

## FAMILIES OF BAKER DOMAINS II

P. J. RIPPON AND G. M. STALLARD

ABSTRACT. Let  $f$  be a transcendental meromorphic function and  $U$  be an invariant Baker domain of  $f$ . We use estimates for the hyperbolic metric to show that there is a relationship between the size of  $U$  and the proximity of  $f$  in  $U$  to the identity function, and illustrate this by discussing how the dynamics of transcendental entire functions of the following form vary with the parameter  $a$ :

$$f(z) = az + bz^k e^{-z}(1 + o(1)) \text{ as } \Re(z) \rightarrow \infty,$$

where  $k \in \mathbf{N}$ ,  $a \geq 1$  and  $b > 0$ .

### 1. INTRODUCTION

Let  $f$  be a meromorphic function which is not rational of degree one and denote by  $f^n$ ,  $n \in \mathbf{N}$ , the  $n$ th iterate of  $f$ . The set of normality,  $N(f)$ , is defined to be the set of points,  $z \in \mathbf{C}$ , such that  $(f^n)_{n \in \mathbf{N}}$  is well-defined, meromorphic and forms a normal family in some neighbourhood of  $z$ . The complement,  $J(f)$ , of  $N(f)$  is called the Julia set of  $f$ . An introduction to the properties of these sets can be found in, for example, [3] for rational functions and in [4] for transcendental meromorphic functions.

The set  $N(f)$  is completely invariant so that, if  $U$  is a component of  $N(f)$ , then, for each  $p \in \mathbf{N}$ , there exists a component  $U_p$  of  $N(f)$  such that  $f^p(U) \subset U_p$ . If  $U_p \neq U_m$ , for each  $p \neq m$ , then we say that  $U$  is a wandering domain. If  $U_p = U$ , then we say that  $U$  is a periodic component of period  $p$  (assuming  $p$  to be minimal) and there are then five possibilities (see, for example, [4]). In particular,  $U$  is called a Baker domain or an essentially parabolic domain if there exists  $z_0 \in \partial U$  such that  $f^{np}(z) \rightarrow z_0$  as  $n \rightarrow \infty$ , for  $z \in U$ , but  $f^p(z_0)$  is not defined.

If  $U$  is a Baker domain, then  $f$  must be transcendental. If  $f$  is in fact a transcendental *entire* function, then  $f^{np}(z) \rightarrow \infty$  as  $n \rightarrow \infty$  for  $z \in U$  and, moreover,  $U$  is simply connected [1, Theorem 3.1]. This is not true in general for transcendental meromorphic functions – for example, in [8] it is shown that the function  $f(z) = z + 2 + e^{-z} + (100(z - (1 + i\pi)))^{-1}$  has a multiply connected Baker domain.

Information about the rate at which iterates tend to infinity in a Baker domain can be obtained by using estimates for the hyperbolic metric. For example, it was shown by Baker (see, for example, [4, Lemma 7]) that, if  $U$  is a simply connected invariant Baker domain, and  $z_0 \in U$ , then for any path  $\Gamma = \bigcup_{n=0}^{\infty} f^n(\Gamma_0)$ , where  $\Gamma_0$  joins  $z_0$  to  $f(z_0)$  in  $U$  and  $0 \notin \Gamma$ , there is a constant  $C$  such that

$$(1.1) \quad |f(z)| \leq C|z|, \text{ for } z \in \Gamma.$$

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Received by the editors January 5, 1999 and, in revised form, April 19, 1999.  
1991 *Mathematics Subject Classification*. Primary 30D05; Secondary 58F08.

In particular,  $|f^n(z_0)| \leq C^n |z_0|$ , for each  $n \in \mathbf{N}$ .

Various examples of functions  $f$  with Baker domains  $U$  are known (see, for example, [2], [5], [9], [10], [11], [13] and [15]) and these examples suggest a connection between the proximity of  $f$  in  $U$  to the identity function and the thinness of  $U$  itself. For example, (see [9] and [15]) the function  $f(z) = z + e^{-z}$  has a Baker domain  $U_m$  in each strip

$$\{z : |\Im(z) - 2m\pi| < \pi\}, m \in \mathbf{Z},$$

and  $U_m$  has asymptotic width  $2\pi$  as  $\Re(z) \rightarrow \infty$ , whereas the function  $f(z) = z + \exp(-e^z)$  has an infinite family of (much thinner) Baker domains in each such strip.

In Section 2 of this paper, we use standard estimates for the hyperbolic metric to obtain the following result which confirms such a connection. We use the following notation throughout the paper:

- $B(z, r) = \{w : |w - z| < r\}$ ;
- $d_U(z) = \inf\{|z - w| : w \in \partial U\}$ ;
- $[z, w]_D$  is the distance from  $z$  to  $w$  with respect to the hyperbolic metric in the domain  $D$ ;
- $z_n = f^n(z_0)$ , where  $z_0$  and  $f$  are given.

