cdot)$ denotes the inner product.

Let $C = [v_1, \bar{v}_1, v_2, \ldots]$ be the matrix whose columns are all the eigenvectors of $A$. Then
\[
\begin{pmatrix}
C^* & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
A & w \\
w^* & 0
\end{pmatrix}
\begin{pmatrix}
C & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
D & C^*w \\
w^*C & 0
\end{pmatrix},
\]
(3.18)

where $D$ is a diagonal matrix of the eigenvalues. It follows from the above considerations of eigenvectors and an elementary calculation that $w^*C = (a_1, \bar{a}_1, a_2, \ldots)$, where each $a_i$ is a complex number, the real and imaginary part of which are $N(0, 1/\sqrt{2})$. Let $p_n(\lambda)$ be the characteristic polynomial of $A$ and say that the eigenvalues of $A$ are $\pm \mu_1, \ldots, \pm \mu_{n/2}$. Of course the eigenvalues of $A$ give the factorisation of $p_n$ as
\[
p_n(\lambda) = (\mu_1^2 - \lambda^2) \ldots (\mu_{n/2}^2 - \lambda^2).
\]
(3.19)

Then, it can be shown, say by expanding along the last row of the RHS of (3.18), that the characteristic polynomial of that larger matrix is such that
\[
p_{n+1}(\lambda) = \frac{p_2(\lambda)}{p_n(\lambda)} = \lambda - \sum_{i=1}^{n/2} \frac{2a_i \bar{a}_i \lambda}{\lambda_i^2 - \lambda^2}.
\]
(3.20)

A computation shows that $2a_i \bar{a}_i$ is exp(1) distributed. So we now need to find the PDF of the zeros of this random rational function, which is precisely what is given by Lemma 3.1. Multiplying the expression that the induction assumption gives us for $n$ with the conditional PDF from Lemma 3.1 proves the statement for $H_{n+1}$ when $n$ is even.

Assume now that $n$ is odd. Do the same construction but the matrix $A$ will now have one eigenvalue which is zero. Performing the same bordering as in (3.18), only this time $w^*C = (a_1, \bar{a}_1, \ldots, a_n, \bar{a}_n, ib)$, where $b$ is $N(0, 1/2)$. As above the characteristic polynomials for the $n \times n$-matrix and the $(n + 1) \times (n + 1)$-matrix
are related by
\[ \frac{p_{n+1}(\lambda)}{p_n(\lambda)} = \lambda - \frac{b^2}{\lambda} = \frac{(n-1)/2}{\lambda^2} \sum_{i=1}^{2n} \frac{2a_i b_i}{\lambda^2 - \lambda^2} \] (3.21)

Apply this time Lemma 3.2 to prove the statement for \( H_{n+1} \) for \( n \) odd. \( \square \)

4. Correlation Functions

4.1. The method of Borodin and collaborators. Consider \( 2n + 1 \) discretisations \( \mathcal{M}_1, \ldots, \mathcal{M}_{2n+1} \) of the half interval \([0, \infty)\) each containing the point 0. On each set \( \mathcal{M}_j \) distribute points \( \{x_j^i\}_{i=1}^{(2n+1)/2} \) with \( x_j^{2i-1} = 0 \). On the configuration of points \( \mathcal{X} := \bigcup_{j=1}^{2n+1} \{x_j^i\} \) define a (possibly signed) measure by

\[
\frac{1}{C} \prod_{i=1}^n \det \left[ W_{2l-1}(x_i^{2l-1}, x_j^{2l}) \right]_{i,j=1,\ldots,n} \times \det \left[ W_{2l}(x_i^{2l}, x_j^{2l+1}) \right]_{i,j=1,\ldots,n} \times \det \left[ q_{j-1}(x_k^{2k+1}) \right]_{j,k=1,\ldots,n} \] (4.1)

for some functions \( \{W_i: \mathcal{M}_i \times \mathcal{M}_{i+1} \to \mathbb{R}\}_{i=1,\ldots,2n} \) and \( \{q_i: \mathcal{M}_{2n+1} \to \mathbb{R}\}_{i=1,\ldots,n} \).

After making use of the Vandermonde determinant expansion one sees that the measures in Corollary 2.11 are of this general form. One viewpoint is that (4.1) specifies a multicomponent system in which species \( l \) \( (l = 1, \ldots, 2n + 1) \) consists of \( [(l + 1)/2] \) particles. But for \( l \) odd one of these particles is fixed at the origin, so the number of mobile particles for species \( l \) is \( [l/2] \). We seek an explicit formula for the correlations of the general multi-component system specified by (4.1).

In fact a generalisation of this very problem has been addressed and solved in a recent work of Borodin and Ferrari, [BF]. This work extends earlier work of Borodin and collaborators [BR, BFPS]. However all these developments are relatively new, so it will be to the benefit of the reader that we give an account of the derivation herein rather than just quote the result (this is further justified as the proof of Theorem 4.2 in [BF] is very brief and calls for a detailed knowledge of the relevant results from [BR, BFPS]).

