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DYNAMICS OF SHIFT-LIKE POLYNOMIAL DIFFEOMORPHISMS OF $\mathbf{C}^N$

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Abstract. We identify a family of polynomial diffeomorphisms of $\mathbf{C}^N$ and show that these mappings may be studied using certain methods (filtration and potential-theoretic) which were developed for the study of polynomial diffeomorphisms of $\mathbf{C}^2$.

0. Introduction

The dynamics of polynomial diffeomorphisms of $\mathbf{C}^2$ have been studied intensively in recent years, starting with the work of Hubbard [H] and Friedland-Milnor [FM]. The methods of pluri-potential theory, i.e., the methods related to plurisubharmonic functions and positive, closed currents, have proven effective in the further study of the dynamics of these maps. This new direction started with the works of [BS] and [FS] and has progressed further in a series of papers by John Smillie and one of the authors. The reader is referred to the survey paper [BuS] for a good overview of this area. The purpose of the present paper is to introduce a family of polynomial diffeomorphisms of $\mathbf{C}^N$, $N \geq 2$, and to show that similar potential-theoretic tools may be developed for them. Our hope is that many of the methods and results from the case $N = 2$ will extend naturally to this more general case.

Let us review some of the features of the dynamics of polynomial diffeomorphisms of $\mathbf{C}^2$. We consider the sets

$$K^{\pm} = \{x \in \mathbf{C}^2 : \{f^{\pm n}(x) : n \geq 0\} \text{ is bounded}\},$$
$$U^{\pm} = \mathbf{C}^N - K^{\pm}, \qquad K = K^+ \cap K^-,$$
$$J^{\pm} = \partial K^{\pm}, \quad \text{and} \quad J = J^+ \cap J^-.$$

Friedland and Milnor [FM] showed that a polynomial diffeomorphism f which is dynamically nontrivial has several interesting properties. One property is that such an f is conjugate to a finite composition of mappings of the form $f : (x, y) \mapsto (y, p(y) - ax)$. For these mappings there are sets V^-, V, and V^+ such that (V^-, V, V^+) forms a filtration for f in the following sense:

1. A point not already in V^- cannot enter V^-, and an f-orbit can remain in V^- for finite positive time,
2. V is compact, and a forward orbit $\{f^n(x) : n \geq 0\}$ is bounded if and only if it is eventually contained in V, and

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3. Every point of V^+ remains in V^+ and tends to infinity in forward time.

Another property of f as given above is that it has minimal degree within its conjugacy class. In fact, if we set $\deg(f) = d$, then $\deg(f^n) = d^n$, where $f^n = f \circ \cdots \circ f$ denotes the n-fold composition. We may use d to measure the (super-exponential) rate of escape to infinity in forward/backward time by defining

$$G^{\pm}(x) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|f^{\pm n}(x)\|.$$

$G^{\pm}$ transforms under composition as: $G^{\pm} \circ f = d^{\pm 1} \cdot G^{\pm}$. Thus the stable/unstable currents, which are defined by $\mu^{\pm} := \frac{1}{2\pi} dd^c G^{\pm}$ may be wedged together to give an invariant measure $\mu := \mu^+ \wedge \mu^-$. The currents $\mu^{\pm}$ and the measure μ have been important in gaining a deeper understanding of f.

For $x = (x_1, \ldots, x_N) \in \mathbf{C}^N$, we set $\|x\| = \max_{1\le j\le N} |x_j|$. For $R < \infty$ large, $1 \le j \le N$, and $0 \le \nu \le N-1$, we define

$$V_j = \{x \in \mathbf{C}^N : |x_j| \ge R, |x_j| = \|x\|\},$$
$$V = \{x \in \mathbf{C}^N : \|x\| \le R\},$$
$$V^- = \bigcup_{j=1}^{N-\nu} V_j, \quad \text{and} \quad V^+ = \bigcup_{j=N-\nu+1}^{N} V_j.$$

In this paper we introduce a family of polynomial diffeomorphisms of $\mathbf{C}^N$, which we call shift-like of type ν. In Lemmas 1, 2 and 3 we show that the sets V^+, V, V^- give a filtration (in the sense of 1, 2, and 3 above) for the dynamical system generated by these mappings.

In Theorem 9 we show that the limits defining the rate of escape functions $G^{\pm}$ converge uniformly on compact subsets of $\mathbf{C}^N$. Thus we may define the corresponding stable/unstable currents $\mu^+ := (\frac{1}{2\pi} dd^c G^+)^{\nu}$ and $\mu^- := (\frac{1}{2\pi} dd^c G^-)^{N-\nu}$, and we define a measure $\mu := \mu^+ \wedge \mu^-$. From Theorem 11 it follows that μ coincides with the harmonic measure of K in the sense of pluri-potential theory.

