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RELATIVELY AND INNER UNIFORM DOMAINS

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ABSTRACT. We generalize the concept of a uniform domain in Banach spaces into two directions. (1) The ordinary metric d of a domain is replaced by a metric $e \geq d$, in particular, by the inner metric of the domain. (2) The uniformity condition is supposed to hold only for certain pairs of points of the domain. We consider neargeodesics and solid arcs in these domains. Applications to the boundary behavior of quasiconformal maps are given. In particular, we study maps between domains of the form $E \times B$, where E is a Banach space and B is a ball.

1. INTRODUCTION

1.1. Uniform domains in euclidean n-space R^n were introduced by O. Martio and J. Sarvas [MaS] in 1979, and independently by P.W. Jones [Jo1]. The notion was generalized for domains in Banach spaces by the author [Vä6]. There are plenty of equivalent characterizations for uniform domains. We recall the definition based on the quasihyperbolic metric. The alternative geometric approach will be considered in 2.16.

Let E be a real Banach space with $\dim E \geq 2$, and let $G \subsetneq E$ be a domain. For $x \in G$ we let $\delta(x) = \delta_G(x)$ denote the distance $d(x, \partial G)$ between x and the boundary ∂G of G. The *quasihyperbolic* (QH) *length* of a rectifiable arc $\gamma \subset G$ is the line integral

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)}. \tag{1.2}$$

The QH *distance* between points $a, b \in G$ is

$$k(a, b) = k_G(a, b) = \inf l_k(\gamma) \tag{1.3}$$

over all rectifiable arcs γ joining a and b in G. The function $k = k_G$ is the QH *metric* of G.

For real numbers r, s we use the notation:

$$r \wedge s = \min(r, s), \qquad r \vee s = \max(r, s).$$

For $a, b \in G$ we write

$$r_G(a, b) = \frac{|a - b|}{\delta(a) \wedge \delta(b)}, \qquad j_G(a, b) = \log(1 + r_G(a, b)). \tag{1.4}$$

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By a basic estimate for the QH metric [GP, 2.1], [Vä5, 2.2(1)], we have always

$$j_G(a,b) \le k_G(a,b). \tag{1.5}$$

Let $c \ge 1$. The domain G is said to be *quasihyperbolically* (QH) *c-uniform* if

$$k_G(a,b) \le c j_G(a,b) \tag{1.6}$$

for all $a, b \in G$.

Replacing the distance $|a-b|$ in (1.4) by the *inner length distance*

$$\lambda(a,b) = \lambda_G(a,b) = \inf l(\gamma) \tag{1.7}$$

over all arcs γ joining a and b in G, we obtain a larger class of domains, called QH *inner c-uniform domains*. These have been recently considered by Z. Balogh and A. Volberg [BV1],[BV2], and by M. Bonk, J. Heinonen and P. Koskela [BHK]. In [BV1], these domains are called *uniformly John domains*, and the definition is slightly different from the above; see 3.13.

In order to handle these and some other properties simultaneously, we assume that e is a metric of G such that

$$|a-b| \le e(a,b) \le \lambda_G(a,b) \tag{1.8}$$

for all $a, b \in G$. We say that G is QH (c,e)*-uniform* if

$$k_G(a,b) \le c j_G(a,b;e) \tag{1.9}$$

for all $a, b \in G$, where

$$j_G(a,b;e) = \log(1 + r_G(a,b;e)), \qquad r_G(a,b;e) = \frac{e(a,b)}{\delta(a) \wedge \delta(b)}. \tag{1.10}$$

The case $e(a,b) = |a-b|$ gives ordinary QH c-uniform domains; the case $e = \lambda_G$ gives the QH inner c-uniform domains.

Throughout the paper, e will be a metric of G satisfying (1.8) (unless $e = 2.718281\ldots$ in obvious cases). For example, e may be the *inner diameter metric* ϱ_G, defined by

$$\varrho_G(a,b) = \inf d(\gamma) \tag{1.11}$$

over all arcs γ joining a and b in G; here $d(\gamma)$ is the diameter of γ in the norm metric. If $e \le e'$ are two metrics, then every QH (c,e)-uniform domain is trivially QH (c,e')-uniform.

For example, balls and half spaces are QH c-uniform with a universal c. A disk in the plane with a radial slit is QH inner uniform but not QH uniform. A parallel strip in the plane is not QH inner uniform.

In this paper we consider *relative* versions of uniformity and inner uniformity. If $A \subset G$ and if (1.6) holds for all $a, b \in A$, we say that G is QH *c-uniform relative to* A, or briefly, QH c-uniform rel A. If $R > 0$ and if G is QH c-uniform rel A for all $A \subset G$ with diameter $d(A) \le R$, we say that G is R*-boundedly* QH *c-uniform*. Equivalently, this means that (1.6) holds whenever $a, b \in G$ and $|a-b| \le R$.

Following the usual practice, we omit the parameters in this and later terminology if we do not want to specify their values. For example, a domain is boundedly QH uniform if it is R-boundedly QH c-uniform for some R and c.

The corresponding relative QH inner and (c,e)-uniformity properties are defined in the obvious way. For example, G is R-boundedly (c,e)-uniform if (1.9) holds whenever $e(a,b) \le R$.

For example, a parallel strip in R^2 is R-boundedly QH c-uniform with some constant $c = c_R$ for all $R > 0$. More generally, let E_1 and E_2 be Banach spaces and let B_2 be the unit ball of E_2. Then the domain $G_0 = E_1 \times B_2$ is R-boundedly QH c_R-uniform for all $R > 0$.

The purpose of this paper is to develop a theory of relatively and boundedly uniform and inner uniform domains. In particular, we study neargeodesics and solid arcs in these domains.

The general theory of uniform domains and their generalizations in a Banach space E is given in Section 2, and the special case $E = R^n$ is considered in Section 3. Since I hope that these sections can be used in the future as a "handbook" on uniform domains and their generalizations, the results are given in a rather general form. We also give in 2.42 a new characterization of uniform domains, based on the invariance of neargeodesics under domain enlarging. Section 4 deals with coarsely quasihyperbolic maps $f : G \to G'$; the definition is recalled in 4.2. In the case $E = R^n$, this class contains all quasiconformal maps and also the pseudo-isometries of Thurston [Th, 5.9]. The main result 4.8 gives a sufficient condition for f to have a limit at a point $b \in \partial G$. In Section 5 we consider the special case where the domain is of the form $G_0 = E_1 \times B_2$. For example, if $f : G_0 \to G_0$ is C-coarsely M-QH and if $\dim E_1 \geq 2$, then f extends to a homeomorphism $\bar{f} : \bar{G}_0 \to \bar{G}_0$ with $\bar{f}(\infty) = \infty$. Moreover, $\bar{f}$ is quasisymmetric rel ∂G_0, and the horizontal and the vertical distortions of f are controlled by (M, C).

As a special case we get results on K-quasiconformal maps of $G_0 = \mathsf{R}^p \times B^q \subset \mathsf{R}^{p+q}$. Alternatively, these results could be obtained by using moduli of suitable path families. However, the author believes that in the theory of quasiconformal maps, one should try to find alternative proofs for various results by replacing the path families by quasihyperbolic arguments, since this would often give stronger results, where (1) quasiconformality is replaced by solidity or by coarse quasihyperbolicity and (2) R^n is replaced by an arbitrary Banach space.

The basic theory of uniform and relatively uniform domains is valid in all normed spaces. However, in the applications to the boundary behavior of maps, the completeness of the space is essential.

1.12. *Notation.* Throughout the paper, E will denote a real Banach space with $\dim E \geq 2$, and G is always a *proper* subdomain of E. The norm of a vector $x \in E$ is written as $|x|$. For balls and spheres we use the fairly standard notation

$$B(a, r) = \{x : |x - a| < r\}, \qquad \bar{B}(a, r) = \{x : |x - a| \leq r\},$$
$$S(a, r) = \{x : |x - a| = r\}.$$

The center a can be omitted if it is the origin. In particular, $B(1)$ is the open unit ball of E.

By an *arc* we mean a set homeomorphic to a real interval, which may be closed, open, or half open. We write $\gamma : a \curvearrowright b$ if γ is an arc with endpoints a and b. This notation also gives an orientation for γ such that a is the first point. If x and y are points of an arc γ, we let $\gamma[x, y]$ denote the compact subarc of γ between x and y. For half open subarcs we use the obvious notation $\gamma[x, y)$.

Notation like ab/xyz means $(ab)/(xyz)$.

If ϱ is a metric of a set X, we let $\varrho(A)$ denote the ϱ-diameter of a set $A \subset X$, and $\varrho(A, B)$ is the ϱ-distance between nonempty subsets A, B of X. In a Banach space E we use the alternative notation $d(x, y) = |x - y|$, and thus $d(A)$ denotes the ordinary diameter of a set $A \subset E$.

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2. Relative, bounded and inner uniformity

2.1. *Summary.* An alternative title for this section could be: Handbook on uniform domains and their generalizations. We extend the theory of uniform and inner uniform domains in a Banach space to domains that are uniform relative to a subset or boundedly uniform. The proofs are often rather easy modifications of the absolute (nonrelative) case. In addition to the QH approach given in the introduction, we define relative and bounded uniformity in terms of cigars. In 2.42 we give a new characterization for uniform domains in terms of neargeodesics. This is applied in 2.44 to show that uniform domains are subinvariant under QH maps.

2.2. **Terminology.** We recall some concepts introduced in [Vä6]. Let $h \geq 0$ and let γ be an arc in a metric space (X, ϱ). The *h-coarse length* of γ is defined as

$$l(\gamma, h) = \sup \sum_{j=1}^{k} \varrho(x_{j-1}, x_j),$$

where the supremum is taken over all finite sequences $(x_0, \ldots, x_k)$ of successive points of γ such that $\varrho(x_{j-1}, x_j) \geq h$ for all $1 \leq j \leq k$. If there is no such sequence, then $l(\gamma, h) = 0$. For $h = 0$ we obtain the ordinary length $l(\gamma) = l(\gamma, 0)$ of γ. If γ is compact and $h > 0$, then $l(\gamma, h) < \infty$. The basic theory of coarse length is given in [Vä6, Sec. 4].

We shall use the coarse length in the case where $X = G \subsetneq E$ is a domain and $\varrho = k$ is the QH metric of G, defined in 1.1. We let $l_k(\gamma, h)$ denote the h-coarse QH length of an arc $\gamma \subset G$. Then $l_k(\gamma, 0) = l_k(\gamma)$ is the QH length of γ, defined by (1.2).

A metric space (X, ϱ) is *c-quasiconvex* if each pair of points $a, b \in X$ can be joined by an arc $\gamma \subset X$ with $l(\gamma) \leq c\varrho(a, b)$. In particular, an arc $\gamma \subset X$ is c-quasiconvex if and only if $l(\gamma[x, y]) \leq c\varrho(x, y)$ for all $x, y \in \gamma$.

An arc γ in a domain G is a *c-neargeodesic* if γ is c-quasiconvex in the QH metric. In other words,

$$l_k(\gamma[x, y]) \leq ck(x, y) \tag{2.3}$$

for all $x, y \in \gamma$. Given points $a, b \in G$ and a number $c > 1$, we can always join a and b by a c-neargeodesic in G; see [Vä6, 3.3]. If $\dim E < \infty$, this is possible with $c = 1$; then γ is a *geodesic* of the QH metric.

More generally, let $h \geq 0$ and $c \geq 1$. An arc $\gamma \subset G$ is (c, h)-*solid* in G if

$$l_k(\gamma[x, y], h) \leq ck(x, y) \tag{2.4}$$

for all $x, y \in \gamma$. Thus $(c, 0)$-solid arcs are c-neargeodesics. Alternatively, the solid arcs can be characterized as coarsely bilipschitz images of real intervals. We prove this in 2.7, but the result is not needed in this paper.

Let e be a metric of G satisfying (1.8). Recall from 1.12 that $e(A)$ is the e-diameter of a set $A \subset G$. If $e \leq \varrho_G$ and A is connected, then $e(A) = d(A)$ is independent of e by [Vä4, 2.13], but $\lambda(A)$ can be larger. On the other hand, the length of an arc $\gamma \subset G$ in each metric e is easily seen to be equal to the ordinary length $l(\gamma)$.

The following two results are λ-versions of [Vä5, 2.2(1)] and [Vä6, 4.5]. In fact, the proofs of the original results also give these stronger versions. Of course, they are true if λ is replaced by any metric $e \leq \lambda$. The first one is well known.

2.5. **Lemma.** *For all $a, b \in G$ we have*

$$\begin{aligned} k(a,b) &\geq j_G(a,b;\lambda) = \log\left(1 + \frac{\lambda_G(a,b)}{\delta(a)\wedge\delta(b)}\right), \\ \lambda_G(a,b) &\leq (\delta(a)\wedge\delta(b))(e^{k(a,b)}-1). \quad \square \end{aligned}$$

2.6. **Lemma.** *Suppose that $r > 0$ and that γ is a (c,h)-solid arc in $G\cap(\partial G+\bar{B}(r))$. Then*

$$\lambda_G(\gamma) \leq Mr(h \vee l_k(\gamma,h)),$$

where $M = M(h) = (e^h - 1)/h$ for $h > 0$ and $M(0) = 1$. If $h = 0$, that is, if γ is a c-neargeodesic, then

$$l(\gamma) \leq r l_k(\gamma). \quad \square$$

2.7. **Proposition.** *Let J be a real interval, and let $g : J \to G$ be an embedding such that*

$$(|x-y| - C)/M \leq k(gx, gy) \leq M|x-y| + C \tag{2.8}$$

for all $x, y \in J$. Then $\gamma = gJ$ is a (c,h)-solid arc in G with $c = 2M(M+1)$, $h = (2M+1)C$. Conversely, if $\gamma \subset G$ is a (c,h)-solid arc, there is an interval $J \subset \mathsf{R}$ and a homeomorphism $g : J \to \gamma$ such that (2.8) *holds with $M = c$, $C = 2h$.*

Proof. The first part of the proposition follows directly from [Vä6, 4.11]. We prove the second part in the case where γ has precisely one endpoint a; the other cases are obtained by easy modifications.

Choose inductively successive points $x_0, x_1, \dots$ of γ such that $x_0 = a$ and x_j is the first point after x_{j-1} with $k(x_{j-1}, x_j) = h$. If no such point exists for $j = N$, the process stops, and we get a finite sequence $x_0, \dots, x_{N-1}$. In this case we set $J = [0, Nh)$, otherwise $J = [0, \infty)$.

Set $\Delta_j = [(j-1)h, jh)$, and choose homeomorphisms $g_j : \Delta_j \to \gamma[x_{j-1}, x_j)$ with $g_j((j-1)h) = x_{j-1}$. In the finite case, we also choose a homeomorphism $g_N : \Delta_N \to \gamma \setminus \gamma[a, x_{N-1})$. We show that the map $g : J \to \gamma$, defined by $g|\Delta_j = g_j$, satisfies (2.8).

Suppose that $x \in \Delta_j$ and $y \in \Delta_k$ with $k = j + s \geq j$. If $s \leq 1$, then $k(gx, gy) \leq 2h$. Assume that $s \geq 2$. Now $(s-1)h \leq |x-y| \leq (s+1)h$, and

$$k(gx, gy) \leq 2h + \sum_{i=j+1}^{k-1} |x_{i-1} - x_i| = (s+1)h \leq |x-y| + 2h,$$

which implies the second inequality of (2.8). Since $k(x_{i-1}, x_i) = h$ for all i, we have $l_k(\gamma[gx, gy], h) \geq (s-1)h$. By the (c,h)-solidity of γ, this implies that $(s-1)h \leq ck(gx, gy)$, and we obtain the first inequality of (2.8) with $M = c$, $C = 2h$. $\square$

2.9. **Quasihyperbolic ψ-uniformity.** Let $\psi : [0,\infty) \to [0,\infty)$ be a homeomorphism. A domain G is QH ψ-*uniform* if

$$k_G(a,b) \leq \psi(r_G(a,b)) \tag{2.10}$$

for all $a, b \in G$. For notation, see (1.3) and (1.4). In the special case $\psi(t) = c\log(1+t)$, we obtain the QH c-uniform domains defined in 1.1. The terminology is somewhat ambiguous, but we shall always use ψ for functions and c for constants.