To begin introduce the \( |\mathcal{M}_{2n+1}| \times n \) matrix \( \Psi = [q_{j-1}(x_i)]_{x_i \in \mathcal{M}_{2n+1}, j=1,\ldots,n} \), the \( |\mathcal{M}_{2l-1}| \times |\mathcal{M}_{2l}| \) matrices \( W_{2l-1} := [W_{2l-1}(x_j, y_k)]_{x_j \in \mathcal{M}_{2l-1}, y_k \in \mathcal{M}_{2l}} \) and the \( |\mathcal{M}_{2l}| \times |\mathcal{M}_{2l+1}| \) matrices \( W_{2l} := [W_{2l}(x_j, y_k)]_{x_j \in \mathcal{M}_{2l}, y_k \in \mathcal{M}_{2l+1}} \). Further introduce the \( n \times |\mathcal{M}_{2l}| \) matrices \( E_l \) with entries in row \( l \) all ones and entries in all other rows all zeros. Finally with \( \mathcal{M} := \{1, \ldots, n\} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{2n+1} \) define the \( |\mathcal{M}| \times |\mathcal{M}| \) matrix \( L \) by

\[
L = \begin{bmatrix}
0 & 0 & E_1 & 0 & E_2 & \ldots & E_n & 0 \\
0 & -W_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & -W_2 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\Psi & 0 & 0 & 0 & \ldots & -W_{2n-1} & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -W_{2n} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{bmatrix}
\] (4.2)

for block zero matrices \( 0 \) of appropriate dimension.
Note that the rows and columns of $L$ are labelled by the elements of the set $\mathcal{M}$ in order. For $Y$ a subset of $\mathcal{M}$, introduce the notation $L_Y$ to denote the restriction of $L$ to the corresponding rows and columns. Then, in accordance with the general theory of $L$-ensembles (see [BR]) one sees that the measure (4.1) can be written
\[
\frac{\det L_{\{1, \ldots, n\}\cup X}}{\det (1^* + L)}
\] (4.3)
where $1^*$ is the $\mathcal{M} \times \mathcal{M}$ identity matrix with the first $n$ ones set to zero. The significance of this structure is the general fact that the correlation function for particles at $Y \in \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{2n+1}$ is given by [BR],
\[
\rho(Y) = \det K_Y, \quad K = 1 - (1 + L_{\mathcal{M}_1\{1, \ldots, n\}})^{-1}
\] (4.4)
for $1$ the identity of appropriate dimension. The correlation functions are thus given by a determinant of size $|Y|$ — the number of particles specified in the correlation — and so by definition the measure (4.1) is a determinantal point process.

Following [BFPS, BF] we will now proceed to isolate special structures in the functional form (4.1) which when present allow $(1 + L)^{-1}$ in (4.3) to be computed explicitly. For this purpose, introduce the matrix
\[
W_{i,j} = \begin{cases} W_i \ldots W_{j-1}, & i < j \\ 0, & i \geq j. \end{cases}
\] (4.5)

For $p = 1, \ldots, 2n$ and $j = 0, \ldots, n - 1$ set
\[
\Psi_{p-2j-2}^p(x) = (W_{[p, 2n+1]} \Psi)_{x, n-j}, \quad x \in \mathcal{M}_p
\] (4.6)
we use the notation $(A)_{i,a,b}$ to denote the element in row $a$ column $b$ of the matrix $A$. Define functions \{\Phi_{p-2l-2}^p\}_{l=0,1,\ldots, p-2l-2\geq0}
\begin{align*}
\text{span}\{\{(E_i W_{[2i,p]})_{2i, x}\}_{i=1, \ldots, (p-1)/2} \cup \{\delta_{p, \text{even}} E_{p/2}\}\}
\end{align*}
by the orthogonality requirement
\[
\sum_{x \in \mathcal{M}_p} \Phi_{p-2j-2}^p(x) \Psi_{p-2k-2}^p(x) = \delta_{j,k}
\] (4.8)
for $j, k = 0, 1, \ldots$ with $p - 2j - 2, p - 2k - 2 \geq 0$. Also set $\Psi_{2n+1}^p(x) = q_j(x)$ ($j = 0, \ldots, n - 1$). In terms of this notation, Theorem 4.2 in [BF] as applied to the measure (4.1) isolates on a special structure which when satisfied allows the elements of $K_Y$ in (4.3) to be made explicit.

**Proposition 4.2** (Theorem 4.2 in [BF]). Define the $[m/2] \times [m/2]$ matrix $B_m$ by
\[
(E_i W_{[2i,m]})_{2i, x} + \delta_{i, m/2} (E_{m/2})_{m, x} = \sum_{l=1}^{[m/2]} (B_m)_{i,l} \Phi_{m-2l}^n(x).
\] (4.9)
Suppose $B_m$ is upper triangular. With the set $Y$ in (4.3) labelled \{(s_l, y_l)\}, where $y_l \in \mathcal{M}_{s_l}$ we then have
\[
K((r, x), (s, y)) := (K_Y)_{(r, x), (s, y)} = -(W_{[r, s]})_{x, y} + \sum_{l=1}^{[s/2]} \Psi_{r-2l}^n(x) \Phi_{s-2l}^n(y).
\] (4.10)
Proof. We follow both the proofs of [BFPS, Lemma 3.4] and [BF, Theorem 4.2], and make use of [BR]. The task is to compute the inverse in (4.3).

