GEOMETRY OF THE FEIGENBAUM MAP

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ABSTRACT. We show that the Cvitanović-Feigenbaum equation can be interpreted as a linearizing equation, and the domain of analyticity of the Feigenbaum fixed point of renormalization as a basin of attraction. As a consequence, we give a combinatorial description of this ramified covering, and we show the surprising result that there exist points in the boundary of this domain with three accesses from inside the domain. Besides, there is a natural decomposition of this basin which makes it possible to recover a result of local connectivity by Hu and Jiang (*The Julia set of the Feigenbaum quadratic polynomial is locally connected*, Preprint, 1993) for the Feigenbaum Julia set.

INTRODUCTION

In 1976, Feigenbaum [F1] [F2], and independently Coullet and Tresser [CT] introduced the notion of renormalization for real dynamical systems. In 1985, the generalization to holomorphic dynamical systems via polynomial-like mappings, was introduced by Douady and Hubbard [DH].

In 1987, Lanford [L] gave a computer assisted proof of the existence of a fixed point for the period-doubling renormalization operator. His result was completed by Eckmann and Wittwer [EW]. Later, Epstein [E1] [E2] gave a proof of the existence which does not require the help of computers. Finally, in the 90's, Sullivan [S] proved the uniqueness of the fixed point for the period-doubling renormalization operator.

This fixed point of renormalization satisfies a functional equation known as the Cvitanović-Feigenbaum equation:

$$f(z) = -\frac{1}{\lambda} f \circ f(\lambda z), \quad 0 < \lambda < 1.$$

We will use this functional equation to study the domain of analyticity of the solution f. Epstein [E2] and McMullen [McM] (7.3) proved that it has a maximal analytic extension – still denoted by f – to a simply connected, open set $\widehat{W}$, dense in $\mathbb{C}$. We will prove the following results.

Theorem A. *The map $f_\lambda(z) = f(\lambda z)$ defined on $\widehat{W}/\lambda$ has a fixed point $x_0 \in \widehat{W}$ with multiplier $-\lambda$. Besides, f_λ is analytic on $\widehat{W}$, $f_\lambda(\widehat{W}) = \widehat{W}$, and $\widehat{W}$ is the basin of attraction of the map f_λ. Finally, f is a linearizer of the map f_λ.*

Theorem B. *All the critical points of f are simple. The critical values of f are all contained in the real axis. Besides, the analytic extension $f : \widehat{W} \to \mathbb{C}$ is a ramified covering, in the sense that for any bounded disk $\mathbb{D}_R \subset \mathbb{C}$, each component of $f^{-1}(\mathbb{D}_R)$ is proper over $\mathbb{D}_R$.*

Since the critical values are real, we will then study the combinatorial structure of $f^{-1}(\mathbb{R})$. We will show that $f : \widehat{W} \to \mathbb{C}$ has the same combinatorial structure as the linearizer of a Blaschke product.

There was a belief that $\partial\widehat{W}$ was a Cantor bouquet. However, the following result proves that such is not the case.

Theorem C. *The boundary of $\widehat{W}$ contains points with three accesses from inside $\widehat{W}$. In particular, this boundary does not have the structure of a Cantor Bouquet.*

We will finally use a natural tiling of $\widehat{W}$ to study the local connectivity of the Julia set of the Feigenbaum polynomial.

1. Renormalization and tuning

1.1. Quadratic-like mappings. To get more familiar with the notions of renormalization and polynomial-like mappings, one can for example consult [DH], [McM] or [S].

Definition 1. Let us recall that a quadratic-like map $f : U' \to U$ is a ramified covering of degree 2 between two topological disks U' and U, with U' relatively compact in U. One can define its filled-in Julia set $K(f)$ and its Julia set $J(f)$ as follows:

$$\begin{cases} K(f) = \{z \in U' \mid (\forall n \in \mathbb{N}) \ f^{\circ n}(z) \in U'\}, \\ J(f) = \partial K(f). \end{cases}$$

Examples. Let $P(z) = z^2 + c$ be a quadratic polynomial. If R is large enough, the restriction of P to the disk $\mathbb{D}_R$ centered at 0 with radius R is a quadratic-like map.

The restriction of the map $f(z) = 2\cos(z) - 1.7 + .9i$ to the square

$$U' = \{z \mid -2 < \mathrm{Re}(z) < 2 \ \text{ and } \ -2 < \mathrm{Im}(z) < 2\}$$

is also a quadratic-like map. Figure 1 shows this quadratic-like map and its filled-in Julia set.

Let us also recall the so-called Straightening Theorem due to Douady and Hubbard.
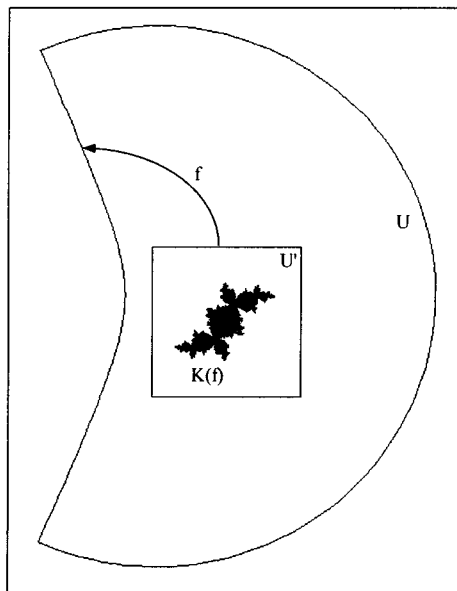
FIGURE 1. The quadratic-like map $z \to 2\cos(z) - 1.7 + .9i$ and its filled-in Julia set.

Theorem 1 (Straightening Theorem). *If $f : U' \to U$ is a quadratic-like map, then there exist*

- *a quadratic polynomial $P(z) = z^2 + c$,*
- *a neighborhood V of the filled-in Julia set $K(P)$ such that $P : P^{-1}(V) = V' \to V$ is a quadratic-like map, and*
- *a quasiconformal homeomorphism $\phi : U \to V$ with $\phi(U') = V'$, such that $\overline{\partial}\phi = 0$ almost everywhere on $K(f)$ and such that on U'*

$$\phi \circ P = f \circ \phi.$$

Moreover, if $K(f)$ is connected, then P is unique.

Definition 2. Two quadratic-like maps f and g are said to be hybrid equivalent if there is a quasi-conformal conjugacy between them, with $\overline{\partial}h = 0$ almost everywhere on the filled-in Julia set $K(f)$.