For example, every convex domain is QH ψ-uniform with $\psi(t) = t$. However, to get interesting results we must assume that

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0.$$

Functions ψ with this property are called *slow*. For example, the function $\psi(t) = c\log(1+t)$ is slow. In fact, if G is QH ψ-uniform with a slow ψ, then G is QH c-uniform with a constant $c = c(\psi)$ [Vä6, 6.16].

If $A \subset G$ and if (2.10) holds for all $a, b \in A$, we say that G is QH *ψ-uniform relative to A*, or briefly QH ψ-uniform rel A. If $R > 0$ and if (2.10) holds for all $a, b \in G$ with $|a - b| \le R$, then G is said to be *R-boundedly* QH *ψ-uniform*. For $\psi(t) = c\log(1+t)$, we obtain the corresponding QH c-uniform properties defined in 1.1.

Inner and e-versions of QH ψ-uniformity are defined in the obvious way. For example, G is QH (ψ, e)-uniform if

$$k_G(a, b) \le \psi(r_G(a, b; e))$$

for all $a, b \in G$, where $r_G(a, b; e)$ is defined in (1.10). Furthermore, G is R-boundedly QH (ψ, e)-uniform if this is true whenever $e(a, b) \le R$. For $e = \lambda$ we obtain the properties QH inner ψ-uniform, QH inner ψ-uniform rel A, and R-boundedly QH inner ψ-uniform.

Given a slow function ψ and numbers $c \ge 1,\ h \ge 0$, we let $M_1(c, h, \psi)$ denote the smallest number $M_1 \ge 1$ such that

$$\begin{cases} t > M(h)c\psi(2t), & \text{if } t > M_1, \\ \psi^{-1}(h/c) \le 2M_1, \end{cases} \tag{2.11}$$

where $M(h) = (e^h - 1)/h$ is the number of 2.6.

2.12. **Solid arcs in uniform domains.** Solid arcs and neargeodesics play an important role in the free quasiworld, since the solidity property of arcs is preserved by freely quasiconformal maps and, more generally, by solid and coarsely QH maps. It is therefore useful to have some knowledge of the geometric properties of these arcs. For uniform domains, such properties were established in [Vä6, Sec. 6]. Our next goal is to obtain relative, bounded, and e-versions of these results.

We start with two central lemmas. Roughly speaking, the first result says that a solid arc must *escape* the boundary of the domain; it cannot travel a long distance near the boundary. In the second lemma we show that if a solid arc *dives* from a distance $\delta(x_0)$ to the distance $\delta(x) = q\delta(x_0)$ with sufficiently small q, then the life of γ after the point x is fairly short.

The proofs of both lemmas follow the proofs of the original results [Vä6, 6.10, 6.11]. Since QH ψ-uniform implies QH inner ψ-uniform and since $d(\gamma) \le \lambda(\gamma)$, they are stronger than the original versions.

2.13. **Escape lemma.** *Suppose that ψ is slow, that G is* QH *inner ψ-uniform rel A, and that $\gamma \subset A \cap (\partial G + \bar{B}(r))$ is a (c, h)-solid arc in G. Then*

$$\lambda_G(\gamma) \le 4M_1 r, \tag{2.14}$$

where $M_1 = M_1(c, h, \psi)$ is defined by (2.11). *This is also true if G is R-boundedly* QH *inner ψ-uniform and if $\gamma \subset G \cap (\partial G + \bar{B}(R/4M_1))$. In this case we have $\lambda_G(\gamma) \le R$.*

If $h = 0$, then (2.14) *is replaced by $l(\gamma) \le 4M_1 r$.*

Proof. Assume first that $h > 0$. We may assume that γ is a compact arc with endpoints a and b. We first show that if $\delta(a) \wedge \delta(b) \geq r/2$, then $\lambda(\gamma) \leq M_1 r$. Setting $t = \lambda(\gamma)/r$ we get

$$l_k(\gamma, h) \leq ck(a, b) \leq c\psi(r_G(a, b; \lambda)) \leq c\psi(2t).$$

If $c\psi(2t) \leq h$, then $2t \leq \psi^{-1}(h/c) \leq 2M_1$ by (2.11). If $c\psi(2t) \geq h$, Lemma 2.6 gives

$$tr = \lambda(\gamma) \leq Mr(h \vee c\psi(2t)) = Mrc\psi(2t).$$

By (2.11) this implies that $t \leq M_1$.

In the general case we choose $a_0 \in \gamma$ such that $\delta(a_0)$ is maximal. We may assume that $\delta(a_0) = r$. Considering a_0 as the first point of the arc $\gamma' = \gamma[a_0, a]$ we divide γ' into subarcs $\gamma_j = \gamma[a_{j-1}, a_j]$, $1 \leq j \leq N$, where a_j is the last point of γ' with $\delta(a_j) = 2^{-j}r$, and $a_N = a$. By the special case considered above we have $\lambda(\gamma_j) \leq 2^{-j+1}M_1 r$ for all j, and hence $\lambda(\gamma') < 2M_1 r$. Similarly $\lambda(\gamma[a_0, b]) < 2M_1 r$, and (2.14) follows.

Observe that if γ is a compact arc, then (2.14) holds as a strict inequality.

Suppose that G is R-boundedly QH inner ψ-uniform and that $\gamma \subset G \cap (\partial G + \bar{B}(R/4M_1))$. If $\lambda(\gamma) \leq R$, then γ is QH inner ψ-uniform rel γ, and (2.14) follows by the first part of the lemma. If $\lambda(\gamma) > R$, there is a compact subarc $\beta \subset \gamma$ with $\lambda(\beta) = R$. Since β is (c, h)-solid, the first part of the lemma gives the contradiction $R = \lambda(\beta) < 4M_1 R/4M_1 = R$.

The proof for the case $h = 0$ is similar but easier. □

2.15. **Diving lemma.** *Suppose that ψ is slow and that $c \geq 1$, $h \geq 0$. Then there is $q = q(c, h, \psi) \in (0, 1)$ with the following property:*

Let G be a QH *inner ψ-uniform domain rel A, and let the arc $\gamma \subset A$ be (c, h)-solid in G with endpoints a_0 and a_1. If $x \in \gamma$ and $\delta(x) \leq q\delta(a_0)$, then for $\gamma_x = \gamma[x, a_1]$ we have*

$$\lambda_G(\gamma_x) \leq 4M_1\delta(x)/q,$$

where $M_1 = M_1(c, h, \psi)$ is defined by (2.11). *This is also true if G is R-boundedly* QH *inner ψ-uniform and $\delta(a_0) \leq R/4M_1$.*

If $h = 0$, then $l(\gamma_x) \leq 4M_1\delta(x)/q$.

Proof. Writing $K = 2(h \vee c\psi(4M_1))$, we show that the lemma holds with $q = e^{-K}$.

In the situation of the lemma we set $r = \delta(x)/q$. It suffices to show that $\gamma_x \subset \partial G + \bar{B}(r)$, since the result then follows from the Escape lemma 2.13.

Assume that $\gamma_x \not\subset \partial G + \bar{B}(r)$. Since $\delta(a_0) \geq r$, there are $x_1, x_2 \in \gamma$ such that $\delta(x_1) = \delta(x_2) = r$ and $x \in \gamma[x_1, x_2] \subset \partial G + \bar{B}(r)$. Suppose first that $h > 0$. Then $\lambda(x_1, x_2) \leq \lambda(\gamma) \leq 4M_1 r$ by 2.13. Since G is QH inner ψ-uniform rel γ and since γ is (c, h)-solid, we obtain

$$l_k(\gamma[x_1, x], h) \leq l_k(\gamma[x_1, x_2], h) \leq ck(x_1, x_2) \leq c\psi(r_G(x_1, x_2; \lambda)) \leq c\psi(4M_1).$$

By 2.5 this implies that

$$\log(1 + |x_1 - x|/qr) \leq k(x_1, x) \leq h \vee l_k(\gamma[x_1, x], h) \leq K/2.$$

On the other hand, $|x_1 - x| \geq \delta(x_1) - \delta(x) = (1 - q)r$, and hence $1 + |x_1 - x|/qr \geq q$. These estimates give the contradiction $K \leq K/2$.

The proof for the case $h = 0$ is similar. □

2.16. **Geometric approach to uniform domains.** We recall the approach to uniform domains based on cigars. Let $\gamma \subset E$ be an arc with endpoints a and b. For $x \in \gamma$ we set

$$\zeta_d(x) = \zeta_d(x, \gamma) = d(\gamma[a, x]) \wedge d(\gamma[x, b]).$$

If γ is rectifiable, we also define the function

$$\zeta_l(x) = \zeta_l(x, \gamma) = l(\gamma[a, x]) \wedge l(\gamma[x, b]).$$

For $c \geq 1$, the sets

$$\begin{aligned} \operatorname{cig}_d(\gamma, c) &= \bigcup \{B(x, \zeta_d(x)/c) : x \in \gamma \setminus \{a, b\}\}, \\ \operatorname{cig}_l(\gamma, c) &= \bigcup \{B(x, \zeta_l(x)/c) : x \in \gamma \setminus \{a, b\}\} \end{aligned}$$

are the *diameter c-cigar* and the *length c-cigar*, respectively, with core γ. The length cigar is only defined for a rectifiable γ.

Let G be a domain and let $\gamma \subset G$ be a rectifiable arc with endpoints a and b. We say that γ satisfies the *uniformity conditions* in G with a constant $c \geq 1$, or briefly, γ is *c-uniform* in G, if

(1) $\operatorname{cig}_l(\gamma, c) \subset G$,
(2) $l(\gamma) \leq c|a - b|$.

Condition (1) is the *cigar condition*, and (2) is the *turning condition*. Alternatively, (1) can be written as

(1′) $\zeta_l(x, \gamma) \leq c\delta(x)$

for all $x \in \gamma$.

A domain G is said to be a *c-uniform domain* if each pair $a, b \in G$ can be joined by a rectifiable arc γ satisfying the c-uniformity conditions (1) and (2). For a set $A \subset G$, we say that G is *c-uniform rel A* if each pair $a, b \in A$ has this property. If $R > 0$ and if each pair $a, b \in G$ with $|a - b| \leq R$ has this property, then G is said to be *R-boundedly c-uniform.*

The (ε, δ) domains considered by Jones [Jo2] are closely related to the R-boundedly c-uniform domains. See 3.14.

The *e-versions* of these properties are obtained by replacing (2) by the condition

(2e) $l(\gamma) \leq ce(a, b)$.

A rectifiable arc $\gamma \subset G$ with endpoints a, b is said to be (c, e)-*uniform* in G if it satisfies (1) and (2e). If $e = \lambda$, then γ is *inner c-uniform* in G. The domain G is (c, e)-*uniform* if each pair $a, b \in G$ can be joined by a (c, e)-uniform arc. The properties (c, e)-uniform rel A, R-boundedly (c, e)-uniform, inner c-uniform, inner c-uniform rel A, and R-boundedly inner c-uniform are defined in the obvious way. For example, G is R-boundedly (c, e)-uniform if each pair a, b with $e(a, b) \leq R$ can be joined by a (c, e)-uniform arc.

The extreme cases $e = d|G$ and $e = \lambda_G$ are the most interesting. The intermediate case $e = \varrho_G$, defined by (1.11), appears in the literature but usually together with a modified cigar condition (1), where cig_l is replaced by cig_d or by the distance cigar defined by the function $\zeta(x, \gamma) = |x - a| \wedge |x - b|$. If $\dim E < \infty$, these variations yield the same class of domains as $e = \lambda$, but this is not true in arbitrary Banach spaces. These questions will be considered in Section 3.

The following result follows immediately from the definitions:

2.17. **Theorem.** *A c-uniform domain is c-quasiconvex and inner c-uniform. Conversely, if G is c_0-quasiconvex and inner c-uniform, then G is cc_0-uniform.* □

2.18. **Examples.** 1. Recall that G is a *c-John domain* if each pair $a, b \in G$ can be joined by an arc γ satisfying the cigar condition (1) of 2.16. Hence an inner c-uniform domain is c-John. If C is any compact subset of the line segment $[0, e_1] \subset \mathsf{R}^2$, then $B^2 \setminus C$ is c-John with a universal c, but it need not be inner uniform. This example is due to J. Heinonen.

2. On the other hand, every simply connected c-John domain in R^2 is inner c_1-uniform with $c_1 = c_1(c)$; cf. [BV1, p. 43].

3. More examples of inner uniform domains can be constructed by locally bilipschitz maps; see Theorem 2.21.

4. Let F be the ray $\{te_1 : t \geq 0\} \subset \mathsf{R}^2$. The domain $\mathsf{R}^2 \setminus F$ is inner uniform but not uniform. Let D be the parallel strip $\{(x, y) : |y| < 1\}$. Then $D \setminus F$ is not inner uniform but it is R-boundedly inner c_R-uniform with some constant c_R for each $R > 0$.

5. It is well known that bounded convex domains are uniform. We shall later make use of the following explicit version of this result.

2.19. **Theorem.** *Suppose that G is a convex domain and that $B(x_0, R_1) \subset G \subset B(x_0, R_2)$. Then G is c-uniform with $c = 2R_2/R_1$.*

Proof. We first show that the function $\delta : G \to \mathsf{R}$ is concave, that is,

$$\delta((1-t)a + tb) \geq (1-t)\delta(a) + t\delta(b) \tag{2.20}$$

for all $a, b \in G$ and $0 \leq t \leq 1$. Write $x = (1-t)a + tb,\ R = (1-t)\delta(a) + t\delta(b)$. Let $w \in E$ with $|w| < 1$. Then the points $a' = a + \delta(a)w$ and $b' = b + \delta(b)w$ lie in G. By convexity, $z = (1-t)a' + tb' \in G$. Since $z = x + Rw$, we obtain $x + B(R) \subset G$, and (2.20) follows.

We may assume that $x_0 = 0$. Let $a, b \in G,\ a \neq b$. Set $r = |a - b|/2$ and $z = (a + b)/2$. We consider two cases.

Case 1. $|z| \geq r$. Set $y = z - rz/|z|$. We show that the arc $\gamma = [a, y] \cup [y, b]$ has the uniformity properties. Since

$$l(\gamma) = |a - y| + |y - b| \leq |a - z| + |z - y| + |y - z| + |z - b| = 4r = 2|a - b|,$$

the turning condition holds with the constant 2.

To prove the cigar condition it suffices to show that $|x - a| \leq c\delta(x)$ for $x \in [a, y]$. Write $x = (1-t)a + ty,\ 0 \leq t \leq 1$. By the concavity of δ we get

$$\begin{aligned}\delta(x) &\geq (1-t)\delta(a) + t\delta(y)\\ &\geq t\delta(y) \geq t(1 - r/|z|)\delta(z) + tr\delta(0)/|z| \geq trR_1/R_2 = 2tr/c.\end{aligned}$$

Hence

$$|x - a| = t|a - y| \leq 2tr \leq c\delta(x).$$

Case 2. $|z| \leq r$. Now we set $\gamma = [a, 0] \cup [0, b]$. Since $|a| \leq |a - z| + |z| \leq 2r$ and $|b| \leq 2r$, we again get $l(\gamma) \leq 2|a - b|$. If $x = (1-t)a \in [a, 0]$, then

$$\delta(x) \geq (1-t)\delta(a) + t\delta(0) \geq tR_1 = 2R_2t/c,$$

and we again obtain

$$|x - a| = t|a| \leq 2rt \leq 2R_2t \leq c\delta(x). \quad \square$$

2.21. **Theorem.** *Suppose that $f : G \to G'$ is a homeomorphism between domains $G \subset E$ and $G' \subset E'$, and that f is locally M-bilipschitz. If G is inner c-uniform, then G' is inner M^2c-uniform.*

Proof. For each rectifiable arc $\alpha \subset G$, we have $l(\alpha)/M \le l(f\alpha) \le Ml(\alpha)$. Since f and f^{-1} are M-Lipschitz on every line segment in the domains and since the spaces are complete, we have $d(x, \partial G)/M \le d(fx, \partial G') \le Md(x, \partial G)$ for all $x \in G$; cf. [Vä5, 4.8]. The theorem follows now easily. □

2.22. **Comparison of the QH and the cigar approach.** We want to compare the concepts defined in 2.16 with the QH uniformity properties introduced in 1.1 and in 2.10. It is well known that the properties "c-uniform" and "QH c-uniform" are quantitatively equivalent for a domain G. This was proved by F.W. Gehring and B.G. Osgood in [GO]; a free version appears in [Vä6, 6.16]. Moreover, these properties are quantitatively equivalent to QH ψ-uniformity with a slow ψ.