Write \( L \) in the structured form
\[
L = \begin{bmatrix} 0 & B \\ C & D - 1 \end{bmatrix},
\]
(4.11)

where
\[
B = \begin{bmatrix} 0, E_1, 0, E_2, \ldots, 0, E_n, 0 \end{bmatrix},
\]
(4.12)
\[
C = \begin{bmatrix} 0, \ldots, 0, \Psi \end{bmatrix}^T.
\]
(4.13)

We know from [BR, Lemma 1.5] that
\[
K = 1 - D^{-1} + D^{-1}CM^{-1}BD^{-1}
\]
(4.14)
with \( M := BD^{-1}C \) and furthermore
\[
D^{-1} = 1 + [W_{i,j}]_{i,j=1,\ldots,2n+1}.
\]
(4.15)

From this latter formula we compute
\[
D^{-1}C = [W_{1,2n+1}]\Psi, \ldots, [W_{2n,2n+1}]\Psi, \Psi^T,
\]
(4.16)
and we compute that the \( m \)-th member of the block row vector \( BD^{-1} \) is equal to
\[
\sum_{k=1}^{[m/2]-1} E_k W_{2k,m} + \delta_{m,\text{even}} E_{m/2}.
\]
(4.17)

These formulas in turn tell us that the \((j, m)\) block of \( K \) is equal to
\[
-W_{j,m} + [W_{j,2n+1}]\Psi M^{-1} \left( \sum_{k=1}^{[m/2]-1} E_k W_{2k,m} + \delta_{m,\text{even}} E_{m/2} \right),
\]
(4.18)
where \( W_{j,2n+1} \) is to be replaced by \( 1 \) for \( j = 2n + 1 \). They tell us too that
\[
M = \left( \sum_{k=1}^{n-1} E_k W_{2k,m} \right) \Psi.
\]
(4.19)

Define the \((2n + 1) \times [m/2]\) matrix \( \Phi^m \) specified by
\[
(\Phi^m)_{i,j} = \begin{cases} 
\Phi^m_{m-2i}(x^m_j), & 1 \leq i \leq [m/2], \\
0, & \text{otherwise}.
\end{cases}
\]
(4.20)

The orthogonality (4.9) used in conjunction with (4.18) shows
\[
\sum_{k=1}^{[m/2]-1} E_k W_{2k,m} + \delta_{m,\text{even}} E_{m/2} = \begin{bmatrix} B_m & 0 \\ 0 & 0 \end{bmatrix} \Phi^m
\]
(4.21)
and furthermore \( M = B_{2n+1} \). These formulas, together with the assumption that \( B_m \) is upper triangular, give that (4.18) reduces to
\[
-W_{j,m} + [W_{j,2n+1}]\Psi \Phi^m.
\]
(4.22)

It follows from this and the definition (4.6) that the element in row \((s, x)\) column \((s, y)\) is equal to (4.10), as required. \( \square \)
As formulated Proposition 4.2 applies in the setting that the domains have been discretised. Minor modification of the definitions gives that (4.10) remains valid in the continuum limit. For this purpose, introduce the notation

\[(a \ast b)(x, y) = \int_0^\infty a(x, z)b(z, y)\,dz,\]  

and note that such an operation is the continuum limit of matrix multiplication. Then specify

\[W_{i,j}(x, y) := \begin{cases} (W_i \ast \cdots \ast W_{j-1})(x, y), & i < j, \\ 0, & i \geq j; \end{cases} \]

and for \(p = 1, \ldots, 2n\) define functions \(\Phi_{p}^{p-2j-2}(x) = (W_{p,2n+1} \ast q_{n-j-1})(x)\)

and for \(p = 1, \ldots, 2n\) define functions \(\Phi_{p}^{p-2l-2} \) constructed from

\[
\text{span}\left\{\left\{\int_0^\infty W_{[2i,p]}(t, x)\right\}_{i=1,\ldots,[(p-1)/2]} \cup \{\delta_{p,\text{even}}\}\right\}
\]

by the orthogonality requirement

\[
\int_0^\infty \Phi_{p-2j-2}^{p-2k-2}(x)\Phi_{p-2k-2}^{p-2j-2}(x)\,dx = \delta_{j,k}.
\]

Consider now the explicit form of these quantities for the PDF in Corollary 2.11. There, independent of \(i\), \(W_{i}(x, y) = \chi_{x<y}\) and so

\[W_{[i,j]}(x, y) = \frac{1}{(j-i-1)!}\chi_{x<y}(y-x)^{j-i-1}.\]

Furthermore, from the Vandermonde determinant identity

\[
\prod_{i=1}^n x_i \prod_{1 \leq j < k \leq n} (x_j^2 - x_k^2) = \det[x_j^{2k-1}]_{j,k=1,\ldots,n} \propto \det[p_{2k-1}(x_j)]_{j,k=1,\ldots,n}
\]

valid for any polynomials \(p_l(x)\) of degree \(l\). We choose the \(p_l(x)\) so that

\[
\int_0^\infty e^{-x^2}p_{2j-1}(x)p_{2k-1}(x)\,dx \propto \delta_{j,k}.
\]

This is achieved by setting \(p_l(x) = H_l(x)\), where \(H_l(x)\) is the Hermite polynomial. Thus \(q_i(x) = e^{-x^2}H_{2i+1}(x)\) and so

\[
\Psi_{p-2j-2}^{p-2j-2}(x) = \frac{1}{(2n-p)!}\int_x^\infty e^{-y^2}H_{2n-2j-1}(y)(y-x)^{2n-p}\,dy.
\]

Making use of the Rodrigues formula

\[H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}\]

it follows from this that for \(j \geq 0\)

\[
\Psi_{j}^{p}(x) = e^{-x^2}H_j(x)
\]
while for $j < 0$

$$
\Psi_j^p(x) = \frac{1}{(-j-1)!} \int_x^\infty (y-x)^{-j-1} e^{-y^2} dy. \tag{4.34}
$$