**Theorem 1.** *Let  $f$  be a transcendental meromorphic function,  $U$  be an invariant Baker domain of  $f$  and  $z_0 \in U$ .*

- (a) *If  $[z_{n+1}, z_n]_U \not\rightarrow 0$ , then there exists  $C > 0$  such that*

$$B(z_n, C|z_{n+1} - z_n|) \cap U^c \neq \emptyset, \text{ for } n \geq 0.$$

- (b) *If  $U$  is simply connected, and  $\Gamma = \bigcup_{n=0}^{\infty} f^n(\Gamma_0)$ , where  $\Gamma_0$  is a path in  $U$  joining  $z_0$  to  $z_1$ , then there exists  $c > 0$  such that*

$$U \supset \bigcup_{z \in \Gamma} B(z, c|f(z) - z|).$$

*Remarks.* 1. Later (Theorem 3) we use Theorem 1 part (b) to show the *non-existence* of a Baker domain under particular circumstances.

2. The sequence  $[z_{n+1}, z_n]_U$  in part (a) is decreasing since, for  $z, w \in U$ ,

$$[f(z), f(w)]_U \leq [f(z), f(w)]_{f(U)} \leq [z, w]_U,$$

by [7, Theorem 4.1]. Whether or not  $[z_{n+1}, z_n]_U \rightarrow 0$  is independent of the choice of  $z_0 \in U$  if  $U$  is simply connected. To see why this is true, we take  $U$  to be the right half-plane  $H$ , without loss of generality. In [14] it was shown that, if  $[z_{n+1}, z_n]_H \not\rightarrow 0$ , for *some*  $z_0 \in H$ , then there exists a function  $g$  that is analytic in  $H$  with  $\Re(g) > 0$  in  $H$  such that

$$g(f(z)) = g(z) + i, \text{ for } z \in H.$$

Thus  $g(z_{n+1}) = g(z_n) + i$ , for each  $n \in \mathbf{N}$  and *any*  $z_0 \in H$  so that

$$[z_{n+1}, z_n]_H \geq [g(z_{n+1}), g(z_n)]_{g(H)} \geq [g(z_{n+1}), g(z_n)]_H \not\rightarrow 0.$$

The example  $f(z) = z + e^{-z}$  mentioned earlier shows that *some* condition such as  $[z_{n+1}, z_n]_U \not\rightarrow 0$  is needed to obtain the conclusion in part (a) of Theorem 1. The condition  $[z_{n+1}, z_n]_U \not\rightarrow 0$  is certainly satisfied if  $U$  is simply connected and  $f$  is univalent in  $U$ . This suggests the question of whether there exists a function  $f$  with Baker domain  $U$  such that  $[z_{n+1}, z_n]_U \not\rightarrow 0$  but  $f$  is *not* univalent in  $U$ . It is

straightforward to check that, if  $f(z) = 2z + e^{-z}$ , then  $f$  has a simply connected invariant Baker domain  $U$  containing  $\{z : \Re(z) > 1\} \cup \{z : \Im(z) = 0\}$ . Similar arguments to those used in the proof of Theorem 2 part (b) below show that  $[z_{n+1}, z_n]_U \not\rightarrow 0$ , if  $z_0 \in U$ . The set of critical points of  $f$  is

$$\{z : z = -\ln 2 + 2m\pi i, \text{ for some } m \in \mathbf{Z}\}$$

and so  $f$  is certainly not univalent in  $U$ , as  $-\ln 2 \in U$ . In fact, since  $f(z) = 2z + \phi(z)$ , where  $\phi(z + 2\pi i) = \phi(z)$ , it follows from the main result of [6] (see also [16, Corollary 1]) that  $N(f)$  is invariant under translation by  $2\pi i$  and so  $U$  contains all of the critical points of  $f$  and, indeed, all of the infinitely many critical values of  $f$ .

There is interest (see, for example, [4]) in establishing the relationship between the Baker domains of a function  $f$  and the set of singularities of  $f^{-1}$ , which consists of the critical values and finite asymptotic values of  $f$ . In Section 3, we use what seems to be a new technique to show that entire functions in a certain large class have Baker domains which contain infinitely many such singularities, and in which the hyperbolic distance between successive iterates of points does not tend to zero.

**Theorem 2.** *Let  $f$  be a transcendental entire function of the form*

$$(1.2) \quad f(z) = az + bz^k e^{-z}(1 + o(1)) \quad \text{as } \Re(z) \rightarrow \infty,$$

where  $k \in \mathbf{N}$ ,  $a > 1$  and  $b > 0$ . Then

- (a)  $f$  has a simply connected invariant Baker domain  $U$  which, for each  $\rho > 0$  and large values of  $R > 0$ , contains an invariant set of the form

$$D_{\rho,R} = \{z : |z^k e^{-z}| < \rho, |z| > R\};$$

- (b)  $[z_{n+1}, z_n]_U \not\rightarrow 0$  as  $n \rightarrow \infty$ , for each  $z_0 \in U$ ;
- (c)  $f$  is not univalent in  $U$  and, moreover, there are infinitely many singularities of  $f^{-1}$  in  $U$ , each of which corresponds to a critical point or an asymptotic path of  $f$  in  $U$ .