This defines an equivalence relation between quadratic-like mappings. The Straightening Theorem says that the hybrid class of a quadratic-like mapping f always contains a quadratic polynomial, and that this polynomial is unique up to affine conjugacy if the Julia set $J(f)$ is connected.

1.2. **Renormalization.** Given a quadratic-like map $f : U' \to U$ with a connected Julia set $J(f)$, there is always a fixed point α which is either non-repelling, or separating (i.e., $J(f) \setminus \{\alpha\}$ has several connected components). This point is called the α fixed point of f. The other fixed point is called the β fixed point of f.

Definition 3. Let $f : U' \to U$ be a quadratic-like map with connected Julia set, and let ω be its critical point. We say that f is renormalizable with period $n > 1$

if there exist an integer $n > 1$, and open disks $V' \subset V \subset \mathbb{C}$ containing ω, such that
- the map $f^{\circ n} : V' \to V$ is a quadratic-like map,
- its Julia set J_0 is connected, and
- the small Julia sets $J_i = f^{\circ i}(J_0)$, $i = 1, \ldots, n - 1$, intersect J_0 at most at its β fixed point.

The quadratic-like map $f^{\circ n} : V' \to V$ is called a renormalization of f. Note that there are several possible choices of domains. Hence, there are several possible renormalizations of a given period. However, when the Julia set of the renormalization is connected, the problem can be solved by defining germs of quadratic-like maps. This will be done later.

Examples. Figure 2 shows the Julia set of the polynomial $P(z) = z^2 - 1.75$, together with the first three forward images of the disk $\Delta_0 = \{z \ / \ |z| < .25\}$. The map $P^{\circ 3} : \Delta_0 \to \Delta_3$ is a quadratic-like map, and its Julia set (which looks like a cauliflower) is connected. Hence, P is renormalizable.
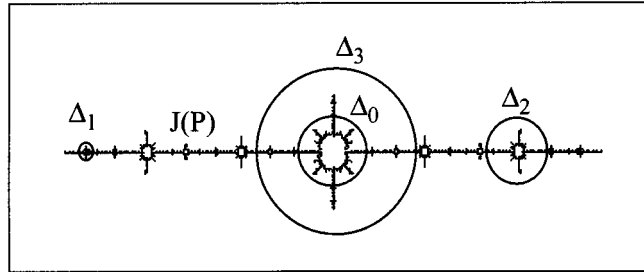

FIGURE 2. The polynomial $P(z) = z^2 - 1.75$ is renormalizable.

Now, given a quadratic polynomial P which is renormalizable; we let $f = P^{\circ n} : V' \to V$ be a renormalization. One can construct a new quadratic polynomial by straightening the map f. The inverse process is called tuning.

1.3. **Tuning.** More precisely, given a parameter c in the Mandelbrot set M, such that 0 is a periodic point of period p for the polynomial $P_c(z) = z^2 + c$, Douady and Hubbard [DH] have constructed a tuning map, $x \to c*x$, which is a homeomorphism of M into itself, sending 0 to c, and such that if $x \neq 1/4$, then f_{c*x} is renormalizable with period p, and the corresponding renormalization is in the same inner class as P_x. This is the way they show there are small copies of the Mandelbrot set within itself (see Figure 3).

The central object in our study is the Feigenbaum polynomial. One can define this polynomial as the unique real polynomial which is a fixed point of tuning by -1. In other words, the Feigenbaum parameter $c_{Feig} = -1.401155...$, is the unique real parameter in the intersection of all the copies of M obtained by tuning by -1. This intersection is not known to consist of a single point, but its real trace is known to be reduced to a single point. Figure 4 shows the location of the parameter c_{Feig} in the Mandelbrot set, and Figure 5 shows the Julia set of the Feigenbaum polynomial.

The Feigenbaum polynomial is the most famous example of a polynomial to be infinitely renormalizable. By construction, the Feigenbaum polynomial, P_{Feig}, is
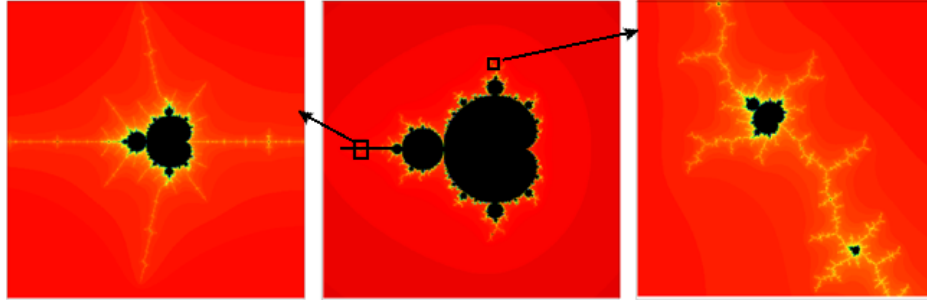
FIGURE 3. The Mandelbrot set and two small copies correspond-
ing to renormalization of period 3.
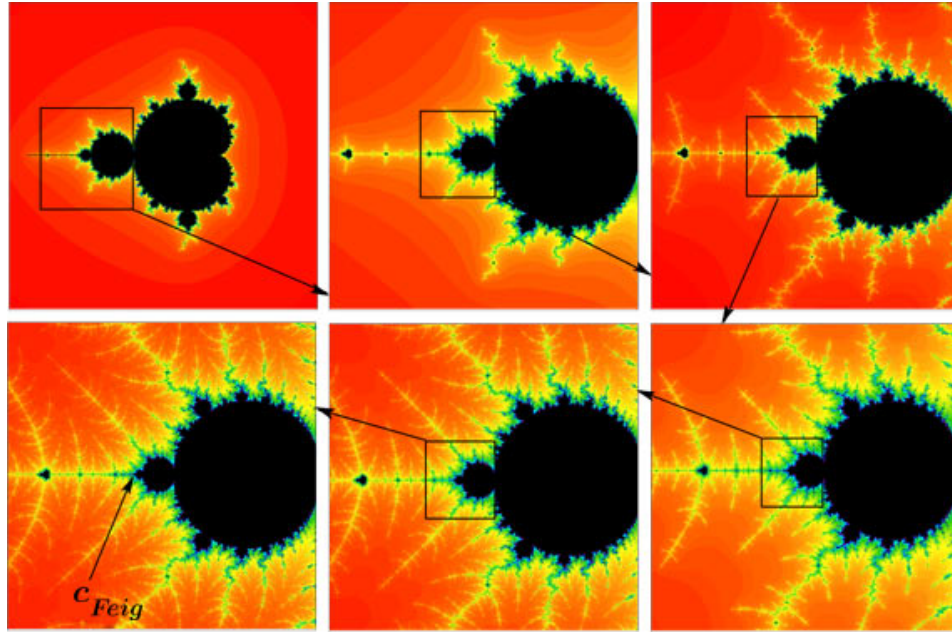
FIGURE 4. The location of the parameter c_{Feig} in the Mandelbrot set.

renormalizable with period 2, i.e., there exist domains U' and U such that $f = P^{\circ 2} : U' \to U$ is a quadratic-like mapping with connected Julia set. There are several possible choices of domains, hence several renormalizations. However, all have the same Julia set, and are equal on this Julia set.