It follows from (4.28) that (4.26) has the explicit form

$$
\text{span}\{\{x^{p-2i}\}_{i=1, \ldots, [(p-1)/2]} \cup \{\delta_{p, \text{even}}\}\} \tag{4.35}
$$

From this set we can construct

$$
\Phi_j^p(x) = \frac{1}{N_j} H_j^p(x) \tag{4.36}
$$

for $i = 1, 2, \ldots$, where

$$
N_j = \int_0^\infty (H_j(x))^2 e^{-x^2} dx = \sqrt{\pi} 2^{j-1} j!, \tag{4.37}
$$

which indeed exhibits the orthogonality (4.27). Furthermore, the continuum limit of the change of basis formula (4.9) reads

$$
x^{m-2i} + \delta_{i, m/2} = \sum_{l=1}^{[m/2]} (B_m)_{i,l} \Phi_{m-2i}^l(x) \tag{4.38}
$$

and this with $\Phi_{m-2l}^m(x)$ a polynomial of degree $m - 2l$ as in (4.36) gives that $B_m$ is an upper triangular matrix. We are therefore justified in applying Proposition 4.2.

**Proposition 4.3.** Let $\Psi_j^p(x)$ be specified by (4.33) and (4.34) for $j \geq 0$ and $j < 0$ respectively. Let $\Phi_j^p(x) = \frac{1}{N_j} H_j^p(x)$ so as to be consistent with (4.36). The $r$ point correlation is given by

$$
\rho_r((s_j, y_j)_{j=1, \ldots, r}) = \det[K((s_j, y_j), (s_k, y_k))]_{j,k=1, \ldots, r} \tag{4.39}
$$

with

$$
K((s, x), (t, y)) = -\frac{1}{(t-s-1)!} \chi_{x<y} (y-x)^{t-s-1} + \sum_{l=1}^{[t/2]} \Psi_{s-2l}^*(x) \Phi_{t-2l}^l(y) \tag{4.40}
$$

Equivalently, for $s \geq t$

$$
K((s, x), (t, y)) = e^{-x^2} \sum_{l=1}^{[t/2]} \frac{H_{s-2l}(x) H_{t-2l}(y)}{N_{t-2l}} \tag{4.41}
$$

while for $s < t$

$$
K((s, x), (t, y)) = -e^{-x^2} \sum_{l=-\infty}^0 \frac{H_{s-2l}(x) H_{t-2l}(y)}{N_{t-2l}}. \tag{4.42}
$$

**Proof.** The only remaining task is to derive (4.42). For $s < t$ consider

$$
-\frac{1}{(t-s-1)!} \chi_{x<y} (\text{sgn} y)^t (|y| - x)^{t-s-1}, \tag{4.43}
$$
which for \( y > 0 \) is equal to the first term in (4.40). For \( t \) even (odd) this is an even (odd) function of \( y \) and so can be expanded in the basis \( \{ \mathcal{H}_{2j+}(y) \}_{j=0,1,\ldots} \), where \( \epsilon = 0, 1 \) for \( t \) even, odd, according to

\[
- \frac{1}{(t-s-1)!} \sum_{k=0}^{\infty} \frac{H_{2k+}(y)}{N_{2k+}} \int_{x}^{\infty} e^{-u^2} (u-x)^{t-s-1} H_{2k+}(u) \, du, \tag{4.44}
\]

valid for \( x > 0 \). Making use of the Rodrigues formula (4.32) and recalling (4.34) this can be rewritten

\[
- e^{-x^2} \sum_{k=-\infty}^{[s/2]} \frac{H_{s-2k}(x)H_{t-2k}(y)}{N_{t-2k}} - \sum_{p=[s/2]+1}^{[l/2]} \Psi_{s-2p}(x)\Phi_{l-2p}(y). \tag{4.45}
\]

Substituting this for the first term in (4.40) and writing in the resulting expression

\[
\sum_{l=1}^{[s/2]} \Psi_{s-2p}(x)\Phi_{l-2p}(y) = e^{-x^2} \sum_{k=1}^{[s/2]} \frac{H_{s-2k}(x)H_{t-2k}(y)}{N_{t-2k}} \tag{4.46}
\]

gives (4.42). \( \square \)

4.4. The method of Nagao and Forrester. In the recent work [FN], the correlations for a class of multicomponent systems generalizing the GUE minor process, in which species \( l \) consisted of \( l \) particles constrained so that they interlace with the particles of species \( l + 1 \), were computed in two ways. One was by using the generalized formula from [BFPS, Lemma 3.4], while the other adapted an approach to multi-component determinantal processes due to Nagao and Forrester [NF]. In this section we will show how Proposition 4.3 can be reclaimed by making use of this latter method.