*Remarks.* 1. If  $f$  is of the form (1.2) with  $k \in \mathbf{Z} \setminus \mathbf{N}$ , then it is easy to check that  $f$  has an invariant Baker domain  $U$  which, for large values of  $R$ , contains an invariant set of the form  $\{z : \Re(z) > R\}$ . Part (c) of Theorem 2 is, however, *not* true in general for such  $k$ . For example,  $f(z) = 2z + 2e^{-2}e^{-z}$  has such an invariant Baker domain  $U$  and  $f$  is univalent in  $U$ . This follows from the corresponding properties of  $f(z) = 2z + 2 - \ln 2 - e^z$  [5, Theorems 1 and 2] by making the change of variable  $w = -z + \ln 2 - 2$ .

2. In [15], we showed that if  $f$  is of the form (1.2) with  $a = 1$ , then for each  $m \in \mathbf{Z}$ , there is an invariant Baker domain  $U_m$  of  $f$  such that, for each  $0 < \theta < \pi$ ,  $U_m$  contains a set of the form

$$V_m(\theta) = \{x + iy : x > v_m(\theta), |y - 2m\pi| < \theta\}.$$

For  $z_0 \in U_m$ ,  $\Re(z_n) \rightarrow \infty$  and  $\Im(z_n) \rightarrow 2m\pi$ , so that  $[z_{n+1}, z_n]_{U_m} \rightarrow 0$  as  $n \rightarrow \infty$ , the  $U_m$  are distinct, and each contains a singularity of  $f^{-1}$ . Thus the change in the dynamics of functions of the form (1.2) as  $a$  decreases to 1 is analogous to the change in dynamical behaviour occurring at a parabolic bifurcation, in that a single basin of attraction at infinity is replaced by infinitely many such basins in which convergence to infinity is much slower.

The following diagrams illustrate this change for  $f(z) = az(1+e^{-z})$  with  $a = 1.01$  and  $a = 1$ . In both cases, points of  $\{x + iy : |x| \leq 12, |y| \leq 12\}$  have been plotted red or yellow if their forward iterates under  $f$  become large very quickly, suggesting that they do not lie in an invariant domain for  $f$ , and black otherwise. Evidence for the location of Baker domains is provided by the forward orbits of many points on  $x = 4$ , which are plotted in white.

Figures 1 and 2 were produced using the software C++Builder, and the authors are grateful to Bob Margolis and Toni Cokayne (Department of Pure Mathematics, Open University) for help with this.

In Section 4, we use Theorem 2 to analyse the dynamics of a particular family of examples. Note that the proof of *uniqueness* in Theorem 3 part (b) below uses Theorem 1 part (b).

**Theorem 3.** *Let  $f(z) = az(1 + e^{-z^p})$ , where  $a > 1$  and  $p \in \mathbf{N}$ . Then, for  $k \in \{0, 1, \dots, p-1\}$ ,*

- (a)  $\{z : \arg z = (2k+1)\pi/p\} \subset J(f)$ ;
- (b)  $f$  has a unique simply connected invariant Baker domain  $U_k$  in

$$A_k = \{z : |\arg z - 2k\pi/p| < \frac{\pi}{p}\},$$

which, for each  $\epsilon$ ,  $0 < \epsilon < \frac{\pi}{2p}$ , contains a set of the form

$$\{z : |\arg z - 2k\pi/p| < \frac{\pi}{2p} - \epsilon, |z| > R\};$$

- (c)  $U_k$  contains infinitely many critical points of  $f$ .

In [15], we showed that the function  $g(z) = e^{2\pi i/p}z(1+e^{-z^p})$ ,  $p \in \mathbf{N}$ , has infinitely many  $p$ -cycles of rather thin Baker domains. These were the first examples of *entire* functions having Baker domains which are not invariant. The following corollary to Theorem 3 shows that a small change to this function gives an entire function with a  $p$ -cycle of Baker domains which are comparable in size to a sector.

**Corollary 1.** *For  $p \in \mathbf{N}$  and  $a > 1$ , let  $g(z) = ae^{2\pi i/p}z(1 + e^{-z^p})$ . Then the sets  $U_k, k = 0, 1, \dots, p-1$ , in Theorem 3 form a  $p$ -cycle of Baker domains for  $g$ , and each  $U_k$  contains infinitely many critical points of  $g$ .*

Indeed,  $g^n(z) = \omega^n f^n(z)$ , for  $n \in \mathbf{N}$ , where  $\omega = e^{2\pi i/p}$  and  $f$  is the function of Theorem 3. In particular,  $g^p = f^p$  so that  $J(g) = J(f)$ , and it follows that the Baker domains of  $g$  and of  $f$  must coincide.

The functions in Corollary 1 are not univalent on their Baker domains. We end, in Section 5, by showing how an approximation theory method used by Eremenko and Lyubich in [10] can be adapted to prove the following result.

**Theorem 4.** *For each  $p \in \mathbf{N}$ , there exists an entire function  $f$  which has a  $p$ -cycle of Baker domains on which  $f$  is univalent.*

The proof of Theorem 4 yields Baker domains of a similar size (namely, comparable to a sector) to those in Theorem 3 and Corollary 1.