Definition 4. We will say that two quadratic-like maps $f : U' \to U$ and $g : V' \to V$ with connected Julia set define the same germ $[f]$ of quadratic-like map if $J(f) = J(g)$ and $f|J(f) = g|J(g)$.

We can define a renormalization operator $\mathcal{R}_2$ in the following way.

Definition 5. Assume $[f]$ is a germ of a quadratic-like map which is renormalizable with period 2. There exist open sets U' and U such that the map $g : U' \to U$ defined by $g = f^{\circ 2}|U$ is a polynomial-like map with connected Julia set. The

FIGURE 5. The Julia set of the Feigenbaum polynomial.

renormalization operator $\mathcal{R}_2$ is defined by

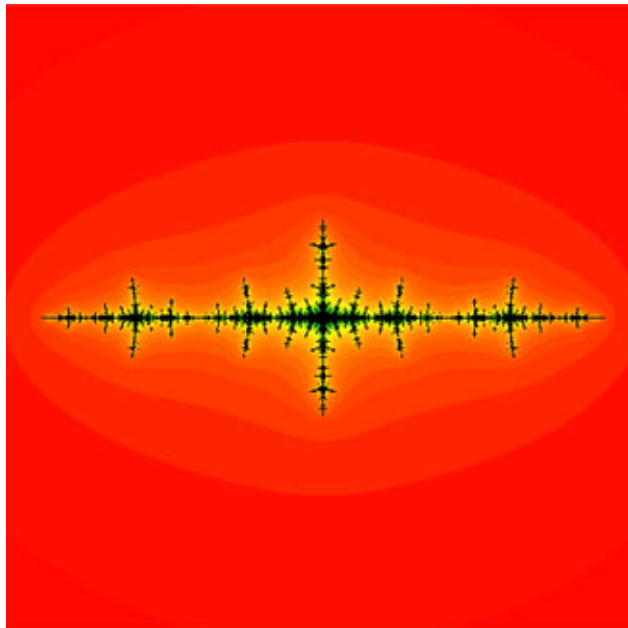$$\mathcal{R}_2([f]) = [\alpha^{-1} \circ g \circ \alpha],$$

with $\alpha = g(0) = f^2(0)$, and $\alpha(z) = \alpha z$.

We have normalized the germ so that the critical value is 1.

Definition 6. Two quadratic-like germs $[f]$ and $[g]$ are said to be hybrid equivalent if there exist representatives $f : U' \to U$ and $g : V' \to V$ which are hybrid equivalent.

The equivalence classes are called hybrid classes of germs.

Now, let P_{Feig} be the Feigenbaum polynomial. The Straightening Theorem says that there is a unique polynomial $Q(z) = z^2 + c$ in the hybrid class of $\mathcal{R}_2([P_{Feig}])$. This polynomial must be renormalizable with period 2^k, for all $k \geq 1$. Besides, it must be a real polynomial. So it is the Feigenbaum polynomial. Hence, we can define a sequence of germs $\mathcal{R}_2^{\circ n}([P_{Feig}])$, $n \in \mathbb{N}$. The following result can be found in [S], [dMvS] or [McM].

Theorem 2. *The sequence of germs* $\mathcal{R}_2^{\circ n}([P_{Feig}])$, $n \in \mathbb{N}$, *converges to a point* $[f]$. *This germ is a fixed point of renormalization:*

$$\mathcal{R}_2([f]) = [f],$$

and is in the hybrid class of the Feigenbaum polynomial. It is the unique fixed point of $\mathcal{R}_2$.

2. THE CVITANOVIĆ-FEIGENBAUM EQUATION

We will first state some results about this fixed point of renormalization. It is just a germ of quadratic-like map. However, Epstein [E2] proved that there exists a natural representative of this germ. We think it is worth making the construction here, to help the reader get accustomed with the tools.

First of all, we would like to recall that the fixed point $[f]$ of $\mathcal{R}_2$ is a solution of the following system of equations (see [E2]).

Definition 7. The Cvitanović-Feigenbaum equation:

$$\begin{cases} f(z) = -\dfrac{1}{\lambda} f \circ f(\lambda z), \\ f(0) = 1, \\ f(z) = F(z^2), \text{ with } F^{-1} \text{ univalent in } \mathbb{C} \setminus (]-\infty, -1/\lambda] \cup [1/\lambda^2, +\infty[). \end{cases}$$

Theorem 3 (Epstein, see [E2]). *Let f be a solution of the equation, and let x_0 be the first positive preimage of 0 by f. Then,*

- *$f(1) = -\lambda$,*
- *$f(\lambda x_0) = x_0$, and*
- *the first critical point in $\mathbb{R}^+$ is x_0/λ, with $f(x_0/\lambda) = -1/\lambda$.*

Figure 6 illustrates Proposition 3. This graph gives the relative positions of some points on the real axis.
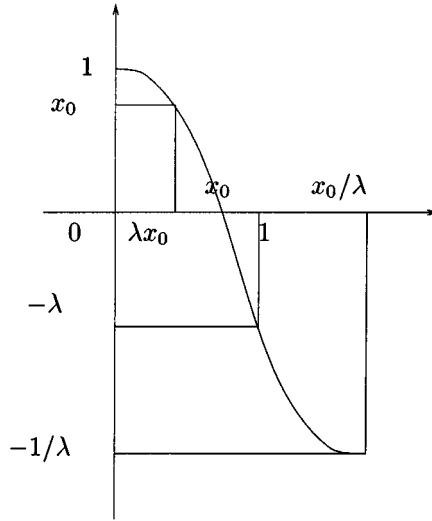
FIGURE 6. The graph of f on $\mathbb{R}^+$.