With \( W_i(x, y) = W(x, y) = \chi_{x<y} \), the first step is to rewrite (4.1) so that it reads

\[
\frac{1}{C} \prod_{l=1}^{n} \det \begin{bmatrix} 1_{(n-l)\times(n-l)} & 0_{(n-l)\times l} \\ 0_{l\times(n-l)} & W(x_i^{2l-1}, x_i^{2l}) - \kappa_l(x_i^{2l-1})_{j=1,\ldots,l}^{j=1,\ldots,l} \end{bmatrix} \\
\times \det \begin{bmatrix} 1_{(n-l)\times(n-l)} & 0_{(n-l)\times l} \\ 0_{l\times(n-l)} & W(x_i^{2l+1}, x_i^{2l+1})_{j=1,\ldots,l}^{j=1,\ldots,l} \end{bmatrix} \tag{4.47}
\]

Here \( \kappa_l(x) \) is arbitrary as the determinant does not depend on \( \kappa_l \), and \( C \) is a normalization which may vary from equation to equation below. Furthermore, \( \kappa_l(x) = e^{-x^2} H_{2j-1}(x) \). Proceeding as in the derivation of the equality between (4.43) and (4.45) we deduce

\[
W(x, y) = \frac{1}{N_0} \int_{x}^{\infty} e^{-t^2} \, dt + e^{-x^2} \sum_{k=1}^{\infty} \frac{H_{2k-1}(x)H_{2k}(y)}{N_{2k}} \tag{4.48}
\]
or alternatively

\[
W(x, y) = e^{-x^2} \sum_{k=1}^{\infty} \frac{H_{2k}(x)H_{2k+1}(y)}{N_{2k+1}} \tag{4.49}
\]
both valid for \( x, y > 0 \). We substitute the first of these in (4.47), after choosing

\[
\kappa_t(x) = \frac{1}{N_0} \int_x^\infty e^{-t^2} \, dt
\]

(4.50)

therein, and the second of these in the second of the determinants in (4.47).

Introduce now the notation

\[
\eta_j(x) = \frac{e^{-x^2/2}}{\gamma_j} H_j(x), \quad \gamma_j := \sqrt{\lambda_j}
\]

(4.51)

so that \( \{\eta_j(x)\}_{j=0,1,...} \) and \( \{\eta_{2j+1}(x)\}_{j=0,1,...} \) each form a set of orthonormal functions on \([0, \infty)\). Set

\[
\phi^{(o)}(x, y) := \sum_{k=0}^{\infty} \frac{\gamma_{2k-1}}{\gamma_{2k}} \eta_{2k-1}(x) \eta_{2k}(y),
\]

(4.52)

\[
\phi^{(e)}(x, y) := \sum_{k=0}^{\infty} \frac{\gamma_{2k}}{\gamma_{2k+1}} \eta_{2k}(x) \eta_{2k+1}(y).
\]

(4.53)

From the above working we then see that (4.47) can be written

\[
\frac{1}{C} \prod_{i=1}^{n} \det \begin{bmatrix} I_{(n-i) \times (n-i)} & 0_{(n-i) \times t} \\ 0_{t \times (n-i)} & [\phi^{(o)}(x_i^{2l-1}, x_i^{2l+1})]_{i=1,...,l-1} \\ \end{bmatrix} \times \det \begin{bmatrix} I_{(n-i) \times (n-i)} & 0_{(n-i) \times t} \\ 0_{t \times (n-i)} & [\phi^{(e)}(x_i^{2l}, x_i^{2l+1})]_{i=1,...,l} \\ \end{bmatrix} \det[\eta_{j-1}(x_k^{2n+1})]_{j,k=1,...,n}.
\]

(4.54)

To proceed further, set

\[
\eta_{j,l}^{(s)} = \begin{cases} \eta_j(x_l^{(s)}), & j > 0, \ l \geq 1, \\ \delta_{j,2l-1}, & \text{otherwise} \end{cases}
\]

(4.55)

and use this to define

\[
A_{j,l}^{(s,t)} = \sum_{k=-n}^{-1} \frac{\gamma_{2k+s}}{\gamma_{2k+t}} \eta_{2k+s, j-n+[s/2]}^{(s)} \eta_{2k+t, j-n+[t/2]}^{(t)},
\]

(4.56)

\[
G_{j,l}^{(s,t)} = \sum_{k=-n}^{\infty} \frac{\gamma_{2k+s}}{\gamma_{2k+t}} \eta_{2k+s, j-n+[s/2]}^{(s)} \eta_{2k+t, j-n+[t/2]}^{(t)}.
\]

(4.57)

One then sees that

\[
det[\eta_{j-1}(x_k^{2n+1})]_{j,k=1,...,l} \propto det[A_{j,l}^{(2n+1,1)}]_{j,l=1,...,n} =: det A^{(2n+1,1)},
\]

(4.58)

\[
det[\phi^{(o)}(x_i^{2l-1}, x_i^{2l+1})]_{i=1,...,l} \propto det[G_{j,k}^{(2l-1,2l)}]_{j,k=1,...,n} =: det G^{(2l-1,2l)},
\]

(4.59)

\[
det[\phi^{(e)}(x_i^{2l}, x_i^{2l+1})]_{i=1,...,l} \propto det[G_{j,k}^{(2l+1,2l)}]_{j,k=1,...,n} =: det G^{(2l+1,2l)}.
\]

(4.60)
Consequently it is possible to rewrite (4.54) as

$$\frac{1}{C} \det \begin{bmatrix} A^{(2n+1,1)} & A^{(2n+1,2)} & A^{(2n+1,3)} & \cdots & A^{(2n+1,2n+1)} \\ 0 & -G^{(1,2)} & -G^{(1,3)} & \cdots & -G^{(1,2n+1)} \\ 0 & 0 & -G^{(2,3)} & \cdots & -G^{(2,2n+1)} \\ 0 & 0 & 0 & \cdots & -G^{(2n,2n+1)} \end{bmatrix}. \quad (4.61)$$