We want to prove relative and bounded e-versions of these results. In 2.23 we show that if G is a domain and if $\gamma \subset G$ is a compact (c, e)-uniform arc, then the endpoints a and b of γ satisfy the QH uniformity condition

$$k(a, b) \le 7c^3 j_G(a, b; e).$$

Hence each (c, e)-uniformity property in the cigar sense implies the corresponding QH (c, e)-uniformity property with $c \mapsto 7c^3$.

In the converse direction, the situation is more complicated. We show in 2.31 that if $\gamma \subset G$ is a c-neargeodesic with endpoints a, b, and if $k(a, b) \le C$, then γ satisfies the uniformity conditions with a constant $c_1 = c_1(c, C)$. From an example in 2.28 we see that c cannot be chosen to be independent of C. On the other hand, it is well known ([GO],[Vä6]) that if G is QH c-uniform, then each compact c_0-neargeodesic satisfies the c_1-uniformity conditions with $c_1 = c_1(c, c_0)$. We show that the corresponding result holds for (c, e)-uniformity rel A, assuming that A satisfies a QH quasiconvexity condition. Moreover, an R-bounded version is true if R is allowed to change.

2.23. **Theorem.** *Suppose that γ is a (c, e)-uniform arc in G with endpoints a, b. Then*

$$k(a, b) \le l_k(\gamma) \le c_1 j_G(a, b; e) \le c_1 k(a, b) \tag{2.24}$$

with $c_1 = c_1(c) \le 7c^3$.

Proof. The first inequality is trivial, and the last one follows from 2.5. It remains to prove the middle inequality. Write $\delta = \delta(a) \wedge \delta(b)$ and $r = r_G(a, b; e) = e(a, b)/\delta$. We consider two cases.

Case 1. $r \le 1/2c$. For every $x \in \gamma$ we have

$$|x - a| \le l(\gamma) \le ce(a, b) = c\delta r \le \delta(a)/2,$$

and hence $\delta(x) \ge \delta(a)/2$. Consequently,

$$l_k(\gamma) \le 2l(\gamma)/\delta(a) \le 2ce(a, b)/\delta(a) \le 2cr.$$

Moreover, since $r \le 1/2c \le 1/2 < 1$, we obtain $r \log 2 \le \log(1 + r) = j_G(a, b; e)$, and hence (2.24) holds with $c_1 = 2c/\log 2 < 3c$.

Case 2. $r > 1/2c$. Write $L = l(\gamma)$, and let $x_0 \in \gamma$ be the point with $l(\gamma[a, x_0]) = L/2$. Since $L \ge \lambda(a, b) \ge e(a, b) = \delta r > \delta/2c$, there is $a_1 \in \gamma[a, x_0]$ with $l(\gamma[a, a_1]) = \delta/4c$. Since $\gamma[a, a_1] \subset \bar{B}(a, \delta(a)/4)$, we can argue as in Case 1 to obtain

$$l_k(\gamma[a, a_1]) \le \frac{2l(\gamma[a, a_1])}{\delta(a)} = \frac{\delta}{2c\delta(a)} \le \frac{1}{2c} \le \frac{1}{2}.$$

For the arc $\beta = \gamma[a_1, x_0]$, the cigar condition implies that

$$l_k(\beta) \le c \int_\beta \frac{|dx|}{l(\gamma[a, x])} = c \int_{\delta/4c}^{L/2} \frac{ds}{s} = c \log \frac{2cL}{\delta}.$$

Considering similarly the second half $\gamma[x_0, b]$ of γ we obtain the estimate $l_k(\gamma) \le 1 + 2c \log(2cL/\delta)$. By the turning condition and by the inequality $\log(Mr) \le M \log(1 + r)$, valid for $M \ge 1$, this yields

$$l_k(\gamma) \le 1 + 2c \log(2c^2 r) \le 1 + 4c^3 j_G(a, b; e).$$

Since $r > 1/2c$, we have

$$j_G(a, b; e) > \log(1 + 1/2c) > 1/3c.$$

These estimates give (2.24) with $c_1 = 3c + 4c^3 \le 7c^3$. □

2.25. **Theorem.** *Suppose that $\gamma \subset G$ is a rectifiable compact arc, that $\mathrm{cig}_l(\gamma, c) \subset G$ and that γ is c-quasiconvex in the inner metric λ_G. Then γ is a c_1-neargeodesic in G with $c_1 = c_1(c) \le 7c^3$.*

Proof. Since each subarc of γ is c-uniform in G, the theorem follows from 2.23. □

2.26. *Remark.* It follows from 2.25 that if $\mathrm{cig}_l(\gamma, c) \subset G$ and if γ is c-quasiconvex in the norm metric, then γ is a $7c^3$-neargeodesic in G.

2.27. **Theorem.** *If G is (c, e)-uniform rel A, then G is* QH *$(7c^3, e)$-uniform rel A. If G is R-boundedly c-uniform, then G is R-boundedly* QH *$(7c^3, e)$-uniform.*

Proof. This is a corollary of 2.23. □

2.28. **Example.** It is natural to ask whether the inequality $k(a, b) \le cj_G(a, b)$ for a pair of points a, b implies that these points can be joined by a c_1-uniform arc in G with $c_1 = c_1(c)$. The following example shows that the answer is negative.

Let $G \subset \mathsf{R}^2$ be the domain $\{x : x_1 < 0 \text{ or } 1 < |x_2| < 3\}$. Let $c \ge 1$, and set $t = e^{-10c},\ a = (2c, 1 + t),\ b = (2c, -1 - t)$. Then $j_G(a, b) = \log(1 + |a - b|/t) > \log(1/t) = 10c$. Let β be the broken line with vertices a, $(2c, 2)$, $(-1, 2)$, $(-1, -2)$, $(2c, -2)$, b. Then

$$k(a, b) \le l_k(\beta) \le 2\log(1/t) + 4c + 6 \le 30c < 3j_G(a, b).$$

In other words, G is QH 3-uniform rel $\{a, b\}$. However, it is easy to see that if $\gamma \subset G$ is an arc joining a and b, then γ satisfies neither of the c-uniformity conditions.

We give several results which show that with additional hypotheses, a QH uniformity condition like $k(a, b) \le cj_G(a, b)$ implies that a neargeodesic with endpoints a, b satisfies the uniformity conditions.

2.29. **Cigar theorem.** *Suppose that G is* QH *(ψ, e)-uniform rel A with a slow ψ and that $\gamma \subset A$ is a compact c-neargeodesic in G. Then γ is (c_1, e)-uniform in G with $c_1 = c_1(c, \psi)$. This is also true if G is R-boundedly* QH *(ψ, e)-uniform and $\gamma \subset G \cap (\partial G + \bar{B}(R/4M_1))$, where $M_1 = M_1(c, 0, \psi)$ is defined by* (2.11).

Proof. Let a_0 and a_1 be the endpoints of γ, and let $x_0 \in \gamma$ be a point with maximal $\delta(x_0)$. Let $q = q(c, 0, \psi)$ be the number given by the Diving lemma 2.15. If $x \in \gamma[a_0, x_0]$ and $\delta(x) \le q\delta(x_0)$, then 2.15 implies that

$$l(\gamma[a_0, x]) \le 4M_1\delta(x)/q.$$

If $x \in \gamma[a_0, x_0]$ and $\delta(x) \ge q\delta(x_0)$, then the Escape lemma 2.13 with $r = \delta(x_0)$ gives

$$l(\gamma[a_0, x]) \le 4M_1\delta(x_0) \le 4M_1\delta(x)/q.$$

Considering similarly the arc $\gamma[a_1, x_0]$ we see that $\operatorname{cig}_l(\gamma, c_1) \subset G$ with $c_1 = c_1(c, \psi) = 4M_1/q$.

To prove the turning condition we may assume that $\delta(a_0) \le \delta(a_1)$. Set $t = e(a_0, a_1)$ and $r = \delta(a_0)$. We must show that $l(\gamma) \le c_2 t$ with $c_2 = c_2(c, \psi)$. We consider two cases.

Case 1. $r \le t$. We may assume that $l(\gamma) > 2t$. Choose points $b_0, b_1 \in \gamma$ such that $l(\gamma[a_0, b_0]) = l(\gamma[a_1, b_1]) = t$. Then $e(a_0, b_0) \le \lambda(a_0, b_0) \le t$ and similarly $e(a_1, b_1) \le t$. Since $\delta(b_j) \ge t/c_1$ by the cigar condition, we obtain

$$e(b_0, b_1) \le e(b_0, a_0) + e(a_0, a_1) + e(a_1, b_1) \le 3t \le 3c_1(\delta(b_0) \wedge \delta(b_1)).$$

Since G is QH (ψ, e)-uniform rel γ, this implies that

$$k(b_0, b_1) \le \psi(r_G(b_0, b_1; e)) \le \psi(3c_1). \tag{2.30}$$

For each $x \in \gamma[b_0, b_1]$ this yields

$$k(x, b_0) \le l_k(\gamma[b_0, b_1]) \le ck(b_0, b_1) \le c\psi(3c_1).$$

By [Vä5, 2.2(1)] or by 2.5 we further obtain $|x - b_0| \le \delta(b_0)(e^{c\psi(3c_1)} - 1)$. Since $\delta(b_0) \le \delta(a_0) + |a_0 - b_0| \le r + t \le 2t$, this gives

$$\delta(x) \le \delta(b_0) + |x - b_0| \le \delta(b_0)e^{c\psi(3c_1)} \le c_3 t$$

with $c_3 = c_3(c, \psi)$. Hence

$$l(\gamma[b_0, b_1]) \le c_3 t l_k(\gamma[b_0, b_1]) \le cc_3 t k(b_0, b_1).$$

This and (2.30) give the turning condition $l(\gamma) \le c_2 t$ with $c_2 = 2 + cc_3\psi(3c_1)$.

Case 2. $r > t$. This case is independent of the uniformity properties of G. The proof of [Vä6, 6.12] gives $l(\gamma) \le c_2|a_0 - a_1|$ with $c_2 = \max\{ue^{2c/u} - 1 : u \ge 1\}$. □

2.31. **Lemma.** *Let γ be a c-neargeodesic in G with endpoints a, b such that $k(a, b) \le C$. Then γ is c_1-uniform in G with $c_1 = c_1(c, C)$.*

Proof. Set $r = \delta(a)$ and $B = B(a, r/2)$. We consider two cases.

Case 1. $\gamma \subset B$. Since $r/2 \le \delta(x) \le 2r$ for all $x \in B$, we have by [Vä5, 2.5]

$$l(\gamma) \le 2rl_k(\gamma) \le 2rck(a, b) \le 4c|a - b|,$$

which is the turning condition. For all $x \in \gamma$, this implies the cigar condition

$$\zeta_l(x, \gamma) \le l(\gamma)/2 \le 2c|a - b| \le cr \le 2c\delta(x).$$

Case 2. $\gamma \not\subset B$. Choose a point $y \in \gamma \setminus B$ and set $t = |a - b|$. By (1.5) we have $k(a, y) \geq j_G(a, y) \geq \log \frac{3}{2} > 1/3$, and hence

$$1/3 < l_k(\gamma) \leq ck(a, b).$$

If $t \leq r/2$, then [Vä5, 2.5] gives $k(a, b) \leq 2t/r$, and hence

$$r \leq 6ct, \tag{2.32}$$

which is trivially true also in the case $t \geq r/2$.

Set $L = l(\gamma)$, and let $\gamma^0 : [0, L] \to \gamma$ be the arclength parametrization with $\gamma^0(0) = a$. For each $0 \leq s \leq L$ we have

$$\delta(\gamma^0(s)) \leq \delta(a) + |a - \gamma^0(s)| \leq r + s,$$

and hence

$$l_k(\gamma) \geq \int_0^L \frac{ds}{r + s} > \log \frac{L}{r}.$$

Since $l_k(\gamma) \leq ck(a, b) \leq cC$, this and (2.32) imply the turning condition $L \leq 6ce^{cC}t$.

For each $x \in \gamma$ we have by [Vä5, 2.2(1)]

$$\log \frac{r}{\delta(x)} \leq k(a, x) \leq l_k(\gamma) \leq ck(a, b) \leq cC.$$

Since $t \leq re^{k(a,b)} \leq re^C$, we obtain the cigar condition

$$\zeta_l(x, \gamma) \leq L/2 \leq 3ce^{cC}t \leq 3ce^{3cC}\delta(x). \quad \square$$

2.33. **Theorem.** *Suppose that $A \subset G$ and that each pair of points in A can be joined by a c_0-neargeodesic $\gamma \subset A$ of G. Let $d|G \leq e \leq \lambda_G$. Then the following conditions are quantitatively equivalent:*

(1) *G is (c, e)-uniform rel A.*
(2) *G is* QH *(c, e)-uniform rel A.*
(3) *$k(x, y) \leq cj_G(x, y; e) + c'$ for all $x, y \in A$.*
(4) *G is* QH *(ψ, e)-uniform rel A with a slow ψ.*

Proof. By quantitativeness we mean that the quantities of each condition depend only on c_0 and on the quantities of another condition.

The implications (3) $\Leftarrow$ (2) $\Rightarrow$ (4) are trivial, (1) $\Rightarrow$ (2) holds for all $A \subset G$ by 2.23, the proof for (3) $\Rightarrow$ (2) is given in [Vä6, 6.15], and (4) $\Rightarrow$ (1) follows from 2.29. $\square$

2.34. *Remarks.* 1. The case $A = G$, $e = d|G$ of 2.33 was given in [Vä6, 6.16]. The case $A = G$, $e = \lambda$ gives four quantitatively equivalent conditions for the inner c-uniformity of G.

A different new characterization of the classical case $e = d|G$, based on subinvariance of neargeodesics, will be given in 2.42.

2. It follows from Theorem 2.33 that one can replace the condition "QH inner ψ-uniform" by "inner c-uniform" in the case $A = G$ of the Escape lemma 2.13 and of the Diving lemma 2.15. Similarly, one can replace the condition "QH (ψ, e)-uniform" in the case $A = G$ of the Cigar theorem 2.29 by "(c, e)-uniform". In the rest of the paper, we often apply these modifications without further notice.

We next turn to the bounded version of 2.33.

2.35. **Lemma.** *Suppose that G is R-boundedly* QH *(ψ, e)-uniform with a slow ψ. Suppose also that γ is a c-neargeodesic in G with endpoints a_0, a_1, and that $e(a_0, a_1) \le R/25M_1$, where $M_1 = M_1(c, 0, \psi)$ is given by* (2.11). *Then γ is a (c_1, e)-uniform arc in G with $c_1 = c_1(c, \psi)$.*

Proof. Set $R_1 = R/25M_1$ and $t = e(a_0, a_1)$. We may assume that $\delta(a_0) \le \delta(a_1)$. If $\delta(a_1) \ge 2R_1$, then $|a_0 - a_1| \le e(a_0, a_1) \le \delta(a_1)/2$. This implies that $k(a_0, a_1) \le 1$, and the assertion follows from 2.31. Thus we may assume that $\delta(a_1) \le 2R_1$. Similarly, we may assume that $t \ge \delta(a_1)/2$, since otherwise we again get $|a_0 - a_1| \le \delta(a_1)/2$.