Theorem 4 (Epstein, see [E2]). *Let F be defined by the Cvitanović-Feigenbaum equation. We then have the following results:*

- *it is possible to extend F^{-1} continuously to the boundary $\mathbb{R}$ of $\mathbb{H}_+$, and even analytically except at points $(-1/\lambda)^n$, $n \geq 1$, which are branching points of type $z^{1/2}$,*
- *the values of F^{-1} are never real except in $[-1/\lambda, 1/\lambda^2]$,*
- *the extension of F^{-1} to the closure of $\mathbb{H}_+$ is injective, and*
- *when z tends to infinity in $\mathbb{H}_+$, $F^{-1}(z)$ tends to a point in $\mathbb{H}_-$ which will be denoted by $F^{-1}(i\infty)$.*

By symmetry, similar statements hold in $\mathbb{H}_-$. Hence, $\mathcal{W} = F^{-1}(\mathbb{C}_\lambda)$ is a bounded domain of $\mathbb{C}$. Those results are summarized in Figure 7.

FIGURE 7. Maximal univalent extension of F.

In the following, we will use the notations:

$$\mathbb{C}_\lambda = \mathbb{C} \setminus (]-\infty, -1/\lambda] \cup [1/\lambda^2, +\infty[),$$
$$\mathcal{W} = F^{-1}(\mathbb{C}_\lambda),$$
$$W = \{z \in \mathbb{C} \mid z^2 \in \mathcal{W}\},$$
$$W_+ = W \cap \{z \in \mathbb{C} \mid 0 < \mathrm{Arg}(z) < \pi/2\}, \text{ and}$$
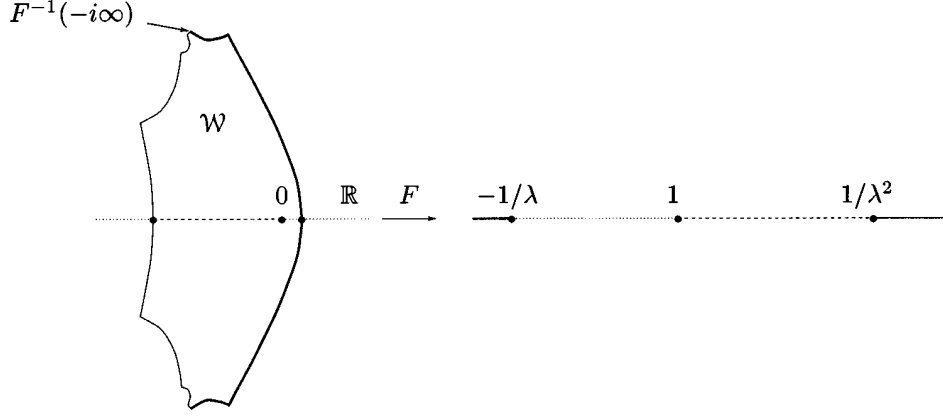$$W_- = W \cap \{z \in \mathbb{C} \mid -\pi/2 < \mathrm{Arg}(z) < 0\}.$$

As $\overline{W}_+ \cap \mathbb{R} = [0, x_0/\lambda]$, we can deduce that $W \cap \mathbb{R} =]-x_0/\lambda, x_0/\lambda[$.

Corollary 1. *The map* $f : W \to \mathbb{C}_\lambda$ *is a quadratic-like map representing the renormalization fixed point.*

Proof. The graph of f (see Figure 6) makes it possible to conclude that $\overline{W} \subset \mathbb{C}_\lambda$, because $x_0 < 1$. Hence $f : W \to \mathbb{C}_\lambda$ is a quadratic-like map. Besides, f has a unique critical point at 0. The corollary follows from the uniqueness of the fixed point. ∎

This quadratic-like map is in the hybrid class of the Feigenbaum polynomial. Thus, to obtain informations like local connectivity, Hausdorff dimension or Lebesgue measure of the Julia set of the Feigenbaum polynomial, it is enough to know the corresponding results for the Julia set $K(f)$ of this quadratic-like map (see Figure 8).

Definition 8. In the following, we will use the notation

$$W_n = f^{-2^n}(\mathbb{C}_\lambda).$$

Observe that since $f : W \to \mathbb{C}_\lambda$ is a polynomial-like map with non-escaping critical point, the set W_n is simply connected, $W_{n+1} \subset W_n$, and $K(f) = \bigcap W_n$.
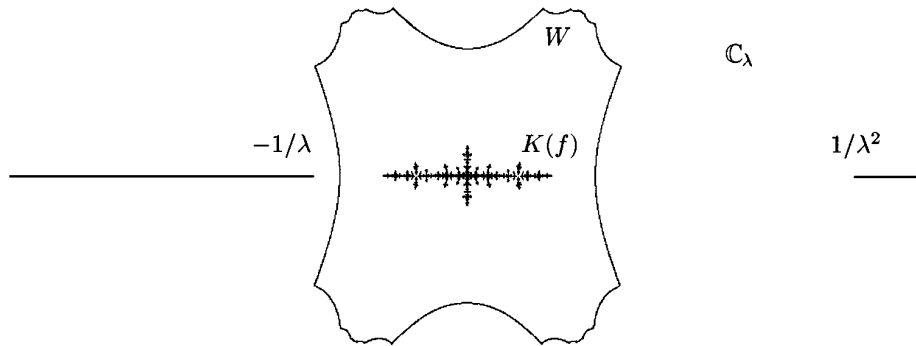
FIGURE 8. A representative of the Feigenbaum fixed point of renormalization, and the Julia set $K(f)$.

3. GEOMETRY OF THE DOMAIN OF ANALYTICITY OF f

Our first goal is to describe the geometry of the maximal domain of analyticity $\widehat{W}$ of f. Epstein [E2] proved that it is an open simply connected subset of $\mathbb{C}$ and McMullen [McM] (7.3) proved that this domain is a dense subset of $\mathbb{C}$. We will show that $\widehat{W}$ is contained in $\widehat{W}/\lambda$, and that it can be seen as the basin of attraction of a map related to f. We will then introduce a tiling and some puzzles. We will use them to give a combinatorial description of f and to prove the local connectivity of the Julia set $K(f)$ at the critical point.