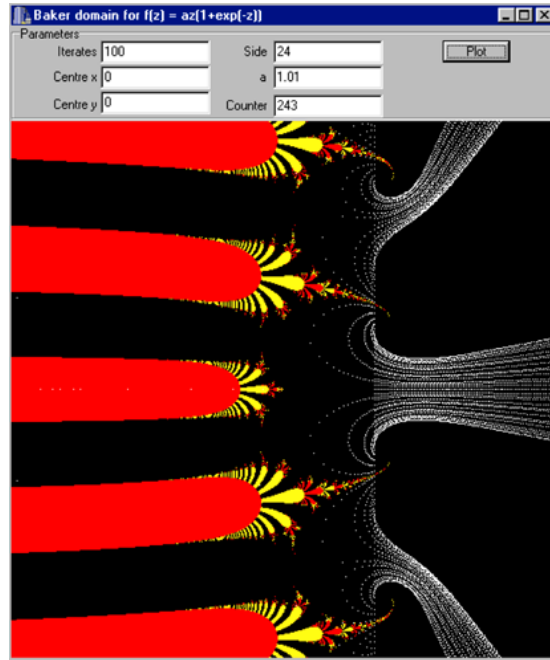


FIGURE 1.  $a = 1.01$

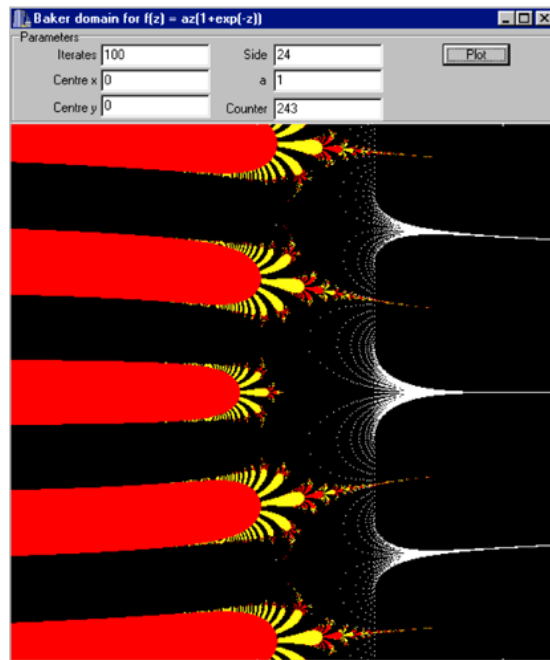


FIGURE 2.  $a = 1$

## 2. PROOF OF THEOREM 1

We deduce Theorem 1 from the following standard estimates for hyperbolic distance.

**Lemma 2.1.** (a) *If  $z, w$  belong to a domain  $U$ , then*

$$|w - z| \geq d_U(z) \left( \frac{2}{[w, z]_U} + 1 \right)^{-1}.$$

(b) *If, furthermore,  $U$  is simply connected, then*

$$|w - z| \leq d_U(z)(\exp(2[w, z]_U) - 1).$$

*Proof.* We begin by supposing that

$$d_U(z) > \left( \frac{2}{[w, z]_U} + 1 \right) |w - z|.$$

If  $L$  is the line segment from  $z$  to  $w$ , then  $L \subset U$  and, for  $\xi \in L$ ,

$$d_U(\xi) \geq d_U(z) - |\xi - z| > \frac{2}{[w, z]_U} |w - z|,$$

so that, by [7, Theorem 4.3], for example,

$$[w, z]_U \leq \int_L \rho_U(\xi) |d\xi| \leq 2 \int_L \frac{|d\xi|}{d_U(\xi)} < \frac{2|w - z|}{2|w - z|} [w, z]_U,$$

where  $\rho_U$  denotes the density of the hyperbolic metric in  $U$ . This contradiction proves part (a).

Now suppose that  $U$  is simply connected and  $\gamma$  is a hyperbolic geodesic in  $U$  from  $z$  to  $w$ . Then, by [7, Theorem 4.3], for example,

$$[w, z]_U = \int_\gamma \rho_U(\xi) |d\xi| \geq \frac{1}{2} \int_\gamma \frac{|d\xi|}{d_U(\xi)} \geq \frac{1}{2} \int_\gamma \frac{|d\xi|}{d_U(z) + |z - \xi|}.$$

Now

$$\gamma \cap \{\xi : |\xi - z| = t\} \neq \emptyset, \text{ for } 0 \leq t \leq |w - z|,$$

and so

$$[w, z]_U \geq \frac{1}{2} \int_0^{|w-z|} \frac{dt}{d_U(z) + t} = \frac{1}{2} \log \left( 1 + \frac{|w - z|}{d_U(z)} \right),$$

as required.

We now show how Theorem 1 follows from Lemma 2.1. From part (a) of Lemma 2.1,

$$d_U(z_n) \leq |z_{n+1} - z_n| \left( \frac{2}{[z_{n+1}, z_n]_U} + 1 \right).$$

So, if  $[z_{n+1}, z_n]_U \neq 0$ , then there exists  $C > 0$  such that

$$d_U(z_n) < C |z_{n+1} - z_n|,$$

for  $n \geq 0$ . This proves part (a) of Theorem 1.