1. SHIFT-LIKE MAPPINGS

We will say that a (holomorphic) polynomial diffeomorphism $f : \mathbf{C}^N \to \mathbf{C}^N$, $N \ge 2$, is *shift-like* if the orbit of a point $x \in \mathbf{C}^N$ under f determines a bi-infinite sequence $(\zeta_j)_{j\in\mathbf{Z}}$ such that

$$f^k(x) = (\zeta_{k+1}, \ldots, \zeta_{k+N}) \in \mathbf{C}^N.$$

Thus the forward iteration of f corresponds to shifting the sequence to the left. In this case it has the form $f(x_1, \ldots, x_N) = (x_2, \ldots, x_N, g(x_2, \ldots, x_N) - ax_1)$ for some polynomial g and some nonzero $a \in \mathbf{C}$, and the sequence ζ_n is generated by the recurrence relation: $\zeta_j = x_j$, for $1 \le j \le N$, and

$$\zeta_{n+N} = g(\zeta_{n+1}, \ldots, \zeta_{n+N-1}) - a\zeta_n \quad \text{for} \quad n \in \mathbf{Z}.$$

Note that this may be used as a recurrence relation for both increasing and decreasing n. We will also refer to a finite composition $f = f_m \circ \cdots \circ f_1$ of such mappings as *shift-like*. In this case we let g_s and a_s denote the polynomial and constant defining f_s. We use the notation $[s]$ for the integer satisfying $1 \le [s] \le m$

and $[s] \equiv s \bmod m$. Thus $f^n = f_{[mn]} \circ \cdots \circ f_{[1]}$. It follows that f generates a sequence $(\zeta_n)_{n\in\mathbf{Z}}$ such that $\zeta_j = x_j$ for $1 \le j \le N$, and

$$\zeta_{N+n} = g_{[n]}(\zeta_{n+1}, \dots, \zeta_{n+N-1}) - a_{[n]}\zeta_n. \tag{1}$$

The action of f on (ζ_n) corresponds to a shift by m units:

$$f^k(x) = (\zeta_{mk+1}, \dots, \zeta_{mk+N}). \tag{2}$$

Let f be a shift-like map. We will say that f has *type* ν, for some $1 \le \nu \le N-1$, if f has the form $f = f_m \circ \cdots \circ f_1$, where each f_s is as follows: there is a polynomial $p_s(z) = \sum_{j=0}^{d_s} c_{s,j} z^j$, $d_s \ge 2$, $c_{s,d_s} \ne 0$, and a nonzero constant $a_s \in \mathbf{C}$ such that

$$f_s(x_1, \dots, x_N) = (x_2, \dots, x_N, p_s(x_{N-\nu+1}) - a_s x_1). \tag{3}$$

By [FM], the polynomial automorphisms of $\mathbf{C}^2$ that are dynamically interesting are all conjugate to shift-like mappings of type 1.

Example. The mapping $h(x, y, z) = (y, z, yz + \beta x)$ is shift-like but is not of type ν for any ν. If $|\beta| = 1$, then the coordinate axes are contained both in K and in the nonwandering set. In this case, neither K nor the nonwandering set is compact.

If f is shift-like of type ν, it is natural to iterate f^ν rather than f; for $1 \le k \le m$ we write

$$f^\nu = g_m \circ \cdots \circ g_1, \quad \text{with} \quad g_k(x) = f_{[k\nu]} \circ \cdots \circ f_{[(k-1)\nu+1]}. \tag{4}$$

We will use the notation $\pi_q(y_1, \dots, y_N) = y_q$. The expression for $g_k(x)$ in (4) is given by

$$g_k(x) = (x_{\nu+1}, \dots, x_N, \pi_{N-\nu+1} g_k(x), \dots, \pi_N g_k(x)), \tag{5}$$

where by (1)

$$\pi_q g_k(x) = p_{[q-N+k\nu]}(x_q) - a_{[q-N+k\nu]} x_{q-(N-\nu)}$$

for $N - \nu + 1 \le q \le N$.

It follows that the degree of the qth coordinate of f^ν is

$$\hat{d}_q = \prod_{k=1}^{m} d_{[q-N+k\nu]}$$

for $N - \nu + 1 \le q \le N$. In §3 we will assume that the numbers $\hat{d}_j$ satisfy

$$d = d_1 \cdots d_m = \hat{d}_{N-\nu+1} = \cdots = \hat{d}_N. \tag{6}$$

This occurs if $d_1 = \cdots = d_m$. Also, if m and ν are relatively prime, then $\{[q - N + k\nu] : 1 \le k \le m\} = \{1, \dots, m\}$; and so (6) holds.