3.1. The domain of analyticity.

Definition 9. Let f and g be two holomorphic functions defined on open connected domains of $\mathbb{C}$: U_f and U_g. We say that g is an analytic extension of f if $f = g$ on some non-empty open set. Moreover, if all such analytic extensions are restrictions of a single map

$$\hat{f}: \widehat{W} \to \mathbb{C},$$

we will say that $\hat{f}$ is the maximal analytic extension of f.

Proposition 1 (see [E2]). *Let f be a solution of the Cvitanović-Feigenbaum equation. It has a maximal analytic extension*

$$\hat{f}: \widehat{W} \to \mathbb{C},$$

where $\widehat{W}$ is a simply connected open subset of $\mathbb{C}$.

Proof. First of all, recall that $f: W \to \mathbb{C}_\lambda$ is a quadratic-like map with non-escaping critical point. Hence, the set $W_n = f^{-2^n}(\mathbb{C}_\lambda)$ is a simply connected open set and the map $f^{\circ 2^n}: W_n \to \mathbb{C}_\lambda$ is a proper map. But the Cvitanović-Feigenbaum equation shows that

$$f_n = \left(-\frac{1}{\lambda}\right)^n f^{\circ 2^n}(\lambda^n z): \ \frac{1}{\lambda^n}W_n \to \frac{1}{(-\lambda)^n}\mathbb{C}_\lambda$$

is an extension of f.

Moreover, as

$$\frac{1}{(-\lambda)^{n-1}}\mathbb{C}_\lambda \subset \frac{1}{(-\lambda)^n}\mathbb{C}_\lambda,$$

the range of f_{n-1} is contained in the range of f_n. Since the domains are connected, we claim that the inclusion also holds for the domains:

$$\frac{1}{\lambda^{n-1}}W_{n-1} \subset \frac{1}{\lambda^n}W_n.$$

Indeed, consider the connected component U of $W_{n-1}/\lambda^{n-1} \cap W_n/\lambda^n$ which contains 0. The two mappings f_{n-1} and f_n coincide on this component. Besides, if the inclusion of the domains does not hold, U must have a boundary point in W_{n-1}/λ^{n-1}, which is mapped by f_{n-1} in the interior of $\mathbb{C}_\lambda/(-\lambda)^n$. This contradicts the properness of f_n.

We can now define

$$\widehat{W} = \bigcup_{n\in\mathbb{N}} \frac{1}{\lambda^n}W_n.$$

The map f has an analytic extension $\hat{f}$ defined on $\widehat{W}$ which coincides with f_n, if $z \in W_n/\lambda^n$.

Next, let us prove that we cannot extend $\hat{f}$ across any point of $\partial\widehat{W}$. Any point $z \in \partial\widehat{W}$ can be approximated by points $z_n \in \partial(W_n/\lambda^n)$. On the other hand, $|\hat{f}(z_n)| = |f_n(z_n)|$ tends to infinity when n tends to infinity. Besides, z cannot be a pole since there is no isolated point in the boundary of $\widehat{W}$. Hence, we cannot extend $\hat{f}$ across any point of $\partial\widehat{W}$. This shows that $\hat{f}$ is a maximal analytic extension of f. ∎

To lighten the notations, we will denote by f – instead of $\hat{f}$ – the maximal analytic extension, when there is no ambiguity. However, $K(f)$ still denotes the Julia set of the quadratic-like restriction $f : W \to \mathbb{C}_\lambda$.

4. THE MAP f IS A LINEARIZER

This section is devoted to the proof of the following result.

Theorem A. *The map $f_\lambda(z) = f(\lambda z)$ defined on $\widehat{W}/\lambda$ has a fixed point $x_0 \in \widehat{W}$ with multiplier $-\lambda$. Besides, f_λ is analytic on $\widehat{W}$, $f_\lambda(\widehat{W}) = \widehat{W}$, and $\widehat{W}$ is the basin of attraction of the map f_λ. Finally, f is a linearizer of the map f_λ.*

Proof. Let us first prove that the map f_λ has an attracting fixed point. In fact, we will prove a little bit more for later purposes.

Lemma 1. *The point x_0 is an attracting fixed point of the map $f_\lambda(z) = f(\lambda z)$. Its multiplier is $-\lambda$. Moreover $f_\lambda : W_+ \to W_-$ and $f_\lambda : W_- \to W_+$ are isomorphisms.*

Remark. The fact that f_λ is an isomorphism from W_+ onto W_- does not mean that f_λ is an automorphism of $W_+ \cup W_-$. Indeed, part of the boundary of $W_\pm$ is mapped inside $W_+ \cup W_-$. Hence f_λ maps $W_+ \cup W_-$ onto $W_+ \cup W_-$ minus a slit. This is the way f_λ can have an attracting fixed point x_0 inside $W_+ \cup W_-$.

Proof. The trick in the proof is that the Cvitanović-Feigenbaum equation can be translated on the following commutative diagram:

$$
\begin{array}{ccc}
(\widehat{W}, x_0) & \xrightarrow{\;f_\lambda\;} & (\widehat{W}, x_0) \\
f \downarrow & & \downarrow f \\
(\mathbb{C}, 0) & \xrightarrow[z \mapsto -\lambda z]{} & (\mathbb{C}, 0).
\end{array}
$$

It is a linearizing equation which proves the first part of the proposition.

Next, in Proposition 1, we obtained the following inclusions:

$$
\lambda W \subset W_1 \subset W.
$$

Hence, $\lambda W_+ \subset W_+$, and since $f : W_+ \to \mathbb{H}_-$ is an isomorphism, we see that f_λ is an isomorphism from W_+ onto its image. On the other hand,

$$
\mathbb{H}_- = f(W_+) = -\frac{1}{\lambda} f \circ f(\lambda W_+).
$$

Hence $f(f_\lambda(W_+)) = \mathbb{H}_+$, and the set $f_\lambda(W_+)$ is a preimage of $\mathbb{H}_+$ under f. As x_0 is in the closure of W_+, $x_0 = f_\lambda(x_0)$ is in the closure of $f_\lambda(W_+)$. Hence, we can immediately deduce that $f_\lambda(W_+) = W_-$. We can use the same arguments to show that $f_\lambda : W_- \to W_+$ is an isomorphism. ∎

Next, let us show that the map f_λ is analytic on $\widehat{W}$. We have to show that $\lambda \widehat{W}$ is contained in the domain of analyticity of f.