If  $z \in \Gamma$ , then  $z = f^n(\xi_0)$ , for some  $\xi_0 \in \Gamma_0$ ,  $n \geq 0$ . Now

$$[f(z), z]_U = [f^{n+1}(\xi_0), f^n(\xi_0)]_U \leq [f(\xi_0), \xi_0]_U \leq \sup_{\xi \in \Gamma_0} [f(\xi), \xi]_U = d,$$

say, and so part (b) of Theorem 1 follows from part (b) of Lemma 2.1, on taking  $c = (e^{2d} - 1)^{-1}$ .

## 3. PROOF OF THEOREM 2

Let  $f$  be a transcendental entire function of the form

$$f(z) = az + bz^k e^{-z}(1 + o(1)) \text{ as } \Re(z) \rightarrow \infty,$$

where  $a > 1$ ,  $b > 0$  and  $k \in \mathbf{N}$ . We begin our proof of Theorem 2 with the following key result.

**Lemma 3.1.** *Let*

$$D_{\rho,R} = \{z : |z^k e^{-z}| < \rho, |z| > R\}.$$

For each  $\rho > 0$ , there exists  $R(\rho) > 0$  such that

$$f(D_{\rho,R}) \subset D_{\rho,R},$$

for each  $R > R(\rho)$ .

*Proof.* First note that

$$(3.1) \quad |z^k e^{-z}| < \rho \iff \Re(z) > k \ln |z| - \ln \rho$$

so that, for any fixed  $\rho > 0$ ,

$$\min\{\Re(z) : z \in D_{\rho,R}\} \rightarrow \infty \text{ as } R \rightarrow \infty.$$

Thus, if  $\rho > 0$  is fixed, then  $R$  can be chosen so large that, for  $z \in D_{\rho,R}$ ,

$$(3.2) \quad |f(z)| \geq a|z| - 2b|z^k e^{-z}| \geq \frac{1}{2}(a+1)|z| > R,$$

and

$$(3.3) \quad |f(z)| \leq a|z| + 2b|z^k e^{-z}| \leq 2a|z|.$$

Now, by (3.1) and (3.2), if  $z = x + iy \in D_{\rho,R}$ , then  $f(z) = X + iY \in D_{\rho,R}$  if and only if

$$X > k \ln |f(z)| - \ln \rho.$$

It follows from (3.1) and (3.3) that, if  $R$  is sufficiently large, then this is true for any  $z \in D_{\rho,R}$ , since

$$\begin{aligned} X &\geq ax - 2b|z^k e^{-z}| \\ &> ax - 2b\rho \\ &> a(k \ln |z| - \ln \rho) - 2b\rho \\ &= k \ln |z| + (a-1)k \ln |z| - a \ln \rho - 2b\rho \\ &> k \ln(2a|z|) - \ln \rho \\ &> k \ln |f(z)| - \ln \rho. \end{aligned}$$

This proves Lemma 3.1.

Theorem 2 part (a) follows from Lemma 3.1, Montel's Theorem and the fact that  $f$  is entire.

To prove part (b), we fix  $\rho > 0$  and take  $z_0 \in D_{\rho,R}$ . If  $R$  is sufficiently large, then  $z_0 \in U$  and it follows from Lemma 3.1 and (3.2) that, for each  $n \in \mathbf{N}$ ,

$$|z_{n+1}| \geq \frac{1}{2}(a+1)|z_n|.$$

Fixing a point  $w \in J(f)$ , we note that, since  $|z_n| \rightarrow \infty$  as  $n \rightarrow \infty$ ,

$$d_U(z_n) \leq |z_n - w| \leq 2|z_n|,$$

for large values of  $n$ . Since  $U$  is simply connected, it follows from Lemma 2.1 part (b) and the above inequalities that, for large  $n$ ,

$$\exp(2[z_{n+1}, z_n]_U) \geq \frac{|z_{n+1} - z_n|}{d_U(z_n)} + 1 \geq \frac{a-1}{4} + 1$$

and so  $[z_{n+1}, z_n]_U \not\rightarrow 0$  as  $n \rightarrow \infty$ . It now follows from the second remark after Theorem 1 that, for any  $z_0 \in U$ ,  $[z_{n+1}, z_n]_U \not\rightarrow 0$ , as required.

The proof of part (c) of Theorem 2 uses the following result.