This has the same structure as [FN, eq. 4.23]. Simple manipulation detailed in this latter reference shows that this in turn can be rewritten to read

$$\frac{1}{C} \det [F^{(s,t)}]_{s,t=1,\ldots,2n+1}, \quad F^{(s,t)} = \begin{cases} A^{(s,t)}, & s \geq t, \\ B^{(s,t)}, & s < t, \end{cases} \quad (4.62)$$

where $B^{(s,t)} := A^{(s,t)} - G^{(s,t)}$. Noting from the definitions of $A^{(s,t)}$ and $G^{(s,t)}$ that for $s \geq t$, $F^{(s,t)}_j \propto \delta_{j,l}$ for $j, l \leq n - \lfloor s/2 \rfloor$, while for $s < t$, $F^{(s,t)} = 0$, $j \leq n - \lfloor s/2 \rfloor$ or $l \leq n - \lfloor t/2 \rfloor$, this reduces to

$$\frac{1}{C} \det [f^{(s,t)}]_{s,t=1,\ldots,2n+1}, \quad (4.63)$$

where $f^{(s,t)}$ is the $s \times t$-matrix with entries

$$f^{(s,t)}_{j,l} = F^{(s,t)}_{j-[s/2]+n,l-[t/2]+n}$$

$$= \begin{cases} \sum_{k=1}^{\lfloor s/2 \rfloor} \frac{2(k-2)!}{s-k} \eta_{s-2k}(x_j) \eta_{t-2k}(x_l), & s \geq t, \\ - \sum_{k=-\infty}^{\lfloor t/2 \rfloor} \frac{2(k-2)!}{t-k} \eta_{s-2k}(x_j) \eta_{t-2k}(x_l), & s < t. \end{cases} \quad (4.65)$$

These exhibit the reproducing property

$$\int_0^\infty f^{(s,t)}_{j,l} f^{(t,u)}_{j,m} dx_l = \begin{cases} f^{(s,u)}_{j,m}, & s \geq t \geq u \text{ or } s < t < u, \\ 0, & \text{otherwise,} \end{cases} \quad (4.66)$$

which allow the integrations required to reduce (4.63) down to the $r$-point correlation function to be performed (see, e.g., [F, Ch. 11]), giving

$$\rho_{(r)}(\{(s_j, x_j)\}_{j=1,\ldots,r}) = \det [f^{(s_j, x_j)}_{j,k}]_{j,k=1,\ldots,r} \quad (4.67)$$

Substituting (4.51) in (4.65) shows

$$f^{(s,t)}_{j,l} = e^{-s^2/2} K((s, x_j), (t, x_l)) \quad (4.68)$$

and this the result (4.39) of Proposition 4.3 is reclamation.

5. Scaling Limits of the Correlations

In the recent study [FN] various scaling limits of the GUE minor process were computed. Two involved species which differed by a fixed amount. In this situation the eigenvalue coordinates were chosen to correspond to the neighbourhood of the spectrum edge in one case, and the bulk of the spectrum in the other. A further
scaling limit, in which the species differ by $O(N^{2/3})$ and the eigenvalue coordinates correspond to the spectrum edge, was also analyzed.

In the first of these situations, the scaled correlation kernel was computed to be the Airy kernel (1.5) independent of the particle species. In the second situation the correlation kernel for the isotropic bead process (1.7) was found. In the final situation, the dynamical extension of the Airy kernel (1.6) was obtained as the scaled correlation kernel.

The anti-symmetric GUE minor process of the present work permits the above three scalings. We will show below that at the soft edge the limiting correlation kernels (3.5) and (1.6) are reclaimed in the cases of the species differing by $O(1)$ and by $O(n^{2/3})$ respectively. In the bulk it will be show that the limiting correlation kernel (1.7) is reclaimed. The antisymmetric GUE minor process also permits the well known hard edge scaling [182], involving eigenvalues in the neighbourhood of the origin. For eigenvalues from the same species, which is an even label, the limiting correlation kernel is

$$K^+(x, y) := 2 \int_0^1 \sin(\pi ux) \sin(\pi uy) \, du$$

(5.1)

while for odd labelled species it is

$$K^-(x, y) := 2 \int_0^1 \cos(\pi ux) \cos(\pi uy) \, du.$$  

(5.2)

In the case of hard edge scaling of the correlation kernel with variable species label, differing by a finite amount, a generalization of the integral representations (5.1) and (5.2) is found.

5.1. Soft edge scaling. In the $N \times N$ GUE the soft edge scaling corresponds to the change of eigenvalue coordinates

$$y_i = \sqrt{2N} + \frac{Y_i}{\sqrt{2N^{1/6}}} ,$$

(5.3)

which has the effect of moving the origin to the neighbourhood of the largest eigenvalue and scaling the distances so the inter-eigenvalue spacings in this neighbourhood are order unity. The same soft edge scaling (5.3) applies for $(2n+1) \times (2n+1)$ anti-symmetric GUE matrices except that $N \mapsto 2n$. We consider first this scaling with the species differing from $2n+1$ by a constant,

$$s_i = 2n + 1 - c_i$$

(5.4)

**Proposition 5.2.** For the soft edge scaling specified by (5.3) with $N \mapsto 2n$ and (5.4),

$$\frac{1}{\sqrt{2n^{1/6}}} K^S((s_j, y_j), (s_l, y_l)) \sim \frac{a_n(c_j, Y_j)}{a_n(c_l, Y_l)} K_{soft}(Y_j, Y_l),$$

(5.5)
where $K^{\text{soft}}$ is given by (3.5) and $a_n(c, Y) = e^{-(2n)^{1/3}Y(2n)^{-c/2}}$. Hence we have the pointwise limit
\[
\lim_{n \to \infty} \left( \frac{1}{\sqrt{2\pi n^{1/6}}} \right)^r \rho(\{(s_j, y_j)\}_{j=1, \ldots, r}) = \det[K^{\text{soft}}(Y_j, Y_k)]_{j,k=1, \ldots, r}^{r}
\]
(5.6)

independent of the particle species.