Lemma 2. *The domain $\lambda \widehat{W}$ is contained in $\widehat{W}$ and $f_\lambda(\widehat{W}) \subset \widehat{W}$.*

Proof. By construction, $W_{n+1} \subset W_n$, and in the proof of Proposition 1, we saw that $W_n/\lambda^n \subset W_{n+1}/\lambda^{n+1}$. Hence,

$$
\lambda \frac{1}{\lambda^n} W_n \subset \lambda \frac{1}{\lambda^{n+1}} W_{n+1} \subset \frac{1}{\lambda^n} W_n.
$$

Since the sets W_n/λ^n give an exhaustion of $\widehat{W}$, this proves that $\lambda \widehat{W} \subset \widehat{W}$. In particular, f is defined on $\lambda \widehat{W}$.

Now, notice that $f^{\circ 2^n}(W_{n+1}) = W_n$. Hence,

$$
f\left(\lambda \frac{1}{\lambda^n} W_n\right) \subset f\left(\lambda \frac{1}{\lambda^{n+1}} W_{n+1}\right) = \left(-\frac{1}{\lambda}\right)^n f^{\circ 2^n}(W_{n+1}) = \frac{1}{\lambda^n} W_n.
$$

This shows that $f(\lambda \widehat{W}) \subset \widehat{W}$. ∎

We will now finish the proof of Theorem A. We will prove that $\widehat{W}$ is that immediate basin of attraction of the map f_λ. Since McMullen proved that $\widehat{W}$ is dense in $\mathbb{C}$, there cannot be any other component in the basin of f_λ.

The commutative diagram we have written tells us that the linearizer of the map f_λ is f. We claim that the domain of analyticity of the linearizer is the immediate basin of the attracting fixed point x_0.

To avoid confusion let us use the name ϕ for the linearizer. We can always extend the linearizer to the immediate basin of attraction using the formula:

$$
\phi(z) = \frac{1}{\lambda^n} f_\lambda^{\circ n}(z),
$$

where n is so chosen large enough so that $f_\lambda^{\circ n}(z)$ is in a neighborhood of x_0 where ϕ is already defined. This proves that the immediate basin of attraction is contained in $\widehat{W}$.

To show the reverse inclusion, we just need to use Lemma 2: the mapping f_λ is analytic on $\widehat{W}$ and maps this domain into itself. Since $\widehat{W}$ is connected, Schwartz's Lemma shows that f_λ is contracting on $\widehat{W}$ for its Poincaré metric. Hence $\widehat{W}$ is contained in the immediate basin of attraction of f_λ. $\blacksquare$

The following two propositions are immediate applications of this result.

Proposition 2. *We have the following inclusion of sets:*

$$\lambda K(f) \subset K(f) \subset \bigcup_{n \in \mathbb{N}} \frac{K(f)}{\lambda^n} \subset \bigcap_{n \in \mathbb{N}} \lambda^n \widehat{W} \subset \widehat{W} \subset \frac{\widehat{W}}{\lambda}.$$

Proof. The set $\lambda K(f)$ is the Julia set $K(f^{\circ 2})$ of the renormalization

$$f^{\circ 2} \; : \; \lambda W \to \lambda \mathbb{C}_\lambda.$$

This Julia set is contained in $K(f)$. The inclusion

$$K(f) \subset \bigcup_{n \in \mathbb{N}} K(f)/\lambda^n$$

follows immediately.

The inclusion $\widehat{W} \subset \widehat{W}/\lambda$ is given by Lemma 2. The inclusion

$$\bigcap_{n \in \mathbb{N}} \lambda^n \widehat{W} \subset \widehat{W}$$

follows immediately from this one.

Lemma 3. *The sets $K(f)/\lambda^n$, $(n \in \mathbb{N})$, are all contained in $\widehat{W}$.*

Proof. The domain $\widehat{W}$ is the union of the sets W_n/λ^n. But $K(f) \subset W_n = f^{-2^n}(\mathbb{C}_\lambda)$. Hence, for all $n \in \mathbb{N}$,

$$K(f)/\lambda^n \subset W_n/\lambda^n \subset \widehat{W}. \quad \square$$

The remaining inclusion

$$\bigcup_{n \in \mathbb{N}} \frac{K(f)}{\lambda^n} \subset \bigcap_{n \in \mathbb{N}} \lambda^n \widehat{W}$$

follows from this lemma. $\blacksquare$

McMullen proved that the union $\bigcup_{n \in \mathbb{N}} K(f)/\lambda^n$ is dense in $\mathbb{C}$. Hence $\widehat{W}$ is a dense open subset of $\mathbb{C}$. As $K(f)$ is a compact set with empty interior, the inclusions are strict because of Baire's Theorem: a countable union of closed sets with empty interior cannot be equal to a countable intersection of dense open sets.

We have used this information to draw the boundary of $\widehat{W}$. Figure 9 shows a blow-up near the critical point of the Feigenbaum Julia set. The points which are colored in black correspond to points in the boundary of $\widehat{W}$. In the next section, we will describe a tiling of the domain $\widehat{W}$. We would like to mention that Epstein has a very nice picture of the domain of analyticity $\widehat{W}$ (see [E2]).