If (6) holds, then for each $N - \nu + 1 \le q \le N$ there exists a constant α_q such that

$$\deg(\pi_q f^\nu(x) - \alpha_q x_q^d) < \hat{d}. \tag{7}$$

The inverse of f is given by $f_1^{-1} \circ \cdots \circ f_m^{-1}$, where the inverse of each f_s is given by

$$f_s^{-1}(x_1, \dots, x_N) = (a_s^{-1}(p_s(x_{N-\nu}) - x_N), x_1, \dots, x_{N-1}). \tag{8}$$

In order to work in negative time, we find it convenient to iterate $f^{-(N-\nu)}$, rather than $f^{-1} = f_1^{-1} \circ \cdots \circ f_m^{-1}$. We write

$$f^{-(N-\nu)} = h_m \circ \cdots \circ h_1, \quad \text{with} \quad h_k = f^{-1}_{[m-k(N-\nu)+1]} \circ \cdots \circ f^{-1}_{[m-(k-1)(N-\nu)]}.$$

Thus we have

$$h_k(x_1, \ldots, x_N) = (\pi_1 h_k(x), \ldots, \pi_{N-\nu} h_k(x), x_1, \ldots, x_{N-\nu}), \tag{9}$$

where by (8) we have

$$\pi_q h_k(x) = a^{-1}_{[m-k(N-\nu)+q]} (p_{[m-k(N-\nu)+q]}(x_q) - x_{q+\nu})$$

for $1 \le q \le N - \nu$. In §3 we will assume that for $1 \le q \le N - \nu$ the numbers

$$\hat{d}_q = \prod_{k=1}^{m} d_{[m-k(N-\nu)+q]}$$

satisfy

$$d = \hat{d}_1 = \cdots = \hat{d}_{N-\nu}. \tag{10}$$

This holds if $d_1 = \cdots = d_m$ or if $(m, N-\nu) = 1$. In this case there are constants α_q, $1 \le q \le N - \nu$ such that

$$\pi_q f^{-(N-\nu)} x = \alpha_q x_q^d + \cdots .$$

Remark 1. The involution $I(x_1, \ldots, x_N) = (x_N, \ldots, x_1)$ conjugates f_s^{-1} to

$$(x_1, \ldots, x_N) \mapsto (x_2, \ldots, x_N, a_s^{-1}(p_s(x_{\nu+1}) - x_1)), \tag{11}$$

which is shift-like of type $(N - \nu)$. This observation allows us to deduce that the results proved for a general map $f = f_m \circ \cdots \circ f_1$ of any type ν will also apply to $f^{-1} = f_1^{-1} \circ \cdots \circ f_m^{-1}$, since this is the general map of type $(N - \nu)$.

Remark 2. If $\delta > 0$ is an integer which divides ν and N, then for $0 \le c < \delta$, the subsequence $\{\zeta_n : n \equiv c \pmod{\delta}\}$ is invariant under each f_s. If we write $\nu' = \nu/\delta$ and $N' = N/\delta$, it follows that the mapping

$$f'_s(y_1, \ldots, y_{N'}) = (y_2, \ldots, y_{N'}, p_s(y_{N'-\nu'+1}) - a_s y_1)$$

is shift-like of type ν' on $\mathbf{C}^{N'}$, and thus each f_s, and the composition $f = f_m \circ \cdots \circ f_1$, are biholomorphically conjugate to a δ-fold product of the mappings $f'_s : \mathbf{C}^{N'} \to \mathbf{C}^{N'}$. Thus there is no loss of generality if we assume that ν and N are relatively prime.

2. Filtration Properties

Since f acts as a shift, it follows that

$$f(V) \subset V \cup V_N \subset V \cup V^+ \text{ and } f(V_j) \subset V_{j-1} \cup V_N \text{ for } 2 \le j \le N. \tag{12}$$

Similarly, since f^{-1} is a shift in the opposite direction, it follows that

$$f^{-1}(V) \subset V \cup V_1 \subset V \cup V^- \text{ and } f^{-1}(V_j) \subset V_{j+1} \cup V_1 \text{ for } 1 \le j \le N-1. \tag{13}$$

We assume that $f = f_m \circ \cdots \circ f_1$, with f_s as in (1) and with $d_s \ge 2$. We let $\rho < 1$ be given and choose $R \ge 1$ sufficiently large that

$$\rho^{-1}|c_{s,d_s}\zeta^{d_s}| > \Big[|p_s(\zeta)| \pm (1 + |a_s|)|\zeta|\Big] > \rho|c_{s,d_s}\zeta^{d_s}| \tag{14}$$

holds for all $|\zeta| \geq R$ and all $1 \leq s \leq m$. We also assume that

$$R > \frac{2(1+|a_s|)}{|c_{d_s}|} \tag{15}$$

holds for $1 \leq s \leq m$. We noted in Remark 1 that f_s^{-1} is also shift-like, although the type is $(N-\nu)$. When we work with f^{-1}, we will assume that the corresponding inequalities (14) and (15) hold for the inverses f_s^{-1}.