**Lemma 3.2.** *Let  $\rho > 6a\pi/b$ . If  $R$  is sufficiently large, then for each  $n \in \mathbf{N}$ , there exists a subarc  $\gamma_n$  of  $\partial D_{\rho,R}$  such that  $f(\gamma_n)$  is a closed curve and  $\min\{\Im(f(z)) : z \in \gamma_n\} \rightarrow \infty$  as  $n \rightarrow \infty$ .*

*Proof.* Let  $C_{\rho,R}$  denote that part of  $\partial D_{\rho,R}$  which lies in the upper half-plane but not on  $\{z : |z| = R\}$ . If  $z = x + iy \in C_{\rho,R}$ , then

$$x = k \ln |z| - \ln \rho,$$

and

$$\theta(z) = k \arg z - y$$

is a continuous argument of  $z^k e^{-z}$  on  $C_{\rho,R}$ . Here  $\arg z$  denotes the principal argument, and we note that  $\arg z \rightarrow \pi/2$  as  $z \rightarrow \infty$  along  $C_{\rho,R}$ . Thus, for  $n \geq n_0$  say, there exist  $\xi_n, \xi'_n, \xi''_n$  on  $C_{\rho,R}$  with

$$(3.4) \quad \theta(\xi_n) = -2\pi n, \quad \theta(\xi'_n) = -2\pi n + 3\pi/2, \quad \theta(\xi''_n) = -2\pi n - 3\pi/2.$$

For  $n \geq n_0$ , we put  $\Gamma_n = \Gamma'_n \cup \Gamma''_n$ , where

$$\Gamma'_n = \{z \in C_{\rho,R} : 0 \leq \theta(z) + 2\pi n \leq 3\pi/2\}$$

and

$$\Gamma''_n = \{z \in C_{\rho,R} : -3\pi/2 \leq \theta(z) + 2\pi n \leq 0\}.$$

We also put

$$f_n(z) = f(z) - a\xi_n = a(z - \xi_n) + bz^k e^{-z} + bz^k e^{-z} \epsilon_n(z).$$

By the hypotheses of Theorem 2, we may assume that  $n_0$  is so large that

$$(3.5) \quad |z - \xi_n| < 2\pi \text{ and } |bz^k e^{-z} \epsilon_n(z)| < \pi a/4, \text{ for } z \in \Gamma_n, n \geq n_0.$$

Since  $\rho > 6a\pi/b$ , it follows from (3.5) that, for  $z \in \Gamma_n, n \geq n_0$ ,

$$(3.6) \quad |f_n(z)| > b\rho - 3\pi a > b\rho/2,$$

and with an appropriate continuous choice of  $\arg(f_n(z))$ ,

$$(3.7) \quad |\arg(f_n(z)) - (\theta(z) + 2\pi n)| < \tan^{-1}\left(\frac{3\pi a}{b\rho}\right) < \tan^{-1}(1/2) < \pi/6.$$

We note that

$$\Im(z - \xi_n) \begin{cases} \leq 0, & \text{for } z \in \Gamma'_n, \\ \geq 0, & \text{for } z \in \Gamma''_n, \end{cases}$$

and so it follows from (3.5) that

$$(3.8) \quad \Im(f_n(z)) \begin{cases} < b\rho + \pi a/4, & \text{for } z \in \Gamma'_n, \\ > -b\rho - \pi a/4, & \text{for } z \in \Gamma''_n. \end{cases}$$

We may also assume that, for  $n \geq n_0$ ,

$$\Im(\xi_n - \xi'_n) \geq \pi/2 \text{ and } \Im(\xi''_n - \xi_n) \geq \pi/2,$$

so that by (3.4) and (3.5),

$$(3.9) \quad \Im(f_n(\xi'_n)) < -b\rho - \pi a/4 \text{ and } \Im(f_n(\xi''_n)) > b\rho + \pi a/4.$$

Now consider the maximal subarc  $l_n$  of  $f_n(\Gamma_n)$  which contains  $f_n(\xi_n) = b\rho(1 + \epsilon_n(\xi_n))$  and lies in the strip  $\{w : |\Im(w)| \leq b\rho + \pi a/4\}$ . In view of (3.7), (3.8) and (3.9),  $l_n$  has endpoints  $w'_n = f_n(\eta'_n)$  and  $w''_n = f_n(\eta''_n)$ , where

$$\Im(w'_n) = -b\rho - \pi a/4, \quad \Im(w''_n) = b\rho + \pi a/4.$$

Thus

$$\arg(w'_n) > \pi \text{ and } \arg(w''_n) < -\pi,$$

so that

$$5\pi/6 < \theta(\eta'_n) + 2\pi n < 3\pi/2 \text{ and } -3\pi/2 < \theta(\eta''_n) + 2\pi n < -5\pi/6.$$

Then let  $L_n$  denote the closed curve consisting of  $l_n$  together with a segment each from the lines

$$\{w : \Im(w) = -b\rho - \pi a/4\}, \quad \{w : \Im(w) = b\rho + \pi a/4\}, \quad \{w : \Re(w) = \mu_n\},$$

where  $\Re(f(z)) < \mu_n$ , for  $z \in \Gamma_n$ . By (3.6) and (3.7),  $L_n$  winds exactly twice round  $\{w : |w| \leq b\rho/2\}$ , and so is not simple. Thus  $l_n$  is not simple and so  $\Gamma_n$  must have a subarc  $\gamma_n$  such that  $f_n(\gamma_n)$  is closed and lies in the strip  $\{w : |\Im(w)| \leq b\rho + \pi a/4\}$ . This is sufficient to prove Lemma 3.2.