Orient γ so that a_0 is the first point. If $\gamma \subset \partial G + \bar{B}(3R_1)$, the assertion follows from the Cigar theorem 2.29, since $3R_1 < R/4M_1$. Hence we may assume that $\delta(x) > 3R_1$ for some $x \in \gamma$. Let x_0 and x_1 be the first and the last point of γ with $\delta(x_0) = \delta(x_1) = 3R_1$. Set $\gamma_j = \gamma[a_j, x_j]$ for $j = 0, 1$. We first show that there is $c_0 = c_0(c, \psi) \ge 1$ such that

$$l(\gamma[a_j, x]) \le c_0 \delta(x) \tag{2.36}$$

for all $x \in \gamma_j$, $j = 0, 1$.

Let $q = q(c, 0, \psi) \in (0, 1)$ be the number given by the Diving lemma 2.15. If $\delta(x) \le q\delta(x_0) = 3qR_1$, then (2.36) follows from 2.15 with $c_0 = 4M_1/q$. Assume that $\delta(x) \ge 3qR_1$. Since the Escape lemma 2.13 gives

$$l(\gamma_j) \le 12M_1R_1, \tag{2.37}$$

(2.36) again holds with $c_0 = 4M_1/q$.

For $j = 0, 1$, let $b_j \in \gamma$ be the point with $l(\gamma[a_j, b_j]) = t$. Since $t \le R_1$, we have $b_j \in \gamma_j$. Since (2.36) gives $\delta(b_j) \ge t/c_0$ and since

$$e(b_0, b_1) \le \lambda(b_0, a_0) + e(a_0, a_1) + \lambda(a_1, b_0) \le 3t < R,$$

we get $r_G(b_0, b_1; e) \le 3c_0$ and $k(b_0, b_1) \le \psi(3c_0)$. Let $c_2 = c_1(c, C_1)$ be the constant of Lemma 2.31 with $C_1 = \psi(3c_0)$. Then the arc $\beta = \gamma[b_0, b_1]$ is c_2-uniform in G. Hence $l(\beta) \le c_2|b_0 - b_1| \le c_2 e(b_0, b_1) \le 3c_2 t$, and we obtain the turning condition

$$l(\gamma) \le (2 + 3c_2)t. \tag{2.38}$$

To prove the cigar condition, let $x \in \gamma$. If $x \in \gamma_0 \cup \gamma_1$, then (2.36) implies the cigar condition $\zeta_l(x, \gamma) \le c_0\delta(x)$. Assume that $x \in \gamma_2 = \gamma[x_0, x_1]$. By (2.37) we have

$$e(x_0, x_1) \le \lambda(x_0, a_0) + e(a_0, a_1) + \lambda(a_1, x_1) \le 24M_1R_1 + R_1 \le 25M_1R_1 = R.$$

Hence $r_G(x_0, x_1; e) < 9M_1$, which implies that $k(x_0, x_1) < \psi(9M_1)$. By 2.31, γ_2 is c_3-uniform in G with $c_3 = c_1(c, \psi(9M_1))$. We show that

$$c_3\delta(x) \ge R_1$$

for all $x \in \gamma_2$. If $|x - x_0| \wedge |x - x_1| \ge R_1$, this follows from the cigar condition for γ_2. If $|x - x_j| \le R_1$, for $j = 0$ or $j = 1$, then $c_3\delta(x) \ge \delta(x) \ge \delta(x_j) - R_1 = 2R_1 > R_1$. In view of (2.38) we get

$$\zeta_l(x, \gamma) \le l(\gamma)/2 < (1 + 2c_2)R_1 \le (1 + 2c_2)c_3\delta(x). \quad \square$$

2.39. **Theorem.** *Suppose that G is R-boundedly* QH *(ψ, e)-uniform with a slow ψ. Then there are $M_2 = M_2(\psi) \ge 1$ and $c = c(\psi) \ge 2$ such that G is R_1-boundedly (c, e)-uniform with $R_1 = R/M_2$.*

Proof. Since each pair of points in G can be joined by a 2-neargeodesic, this follows from 2.35 with $M_2 = 25M_1(2, 0, \psi)$ and $c = c_1(2, \psi)$. □

2.40. **Theorem.** *For a domain G and for a metric e, the following conditions are quantitatively equivalent:*

(1) *G is R-boundedly (c, e)-uniform.*
(2) *G is R-boundedly* QH *(c, e)-uniform.*
(3) *G is R-boundedly* QH *(ψ, e)-uniform with a slow function ψ.*

Proof. Trivially (2) implies (3) with the same R and with $\psi(t) = c \log(1 + t)$. By 2.39, (3) implies (1) with $R \mapsto R/M_2(\psi)$ and $c = c(\psi)$. By 2.27, (1) implies (2) with the same R and with $c \mapsto 7c^3$. □

2.41. **Enlarging the domain.** Suppose that $G \subset D$ are proper subdomains of E. A neargeodesic in G need not be a neargeodesic in D. We next show that this is true if G is a uniform domain, and that this property in fact characterizes uniform domains. Contrary to the theory hitherto, this result seems to have no inner version.

2.42. **Theorem.** *For a domain G, the following conditions are quantitatively equivalent:*

(1) *G is c-uniform.*

(2) *If $c_0 \geq 1$ and if γ is a c_0-neargeodesic in G, then γ is a c_1-neargeodesic in every domain $D \supset G$ with $c_1 = c_1(c_0)$.*

(3) *If γ is a 2-neargeodesic in G and if $y \in E \setminus G$, then γ is a c-neargeodesic in $E \setminus \{y\}$.*

Proof. Trivially (2) implies (3) with $c = c_1(2)$. It suffices to prove the quantitative implications (3) ⇒ (1) ⇒ (2).

(1) ⇒ (2): Suppose that γ is a c_0-neargeodesic in G and that $G \subset D \subsetneq E$. Each subarc β of γ is also a c_0-neargeodesic in G. By the Cigar theorem 2.29 (and by 2.34), β is c_2-uniform in G with a constant $c_2 = c_2(c, c_0)$. The turning condition implies that γ is c_2-quasiconvex in the norm metric. By 2.26, γ is a $7c_2^3$-neargeodesic in D.

(3) ⇒ (1): Let $a, b \in G$. Choose a 2-neargeodesic γ joining a and b in G. Let $y \in E \setminus G$. Since γ is a c-neargeodesic in $D = E \setminus \{y\}$ and since D is a c'-uniform domain with a universal c' by [Vä6, 6.5], γ is c_2-uniform in D with $c_2 = c_2(c)$ by 2.29. Since this holds for all $y \in E \setminus G$, the arc γ is c_2-uniform in G. □

2.43. **Subinvariance.** It is well known that uniform domains in R^n are subinvariant under quasiconformal maps. By this we mean that if $f : G \to G'$ is a K-quasiconformal map between domains in R^n, and if $D \subset G$ and G' are c-uniform, then fD is c'-uniform with $c' = c'(c, K, n)$. This follows from the corresponding result for QED domains [FHM, p. 121] and from [Vä2, 5.6]. An alternative proof is based on the fact that the uniform domains in R^n are precisely the BMO extension domains.

We do not know whether the corresponding result holds for freely quasiconformal maps in Banach spaces. However, we can use 2.42 to prove that uniform domains are subinvariant under QH maps. Recall that a homeomorphism $f : G \to G'$ is M-QH if f is M-bilipschitz in the QH metric.

2.44. **Theorem.** *Suppose that $f : G \to G'$ is M-*QH*, that the domain $G' \subset E'$ is c-uniform, and that $D \subset G$ is c-uniform. Then fD is c'-uniform with $c' = c'(c, M)$.*

Proof. Let γ be a 2-neargeodesic in fD, and let $y \in E' \setminus fD$. By 2.42, it suffices to show that γ is a c'-neargeodesic in $E' \setminus \{y\}$ with $c' = c'(c, M)$. We let $c_1, c_2, \dots$ denote constants depending only on (c, M). The restriction $f_D : D \to fD$ of f is $4M^2$-QH by [Vä5, 4.7]. Hence the arc $\alpha = f^{-1}\gamma$ is a c_1-neargeodesic in D with $c_1 = 32M^4$. By 2.42, α is a c_2-neargeodesic in $G \setminus f^{-1}\{y\}$. Hence γ is a c_3-neargeodesic in $G' \setminus \{y\}$ with $c_3 = 16M^4 c_2$. Since $G' \setminus \{y\}$ is c_4-uniform by [Vä6, 6.7], γ is a c_5-neargeodesic in $E' \setminus \{y\}$ by 2.42. □

2.45. **Coarse cigar theorem.** *Suppose that G is* QH *inner ψ-uniform rel A with a slow ψ and that $\gamma \subset A$ is a (c, h)-solid arc in G with endpoints a_0, a_1. Then*

(1) $\mathrm{cig}_d(\gamma, c_1) \subset G$,

(2) $d(\gamma) \le c_1|a_0 - a_1| \vee 2r(e^h - 1)$,

where $c_1 = c_1(c, h, \psi)$ and $r = \delta(a_0) \wedge \delta(a_1)$. This is also true if G is R-boundedly QH *inner ψ-uniform and $\gamma \subset G \cap (\partial G + \bar{B}(R/4M_1))$, where $M_1 = M_1(c, h, \psi)$ is defined by* (2.8).

Proof. The first part is again proved almost verbatim as the absolute case in [Vä6, 6.22]. Note, however, that [Vä6, 6.22] has unnecessary parentheses in (2). The second part follows from the first part and from 2.13, which gives $d(\gamma) \le R$. □

2.46. **Uniform spaces.** Suppose that X is a metric space. If $\gamma \subset X$ is a rectifiable compact arc, we can define $\mathrm{cig}_l(\gamma, c)$ as in 2.16. We say that X is a c-*uniform space* if each pair of points in X can be joined by an arc γ such that the closure of $\mathrm{cig}_l(\gamma, c)$ is complete.

If e is a metric of a domain $G \subset E$ such that $d|G \le e \le \lambda_G$ and if (G, e) is a c-uniform space, then G is a c-uniform domain in the sense of 2.16. Conversely, if G is a c-uniform domain, then (G, e) is a c'-uniform space for all $c' > c$.

3. Finite-dimensional spaces

3.1. *Summary.* In this section we mainly consider the case $\dim E = n < \infty$, with the exception of 3.12. Remember that we always assume that $n \ge 2$. By a result of F. John (see [MiS, 3.3]), there is a linear bijection $f : E \to R^n$ with $|x| \le |fx| \le \sqrt{n}|x|$ for all $x \in E$. Hence we shall assume without an essential loss of generality that $E = R^n$.

It is well known that uniform domains in R^n can be characterized in terms of diameter and distance cigars. With rather obvious modifications, the proofs extend to (c, e)-uniform domains, in particular, to inner uniform domains. We also show that in the turning condition, one can replace the inner length metric λ_G by the inner diameter metric ϱ_G. In fact, we show that the inner metrics ϱ_G and λ_G are bilipschitz equivalent in a class of domains, called airy domains, which contains all John domains. We also show that the uniform domains are precisely the domains which remain inner uniform under all Möbius maps.

For simplicity, we do not formulate relative and bounded versions of these results.

3.2. **Airy sets.** We say that a set $A \subset E$ is c-*airy*, $c \ge 1$, if for each $a \in A$ and $0 < t < d(A)/2$ there is a rectifiable arc $\gamma \subset A$ joining a to a point b such that $l(\gamma) \le t$ and $B(b, t/c) \subset A$.

John domains, and hence inner uniform domains are airy by Lemma 3.3 below. The planar domain $\{(x, y) : y \ne 0 \text{ or } 0 < x < 1\}$ is 1-airy but not a John domain.

3.3. **Lemma.** *A c-John domain is $2c$-airy.*

Proof. Let $a \in G$ and let $0 < t < d(G)/2$. Then there is $b \in G$ with $|a-b| > t$. Since G is c-John, we can join a and b by an arc γ with $\mathrm{cig}_l(\gamma, c) \subset G$. Since $l(\gamma) \geq |a-b| > t$, there is $x \in \gamma$ with $l(\gamma[a,x]) = t/2$, and then $\delta(x) \geq t/2c$. □

3.4. **Theorem.** *Suppose that $G \subset R^n$ is a c-airy domain. Then $\lambda_G \leq c' \varrho_G$ with $c' = c'(c,n)$.*

Proof. Let $a, b \in G$, and set $M = \varrho_G(a,b)$. We must find an arc $\gamma : a \curvearrowright b$ with $l(\gamma) \leq c'M$.

Choose an arc $\alpha : a \curvearrowright b$ with $d(\alpha) < 2M$ and successive points $a = x_0, \dots, x_N = b$ of α such that $[x_{j-1}, x_j] \subset G$ for all j. Setting $t = M/2$ we have $0 < t < d(G)/2$. Since G is c-airy, we can join each x_j to a point y_j by an arc α_j such that

$$l(\alpha_j) \leq t, \qquad \delta(y_j) \geq t/c.$$

Set $Y = \{y_j : 0 \leq j \leq N\}$ and $U = B(Y, t/c)$. Let $\mathcal{P}$ be the family of all components of U.

Fact 1. $\#\mathcal{P} \leq (6c)^n$.

To prove this, observe that the volume of each member of $\mathcal{P}$ is at least $\Omega_n(M/2c)^n$, where Ω_n is the volume of the unit n-ball. Since $\mathcal{P}$ is disjoint, this implies that $m(U) \geq (\#\mathcal{P})\Omega_n(M/2c)^n$. On the other hand, U is contained in the ball $B(a,R)$ with

$$R = d(\alpha) + t + t/c \leq 2M + M/2 + M/2 = 3M.$$

Hence $m(U) \leq \Omega_n(3M)^n$, and Fact 1 follows.

Fact 2. If $V \in \mathcal{P}$ and if $u, v \in V \cap Y$, then there is an arc $\beta : u \curvearrowright v$ in V such that $l(\beta) \leq c_1 M$ with a constant $c_1 = c_1(c,n)$.

For $z \in Y \cap V$ we write $B(z) = B(z, t/c)$. Since V is connected, there is a finite sequence $u = z_0, \dots, z_k = v$ in $Y \cap V$ such that $B(z_{j-1})$ meets $B(z_j)$ for all $1 \leq j \leq k$. Passing to a subsequence we may assume that $B(z_i)$ and $B(z_j)$ are disjoint whenever $|i-j| \geq 2$. Then the balls $B(z_j)$ with even j are disjoint. Since the number of these balls is at least $k/2$, we get

$$(k/2)\Omega_n(t/c)^n \leq m(V) \leq m(U) \leq \Omega_n(3M)^n,$$

and hence $k \leq 2(6c)^n$. Now the union of the segments $[z_{j-1}, z_j]$ is an arc $\beta : u \curvearrowright v$ with

$$l(\beta) \leq 2tk/c = kM/c \leq c_1 M,$$

where $c_1 = 2 \cdot 6^n c^{n-1}$, and Fact 2 is proved.

We define integers $-1 = j_0 < j_1 < \cdots < j_s = N$ and distinct members $V_1, \dots, V_s$ of $\mathcal{P}$ as follows. Let V_1 be the member containing y_0, and let j_1 be the largest number j such that $y_j \in V_1$. If $j_1 = N$, then $s = 1$ and the process stops. Otherwise we let $V_2 \in \mathcal{P}$ be the set containing y_{j_1+1}. We set $j_2 = \max\{j : y_j \in V_2\}$ and continue in this manner until we obtain $j_s = N$.