Remark 3. In this section on filtrations, we will often treat the iteration $f^n = (f_m \circ \cdots \circ f_1) \circ \cdots \circ (f_m \circ \cdots \circ f_1)$ as part of the composition $f_{s_j} \circ f_{s_{j-1}} \circ \cdots \circ f_{s_1}$ of an infinite sequence of mappings $f_{s_1}, f_{s_2}, f_{s_3}, \ldots$, such that the degrees d_{s_j} are uniformly bounded, and the conditions (14) and (15) hold uniformly. In this case, orbits correspond to sequences $\{\zeta_n : n \in \mathbf{Z}\}$ such that

$$\zeta_n = p_{s_{(n-N)}}(\zeta_{n-\nu}) - a_{s_{(n-N)}}\zeta_{n-N} \tag{16}$$

for all n. The arguments given below concerning the existence of a filtration continue to apply in this more general situation.

Lemma 1. *If* (14) *holds, then* $f_s V_{N-\nu+1} \subset V_N$ *holds for each* $1 \leq s \leq m$, *and thus* $fV^+ \subset V^+$.

Proof. If $x \in V_{N-\nu+1}$, then $\|x\| = |x_{N-\nu+1}| > R$. Thus, by (14), it follows that $|x_{N+1}| > \|x\|$. Thus $fx \in V_N \subset V^+$. ☐

Under the involution $I(x_1, \ldots, x_N) = (x_N, \ldots, x_1)$, the sets $V^\pm$ of type $(N-\nu)$ are taken to $V^\mp$ of type ν. If we apply the argument of Lemma 1 to f^{-1} we obtain:

Lemma 2. *If* (14) *holds for* f^{-1}, *then* $f_s^{-1}V_{N-\nu} \subset V_1$, *holds for* $1 \leq s \leq m$, *and thus* $f^{-1}V^- \subset V^-$.

We begin by giving a weak estimate which shows that points of V^+ escape to infinity in forward time.

Lemma 3. *There exists* $c' > 0$, *depending only on* f, *such that if* R *is large,*

$$c'||x||^2 \leq ||f^\nu(x)||$$

for all $x \in V^+$. *In particular, if we take* R *such that* $Rc' > 1$, *then there exists* $\kappa > 1$ *such that for every* $x \in V^+$, *we have* $||f^{n\nu}x|| \geq \kappa^{2^n}/c'$.

Proof. We write $f^\nu = g_m \circ \cdots \circ g_1$ as above. It suffices to prove the Lemma for each of these mappings g_k. If $x \in V^+$, then there exists $N-\nu+1 \leq j \leq N$ such that $|x_j| = ||x||$. By (5), we have that the size of the j-th component of $g_k(x)$ is

$$|p_s(x_j) - a_s x_i| \geq |c_s x_j^{d_s}| - |a_s x_i| \geq c(|x_j|^{d_s} - ||x||) \geq c_k ||x||^2,$$

since $|x_j| = ||x|| \geq |x_i|$.

If $Rc' > 1$, we may write $R = {c'}^{-1}\kappa$ with $\kappa > 1$. The final inequality follows by repeatedly substituting $||x|| \geq {c'}^{-1}\kappa$ into the first inequality. ☐

Lemma 4. *There exists* c, *depending only on* f *such that if* $R < \|x\| \leq M$, *and if* $f_{s_j} \circ \cdots \circ f_{s_1} x \in V^-$ *for* $0 \leq j \leq T + N + \nu$, *then*

$$\|f_{s_T} \circ \cdots \circ f_{s_1} x\|^2 \leq \max_s \{\frac{2(1+|a_s|)}{|c_{d_s}|} M, R^2\}.$$

Proof. Let $(\zeta_n)_{n\in\mathbf{Z}}$ denote the sequence associated with the orbit of x under the family of mappings $f_{s_1}, f_{s_2}, \dots$ as in (16). From the condition $f_{s_j}\circ\cdots\circ f_{s_1}x = (\zeta_{j+1},\dots,\zeta_{j+N}) \in V^-$, we have $\max_{1\le q\le N-\nu}|\zeta_{j+q}| \ge |\zeta_{j+k}|$ for $1\le k\le N$. Applying this inductively, starting with $j=0$, and extending to $0\le j\le T+N$, we have

$$\max_{1\le j\le N-\nu}|\zeta_j| \ge |\zeta_k| \quad \text{for} \quad k\le T+N+\nu.$$