Lemma 3.2 shows that  $f$  is not univalent in  $U$ . To complete the proof of Theorem 2 part (c), we now take a sequence of arcs  $\gamma_n$  satisfying the conditions of Lemma 3.2 together with an  $\epsilon$ -neighbourhood  $G_n$  of each arc  $\gamma_n$  such that  $G_n \subset U$ . Since  $U$  is simply connected, the union  $\Omega_n$  of  $f(G_n)$  with the bounded complementary components of  $f(G_n)$  is a bounded, simply connected subset of  $U$ . We claim that  $\Omega_n$  contains a singularity of  $f^{-1}$  corresponding to a critical point or asymptotic path of  $f$  in  $U$ . Otherwise, the branch of  $f^{-1}$  mapping  $f(\alpha_n)$  to  $\alpha_n$ , where  $\alpha_n$  is an endpoint of  $\gamma_n$ , can be continued along all paths in  $\Omega_n$  to give a single-valued analytic function in  $\Omega_n$  with values in  $U$ , and this is impossible since  $f(\gamma_n) \subset \Omega_n$ . This completes the proof of Theorem 2.

#### 4. PROOF OF THEOREM 3

Let  $f(z) = az(1 + e^{-z^p})$ , where  $a > 1$  and  $p \in \mathbf{N}$ . To prove part (a) of Theorem 3, note that, if  $z = te^{i(2k+1)\pi/p}$ , where  $t > 0$  and  $k \in \{0, 1, \dots, p-1\}$ , then

$$f(z) = ate^{i(2k+1)\pi/p}(1 + e^{t^p}).$$

Thus

$$f^n(z) = \phi^n(t)e^{i(2k+1)n\pi/p},$$

where

$$\phi(t) = at(1 + e^{t^p}).$$

A simple calculation shows that  $\phi'(t) \geq \frac{2\phi(t)\ln\phi(t)}{t\ln t}$ , for large values of  $t$ , and it follows by integration (see [16]) that, if  $t_1$  is sufficiently large and  $t_2 > t_1$ , then

$$\frac{\ln\phi^n(t_2)}{\ln\phi^n(t_1)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Thus, if  $z_1 = t_1 e^{i(2k+1)\pi/p}$  and  $z_2 = t_2 e^{i(2k+1)\pi/p}$ , where  $t_2 > t_1 > 0$  and  $k \in \{0, 1, \dots, p-1\}$ , then

$$\frac{\ln(|f^n(z_2)| + 1)}{\ln(|f^n(z_1)| + 2)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

It now follows from (1.1) that  $z_1$  cannot belong to a Baker domain and, from hyperbolic metric estimates (see [16]), that  $z_1$  cannot belong to any component of  $N(f)$ . This proves part (a).

To prove part (b), we use the fact that  $(f(z))^p = g(z^p)$ , where

$$g(z) = a^p z(1 + e^{-z})^p = a^p z + pa^p z e^{-z}(1 + o(1)),$$

as  $\Re(z) \rightarrow \infty$ . By Theorem 2,  $g$  has a simply connected invariant Baker domain  $U$  which, for each  $\rho > 0$  and large values of  $R > 0$ , contains a set of the form

$$D_{\rho,R} = \{z : |ze^{-z}| < \rho, |z| > R\}.$$

Also, since  $g$  has no finite asymptotic values, there are infinitely many critical points of  $g$  in  $U$ .

It follows that  $f$  has a simply connected invariant Baker domain  $U_0$  in  $A_0$  which, for each  $\epsilon$ ,  $0 < \epsilon < \frac{\pi}{2p}$ , contains a set of the form

$$(4.1) \quad E_{\epsilon,R} = \{z : |\arg z| < \frac{\pi}{2p} - \epsilon, |z| > R\},$$

and that there are infinitely many critical points of  $f$  in  $U_0$ . Also, for  $k = 0, 1, \dots, p-1$ , each  $U_k = e^{2\pi ki/p} U_0$  is such an invariant Baker domain in  $A_k$ .

If  $f$  has other invariant Baker domains, then by symmetry, part (a) and the fact that the positive real axis lies in  $U$ , there is one,  $V$  say, which lies between  $U_0$  and  $\{z : \arg z = \pi/p\}$ . Since  $f$  is entire,  $V$  is simply connected and so, by (1.1),  $V$  contains a path  $\Gamma$  tending to  $\infty$  of the form

$$\Gamma = \bigcup_{n=0}^{\infty} f^n(\Gamma_0),$$

where  $\Gamma_0$  joins  $z_0$  to  $z_1 = f(z_0)$ , such that, for some  $C > 0$ ,

$$(4.2) \quad |f(z)| \leq C|z|, \text{ for } z \in \Gamma.$$

Now let

$$B_\epsilon = \{z : |\arg z - \frac{\pi}{p}| < \frac{\pi}{2p} - \epsilon\}, \quad 0 < \epsilon < \frac{\pi}{2p}.$$

For  $z \in B_\epsilon$ , we have

$$\frac{\pi}{2} + p\epsilon < \arg(z^p) < \frac{3\pi}{2} - p\epsilon,$$

so that, for such  $z$ ,

$$|f(z)| \geq a|z|(\exp(\Re(-z^p)) - 1) \geq a|z|(\exp(\frac{2\epsilon p}{\pi}|z|^p) - 1).$$