For each $1 \leq i \leq s$ we apply Fact 2 to find an arc $\beta_i \subset V_i$ joining the points $y_{j_{i-1}+1}$ and y_{j_i}. Let γ_0 be the union of these arcs, the line segments $[x_{j_i}, x_{j_i+1}]$, $1 \leq i \leq s-1$, and the arcs α_r for $r = 0, j_1, j_1+1, j_2, j_2+1, \dots, j_s = N$. Then γ_0 is connected and contains a and b. Since $s \leq (6c)^n$ by Fact 1, the total length of all these arcs is at most

$$sc_1 M + sd(\alpha) + 2st \leq (6c)^n(3 + c_1)M = c'M.$$

Hence γ_0 contains an arc $\gamma : a \curvearrowright b$ with $l(\gamma) \le c'M$. □

3.5. **Theorem.** *Suppose that $G \subset R^n$ is an inner c-uniform domain. Then there is $c_1 = c_1(c, n)$ such that G is (c_1, e)-uniform for all metrics e with $\varrho_G \le e \le \lambda_G$.*

Proof. As a c-John domain, G is $2c$-airy by 3.3. Hence $\lambda_G \le c'\varrho_G$, where $c' = c'(2c, n)$ is given by 3.4. Consequently, the theorem holds with $c_1 = c'$. □

3.6. **Example.** We show that 3.4 does not hold in the Hilbert space $E = l_2$. Let $e_1, e_2, \ldots$ be the orthonormal basis of E, and let Z be the union of all segments $[e_j, e_{j+1}]$, $j \in \mathsf{N}$. Then the neighborhood $G = B(Z, 1/10)$ is airy, and $\varrho_G(G) < \infty$, $\lambda_G(G) = \infty$.

A more convincing example is given by the union D of this G and all cones $V_j = \{x : d(x, L_j) < |x - e_j|/2\}$, $j \in \mathsf{N}$, where $L_j = \{te_j : t \ge 1\}$. Now D is airy and unbounded, $\varrho_D(e_1, e_j) = \sqrt{2}$, and $\lambda_D(e_1, e_j) \to \infty$ as $j \to \infty$.

3.7. **Lemma.** *A c-John domain $G \subset E$ satisfies the condition $(2c + 1)$-LLC$_2$: If $x_0 \in G$ and $r > 0$, then each pair of points in $G \setminus B(x_0, (2c + 1)r)$ can be joined by an arc in $G \setminus B(x_0, r)$.*

Proof. Let $a, b \in G \setminus B(x_0, (2c+1)r)$ and choose an arc $\alpha : a \curvearrowright b$ with $\operatorname{cig}_l(\alpha, c) \subset G$. If there is $x \in \alpha \cap B(x_0, r)$, then $\zeta_l(x, \alpha) > 2cr$, and hence $\delta(x) > 2r$. This implies that $\delta(x_0) > r$, and we obtain the desired arc by replacing a subarc of α by an arc in $S(x_0, r)$. □

3.8. **Bounded turning.** A domain G is of *c-bounded turning* if each pair $a, b \in G$ can be joined by an arc $\gamma \subset G$ such that $d(\gamma) \le c|a - b|$. Then $\varrho_G \le cd|G$.

3.9. **Theorem.** *For a domain $G \subset R^n$, the following conditions are n-quantitatively equivalent:*

(1) *G is c-uniform.*

(2) *G is inner c-uniform and of c-bounded turning.*

(3) *If g is a Möbius map of $R^n \cup \{\infty\}$ with $gG \subset R^n$, then gG is inner c-uniform.*

Proof. By n-quantitativeness we mean that the constants c in the conditions depend only on each other and on n.

(1) ⇒ (3) follows from the fact that a Möbius image of a c-uniform domain is c_1-uniform with $c_1 = c_1(c)$ (see [Vä6, 6.24]).

(2) ⇒ (1): By 3.5, G is (c_1, ϱ_G)-uniform with $c_1 = c_1(c, n)$. Since $\varrho_G \le cd|G$, G is cc_1-uniform.

(3) ⇒ (2): Trivially, G is inner c-uniform. To prove the bounded turning let $a, b \in G$. We may assume that there is $x_0 \in [a, b] \setminus G$. Set $r = |a - x_0| \vee |b - x_0|$ and let g be the inversion in the sphere $S(x_0, r)$. Since gG is inner c-uniform and hence c-John, it follows from 3.7 that there is an arc $\alpha : ga \curvearrowright gb$ in $gG \setminus B(x_0, r/c_1)$ where $c_1 = 2c + 1$. Then $g\alpha : a \curvearrowright b$ in $G \cap \bar{B}(x_0, c_1 r)$, and $d(g\alpha) \le 2c_1 r \le 2c_1|a - b|$. Hence G is of $2c_1$-bounded turning. □

3.10. **Diameter and distance cigars.** Let $\gamma \subset E$ be an arc with endpoints a and b. For $c \ge 1$, the length cigar $\operatorname{cig}_l(\gamma, c)$ and the diameter cigar $\operatorname{cig}_d(\gamma, c)$ were defined in 2.16; the former is only defined if γ is rectifiable. In addition, we consider the *distance cigar*

$$\operatorname{cig}_{\mathrm{dist}}(\gamma, c) = \bigcup \{B(x, (|x - a| \wedge |x - b|)/c) : x \in \gamma\}.$$

We always have

$$\mathrm{cig}_{\mathrm{dist}}(\gamma, c) \subset \mathrm{cig}_d\,(\gamma, c) \subset \mathrm{cig}_l\,(\gamma, c).$$

It is well known that uniform domains in R^n can be characterized in terms of diameter cigars [Ma, 4.5] or distance cigars [Vä3, 2.10]. The turning condition $l(\gamma) \le c|a-b|$ is then replaced by the inequality $d(\gamma) \le c|a-b|$. In arbitrary Banach spaces, diameter and distance cigars still give the same class of domains, but this is strictly larger than the class of uniform domains. For example, the domain $G \subset l_2$ of Example 3.6 is not uniform although it satisfies the corresponding condition with diameter cigars.

We next give the corresponding results for inner uniformity, using the turning condition $d(\gamma) \le c\varrho_G(a, b)$.

3.11. **Theorem.** *For a domain $G \subset R^n$, the following conditions are n-quantitatively equivalent:*

(1) *G is inner c-uniform.*

(2) *Each pair of points $a, b \in G$ can be joined by an arc γ such that* $\mathrm{cig}_d\,(\gamma, c) \subset G$ *and $d(\gamma) \le c\varrho_G(a, b)$.*

Proof. Suppose that (1) is true and that $a, b \in G$. By 3.4 there is $c_1 = c_1(c, n)$ such that $\lambda_G \le c_1\varrho_G$. Let $\gamma : a \curvearrowright b$ be an arc satisfying the c-uniformity conditions in G. Since $\mathrm{cig}_d\,(\gamma, c) \subset \mathrm{cig}_l\,(\gamma, c)$ and since

$$d(\gamma) \le l(\gamma) \le c\lambda_G(a, b) \le cc_1\varrho_G(a, b),$$

(2) holds with the constant cc_1.

Conversely, assume that (2) is true and that $a, b \in G$. Choose an arc $\gamma : a \curvearrowright b$ satisfying (2). Using a straightening technique of Martio and Sarvas [MaS, 2.7] we can replace γ by an inscribed polygonal arc α such that $\mathrm{cig}_l\,(\alpha, c_2) \subset \mathrm{cig}_d\,(\gamma, c)$ and $l(\alpha) \le c_2 d(\gamma)$ with some constant $c_2 = c_2(c, n)$. Now $l(\alpha) \le c_2 d(\gamma) \le cc_2\varrho_G(a, b) \le cc_2\lambda_G(a, b)$, and hence (1) holds with the constant cc_2. □

3.12. **Theorem.** *For a domain $G \subset E$, the following conditions are quantitatively equivalent:*

(1) *Each pair of points $a, b \in G$ can be joined by an arc γ such that* $\mathrm{cig}_d\,(\gamma, c) \subset G$ *and $d(\gamma) \le c\varrho(a, b)$.*

(2) *Each pair of points $a, b \in G$ can be joined by an arc γ such that* $\mathrm{cig}_{\mathrm{dist}}(\gamma, c) \subset G$ *and $d(\gamma) \le c\varrho(a, b)$.*

Proof. Observe that contrary to the other results of this section, the theorem holds in all Banach spaces E.

Trivially (1) implies (2) with the same c. Conversely, assume that (2) is true. Since the argument is well known (see [Vä3, 2.18]), we omit some details. Let $a, b \in G$, and set $2r = \delta(a) \wedge \delta(b)$, $M = \varrho_G(a, b)$. Fix points $a_1 \in S(a, r)$, $b_1 \in S(b, r)$. If $M \le 2r$, we can choose $\gamma = [a, b]$. If $2r \le M \le 12r$, we choose an arc $\beta : a_1 \curvearrowright b_1$ satisfying (2). Then $\gamma = [a, a_1] \cup \beta \cup [b, b_1]$ satisfies in the obvious sense the path version of (1) with the constant $72c$.

We can thus assume that $M \ge 12r$. Since $d(G) > \varrho_G(a, b) = M$, there is $z \in G$ with $|z - a| \wedge |z - b| > M/4$. Let $N \ge 2$ be the unique integer with $3 \cdot 2^N r \le M < 3 \cdot 2^{N+1} r$. Set $a_0 = a$, $A_1 = [a_0, a_1]$, and choose inductively points a_j and arcs $A_j : a_{j-1} \curvearrowright a_j$ for $1 \le j \le N$ as follows: Assuming that $a_i \in S(a, 2^{i-1}r)$ and $A_i \subset \bar{B}(a, 2^{i-1}r)$ have been defined for $i \le j$, we choose an arc $\alpha_j : a_j \curvearrowright z$ satisfying (2) and let a_{j+1} be the first point of α_j in $S(a, 2^j r)$; then

$A_{j+1} = \alpha_j[a_j, a_{j+1}]$. A similar construction gives points $b_j \in S(b, 2^{j-1}r)$ and arcs $B_j : b_{j-1} \curvearrowright b_j,\ 1 \le j \le N$. Choosing an arc $\beta : a_N \curvearrowright b_N$ satisfying (2) we get in the obvious way a path γ joining a and b in $A_1 \cup \cdots \cup A_N \cup \beta \cup B_N \cup \cdots \cup B_1$. This satisfies the path version of (1) with the constant $96c^3$. □

3.13. *Remark.* Balogh and Volberg [BV1] say that a domain $G \subset R^2$ is *uniformly John* if there is c such that condition (2) of 3.12 holds. By 3.11 and 3.12 we see that for domains $G \subset R^n$, this is n-quantitatively equivalent to inner c-uniformity.

3.14. *Remark.* Jones [Jo2] says that a domain $G \subset R^n$ is an (ε, δ) *domain* with $\varepsilon > 0,\ \delta > 0$ if each pair $a, b \in G$ with $|a - b| < \delta$ can be joined by a rectifiable arc γ such that $l(\gamma) \le |a - b|/\varepsilon$ and

$$\zeta_J(x) = \frac{|x-a||x-b|}{|a-b|} \le d(x, \partial G)/\varepsilon$$

for all $x \in \gamma$. We show that this property is n-quantitatively equivalent to R-bounded c-uniformity. Setting $\zeta(x) = |x - a| \wedge |x - b|$ we have

$$1/\zeta_J(x) \le 1/|x-b| + 1/|x-a| \le 2/\zeta(x). \tag{3.15}$$

In the other direction, assuming the turning condition $l(\gamma) \le c|a - b|$ we get

$$|x-a| \vee |x-b| \le l(\gamma) \le c|a-b|,$$

and hence

$$\zeta_J(x) \le c\zeta(x) \le c\zeta_l(x).$$

From this it follows that an R-boundedly c-uniform domain is an (ε, δ) domain with $\delta = R$, $\varepsilon = c^{-2}$. Conversely, an (ε, δ) domain is R-boundedly c-uniform with $R = \delta/2,\ c = c(\varepsilon, n)$. To see this, let $a, b \in G$ with $|a - b| \le R$, and let $\gamma : a \curvearrowright b$ be an arc satisfying the (ε, δ) condition. Then $\mathrm{cig}_{\mathrm{dist}}(\gamma, 2/\varepsilon) \subset G$ by (3.15). A variation of the proof of 3.9 gives an arc $\beta : a \curvearrowright b$ such that $\mathrm{cig}_d(\beta, c_1) \subset G$ and $d(\beta) \le c_1|a - b|$ with some $c_1 = c_1(\varepsilon)$; the proof only makes use of pairs with distance less than $2R = \delta$. Finally, the straightening technique of [MaS, 2.7] again gives the desired c-uniform arc $\alpha : a \curvearrowright b$ with $c = c(\varepsilon, n)$.

4. Boundary extension

4.1. *Summary.* We study the question: When does a coarsely quasihyperbolic map $f : G \to G'$ have a limit at a given boundary point? This question has been rather extensively studied for quasiconformal maps in R^n. In the free quasiworld, it is known that CQH maps between uniform domains have homeomorphic extensions to the closures. For nonuniform domains, there are hardly any earlier results.

4.2. **Definitions.** We first recall the basic terminology for maps in the free quasiworld. Let E and E' be Banach spaces, and let $G \subset E$ and $G' \subset E'$ be domains with QH metrics $k = k_G$ and $k' = k_{G'}$. Let $f : G \to G'$ be a homeomorphism. If $M \ge 1,\ C \ge 0$, and

$$(k(x, y) - C)/M \le k'(fx, fy) \le Mk(x, y) + C,$$

then f is said to be C-*coarsely* M-*quasihyperbolic* or briefly (M, C)-CQH. The map is M-*quasihyperbolic* if M-QH and if this holds with $C = 0$. In other words, f is M-bilipschitz in the QH metric. If $\varphi : [0, \infty) \to [0, \infty)$ is a homeomorphism and if

$$\varphi^{-1}(k(x, y)) \le k'(fx, fy) \le \varphi(k(x, y))$$

for all $x, y \in G$, then f is *φ-solid*. If f defines a φ-solid map $f_D : D \to fD$ for each proper subdomain $D \subsetneq G$, then f is *freely φ-quasiconformal*. This notion makes sense also in the case $G = E,\ G' = E'$. Between these classes, we have the following quantitative implications:

$$M\text{-QH} \ \Rightarrow\ \text{freely } \varphi\text{-quasiconformal} \ \Rightarrow\ \varphi\text{-solid} \ \Rightarrow\ (M, C)\text{-CQH};$$

see [Vä6, 4.14]. Alternatively, the freely quasiconformal maps can be characterized by the property that for some (M, C), each $f_D : D \to fD$ is (M, C)-CQH [Vä7, 2.21]. In the case $E = E' = \mathsf{R}^n$, the freely quasiconformal maps are just the quasiconformal maps in the ordinary sense.

We shall consider the boundary behavior of CQH maps. Hence all results hold for the other three classes as well, and for quasiconformal maps of domains in R^n.

The closure $\bar{A}$ and the boundary ∂A of a set $A \subset E$ is always taken in the extended space $\dot{E} = E \cup \{\infty\}$, where the neighborhoods of ∞ are the complements of closed bounded sets. In particular, $\infty \in \partial A$ if and only if A is unbounded.

Let $c \ge 1$ and $h \ge 0$. A domain G is said to be *locally (c, h)-solidly connected*, or (c, h)-LSC, at a point $b \in \partial G$ if each neighborhood U of b contains a neighborhood V of b such that every pair of points in $V \cap G$ can be joined by an arc $\gamma \subset U \cap G$ such that γ is (c, h)-solid in G.

For example, a Jordan domain $G \subset \mathsf{R}^2$ is $(4, 0)$-LSC at each boundary point. This is seen by mapping G conformally onto the unit disk and using the fact that hyperbolic geodesics are 4-neargeodesics. More examples of the LSC property are given by the following result:

4.3. **Theorem.** *An R-boundedly c-uniform domain is* $(c', 0)$-LSC *at each finite boundary point for all $c' > 1$. An unbounded c-uniform domain is* $(c_0, 0)$-LSC *at ∞ with a universal constant $c_0 < 145$.*

Proof. Suppose that G is R-boundedly c-uniform. By 2.40, G is R-boundedly QH ψ-uniform with the slow function $\psi = 7c^3 \log(1 + t)$. Let $M_1 = M_1(c, 0, \psi)$ be the number given by (2.11), and let $0 < r < R/50M_1$. Suppose that $1 < c' \le 2$, that $b \in \partial G,\ b \ne \infty$, and that $x, y \in G \cap B(b, r)$. Choose a c'-neargeodesic $\gamma : x \curvearrowright y$. By 2.35, γ is c_1-uniform in G with $c_1 = c_1(c)$. The turning condition implies that $\gamma \subset B(b, (c_1 + 1)r)$. Hence G is $(c', 0)$-LSC at b.