Now we use the fact that $M\ge \|x\|$ and (16) to obtain

$$M \ge |\zeta_k| = |p_s(\zeta_{k-\nu}) - a_s\zeta_{k-N}|.$$

Now either $|\zeta_{k-\nu}|\le R$, or we may apply (14) to obtain

$$M \ge \frac{|c_{d_s}|}{2}|\zeta_{k-\nu}|^{d_s} - |a_s\zeta_{k-N}| \ge \frac{|c_{d_s}|}{2}|\zeta_{k-\nu}|^2 - |a_s|M.$$

This gives

$$\frac{2(1+|a_s|)}{|c_{d_s}|}M \ge |\zeta_{k-\nu}|^2.$$

Finally, since $f_{s_T}\circ\cdots\circ f_{s_1}x = (\zeta_{T+1},\dots,\zeta_{T+N})$, and since the previous inequality holds for $k\le T+N+\nu$, we have the desired estimate. □

Lemma 5. *Any orbit $f^n x$ can remain in V^- for only finitely many values of $n\ge 0$.*

Proof. Let us suppose that the forward orbit of $f^n x$ remains in V^- for all $n\ge 0$. It follows from Lemma 1 that $f_{[T]}\circ\cdots\circ f_{[1]}x\in V^-$ for all $T\ge 0$. Let $c=\max_s \frac{2(1+|a_s|)}{|c_{d_s}|}$, and define M_j by $M_0 := \|x\|$ and $M_{j+1} := (cM_j)^{1/2}$. Since $f^{j(N-\nu)}x\in V^-$, we have $\|f^{j(N-\nu)}x\| > R$. We apply Lemma 4 inductively in j to the map $f_{[T]}\circ\cdots\circ f_{[1]}$ with $T = j(N-\nu)m$ and obtain that

$$\|f^{(j+1)(N-\nu)}x\| \le \sqrt{c\|f^{j(N-\nu)}x\|} \le \sqrt{cM_j} = M_{j+1}.$$

On the other hand, it is easily seen that the sequence $\{M_j\}$ decreases to c, and $c<R$ by (15), which is a contradiction. □

We may summarize our work so far with the following:

Theorem 6. *If f is a shift-like mapping of type ν, and if R is chosen sufficiently large, then the sets V^-, V, and V^+ have the filtration properties 1, 2, and 3 for f as given above. Further, V^+, V, and V^- have the same filtration properties for the mapping f^{-1}.*

Proposition 7. *$U^\pm = \bigcup_{n=0}^\infty f^{\mp n}V^\pm$, and this union is increasing.*

Proof. By Lemma 1, $fV^+\subset V^+$, so $f^{-n}V^+\subset f^{-n-1}V^+$ for $n\ge 0$, so the union is increasing. By Lemma 3, if $x\in V^+$, then $\lim_{n\to\infty}\|f^{\nu n}x\| = \infty$. Thus $U^+\supset V^+$. By the invariance of U^+ under f, we obtain $U^+\supset f^{-n}V^+$. On the other hand, if $x\in U^+$, the forward orbit is unbounded. Since a forward orbit cannot remain in V^- for all positive time, we must have $f^n x\in V^+$, which is to say that $x\in f^{-n}V^+$. The arguments for V^- are analogous. □

Corollary 8. *$K\subset V$ and $K^\pm\cap V^\pm = \emptyset$.*

Homology of $U^{\pm}$. For $N-\nu+1 \le j \le N$, the circle $\sigma_j : \theta \mapsto (R, \ldots, e^{2\pi i\theta}R, \ldots, R)$ (with the exponential in the j-th coordinate) generates $H_1(V_j; \mathbf{Z})$. The action of one of the component mappings f_s on homology is: $f_{s*}\sigma_j = \sigma_{j-1}$ for $N-\nu+1 < j \le N$, and $f_{s*}\sigma_{N-\nu+1} = d_s \cdot \sigma_N$. The inverse gives a homeomorphism $f^{-1} : V^+ \to f^{-1}V^+$. The action of $f_*^{-\nu}$ on the homology of V^+ is to divide all ν of the generators $\sigma_{N-\nu+1}, \ldots, \sigma_N$ by $d = d_1 \cdots d_m$. By Proposition 7, $\bigcup_{n\ge 0} f^{-n}V^+ = U^+$, so the homology of the limit U^+ is given as the ν-fold product:

$$H_1(U^+; \mathbf{Z}) = \mathbf{Z}[\frac{1}{d}] \times \cdots \times \mathbf{Z}[\frac{1}{d}].$$

A similar argument gives $H_1(U^-; \mathbf{Z})$ as the $(N-\nu)$-fold product of $\mathbf{Z}[\frac{1}{d}]$.