Finally, the intersection of the contractions $\lambda^n \widehat{W}$ can be defined dynamically.
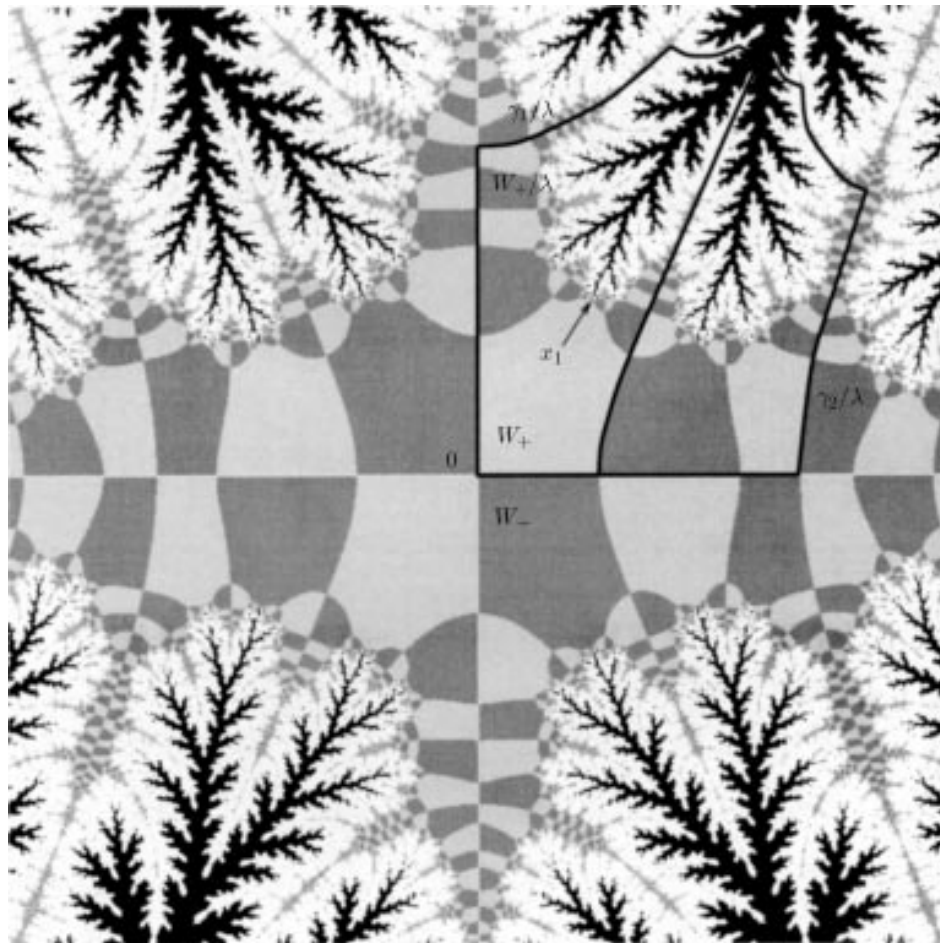
FIGURE 9. The domain of analyticity of the Feigenbaum fixed
point of renormalization. Everything which is not black, is in the
domain of analyticity. The map f sends light tiles to $\mathbb{H}_-$ and dark
tiles to $\mathbb{H}_+$.

Proposition 3. *The intersection of all the sets $\lambda^n \widehat{W}$ is the set of points whose
orbit under iteration of $f : \widehat{W} \to \mathbb{C}$ stay in $\widehat{W}$:*

$$\bigcap_{n \in \mathbb{N}} \lambda^n \widehat{W} = \widehat{K} = \{ z \in \widehat{W} \mid (\forall n \in \mathbb{N}) \ f^{\circ n}(z) \in \widehat{W} \}.$$

Proof. To show this equality, note that

$$\widehat{K} = \bigcap_{n \in \mathbb{N}} f^{-n}(\widehat{W}).$$

We have noticed that $f(\lambda \widehat{W}) = f_\lambda(\widehat{W}) = \widehat{W}$. Moreover, in a neighborhood of the
origin,

$$f(z) = -\frac{1}{\lambda} f \circ f(\lambda z).$$

This equality has to be true whenever both sides are simultaneously defined. The left-hand side is defined on $\widehat{W}$. When $z \in \widehat{W}$, $f_\lambda(z) \in \widehat{W}$, and the right-hand side of the equality is defined. The Cvitanović-Feigenbaum equation holds on the whole domain $\widehat{W}$.

As $f^{-1}(\widehat{W}) = \lambda\widehat{W}$, it follows by induction on n that $f^{-(2^n-1)}(\widehat{W}) = \lambda^n\widehat{W}$. Indeed, the Cvitanović-Feigenbaum equation shows that

$$f^{-1}(z) = \frac{1}{\lambda^n} f^{-2^n}(\lambda^n z).$$

Hence,

$$f^{-2^n}(\lambda^n\widehat{W}) = \lambda^n f^{-1}(\widehat{W}) = \lambda^{n+1}\widehat{W}.$$

Replacing $\lambda^n\widehat{W}$ by $f^{-(2^n-1)}(\widehat{W})$ shows that the induction property holds for $n+1$. The Theorem follows immediately from this observation. ∎

5. Combinatorial description of the map $f : \widehat{W} \to \mathbb{C}$

5.1. The map $f : \widehat{W} \to \mathbb{C}$ is a ramified covering.

Theorem B. *All the critical points of f are simple. The critical values of f are all contained in the real axis. Besides, the analytic extension $f : \widehat{W} \to \mathbb{C}$ is a ramified covering, in the sense that for any bounded disk $\mathbb{D}_R \subset \mathbb{C}$, each component of $f^{-1}(\mathbb{D}_R)$ is proper over $\mathbb{D}_R$.*

Proof. Remember that on W_n/λ_n, the map f coincides with

$$\left(-\frac{1}{\lambda}\right)^n f^{\circ 2^n}(\lambda^n z).$$

The critical points of $f^{\circ 2^n}$ in W_n are preimages of 0 by the polynomial-like map $f : W \to \mathbb{C}_\lambda$. Those are all simple critical points. By rescaling with a factor $(-1/\lambda)^n$ we see that the critical points of f in W_n/λ_n are simple. Since $\widehat{W}$ is the union of the sets W_n/λ_n, the statement for critical points follows.

The critical orbit of the polynomial-like map $f : W \to \mathbb{C}_\lambda$ is contained in the real axis. Hence, rescaling with a factor $(-1/\lambda)^n$, we see that the critical values of f are all contained in the real axis.

To finish the proof of Theorem B, we need the following lemma.

Lemma 4. *If $U \subset \widehat{W}$ is not relatively compact in $\widehat{W}$, then $f(U)$ is not relatively compact in $\mathbb{C}$.*

Remark. This lemma implies for example that f has no singular value in $\mathbb{C}$.

Proof. Remember that the set W_n has been defined via the quadratic-like map $f : W \to \mathbb{C}_\lambda$ as

$$W_n = \left\{ z \in W \mid f^{\circ 2^n}(z) \in \mathbb{C}_\lambda \right\};$$

in particular, W_n is relatively compact in W_{n-1}. Hence, W_n/λ^n is relatively compact in the domain of analyticity of $f_\lambda \colon \widehat{W}/\lambda$. Besides,

$$f_\lambda\left(\frac{1}{\lambda^{n+1}}W_{n+1}\right) = \left(-\frac{1}{\lambda}\right)^n f^{\circ 2^n}(W_{n+1}) = \frac{1}{\lambda^n}W_n.$$

Thus, $f_\lambda^{\circ n}$ is analytic on a neighborhood of W_n/λ_n, and maps W_n/λ_n to W.