Proof. Substituting (4.36) in (4.40) shows that for $s_j \geq s_l \ (c_j \leq c_l)$
\[
K((s_j, y_j), (s_l, y_l)) = \frac{2e^{-c_j^2}}{\sqrt{\pi}} \sum_{k=1}^{[s_j/2]} \frac{1}{2^{n-2k}(s_l - 2k)!} H_{s_j - 2k}(y_j) H_{s_l - 2k}(y_l).
\]
(5.7)

As done for the derivation of the corresponding result in the case of the GUE, [FN], we make use of the uniform large $N$ expansion [O]
\[
e^{x^2/2} H_{N-k}(x) = \pi^{1/4} 2^{(N-k)/2+1/4} ([N-k]!)^{1/2} \sqrt{N-1/12}
\]
\[
\times \left( \text{Ai} \left( X + \frac{k}{N^{1/3}} \right) + O(N^{-2/3}) \right) \begin{cases} O(e^{-k/N^{1/3}}), & k \geq 0, \\ O(1), & k < 0 \end{cases}
\]
(5.8)

in the summand of (5.7) and so obtain
\[
K((s_j, y_j), (s_l, y_l)) \sim 2e^{-(2n)^{1/3}(Y_j - Y_k)} 2^{-(c_j - c_l)/2} \sqrt{2}(2n)^{-1/6}
\]
\[
\times \sum_{k=1}^{n} \frac{(2n + 1 - c_j - 2k)!}{(2n + 1 - c_l - 2k)!} \frac{1}{2^{n-2k}(s_l - 2k)!} H_{s_j - 2k}(y_j) H_{s_l - 2k}(y_l) \text{Ai} \left( Y_j + \frac{2k}{(2n)^{1/3}} \right) \text{Ai} \left( Y_l + \frac{2k}{(2n)^{1/3}} \right).
\]
(5.9)

We observe that the leading contribution to the sum comes from terms $k = O(n^{1/3})$. In this regime
\[
\frac{(2n + 1 - c_j - 2k)!}{(2n + 1 - c_l - 2k)!} \sim (2n)^{(s_l - c_l)/2}
\]
(5.10)

and so this term can be factored out of the summand, leaving us with a Riemann sum approximation to the definite integral (3.5), which implies (5.5)

We know from [FN] that in the case $s_j < s_l \ (c_j > c_l)$ the form (4.42) is not appropriate, due to the resulting Riemann sum not being convergent. Instead, we make use of (4.40), breaking the sum over $l$ therein into the ranges $l \in [1, [s/2]]$, $l \in [[s/2] + 1, \lfloor t/2 \rfloor]$ to deduce that for the given scaling $K((s_j, y_j), (s_l, y_l))$ is to leading order given by the right hand side of (5.7), but with the upper terminal replaced by $[s_k/2]$. This latter difference does not effect the leading asymptotic form, and so (5.5) is valid in all cases.

Proposition 5.2 tells us that when measured according to the scale (5.3), the largest eigenvalues of the successive minors are indistinguishable in position. On the other hand we know the eigenvalues of the successive minors must interlace. A question for future study then is to quantify the $N$-dependent scale which distinguishes the largest eigenvalues in this setting.

We turn our attention now to soft edge scaling with the species separated by $O((2n)^{2/3})$. 

Proposition 5.3. Scale $s_i$ according to
\[ s_i = 2n - 2c_i(2n)^{2/3} \]  
and scale $y_i$ according to
\[ y_i = (2s_i)^{1/2} + \frac{Y_i}{\sqrt{2}s_i^{1/6}} \]  
For large $n$ and $\alpha_n(c, y) := e^{-(2n)^{1/3}Y(4n)^{-c(2n)^{2/3}}e(2n)^{1/3}e^{2c^{3/3}}}$,
\[ \frac{1}{\sqrt{2}(2n)^{1/6}}K((s_j, y_j), (s_l, y_l)) \sim \frac{\alpha_n(c_j, Y_j)}{\alpha_n(c_l, Y_l)} K^{\text{soft}}((c_j, Y_j), (c_l, Y_l)), \]  
where $K^{\text{soft}}$ is given by (3.5), and hence
\[ \lim_{n \to \infty} \left( \frac{1}{\sqrt{2}(2n)^{1/6}} \right)^r \rho(r) \{ (s_j, y_j) \}_{j=1,\ldots,r} = \det[K^{\text{soft}}((c_j, Y_j), (c_l, Y_l))]_{j,l=1,\ldots,r}. \]  

Proof. In the case $c_j \leq c_l$, the formula (5.9) still applies, but now with $c_i \mapsto 2c_i(2n)^{2/3}$. This latter point means that (5.10) is now inappropriate, following [FN, eq. (5.40)] we use instead the large $s$ expansion
\[ \left( \frac{(s - k_j)!}{(s - k_l)!} \right)^{1/2} \sim s^{(k_l - k_j)/2} e^{(k_j^2 - k_l^2)/4s} / 12s^2 \]  
with $s = 2n, k_i = c_i + 2k$. A Riemann sum approximating the first integral in (1.6) is obtained, establishing (5.13) in the case $c_j \leq c_l$.