Next assume that G is unbounded and c-uniform. We may assume that $0 \in \partial G$. Let $u : E \setminus \{0\} \to E \setminus \{0\}$ be the inversion $ux = x/|x|^2$. Since u is η-quasimöbius with $\eta(t) = 81t$ by [Vä2, 1.6], the domain uG is uniform by [Vä6, 6.26]. By the first part of the theorem, uG is $(c', 0)$-LSC at the origin for each $c' > 1$. By [Vä8, 2.9], u defines a 12-QH map $f : G \to uG$. Hence f^{-1} maps each c'-neargeodesic of uG onto a c_0-neargeodesic of G with $c_0 = 144c'$, and the theorem follows. □

4.4. *Remarks.* 1. If G is (c, h)-LSC at $b \in \partial G$, then G is locally connected at b, that is, b has arbitrarily small neighborhoods U such that $U \cap G$ is connected. In 4.6 we show that the converse is not true.

2. A parallel strip in R^2 is R-boundedly c_R-uniform for all $R > 0$, but it is not even locally connected at ∞. However, domains $G_0 = E_1 \times B_2$ are LSC at ∞, where B_2 is a ball in a space E_2 and dim $E_1 \ge 2$. The proof will be given in 5.11. For example, a domain between two parallel planes in R^3 is LSC at ∞.

3. We get more examples of the LSC property by auxiliary maps and by the following result:

4.5. **Theorem.** *Suppose that* $f : G \to G'$ *in* (M, C)-CQH *and that* $b \in \partial G$ *and* $b' \in \partial G'$ *are points such that* f *extends to a homeomorphism* $G \cup \{b\} \to G' \cup \{b'\}$. *If* G *is* (c, h)-LSC *at* b, *then* G' *is* (c', h')-LSC *at* b' *with* (c', h') *depending only on* (c, h, M, C).

Proof. Since f maps (c, h)-solid arcs onto (c', h')-solid arcs by [Vä6, 4.15], the result follows from the definition of the LSC property. □

4.6. **Example.** We construct a domain $G \subset \mathsf{R}^3$, which is locally connected at $0 \in \partial G$ but not LSC at 0. Define $g : (0, 1] \to \mathsf{R}$ by $g(t) = e^{-1/t}$, and set $A = \bar{B}^2 \setminus \{(x, y) : 0 < x \le 1,\ |y| < g(x)\}$. The domain $G = \mathsf{R}^3 \setminus A$ is locally connected at the origin.

Let $c \ge 1$ and $h \ge 0$. Then there is $r = r(c, h) > 0$ such that no arc in $B(r) \cap G$ is (c, h)-solid and joins a point in $[0, e_3]$ to a point in $[0, -e_3]$. We omit the elementary but lengthy proof. Thus G is not (c, h)-LSC at the origin.

4.7. *Remark.* Fattening slightly the set A in 4.6 we obtain an example where A is a topological 3-cell. With a Möbius map we get a Jordan domain in R^3 that is not LSC at one boundary point. Recall from 4.2 that Jordan domains in R^2 are LSC at all boundary points.

We next give the main result of this section. We let $\lambda' = \lambda_{G'}$ denote the inner metric of G'.

4.8. **Theorem.** *Suppose that* $f : G \to G'$ *is* CQH *and that* G *is* LSC *at* $b \in \partial G$. *Suppose also that* $d|G' \le e \le \lambda'$ *and that*

(1) *for each bounded set* $A \subset G'$ *there is* $c_A \ge 1$ *such that* G' *is* QH (c_A, e)-*uniform rel* A,

(2) $e(A) < \infty$ *for each bounded* $A \subset G'$.

Then f *has a limit at* b.

Condition (2) *holds automatically if* $e = d|G'$ *or if* G' *is bounded or if* $\dim E' < \infty$.

Proof. Suppose that f is (M, C)-CQH and that G is (c, h)-LSC at b. Assume that f does not have a limit at b.

Suppose first that $b \ne \infty$. For $r > 0$ we write $U(r) = B(b, r)$. Fix a point $x_0 \in G$. Since ∞ is not a limit of f at b, there is $R > 0$ such that $fU(r)$ meets $B(fx_0, R)$ for all $r > 0$. Furthermore, there is $t > 0$ such that $d(fU(r)) > t$ for all $r > 0$, since otherwise $\lim_{x \to b} f(x)$ exists by the completeness of E'.

Let $r > 0$. Since G is (c, h)-LSC at b, there is $s \in (0, r)$ such that points of $U(s)$ can be joined by a (c, h)-solid arc of G in $U(r)$. Pick $x \in U(s)$ with $fx \in B(fx_0, R)$ and then $y \in U(s)$ with $|fx - fy| \ge t/2$. Let $\alpha \subset U(r)$ be a (c, h)-solid arc in G with endpoints x and y. By [Vä6, 4.15], the arc $f\alpha$ is (c', h')-solid in G' with (c', h') depending on (c, h, M, C).

Orient α so that x is its first point. Let z be the first point of α with $|fx - fz| = t/2$. Let $z_0 \in \alpha[x, z]$ be a point where $\delta'(fz_0) = d(fz_0, \partial G')$ is maximal. Write $r' = \delta'(fz_0)$, $\gamma = f\alpha[x, z]$, and $A = G' \cap B(fx_0, R + t/2)$. Since G' is (c_A, e)-uniform rel A and since $\gamma \subset G' \cap \bar{B}(fx, t/2) \subset A$, we can apply 2.27 and the Escape lemma 2.13 with $G \mapsto G'$, $r \mapsto r'$ to obtain an estimate $d(\gamma) \le M_2 r'$ with a constant M_2 depending on (c, h, M, C, c_A) but not on r. Hence $r' \ge d(\gamma)/M_2 \ge t/2M_2$, which yields

$$r_{G'}(fx_0, fz_0; e) \le \frac{e(A)}{\min(t/2M_2, \delta'(f(x_0)))} = c_1 < \infty,$$

by (2), where c_1 does not depend on r. Since G' is (c_A, e)-uniform rel A, and since $fx_0, fz_0 \in A$, this and 2.27 imply that $k'(fx_0, fz_0) \le c_A \log(1 + c_1) = c_2$. Since f is (M, C)-CQH, we obtain

$$k(x_0, z_0) \le Mc_2 + C.$$

On the other hand,

$$k(x_0, z_0) \ge \log \frac{\delta(x_0)}{\delta(z_0)} \ge \log \frac{\delta(x_0)}{r}.$$

As $r \to 0$, these inequalities give a contradiction.

If $b = \infty$, we can use the same argument replacing $U(r)$ by $G \setminus \bar{B}(1/r)$. For $r < 1/|x_0|$ we then have

$$k(x_0, z_0) \ge \log \frac{|x_0 - z_0|}{\delta(x_0)} \ge \log \frac{1/r - |x_0|}{\delta(x_0)} \to \infty$$

as $r \to 0$.

If $e = d|G'$, then (2) is trivially true. Suppose that G' is bounded and that $a, b \in G'$. Since G' is (c_0, e)-uniform with $c_0 = c_{G'}$ by (1), we can join a and b by an arc β such that $\operatorname{cig}_l(\beta, c_0) \subset G'$. Let $x \in \beta$ be the point with $l(\beta[a, x]) = l(\beta)/2$. Then $B(x, l(\beta)/2c_0) \subset G'$, which implies that

$$e(a, b) \le \lambda'(a, b) \le l(\beta) \le c_0 d(G').$$

Hence $e(G') \le c_0 d(G') < \infty$. If $\dim E' < \infty$, then (2) follows from (1) and from Lemma 4.9 below. □

4.9. **Lemma.** *Suppose that* $\dim E < \infty$, *that* $G \subset E$ *is a domain, that* $c \ge 1$, *and that* $A \subset G$ *is a bounded set such that each pair of points in* A *can be joined by an arc* γ *with* $\operatorname{cig}_l(\gamma, c) \subset G$. *Then* A *is bounded in the metric* λ_G.

Proof. Assume that $\lambda(A) = \infty$. Choose a ball $B(R)$ containing A and a sequence of points $x_0, x_1, \dots$ in A such that $\lambda(x_i, x_j) \ge 2R$ for $i \ne j$. For $j \ge 1$ join x_j and x_0 by an arc γ_j with $\operatorname{cig}_l(\gamma_j, c) \subset G$. Then $l(\gamma_j) \ge 2R$ for all j. Let $y_j \in \gamma_j$ be the point with $l(\gamma[x_j, y_j]) = R/4$. Then the balls $B_j = B(y_j, R/4c)$ lie in G. If B_i meets B_j, we can join x_i and x_j by an arc $\alpha \subset \gamma[x_i, y_i] \cup [y_i, y_j] \cup \gamma[y_j, x_j] \subset G$, and we obtain the contradiction

$$\lambda(x_i, x_j) \le l(\alpha) \le R/2 + R/2c \le R < 2R \le \lambda(x_i, x_j).$$

Hence the balls B_j are disjoint. Since they lie in $B(2R)$ and since $\dim E < \infty$, this leads to a contradiction, and the lemma is proved. □

4.10. **Theorem.** *Suppose that* G *and* G' *are domains in* R^n *with* G' *inner uniform, and that* $f : G \to G'$ *is* CQH, *for example, quasiconformal. If* G *is* LSC *at* $b \in \partial G$, *then* f *has a limit at* b. *If* G *is uniform, then* f *has a continuous extension to* $\bar{G}$.

Proof. This follows from 4.8 and 4.3. □

4.11. **Theorem.** *Suppose that* $f : G \to G'$ *is* CQH, *that* G *is* LSC *at* $b \in \partial G$, *and that* G' *is convex. Then* f *has a limit at* b.

Proof. Let $A \subset G'$ be bounded. Choose a ball B containing A. The domain $G' \cap B$ is uniform by 2.19, and hence QH ψ-uniform with a slow ψ. The theorem follows from 4.8 with $e = d|G'$. □

4.12. **Example.** We show that Theorem 4.8 does not hold without condition (2). Let E be a separable Hilbert space with orthonormal base $e_1, e_2, \dots$. We consider e_1 as vertical, and let G denote the lower half space $\{x : x \cdot e_1 < 0\}$. Set $T = \{x : x \cdot e_1 = 0\}$. For $k \geq 2$, let B_k be the disk $T \cap B(4ke_k, k)$. Define a 4-bilipschitz function $u : T \to \mathsf{R}$ by $u(x) = 4(k - |x - 4ke_k|)$ for $x \in B_k,\ k \geq 2$, and by $u(x) = 0$ elsewhere. Let $P : E \to T$ be the orthogonal projection. Define a homeomorphism $f_1 : G \to G_1$ by $f_1(x) = x + u(Px)e_1$. The image domain G_1 is the union of G, the disks B_k, and the conical domains V_k with base B_k and vertex $v_k = 4k(e_k + e_1)$. Since $f^{-1}(x) = x - u(Px)e_1$, we see that f_1 is 5-bilipschitz.

For each $k \geq 2$, we can easily find a homeomorphism $g_k : \bar{V}_k \to \bar{V}'_k$ such that $g_k|B_k = \text{id}$, $g_k v_k = 0$, the domains V'_k are disjoint subdomains of the upper half space, and the maps g_k are M-bilipschitz for some M independent of k. The maps $g_k|V_k$ extend by identity to a homeomorphism $g : G_1 \to G'$, where G' is the union of G and all B_k and V'_k. Since g is locally M-bilipschitz, we obtain a locally $5M$-bilipschitz homeomorphism $f = g \circ f_1 : G \to G'$. By [Vä5, 4.8], f is $25M^2$-quasihyperbolic. The half space G is uniform. By 2.21, the domain G' is inner uniform. Hence the conditions of 4.8 are satisfied with $e = \lambda'$, except for (2). Since f has no limit at ∞, we see that condition (2) cannot be omitted.

4.13. **Local uniformity.** We say that a domain G is *locally c-uniform* at a point $b \in \partial G$ if b has a neighborhood U such that G is c-uniform rel $U \cap G$, that is, each pair $x, y \in U \cap G$ can be joined by an arc γ which is c-uniform in G.

A c-uniform domain is trivially locally c-uniform at each boundary point. Conversely, if G is bounded and locally c-uniform at each boundary point, then G is a uniform domain. This was proved by P. Alestalo [Al], but the result is not needed in this paper. The domain $G_0 = E_1 \times B_2$ is locally c-uniform with a universal c at each finite boundary point but not at ∞. This difference will be used in Section 5 to prove that if dim $E_1 \geq 2$, then every CQH map $G_0 \to G_0$ must fix the point at infinity. The proof is based on 4.15 below.

4.14. **Theorem.** *Suppose that G is locally c_0-uniform at a finite boundary point b. Then there is $r_0 > 0$ such that if $0 < r \leq r_0$ and if γ is a (c, h)-solid arc in G with endpoints in $B(b, r)$, then $\gamma \subset B(b, M_2 r)$, where M_2 depends on (c, h, c_0). Moreover, G is $(c, 0)$-LSC at b for each $c > 1$.*

Proof. Choose a neighborhood U of b such that G is c_0-uniform rel $U \cap G$. Then G is QH $7c_0^3$-uniform rel $U \cap G$ by 2.27. Let $r_1 > 0$ be such that $\bar{B}(b, r_1) \subset U$, and let $M_1 = M_1(c, h, \psi)$ be the constant given by (2.11) for $\psi(t) = 7c_0^3 \log(1 + t)$. We show that the theorem holds with $r_0 = r_1/(1 + 4M_1)$ and with a suitable M_2.

Let $0 < r \leq r_0$, and let γ be a (c, h)-solid arc in G with endpoints $a_0, a_1 \in B(b, r)$. We let $c_1, c_2, \dots$ denote constants depending only on (c, h, c_0).

We may assume that $\gamma \not\subset B(b, (1 + 4M_1)r)$, since otherwise the theorem holds with $M_2 = 1 + 4M_1$. Let x_0 be the first point of γ after a_0 such that $|x_0 - b| = (1 + 4M_1)r$. If $\gamma[a_0, x_0] \subset \partial G + \bar{B}(r)$, it follows from the Escape lemma 2.13 that $d(\gamma[a_0, x_0]) \leq 4M_1 r$. This gives the contradiction

$$|b - x_0| \leq |b - a_0| + |a_0 - x_0| < r + 4M_1 r = |b - x_0|.$$

Consequently, $\gamma[a_0, x_0] \not\subset \partial G + \bar{B}(r)$. Considering similarly the other endpoint a_1, we find points $b_0, b_1 \in \gamma$ such that $\delta(b_j) = r$ and such that $\gamma[a_j, b_j] \subset \bar{B}(b, (1 + 4M_1)r)$, $j = 0, 1$. Then

$$r_G(b_0, b_1) = \frac{|b_0 - b_1|}{r} \leq 2(1 + 4M_1) = c_1.$$

Since $b_0, b_1 \in U$, this implies that $k(b_0, b_1) \leq \psi(c_1)$. Setting $\beta = \gamma[b_0, b_1]$ we thus have $l_k(\beta, h) \leq c\psi(c_1) = c_2$.