From this, (4.1), the fact that  $U_0 \cap V = \emptyset$  and (4.2), we deduce that

$$(4.3) \quad \arg z \rightarrow \frac{\pi}{2p} \text{ as } z \rightarrow \infty \text{ along } \Gamma.$$

Hence

$$\arg(-z^p) \rightarrow -\frac{\pi}{2} \text{ as } z \rightarrow \infty \text{ along } \Gamma.$$

Thus, if  $h(z) = e^{-z^p}$ , then  $h(\Gamma)$  winds infinitely often round 0. In particular, there is a sequence  $\xi_n \in \Gamma$  tending to  $\infty$  such that  $e^{-\xi_n^p}$  is real and positive. Now

$$|f(\xi_n) - \xi_n| = |a\xi_n(1 + e^{-\xi_n^p}) - \xi_n| \geq (a - 1)|\xi_n|.$$

Thus, by Theorem 1 part (b),

$$V \supset \bigcup_{n=0}^{\infty} B(\xi_n, c(a - 1)|\xi_n|),$$

for some  $c > 0$ . Together with (4.3), this implies that, if  $\epsilon > 0$  is sufficiently small, then  $V \cap E_{\epsilon, R} \neq \emptyset$  and hence  $V \cap U_0 \neq \emptyset$ . This, however, is a contradiction, and so the proof of Theorem 3 is complete.

#### 5. PROOF OF THEOREM 4

Recall that Theorem 4 states that, for each  $p \in \mathbf{N}$ , there exists an entire function  $f$  which has a  $p$ -cycle of Baker domains on which  $f$  is univalent. To prove this result we consider, for  $k \in \{0, 1, \dots, p - 1\}$ , the truncated sector

$$S_k = \{z : |z| \geq \frac{3}{4}, |\arg z - 2k\pi/p| \leq \frac{\pi}{2p}\},$$

and put  $\omega_k = e^{2\pi ik/p}$ ,

$$f_k(z) = a(z - \omega_k) + \omega_k, \quad g_k(z) = b(z - \omega_k) + \omega_k, \quad h_k(z) = c(z - \omega_k) + \omega_k,$$

where the constants  $a, b$  and  $c$  are chosen so that  $1 < c < b < a < 3/2$  and the sets  $f_k(S_k)$  are mutually disjoint. Then put

$$S = \mathbf{C} \setminus \bigcup_{k=0}^{p-1} g_k(S_k),$$

$$T_k = \{z : |\arg(z - 2\omega_k) - 2k\pi/p| \leq \frac{\pi}{2p}\},$$

and

$$\epsilon = \min\{\text{dist}(f_0(\partial S_0), g_0(\partial S_0)), \text{dist}(f_0(\partial T_0), g_0(\partial T_0)), \text{dist}(\partial S_0, h_0(\partial S_0))\}.$$

In particular,  $0 < \epsilon < a - 1 < 1/2$ .

It follows from Arakelyan's Theorem [12] that there exists an entire function  $f$  such that

$$(5.1) \quad |f(z) - f_k(z)| < \epsilon, \text{ for } z \in h_k(S_k), \quad k = 0, 1, \dots, p - 1,$$

and

$$(5.2) \quad |f(z)| < \epsilon, \text{ for } z \in S.$$

Since we may assume that  $f$  is symmetric under rotation by  $\omega_1$ , we need only consider the case  $k = 0$  from now on. Condition (5.1) implies that  $f(\partial S_0) \subset S$  and condition (5.2) then implies that  $\partial S_0$  is contained in an attracting component of  $N(f)$ . On the other hand, condition (5.1) and the fact that  $b > 1$  implies that  $f(T_0) \subset g_0(T_0) \subset T_0$  and, for  $z \in T_0$ ,

$$\begin{aligned} \Re(f(z) - z) &\geq \Re(f_0(z) - z) - |f_0(z) - f(z)| \\ &\geq a - 1 - \epsilon > 0. \end{aligned}$$

Thus  $T_0$  must be a subset of an invariant Baker domain  $U_0$  for  $f$ , which is disjoint from  $\partial S_0$ .

We now check that  $f$  is univalent on  $S_0$  and hence on  $U_0$ . If we write  $f(z) = a(z-1) + 1 + \phi(z)$ , then  $|\phi(z)| < \epsilon$  on  $h_0(S_0)$  and so  $|\phi'(z)| < \epsilon/\epsilon = 1$  on  $S_0$ . Thus, if  $z_1, z_2 \in S_0$  with  $f(z_1) = f(z_2)$ , then

$$a|z_1 - z_2| = |\phi(z_1) - \phi(z_2)| \leq |z_1 - z_2|$$

and so  $z_1 = z_2$  as required.

We end by observing that the function  $\omega_1 f$  has the required properties.

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DEPARTMENT OF PURE MATHEMATICS, THE OPEN UNIVERSITY, WALTON HALL, MILTON KEYNES, MK7 6AA ENGLAND

*E-mail address*: p.j.rippon@open.ac.uk

*E-mail address*: g.m.stallard@open.ac.uk