3. Green Functions

In this Section we study the rate of escape functions for the forward iterates of f^ν and the backward iterates of $f^{N-\nu}$. For the rest of this paper we assume that (6) and (10) hold.

Example. If $f = f_2 \circ f_1$ with $f_1(x, y, z) = (y, z, y^3 + x)$ and $f_2(x, y, z) = (y, z, y^2 + x)$, then $f(x, y, z) = (z, y^3 + x, z^2 + y)$. The condition (6) does not hold, and the arguments below do not apply to this function.

For $n \ge 0$, we define:

$$G_n^+(x) := \frac{1}{d^n} \log^+ ||f^{n\nu}(x)||,$$
$$G_n^-(x) := \frac{1}{d^n} \log^+ ||f^{-n(N-\nu)}(x)||.$$

Theorem 9. *The limits $G^{\pm} := \lim_{n\to\infty} G_n^{\pm}$ are uniform on compact subsets of $\mathbf{C}^N$, and $G^{\pm}$ are continuous and pluri-subharmonic on $\mathbf{C}^N$. We have*

$$G^+ \circ f^\nu = d \cdot G^+ \quad \text{and} \quad G^- \circ f^{N-\nu} = d^{-1} \cdot G^-. \tag{17}$$

Further, $K^{\pm} = \{G^{\pm} = 0\}$, and

$$G^{\pm}(x) = \log \|x\| + O(1)$$

holds uniformly on $V^{\pm}$ as $x \to \infty$.

Proof. Without loss of generality we consider only G^+. We will show that the limit defining G^+ converges uniformly on compact sets. Thus G^+ is continuous and pluri-subharmonic. By Lemma 5, any compact subset of V^- will be mapped to $V \cup V^+$ in finite positive time. Thus it suffices to show that the series $\sum(G_{n+1}^+ - G_n^+)$ converges uniformly on compact subsets of $V \cup V^+$.

We may assume that V is contained in the polydisk of radius R. For the points x such that $f^{n\nu}x \in V$ for all $n \ge 0$, we have

$$G_{n+1}^+(x) - G_n^+(x) \le \frac{1}{d^n} \log R.$$

If $f^{n\nu}x \notin V$ for some $n \ge 0$, then $f^{n\nu}x \in V^+$ for $n \ge n_0$. Thus it will suffice to show that the series converges uniformly on V^+. In order to estimate $G_{n+1}^+ - G_n^+$ on V^+, let us write $y = f^{n\nu}(x)$ and $z = f^\nu(y)$. Let $N-\nu+1 \le m, k \le N$ be indices such that $|z_k| = ||z||$ and $|y_m| = ||y||$. Thus

$$G_{n+1}^+(x) - G_n^+(x) = \frac{1}{d^{n+1}} \log \frac{|z_k|}{|y_m|^d}.$$

By (7) we have $z_i = \alpha_i y_i^d + O(||y||^{d-1})$ for $N - \nu + 1 \leq i \leq N$. Thus since $|y_k| \leq |y_m| = ||y||$, we have

$$\frac{|z_k|}{|y_m|^d} \leq \frac{|\alpha_k y_k^d| + O(||y||^{d-1})}{|y_m|^d} \leq |\alpha_k| + O(||y||^{-1}).$$

Now we only need to bound $|z_k|/|y_m|^d$ from below. Equation (7) and $||z|| = |z_k| \geq |z_m|$ give

$$|\alpha_k y_k^d| + O(||y||^{d-1}) = |z_k| \geq |z_m| \geq |\alpha_m y_m^d| + O(||y||^{d-1}) = |\alpha_m| ||y||^d + O(||y||^{d-1}),$$

from which we conclude that

$$\frac{|z_k|}{|y_m|^d} \geq |\alpha_m| + O(||y||^{-1}).$$

Thus

$$G_{n+1}^+(x) - G_n^+(x) = O(d^{-(n+1)})$$

on V^+, and the series converges uniformly.

The asymptotic behavior of $G^+ = \log ||x|| + \sum_{n=0}^{\infty} (G_{n+1}^+ - G_n^+)$ is given by the fact that the series converges uniformly on $V \cup V^+$. Further, it follows from the definition that $d \cdot G_{n+1}^+ = G_n^+ \circ f^\nu$. Thus (17) follows from the uniform convergence of $\{G_n^+\}$.