Let $y \in \beta$. If $k(y, b_0) \leq h$, then [Vä5, 2.2(1)] gives $|y - b_0| \leq \delta(b_0)e^{k(y,b_0)} \leq re^h$. If $k(y, b_0) \geq h$, then

$$l_k(\beta, h) \geq k(y, b_0) \geq \log \frac{|y - b_0|}{r}.$$

Hence $|y - b_0| \leq re^{c_2}$. Setting $c_3 = e^{h \vee c_2}$ we thus have $|y - b_0| \leq c_3 r$ in each case. Consequently,

$$|y - b| \leq |y - b_0| + |b_0 - b| < (c_3 + 1 + 4M_1)r,$$

and the first part of the theorem is proved with $M_2 = c_3 + 1 + 4M_1$. The second part is a corollary of the first part. □

4.15. **Theorem.** *Suppose that G is locally c-uniform at a finite boundary point b. Then there is no open solid arc $\gamma \subset G$ such that $\gamma \cup \{b\}$ is a Jordan curve. If G is uniform, this also holds in the case $b = \infty$.*

Proof. The first part follows from 4.14. If G is uniform, we can make use of an auxiliary inversion to reduce the case to the first part. □

5. Maps of domains $G_0 = E_1 \times B_2$

5.1. *Summary.* Throughout this section we assume that E_1 and E_2 are Banach spaces of dimensions ≥ 1, and we let B_j denote the unit ball of E_j, $j = 1, 2$. Other balls in E_j are written as $B_j(x, r)$. We set $E = E_1 \times E_2$ and $G_0 = E_1 \times B_2$. For example, G_0 may be the infinite tube $\mathsf{R} \times B^2$ or the domain $\mathsf{R}^2 \times B^1$ between two parallel planes in R^3. In R^n, the domain G_0 is $\mathsf{R}^p \times B^{n-p}$ with $1 \leq p \leq n - 1$. The author knows of no previous article dealing with the intermediate case $2 \leq p \leq n-2$.

We study CQH maps between these domains. If $G_0' = E_1' \times B_2'$ is another such domain, and if there is a CQH map $f : G_0 \to G_0'$, we show that $\dim E_1 = \dim E_1'$ and $\dim E_2 = \dim E_2'$. Hence we mainly consider self maps $f : G_0 \to G_0$. If $\dim E_1 \geq 2$, then f extends to a homeomorphism $\bar{f} : \bar{G}_0 \to \bar{G}_0$ with $\bar{f}(\infty) = \infty$. We also show that $\bar{f}$ is quasisymmetric rel ∂G_0 and estimate the horizontal and vertical distortions of f.

5.2. **Norm of E.** We say that a norm $|x|$ of $E = E_1 \times E_2$ is *admissible* if

$$|x_1| \vee |x_2| \leq |x| \leq |x_1| + |x_2| \tag{5.3}$$

for all $x_1, x_2 \in E$. Throughout this section we assume that the norm of E is admissible. If $E_1 = \mathsf{R}^p$ and $E_2 = \mathsf{R}^q$, we use the euclidean norm $|x| = (x_1^2 + x_2^2)^{1/2}$, and we identify $E = \mathsf{R}^{p+q}$. Since $|x_1| + |x_2| \leq 2(|x_1| \vee |x_2|)$, all admissible norms are bilipschitz equivalent.

Let $P_1 : E \to E_1$ and $P_2 : E \to E_2$ be the projections. From (5.3) it follows that the operator norms $|P_1|$ and $|P_2|$ are equal to 1. Moreover, the natural embeddings $E_1 \to E$ and $E_2 \to E$ are isometries. Hence we can identify E_1 and E_2 with the subspaces $E_1 \times \{0\}$ and $\{0\} \times E_2$ of E. Then we can write $x = P_1x + P_2x$ for each $x \in E$.

5.4. **Quasihyperbolic metric of G_0.** Since $\partial G_0 = (E_1 \times \partial B_2) \cup \{\infty\}$, we have $\delta(x) = d(x, \partial G_0) = 1 - |x_2| \le 1$ for each $x = (x_1, x_2) \in G_0$. Let $k = k_{G_0}$ be the QH metric of G_0. We list some obvious properties of k. Let $a \in E_1$ and $u \in \partial B_2$.

(1) If $0 \le s \le t < 1$, then

$$k(a + su, a + tu) = \log \frac{1 - s}{1 - t}.$$

In particular, $k(a, a + tu) = \log(1/(1 - t))$.

(2) The segment $[a, a + u)$ is a QH geodesic.

(3) If $a, b \in E_1$, then $k(a, b) = |a - b|$.

(4) If $x, y \in G_0$ and if $\delta(x), \delta(y) \ge r$, then $rk(x, y) \le |x - y| \le k(x, y)$.

(5) Every line $L \subset E_1$ is a QH geodesic of G_0. If $x \in G_0$, the line $L + x$ is a c-neargeodesic with $c = 1/\delta(x)$.

5.5. **Lemma.** *Suppose that $\alpha \subset E_1$ is a c-quasiconvex arc with endpoints a, a' and that $u \in \partial B_2$. Then the arc $\gamma = [a, a + u) \cup \alpha$ is a $2c$-neargeodesic of G_0.*

Proof. Let $0 < t < 1$ and set $\gamma_0 = \alpha \cup [a, a + tu]$. In view of (2) and (3) of 5.4, it suffices to show that

$$l_k(\gamma_0) \le 2ck(a', a + tu). \tag{5.6}$$

Let $\beta : a' \curvearrowright a + tu$ be a rectifiable arc in G_0. Since $\delta(x) \le 1$ in G_0, we have $l_k(\beta) \ge l(\beta) \ge |a - a'|$. Moreover,

$$l_k(\beta) = \int_\beta \frac{|dx|}{1 - |P_2x|} \ge \int_0^t \frac{dr}{1 - r} = \log \frac{1}{1 - t};$$

a rigorous proof of the inequality can be carried out by 5.3 and 5.7 of [Vä1]. Hence

$$2l_k(\beta) \ge |a - a'| + \log \frac{1}{1 - t} \ge l(\alpha)/c + \log \frac{1}{1 - t} \ge l_k(\gamma_0)/c,$$

and (5.6) follows. □

5.7. **Lemma.** *Suppose that $\alpha \subset E_1$ is a c-quasiconvex arc with endpoints a, a' and that $u, u' \in \partial B_2$. If $|a - a'| \ge 2$, then $\gamma = [a, a + u) \cup \alpha \cup [a', a' + u')$ is a $2c$-neargeodesic of G_0.*

Proof. Let $t, t' \in [0, 1)$ and set $b = a + tu,\ b' = a' + t'u'$. Write $\gamma_0 = \gamma[b, b']$ and suppose that $\beta : b \curvearrowright b'$ is a rectifiable arc in G_0. In view of 5.5, it suffices to show that $l_k(\gamma_0) \le 2cl_k(\beta)$.

Let $x_0 \in \beta$ be a point where $\delta(x_0)$ is maximal, and set $r = \delta(x_0)$. Then $(1 - t) \vee (1 - t') \le r \le 1$. For the arcs $\sigma = \beta[b, x_0]$ and $\sigma' = \beta[x_0, b']$ we have as in 5.5

$$l_k(\sigma) \ge \log \frac{r}{1 - t}, \qquad l_k(\sigma') \ge \log \frac{r}{1 - t'}.$$

Moreover, $l_k(\beta) \ge |a - a'|/r$. Hence $2l_k(\beta) \ge g(r)$ where

$$g(r) = \frac{|a - a'|}{r} + \log \frac{r}{1 - t} + \log \frac{r}{1 - t'}.$$

Since $|a-a'|\ge 2$, we see by elementary calculus that g is decreasing on $(0,1]$. Hence

$$2cl_k(\beta)\ge cg(1)\ge l(\alpha)+\log\frac{1}{1-t}+\log\frac{1}{1-t'}=l_k(\gamma_0).\quad\square$$

5.8. **Lemma.** *There is a universal constant c_0 such that each pair $a,b\in G_0$ can be joined by an arc γ that is c_0-quasiconvex both in the norm metric and in the* QH *metric of G_0.*

Proof. If $|P_1a-P_1b|\ge 2$, the arc $\gamma=[a,P_1a]\cup[P_1a,P_1b]\cup[P_1b,b]$ is a 2-neargeodesic of G_0 by 5.7. It is easy to verify that γ is 2-quasiconvex in the norm metric.

Suppose that $|P_1a-P_1b|<2$, and set $z=(P_1a+P_1b)/2$. Then a and b lie in the convex domain $D=B_1(z,1)\times B_2$. Since $B(z,1)\subset D\subset B(z,2)$, the domain D is 4-uniform by 2.19. Let $\gamma:a\curvearrowright b$ be a 2-neargeodesic of D. Applying the part (1) $\Rightarrow$ (2) of 2.42 with the substitution $G\mapsto D,\ D\mapsto G_0$ we see that γ is a c_0-neargeodesic of G_0 with a universal c_0. Moreover by the Cigar theorem 2.29, each subarc of γ is c_1-uniform in D with a universal c_1, and hence γ is c_1-quasiconvex in the norm metric. $\square$

5.9. **Lemma.** *Let F be a normed vector space with $\dim F\ge 2$, and let B be a ball (open or closed) in F. Then each pair of points in $F\setminus B$ can be joined by a 5-quasiconvex arc in $F\setminus B$.*

Proof. We may assume that B is centered at the origin. Let $a,b\in F\setminus B$ with $|a|=r\le R=|b|$. Setting $y=rb/R$ we have $|y|=r$. Since the sphere $S(r)$ is 2-quasiconvex by [Sc, 5F], there is an arc $\alpha:a\curvearrowright y$ in $S(r)$ with $l(\alpha)\le 2|a-y|$. It suffices to show that $l(\gamma)\le 5|a-b|$ for $\gamma=\alpha\cup[y,b]$. We have

$$l(\gamma)\le 2|a-y|+|y-b|\le 2|a-b|+3|y-b|.$$

Since $|y-b|=R-r\le|a-b|$, we obtain $l(\gamma)\le 5|a-b|$. $\square$

5.10. **Proposition.** *For each $R>0$, the domain G_0 is R-boundedly $(R+2)$-uniform.*

Proof. Let $a,b\in G_0$ with $|a-b|\le R$. For $z=(P_1a+P_1b)/2$, the domain $D=G_0\cap B(z,1+R/2)$ is convex and contains a and b. Moreover, $B(z,1)\subset D\subset B(z,1+R/2)$. Hence D is $(R+2)$-uniform by 2.19, and the proposition follows. $\square$

5.11. **Proposition.** *The domain G_0 is locally 4-uniform and $(c,0)$-LSC at each finite boundary point for all $c>1$. If $\dim E_1\ge 2$, then G_0 is $(c_0,0)$-LSC at ∞ with a universal constant c_0.*

Proof. Let $b=(b_1,b_2)\in\partial G_0\setminus\{\infty\}$, and set $D=B_1(b_1,1)\times B_2=P_1^{-1}B_1(b_1,1)\cap G_0$. Then $B(b_1,1)\subset D\subset B(b_1,2)$, and hence D is 4-uniform by 2.19. Consequently, G_0 is locally 4-uniform at b. The LSC property follows from 4.3 and 5.10.

Suppose that $\dim E_2\ge 2$. Let $R>0$ and let $x,y\in G_0\setminus(\bar B_1(R+2)\times B_2)$. It suffices to show that x and y can be joined by a c_0-neargeodesic of G_0 in $G_0\setminus(\bar B_1(R)\times B_2)$. Writing $x=(x_1,x_2)$ and $y=(y_1,y_2)$ we consider two cases.

Case 1. $|x_1-y_1|<2$. The domain $D=B_1(x_1,2)\times B_2$ is convex and 6-uniform by 2.19. Let $\gamma:x\curvearrowright y$ be a 2-neargeodesic of D. By 2.42, we again see that γ is a c_0-neargeodesic of G_0 with a universal c_0. Moreover, $\gamma\subset G_0\setminus(\bar B_1(R)\times B_2)$.

Case 2. $|x_1 - y_1| \geq 2$. By 5.9, we can join x_1 and y_1 by a 5-quasiconvex arc $\alpha \subset E_1 \setminus B_1(R)$. Then $\gamma = [x, x_1] \cup \alpha \cup [y_1, y]$ joins x and y in $G_0 \setminus (B_1(R) \times B_2)$, and γ is a 10-neargeodesic by 5.7. □

5.12. **Theorem.** *The domain* G_0 *is* CQH *equivalent to a uniform domain if and only if* $\dim E_1 = 1$.

Proof. Suppose that $\dim E_1 = 1$, let $a \in E_1$ be a unit vector, and let H be the half space $\{(ta, x) : t > 0, x \in E_2\}$ of $E = E_1 \times E_2$. Then H is a uniform domain, and the map $f : H \to G_0$, defined by

$$fx = \frac{x}{|x|} + a \log \frac{1}{|x|},$$

is 3-QH in the norm $|x| = |x_1| + |x_2|$ (see [Vä9, 8.14]).

Next assume that $\dim E_2 \geq 2$ and that $f : G_0 \to G$ is a CQH map onto a uniform domain G. Since G_0 and G are LSC at all boundary points by 4.3 and 5.11, and since G_0 is boundedly uniform by 5.10, it follows from 4.8 that f extends to a homeomorphism $\bar{f} : \bar{G}_0 \to \bar{G}$. Choose a line $L \subset E_1$. Since L is a geodesic of G_0, fL is a solid open arc in G. Moreover, $fL \cup \{\bar{f}(\infty)\}$ is a Jordan curve. This is a contradiction by 4.15. □

5.13. **Theorem.** *Suppose that* $G_0 = E_1 \times B_2$ *and* $G_0' = E_1' \times B_2'$ *are domains as in 5.1 and that* $f : G_0 \to G_0'$ *is* CQH. *Then* f *has a limit at every finite boundary point of* G_0, *and* $\dim E_1 = \dim E_1'$, $\dim E_2 = \dim E_2'$. *If* $\dim E_1 \geq 2$, *then* f *extends to a homeomorphism* $\bar{f} : \bar{G}_0 \to \bar{G}_0'$ *with* $\bar{f}(\infty) = \infty$.

Proof. Let $b \in \partial G_0$, $b \neq \infty$. Then G is LSC at b by 5.11, and G_0' is boundedly uniform by 5.10. Hence f has a limit at b by 4.8.

If $\dim E_1 = 1$, then $\dim E_1' = 1$ by 5.12, and hence $\dim E_2 = \dim E_2'$ (either both are infinite or both are finite and equal).

Suppose that $\dim E_1 \geq 2$. Then $\dim E_1' \geq 2$. Since G_0 and G_0' are now LSC also at ∞ by 5.11, f extends to a homeomorphism $\bar{f} : \bar{G}_0 \to \bar{G}_0'$ by 4.8. Let $L \subset E_1$ be a line. As in the proof of 5.12, fL is an open solid arc in G_0', and $fL \cup \{\bar{f}(\infty)\}$ is a Jordan curve. From 4.15 and 5.11 it follows that $\bar{f}(\infty) = \infty$.

Since $\bar{f}$ defines a homeomorphism of $E_1 \times \partial B_2$ onto $E_1' \times \partial B_2'$, the sets ∂B_2 and $\partial B_2'$ are homotopy equivalent. Hence $\dim E_2 = \dim E_2'$, which implies that $\dim E_1 = \dim E_1'$. □

5.14. **Corollary.** *If* $1 \leq p < q < n$, *the domains* $\mathsf{R}^p \times B^{n-p}$ *and* $\mathsf{R}^q \times B^{n-q}$ *are not quasiconformally equivalent.* □

5.15. *Remark.* Consider the case $\dim E_1 = 1$ of 5.13. Replacing the one-point extension $\bar{G}_0 = (E_1 \times \bar{B}_2) \cup \{\infty\}$ by the two-point extension $G^* = (E_1 \times \bar{B}_2) \cup \{-\infty, \infty\}$ in the usual way, it is easy to see that f extends to a homeomorphism $f^* : G^* \to G'^*$. For example, one can make use of auxiliary QH maps of G_0 and G_0' onto half spaces; see the proof of 5.12.

5.16. **Conventions.** In the rest of the paper we consider self maps of G_0. In all results henceforth we assume that:

(1) $G_0 = E_1 \times B_2$ as in 5.1.
(2) $f : G_0 \to G_0$ is an (M, C)-CQH map.
(3) Either $\dim E_1 \geq 2$ or $|fx| \to \infty$ as $|x| \to \infty$.

From (3) it follows that f extends to a homeomorphism $\bar{f} : \bar{G}_0 \to \bar{G}_0$ with $\bar{f}(\infty) = \infty$.