In the case $c_j > c_l$, it follows from (4.42) that (5.9) remains true, but with $k$ now ranging from $-\infty$ to 0, and the RHS multiplied by $-1$. Because the resulting Riemann integral is convergent, and is precisely the same as that obtained for $c_j \leq c_l$ except that the range is over $(-\infty, 0)$, the second integral in (1.6) is obtained, establishing (5.13) for $c_j > c_l$. \qed

5.4. Hard edge and bulk. The hard edge scaling is obtained by the change of variables
\[ y_i = \frac{\pi Y_j}{2\sqrt{n}}. \]  
so that the mean particle density in the neighbourhood of the origin is of order unity. We seek to analyze the correlations with this scaling, and the species differing from $2n$ as specified by (5.4). Taking $Y_j \to \infty$ in the expression, with the differences between the $Y_j$ fixed, the bulk correlations follow as a limit of the hard edge correlations.

Proposition 5.5. For the hard edge scaling specified by (5.16), (5.4),
\[ \frac{\pi}{2\sqrt{n}} K((s_j, y_j; s_l, y_l) \sim 2^{2-c_l} n^{(c_l-c_j)/2} K^{\text{hard}}((s_j, y_j), (s_l, y_l)). \]
Furthermore, with

to deduce

for large \( n \) and in this case \( c_l > c_j \).

Hence

\[
\lim_{n \to \infty} \left( \frac{\pi}{2\sqrt{n}} \right)^r \rho(\{(s_j, y_j)\}_{j=1,\ldots,r}) = \det[K_{\text{hard}}((c_j, Y_j), (c_l, Y_l))]_{j,l=1,\ldots,r}. \tag{5.19}
\]

Furthermore, with \( s_j \) again given by (5.4), the bulk scaling is obtained by the change of variables

\[
y_i = \frac{\pi Y_j}{2\sqrt{n}} + a, \quad 0 < a < 2\sqrt{n} \tag{5.20}
\]

and in this scaling limit

\[
\lim_{n \to \infty} \left( \frac{\pi}{2\sqrt{n}} \right)^r \rho(\{(s_j, y_j)\}_{j=1,\ldots,r}) = \det[K_{\text{bead}}((c_j, Y_j), (c_l, Y_l))]_{j,l=1,\ldots,r}. \tag{5.21}
\]

Proof. Consider the hard edge scaling, and suppose \( c_l > c_j \). In (5.7) we substitute for the Hermite polynomials the uniform asymptotic expansion [8]

\[
\frac{\Gamma(n/2 + 1)}{\Gamma(n + 1)} e^{-x^2/2} H_n(x) = \cos(\sqrt{2n+1} x - n\pi/2) + O(n^{-1/2}) \tag{5.22}
\]

to deduce

\[
K((s_j, y_j), (s_l, y_l)) \sim \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\lfloor s_j/2 \rfloor} \frac{1}{2^{n-k} k!} \frac{(s_j - 2k)!}{(s_j/2 - k)! (s_j/2 - k)!} \times \cos \left( \pi \sqrt{\frac{n-k}{n} Y_j - \pi \left( \frac{1-c_j}{2} \right)} \right) \cos \left( \pi \sqrt{\frac{n-k}{n} Y_l - \pi \left( \frac{1-c_l}{2} \right)} \right). \tag{5.23}
\]

Recalling (5.3), and scaling the summand by

\[
2^{2(n-k)}(\pi(n-k))^{-1/2} \left( \frac{(n-k)!}{(2n-2k)!} \right)^2 \tag{5.24}
\]

(which for large \( n-k \) tends to unity) shows

\[
K((s_j, y_j), (s_l, y_l)) \sim \frac{2}{\pi} 2^{c_l-c_j} n^{(-1+c_l-c_j)/2} \sum_{k=1}^{n} \left( \frac{n-k}{n} \right)^{(-1+c_l-c_j)/2} \times \cos \left( \pi \sqrt{\frac{n-k}{n} Y_j - \pi \left( \frac{1-c_j}{2} \right)} \right) \cos \left( \pi \sqrt{\frac{n-k}{n} Y_l - \pi \left( \frac{1-c_l}{2} \right)} \right). \tag{5.25}
\]

This is a Riemann sum to the first integral in (5.18), implying the leading asymptotics (5.17) in this case \( c_l > c_j \).
Consider next the hard edge scaling in the case $c_l < c_j$. As in the analogous stage of the proof of Proposition 5.2, it follows from (4.42) that (5.23) remains true but with the summation now over $(-\infty, 0]$, and the RHS multiplied by $-1$. This gives a Riemann sum approximation to the second integral in (5.18) and establishes (5.17) in the case $c_l < c_j$.

The result (5.21) can be deduced from (5.19) by noting that the sought bulk scaling correlation kernel must be the limit $x, y \to \infty, |x-y|$ fixed of $K_{\text{hard}}((s, x); (t, y))$.

A simple rewrite of the resulting integrals gives (1.7). □

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