Finally, let us note that $G^+ > 0$ on V^+. For each $x \in U^+$, it follows by Proposition 7 that $f^n x \in V^+$ for some $n \geq 0$. Thus $G^+(f^n(x)) > 0$, so it follows from (17) that $G^+(x) > 0$. This shows that $G^+ > 0$ on $U^+ = \mathbf{C}^N - K^+$. Conversely, it is evident that $G^+ = 0$ on K^+. □

Remark 4. If $m = 1$, i.e. if $f = f_1$, then we actually have

$$G^+(x) = \log \|x\| + (d-1)^{-1} \log |c_d| + o(1),$$
$$G^-(x) = \log \|x\| + (d-1)^{-1} \log |a^{-1} c_d| + o(1),$$

on $V^{\pm}$ as $x \to \infty$, where $a = a_1$ and $c_d = c_{1,d_1}$ in the notation of (3).

4. Invariant Currents

Let us begin with some general computations involving currents. We recall (see [BT]) that if U is a continuous, psh function, and if T is a positive, closed (p,p)-current, then $dd^c U \wedge T$ is a $(p-1, p-1)$-current, whose action on a test form φ of degree $(p-1, p-1)$ is defined by $dd^c U \wedge T(\varphi) = T(U dd^c \varphi)$. If τ is a Borel measure, and if $t \mapsto S_t$ is a Borel measurable family of (p,p)-currents, we will define a new (p,p)-current, which acts on a test form φ of degree (p,p) as $(\int \tau(t) S_t)(\varphi) := \int (S_t \varphi) \tau(t)$.

We define

$$L^-(\zeta_1, \ldots, \zeta_N) = \max_{j=1,\ldots,N-\nu} \log^+ |\zeta_j|$$

and

$$L^+(\zeta_1, \ldots, \zeta_N) = \max_{j=N-\nu+1,\ldots,N} \log^+ |\zeta_j|.$$

Lemma 10. *We have*

$$(dd^cL^-)^{N-\nu} = dd^cL^- \wedge \cdots \wedge dd^cL^- = \int \tau_{N-\nu}(\zeta')\ [\{\zeta'\} \times \mathbf{C}^\nu] \tag{18}$$

and

$$(dd^cL^+)^\nu = dd^cL^+ \wedge \cdots \wedge dd^cL^+ = \int \tau_\nu(\zeta'')\ [\mathbf{C}^{N-\nu} \times \{\zeta''\}], \tag{19}$$

where τ_j *denotes the* j*-dimensional Hausdorff measure on the* j*-torus* $\{|\zeta_1| = \cdots = |\zeta_j| = 1\}$ *in* $\mathbf{C}^j$*, and* $[X]$ *denotes the current of integration over the complex manifold* X*. In particular,*

$$(dd^c \max(L^+, L^-))^N = (dd^cL^-)^{N-\nu} \wedge (dd^cL^+)^\nu. \tag{20}$$

Proof. First we consider L^- and show that (18) holds; the proof of (19) is similar. Let us introduce the variables $z_j = x_j + iy_j = \log \zeta_j$ for $1 \le j \le N - \nu$ and $z_j = \zeta_j$ for $N - \nu + 1 \le j \le N$. Then $\log^+ |\zeta_j| = \max(x_j, 0)$ for $1 \le j \le N - \nu$. Since the operator dd^c is invariant under holomorphic coordinates, we may compute in the z coordinates to obtain

$$(dd^c \max_{1 \le j \le N-\nu}(x_j, 0))^{N-\nu} = \int_{y \in \mathbf{R}^{N-\nu}} dy\, [\{y\} \times \mathbf{C}^\nu],$$

where dy denotes Lebesgue measure. This identity is remarked in [BT], and the computation is carried out in [HP, Lemma 3.5]. We obtain the formula (18) by transforming this identity (locally) under the exponential map $y \mapsto e^{iy}$; we observe that under the exponential map, the current of integration over $[\{y\} \times \mathbf{C}^\nu]$ is taken to the current of integration $[\{\zeta\} \times \mathbf{C}^\nu]$ and that $(N - \nu)$-dimensional Lebesgue measure on $\mathbf{R}^{N-\nu}$ is taken (locally) to the measure $\tau_{N-\nu}$.

For (20) we recall that the wedge product of currents of integration corresponds to the current of integration over the intersection. Thus $\delta_{(\zeta',\zeta'')} = [\{\zeta'\} \times \mathbf{C}^{N-\nu}] \wedge [\mathbf{C}^\nu \times \{\zeta''\}]$ is the unit point mass at (ζ', ζ''). Integrating this observation with respect to $\tau_{N-\nu}$ in the variable ζ' and τ_ν in ζ'', and applying (18) and (19), we have that

$$\tau_\nu \otimes \tau_{N-\nu} = (dd^cL^+)^\nu \wedge (dd^cL^-)^{N-\nu}.$$

Since $L = \log^+ \|\zeta\| = \max(L^-, L^+)$ is equal to L^- in the case $\nu = 0$, we see by (18) that $(dd^cL)^N$ is also equal to the measure $\tau_N = \tau_\nu \otimes \tau_{N-\nu}$. Thus $(dd^cL)^N$ is equal to the left hand side of this identity, which gives (20). □