We estimate the vertical and horizontal distortions of f in 5.17 and 5.18, and show that f is coarsely bilipschitz in the norm metric in 5.19. In 5.22 we prove a result called the Fundamental lemma, which is used in 5.24 to show that $\bar{f}$ is quasisymmetric rel ∂G_0. In 5.26 we prove that if f is freely quasiconformal, then f is quasisymmetric.

5.17. **Theorem.** *If $a \in G_0$, then $\delta(fa) \ge \mu(\delta(a), M, C) > 0$.*

Proof. Set $r = \delta(a)$ and choose a line L through a, parallel to E_1. Then L is a $(1/r)$-neargeodesic of G_0 by 5.4(5). Hence fL is a (c,h)-solid arc with (c,h) depending on (r, M, C). By 5.10, G_0 is 1-boundedly 3-uniform and hence, by 2.27, 1-boundedly QH ψ-uniform with $\psi(t) = 189\log(1+t)$. Let $M_1 = M_1(c,h,\psi)$ be the number given by (2.11).

If $\delta(fa) \ge 1/4M_1$, there is nothing to prove. Assume that $\delta(fa) \le 1/4M_1$. Since $\bar{f}(\infty) = \infty$, the arc fL converges to ∞ at both ends. If $\delta(x) < 1/4M_1$ for all $x \in fL$, then fL is bounded by the Escape lemma 2.13. Hence there is $x_0 \in fL$ with $\delta(x_0) \ge 1/4M_1$. Let $q = q(c,h,\psi) > 0$ be the number given by the Diving lemma 2.15. If $\delta(x) \le q\delta(x_0)$ for some $x \in fL$, then the x-component of $fL \setminus \{x_0\}$ is bounded by 2.15. Hence $\delta(x) > q\delta(x_0)$ for all $x \in fL$, and the theorem holds with $\mu = q/4M_1$. □

5.18. **Theorem.** *If $a, b \in G_0$ with $P_1a = P_1b$, then $|fa - fb| \le c_1(M, C)$.*

Proof. By the triangle inequality we may assume that $P_2a = 0$. Set $x = f^{-1}P_1fb$, $\gamma' = [fx, fb]$, $\gamma = f^{-1}\gamma'$. Then γ' is a QH geodesic, and hence γ is (c,h)-solid with (c,h) depending on (M,C). Let $M_1 = M_1(c,h,\psi)$ be as in the preceding proof, and set $r = 1/5M_1$. Since $\delta(fx) = 1$, 5.17 gives $\delta(x) \ge \mu = \mu(1, M, C) > 0$. We consider three cases.

Case 1. $|b - x| \le 1$. Now $|a - x| \le |a - b| + |b - x| \le 2$, and hence

$$\begin{aligned} |fa - fb| &\le |fa - fx| + 1 \le k(fa, fx) + 1 \le Mk(a,x) + C + 1 \\ &\le M|a - x|/\mu + C + 1 \le 2M/\mu + C + 1. \end{aligned}$$

Case 2. $\delta(b) \ge r$. Now $k(a,b) \le |a - b|/r \le 1/r = 5M_1$, and hence

$$|fa - fb| \le k(fa, fb) \le Mk(a,b) + C \le 5MM_1 + C.$$

Case 3. $|b - x| \ge 1$ and $\delta(b) \le r$. If $z \in \gamma$ and if $\gamma[b,z] \subset \partial G_0 + \bar{B}(r)$, then the Escape lemma 2.13 gives

$$|b - z| \le 4M_1r = 4/5 \le |b - x|.$$

Hence there is the first point z of γ such that $\delta(z) = r$. Since $|a-z| \le |a-b|+|b-z| < 2$, we get

$$k(a,z) \le |a - z|/r < 2/r = 10M_1,$$

which yields

$$\begin{aligned} |fa - fb| &\le |fa - fz| + 1 \le k(fa, fz) + 1 \\ &\le Mk(a,z) + C + 1 \le 10MM_1 + C + 1. \quad \square \end{aligned}$$

5.19. **Theorem.** *The map $f : G_0 \to G_0$ is C_1-coarsely M-bilipschitz in the norm metric with $C_1 = C_1(M, C)$. That is,*

$$(|x - y| - C_1)/M \le |fx - fy| \le M|x - y| + C_1$$

for all $x, y \in G_0$.

Proof. Let $x = (x_1, x_2),\ y = (y_1, y_2) \in G_0$. By 5.18 we obtain

$$\begin{aligned} |fx - fy| &\le |fx - fx_1| + |fx_1 - fy_1| + |fy_1 - fy| \le k(fx_1, fy_1) + 2c_1 \\ &\le Mk(x_1, y_1) + C + 2c_1 \le M|x - y| + C + 2c_1. \end{aligned}$$

Since f^{-1} is (M, C)-CQH, the theorem follows. □

5.20. **Quasisymmetry.** We recall the basic definitions of quasisymmetry. Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. By a *triplet* in a metric space (X, ϱ) we mean a triple $T = (x, y, z)$ of distinct points in X, and we write

$$|T| = \varrho(x, y)/\varrho(x, z).$$

An embedding $f : X \to Y$ into a metric space Y is *η-quasisymmetric* if $|fT| \le \eta(|T|)$ for each triplet T in X. If $\dot{X} = X \cup \{\infty\}$ is the one-point extension of X and if $f : \dot{X} \to \dot{Y}$ is an embedding such that $f(\infty) = \infty$ and f defines an η-quasisymmetric map $f : X \to Y$, we also say that f is η-quasisymmetric.

If $A \subset X$, a triplet (x, y, z) in X is said to be a *triplet in* (X, A) if either $x \in A$ or $\{y, z\} \subset A$. An embedding $f : X \to Y$ is *η -quasisymmetric rel A* if $|fT| \le \eta(|T|)$ for each triplet T in (X, A).

We want to show that the map $\bar{f} : \bar{G}_0 \to \bar{G}_0$ is quasisymmetric rel ∂G_0. The corresponding result for uniform domains was given in [Vä6, 7.9]. The proof was based on [Vä6, Fundamental lemma 7.3]. We give a variation of the latter result in 5.22. First we prove a simple consequence of 5.19.

5.21. **Lemma.** *Let $T = (x, y, z)$ be a triplet in G_0 with $|x - z| \ge 2C_1$, where C_1 is given by* (5.19)*. Then $|fT| \le 2M^2|T| + M$.*

Proof. By 5.19 we get

$$\begin{aligned} |fx - fy| &\le M|x - y| + C_1 \le M|T||x - z| + |x - z|/2, \\ |fx - fz| &\ge (|x - z| - C_1)/M \ge |x - z|/2M, \end{aligned}$$

and the lemma follows. □

5.22. **Fundamental lemma.** *Suppose that $T = (x, y, z)$ is a triplet in G_0 such that $|T| \le 1$ and $k(x, z) \ge 2C \vee \frac{1}{2}$. Then $|fT| \le H(M, C)$.*

Proof. Let $C_1 = C_1(M, C)$ be the number given by 5.19. If $|x - z| \ge 2C_1$, the result follows from 5.21. Hence we may assume that $|x - z| \le 2C_1$.

Write $|x - z| = t,\ |fx - fz| = t'$. We want to find an estimate $|fx - fy| \le Ht'$. Let $L_0 \subset E_1$ be a line containing $P_1 fx$ and $P_1 fy$, and let L be a component of $L_0 \setminus \{P_1 fy\}$, not containing $P_1 fx$. The arc $\alpha' = [fy, P_1 fy] \cup L$ is a 2-neargeodesic of G_0 by 5.5.

By 5.8 there is an arc $\beta' : fx \curvearrowright fz$, which is c_0-quasiconvex both in the norm metric and in the QH metric of G_0 with a universal constant c_0. The arcs $\alpha = f^{-1}\alpha'$ and $\beta = f^{-1}\beta'$ are (c, h)-solid with (c, h) depending on (M, C).

By 5.10, G_0 is $2C_1$-boundedly $(2C_1 + 2)$-uniform, and hence, by 2.27, $2C_1$-boundedly QH ψ-uniform with a slow function ψ depending only on (M, C). Let

$M_1 = M_1(c, h, \psi)$ be the number given by (2.11), and set $r = t/5M_1$. Let z_1 be the first point of β with $|x - z_1| = t$, and set $\beta_0 = \beta[x, z_1]$. Let y_1 be the first point of α with $|y - y_1| = t$, and set $\alpha_0 = \alpha[y, y_1]$.

Fact 1. $\alpha_0 \not\subset \partial G_0 + \bar{B}(r)$ and $\beta_0 \not\subset \partial G_0 + \bar{B}(r)$.

Assume that $\alpha_0 \subset \partial G_0 + \bar{B}(r)$. Since $r \le 2C_1/5M_1$, the Escape lemma 2.13 implies that $d(\alpha_0) \le 4M_1 r = 4t/5 < t$. Since $d(\alpha_0) \ge |y - y_1| = t$, this is a contradiction. The second part is proved similarly.

By Fact 1 we can choose points $y_0 \in \alpha_0$ and $z_0 \in \beta_0$ such that $\delta(y_0) \wedge \delta(z_0) \ge r$. Then

$$|y_0 - z_0| \le |y_0 - y| + |y - x| + |x - z_0| \le 3t.$$

Hence $k(y_0, z_0) \le |y_0 - z_0|/r \le 3t/r = 15M_1$, which yields

$$k(fy_0, fz_0) \le 15MM_1 + C = c_1 \tag{5.23}$$

with $c_1 = c_1(M, C)$.

Fact 2. $\delta(fx) \le 8Mt'$.

Assume that $\delta(fx) > 8Mt'$. Since $\delta(fx) > 2t' = 2|fx - fz|$, we have $k(fx, fz) \le 2t'/\delta(fx) < 1/4M$, and we get the contradiction

$$1/4 + C \le (2C) \vee (1/2) \le k(x, z) \le Mk(fx, fz) + C < 1/4 + C.$$

Fact 3. $|fx - fy| \le (32Mt') \vee 6|fx - fy_0|$.

We assume that $|fx - fy| > 32Mt'$ and show that $|fx - fy| \le 6|fx - fy_0|$. We consider two cases.

Case 1. $|fy_0 - P_1 fy| \le 1$. If $fy_0 \in [fy, P_1 fy]$, then $|fy_0 - fy| < \delta(fy_0)$. If $fy_0 \in L$, then $|fy_0 - fy| \le |fy_0 - P_1 fy| + |P_1 fy - fy| \le 2 = 2\delta(fy_0)$, and thus we always have $|fy_0 - fy| \le 2\delta(fy_0)$. Since Fact 2 gives

$$\delta(fy_0) \le \delta(fx) + |fx - fy_0| \le 8Mt' + |fx - fy_0| \le |fx - fy|/4 + |fx - fy_0|,$$

we obtain

$$|fx - fy| \le |fx - fy_0| + |fy_0 - fy| \le |fx - fy|/2 + 3|fx - fy_0|,$$

and hence $|fx - fy| \le 6|fx - fy_0|$.

Case 2. $|fy_0 - P_1 fy| \ge 1$. Now $P_1 fy \in [P_1 fx, fy_0] \subset L_0$, and hence

$$\begin{aligned} |fx - fy| &\le |P_1 fx - P_1 fy| + 2 \le 2|P_1 fx - P_1 fy| + 2|P_1 fy - fy_0| \\ &= 2|P_1 fx - fy_0| \le 2|fx - fy_0|, \end{aligned}$$

and Fact 3 is proved.

We turn to the proof for the estimate $|fx - fy| \le Ht'$. If $|fx - fy| \le 32Mt'$, it holds with $H = 32M$. By Fact 3 we may thus assume that $|fx - fy| \le 6|fx - fy_0|$. Using [Vä5, 2.2(1)], (5.23), Fact 2, and the c_0-quasiconvexity of β' we obtain

$$\begin{aligned} |fx - fy_0| &\le |fy_0 - fz_0| + |fx - fz_0| \le \delta(fz_0)e^{k(fy_0, fz_0)} + |fx - fz_0| \\ &\le (\delta(fx) + |fx - fz_0|)e^{c_1} + |fx - fz_0| \\ &\le 8Me^{c_1}t' + (e^{c_1} + 1)c_0 t' = c_2 t' \end{aligned}$$

with $c_2 = c_2(M, C)$. Hence $|fx - fy| \le 6c_2 t'$. □

Recall that in the following results, we assume that f and G_0 satisfy the conditions of 5.16.

5.24. **Theorem.** *The map* $\bar{f} : \bar{G}_0 \to \bar{G}_0$ *is* η*-quasisymmetric rel* ∂G_0 *with* η *depending only on* (M, C).

Proof. Let $T = (x, y, z)$ be a triplet in $(\bar{G}_0 \setminus \{\infty\}, \partial G_0 \setminus \{\infty\})$. By symmetry and by [Vä6, 5.8], it suffices to find an estimate $|fT| \le \eta(|T|)$, where $\eta : (0, \infty) \to (0, \infty)$ is increasing, but $\eta(t)$ need not converge to 0 as $t \to 0$.

Choose a sequence of triplets $T_n = (x_n, y_n, z_n)$ in G_0 such that $x_n \to x,\ y_n \to y,\ z_n \to z$. Write $R = 2C \vee \frac{1}{2}$. Since $\delta(x_n) \to 0$ or $\delta(z_n) \to 0$, we may assume that $|x_n - z_n| \ge e^R(\delta(x_n) \wedge \delta(z_n))$ for all n. This implies that $k(x_n, z_n) \ge R$. Fix n and choose successive points $x_n = a_0, \dots, a_N = y_n$ of the line segment $[x_n, y_n] \subset G_0$ such that a_1 is the last point with $|a_1 - x_n| \le |x_n - z_n|$, and for $j \ge 2,\ a_j$ is the last point with $|a_j - a_{j-1}| \le |a_{j-1} - x_n|$. By the Fundamental lemma 5.22 we have $|fa_1 - fx_n| \le H|fx_n - fz_n|$ with a constant $H = H(M, C)$. If $N = 1$, this gives $|fT_n| \le H$. If $N \ge 2$, the Fundamental lemma gives by induction

$$\begin{aligned} |fa_{j+1} - fx_n| &\le |fa_{j+1} - fa_j| + |fa_j - fx_n| \le (H+1)^j |fa_1 - fx_n| \\ &\le H(H+1)^j |fx_n - fz_n| \end{aligned}$$

for $1 \le j \le N - 1$, and hence $|fT_n| \le H(H+1)^{N-1}$. Since

$$|x_n - y_n| = \sum_{j=1}^{N} |a_{j-1} - a_j| \ge (N-1)|a_1 - x_n| = (N-1)|x_n - z_n|,$$

this implies that $|fT_n| \le H(H+1)^{|T_n|}$. As $n \to \infty$, this gives the desired estimate $|fT| \le \eta(|T|)$ with $\eta(t) = H(H+1)^t$. □

5.25. **Corollary.** *The induced map* $\partial G_0 \to \partial G_0$ *is* η*-quasisymmetric with* η *depending only on* (M, C). □

5.26. **Theorem.** *If* f *is freely quasiconformal, then* f *is* η*-quasisymmetric with* $\eta = \eta_\varphi$.

Proof. Since G_0 is convex, it suffices to show that f is weakly H-quasisymmetric with $H = H(\varphi)$ (see [Vä6, 5.5]). Let $T = (x, y, z)$ be a triplet in G_0 with $|T| \le 1$. We must find an estimate $|fT| \le H(\varphi)$.

By [Vä6, 4.14], f is $(M, 1/4)$-CQH with some $M = M(\varphi)$. If $k(x, z) \ge 1/2$, the desired estimate follows from the Fundamental lemma 5.22. If $k(x, z) \le 1/2$, then

$$|x - y|/\delta(x) \le e^{k(x,y)} - 1 \le \sqrt{e} - 1 = q < 1.$$

Since f is q-locally η_φ-quasisymmetric by [Vä5, 5.10], we have $|fT| \le \eta(1)$. □

5.27. **Corollary.** *If* $f : \mathsf{R}^p \times B^q \to \mathsf{R}^p \times B^q$ *is* K*-quasiconformal and if* $p \ge 2$, *then* f *is* η*-quasisymmetric with* $\eta = \eta_{K,p,q}$. □

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