Since G^+ and G^- are continuous, we may define $\mu^+ := (\frac{1}{2\pi} dd^cG^+)^\nu$ and $\mu^- := (\frac{1}{2\pi} dd^cG^-)^{N-\nu}$. It follows from Theorem 9 that

$$(f^\nu)^*\mu^+ = d^\nu\mu^+ \text{ and } (f^{N-\nu})^*\mu^- = d^{\nu-N}\mu^-.$$

We take the wedge product $\mu := \mu^+ \wedge \mu^-$ and obtain a Borel measure, which then satisfies

$$f_*^{\nu(N-\nu)}(\mu) = \mu.$$

We define $G := \max(G^+, G^-)$.

Theorem 11. $\mu^+ = 0$ *on* U^+*;* $\mu^- = 0$ *on* U^-*;* $(dd^cG)^N = 0$ *on* $\mathbf{C}^N - K$*; and*

$$(\frac{1}{2\pi} dd^cG)^N = \mu.$$

Proof. It follows from (18) that the support of $(dd^cL^-)^{N-\nu}$ is disjoint from $\{L^- > 0\}$, and so $(dd^cL^-)^{N-\nu} = 0$ there. Similarly, $(dd^cL^+)^\nu = 0$ on $\{L^+ > 0\}$. We restrict ourselves now to the case of G^+; the case of G^- is similar. By Theorem 9, we have that $d^{-n}L^+(f^{(N-\nu)n})$ converges uniformly on compact sets to G^+ as $n \to \infty$. It then follows that $\mu^+ = (dd^cG^+)^\nu = 0$ on $\{G^+ > 0\}$.

To work with G, we note that the sequence

$$G_n := d^{-n} \max(L^-(f^{(N-\nu)n}), L^+(f^{\nu n}))$$

converges uniformly on compact sets to G. Arguing as from (18) with $\nu = 0$, we have that

$$(dd^c \max(L^+, L^-))^N = 0$$

on $\{\zeta \in \mathbf{C}^N : \|\zeta\| > 1\}$, and so $(dd^cG_n)^N = 0$ on $\{G_n > 0\}$. Taking the limit as $n \to \infty$, we have that $(dd^cG)^N = 0$ on $\{G > 0\} = \mathbf{C}^N - K$.

Finally, we note that by equation (20), we have that

$$(dd^cG_n)^N = \frac{1}{d^{nN}}(dd^cL^-(f^{(N-\nu)n}))^\nu \wedge (dd^cL^+(f^{\nu n}))^{N-\nu}.$$

The last assertion follows upon taking the limit as $n \to \infty$. □

Remark 5. Let us recall (see Klimek [K]) that the pluri-complex Green function G_K is characterized as the psh function on $\mathbf{C}^N$ such that $G_K = \log\|x\| + O(1)$ at infinity, $G_K = 0$ on K, and $(dd^cG_K)^N = 0$ on $\mathbf{C}^N - K$. It follows from Theorems 9 and 11 that $G := (G^+, G^-)$ coincides with G_K. Since G is continuous, it follows by definition that K is pluri-regular. By a Theorem of Siciak, it follows that G is equal to the supremum of $\deg(q)^{-1}\log|q|$, taken over all polynomials q with $|q|_K \leq 1$, and degree equal to $\deg(q)$. By Theorem 11, it follows that μ is the pluri-complex equilibrium measure of K, normalized to have total mass one.

References

[BS] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity, Invent. Math., 103 (1991), 69–99. MR **92a:**32035

[BT] E. Bedford and B.A. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math. 37 (1976), 1–44. MR **56:**3351

[BuS] G. Buzzard and J. Smillie, Complex dynamics in several variables, *Flavors of Geometry*, MSRI Publications, Vol. 31, 1997. CMP 98:06

[FS] J.-E. Fornæss and N. Sibony, Complex Hénon mappings in $\mathbf{C}^2$ and Fatou-Bieberbach domains, Duke Math. J. 65 (1992), 345–380. MR **93d:**32040

[FM] S. Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms. Ergodic Theory Dyn. Syst. 9, 67–99 (1989). MR **90f:**58163

[H] J.H. Hubbard, The Hénon mapping in the complex domain. In: Chaotic Dynamics and Fractals. Barnsley, M., Demko, S. (eds.), pp. 101–111. New York: Academic Press 1986. CMP 19:01

[HP] J.H. Hubbard and P. Papadopol, Superattractive fixed points in $\mathbf{C}^n$, Indiana Univ. Math. J., 43 (1994), 321–365. MR **95e:**32025

[K] M. Klimek, *Pluripotential theory*, London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991. MR **93h:**32021

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