Now, if $U \subset \widehat{W}$ is not relatively compact in $\widehat{W}$, we have only two possibilities:

- either U is entirely contained in one of the sets W_n/λ^n; in that case $\overline{U}$ contains a point in $\overline{W_n/\lambda^n} \cap \partial\widehat{W}$. Those points are mapped by $f_\lambda^{\circ n}$ to x_1, $-x_1$, $\overline{x}_1$ or $-\overline{x}_1$. Hence $f_\lambda^{\circ n}(\overline{U})$ is contained in $\overline{W}$ and contains one of those points, let's say x_1. Since $f(z)$ tends to infinity when $z \in W$ tends to x_1, and since

$$f(U) = \left(-\frac{1}{\lambda}\right)^n f\left(f_\lambda^{\circ n}(U)\right),$$

  we see that U cannot be compact in $\mathbb{C}$;
- or U intersects the boundary of infinitely many sets W_n/λ^n; in that case, its image has to intersect the boundary of infinitely many sets $\mathbb{C}_\lambda/\lambda^n$ and is not compact in $\mathbb{C}$.

∎

We now come back to the proof of Theorem B. The critical points of f are discrete in $\widehat{W}$ (since otherwise f' would be identically 0). Hence, to see that f is a ramified cover, it is enough to show that for any disk $\mathbb{D}_R \subset \mathbb{C}$ centered at 0 with radius R, the connected components of $f^{-1}(\mathbb{D}_R)$ are relatively compact in $\widehat{W}$.

So, we let $V \subset \widehat{W}$ be a neighborhood of x_0 such that $f : V \to \mathbb{C}$ is an isomorphism onto the disk centered at 0 of radius r, and we suppose that $f^{-1}(\mathbb{D}_R)$ has a connected component U which is not relatively compact in $\widehat{W}$. By Lemma 4, the sets $f_\lambda^{\circ n}(U)$ are never relatively compact in $\widehat{W}$. Indeed,

- either U is relatively compact in the domain $\widehat{W}/\lambda$ of f_λ; in that case, $\overline{U}$ contains a point of the boundary of the basin, $f_\lambda(\overline{U})$ also since f_λ is analytic on a neighborhood of $\overline{U}$,
- or U is not relatively compact in $\widehat{W}/\lambda$ and $f_\lambda(U)$ is not compact in $\mathbb{C}$.

Besides, since every point in U is attracted by $x_0 \in V$, there exists an integer n_0 such that $f_\lambda^{\circ n_0}(W) \cap V \neq \emptyset$. Since $f_\lambda(V) \subset V$, for every $n \geq n_0$, $f_\lambda^{\circ n}(W) \cap V \neq \emptyset$, and since W is connected there exists a point $w_n \in W$ such that $f_\lambda^{\circ n}(w_n) \in \partial V$. Hence,

$$R \geq |f(w_n)| = \left|\frac{f(f_\lambda^{\circ n}(w_n))}{(-\lambda)^n}\right| = \frac{r}{\lambda^n},$$

which gives a contradiction if n is large enough so that $R < r/\lambda^n$. ∎

5.2. Defining a tiling of the domain of analyticity. Given a point $x \in \widehat{W}$, the orbit of x under f_λ is attracted by the fixed point x_0. Hence, either the orbit is eventually contained in the real axis, or the orbit eventually enter the pieces $W_\pm$.

Definition 10. For any $x \in \widehat{W}$, if $f_\lambda^{\circ n}(x) \in W_\pm$, then the puzzle piece P_x around x is the connected component of $f_\lambda^{-n}(W_\pm)$ that contains x. We will say that p is a tile of $\widehat{W}$ if and only if $p = \overline{P}_x \cap \widehat{W}$ for some $x \in \widehat{W}$. The set of tiles will be denoted by $\mathcal{P}_{\widehat{W}}$ and we will call it a tiling of $\widehat{W}$.

Remark. Since f_λ exchanges the two sets W_+ and W_-, the definition of the puzzle pieces P_x does not depend on the choice of the integer n such that $f_\lambda^{\circ n}(x) \in W_\pm$. In particular, if $p_1 \in \mathcal{P}_{\widehat{W}}$ and $p_2 \in \mathcal{P}_{\widehat{W}}$ are two tiles, then

$$\mathring{p}_1 \cap \mathring{p}_2 = \emptyset.$$

Figure 9 shows the tiling of $\widehat{W}$. The purpose of this section is to give a combinatorial description of this tiling. Notice that since the critical values of f – hence of f_λ – are contained in $\mathbb{R}$, if P_x is a puzzle piece around $x \in \widehat{W}$, then, for n large enough, $f_\lambda^{\circ n} : P_x \to W_\pm$ is a non-ramified covering, and since $W_\pm$ is simply connected, it is an isomorphism. Observe also that $\overline{W}_+ \cup \overline{W}_-$ is a neighborhood of the fixed point x_0 of f_λ. Hence, every point in the basin $\widehat{W}$ of f_λ is contained in a tile. In other words,

$$\widehat{W} = \bigcup_{p \in \mathcal{P}_{\widehat{W}}} p.$$

Besides, the tiling is defined by pulling by the mapping f_λ, but since f is its linearizer, it will give some information about f.

Lemma 5. *Let P_x be a puzzle piece around $x \in \widehat{W}$. Then $f(P_x) = \mathbb{H}_\pm$ and $f : P_x \to \mathbb{H}_\pm$ is an isomorphism.*

Proof. Since the linearizer of f_λ is the mapping f itself, we can write:

$$f(z) = \left(-\frac{1}{\lambda} \right)^n f \circ f_\lambda^{\circ n}(z).$$

But, if P_x is a puzzle piece, there exists an integer n such that $f_\lambda^{\circ n} : P_x \to W_\pm$ is an isomorphism (observe also that if $f_\lambda^{\circ n}$ is an isomorphism from P_x onto W_+, then $f_\lambda^{\circ (n+1)}$ is an isomorphism from P_x onto W_-). Hence,

$$f(P_x) = \left(-\frac{1}{\lambda} \right)^n f(W_\pm) = \mathbb{H}_\mp.$$

The result follows, since $f : W_\pm \to \mathbb{H}_\mp$ is an isomorphism. ∎

This shows that we could have defined the puzzle pieces as the connected components of $\widehat{W} \setminus f^{-1}(\mathbb{R})$. Hence, if we give a combinatorial description of the tiling, we will at the same time give a combinatorial information on how the critical points of $f : \widehat{W} \to \mathbb{C}$ are related.

Remark. This won't give any information about the relative position of the critical values.

Observe also that the points that are in several tiles are mapped by f to $\mathbb{R}$. Since f has only simple critical points (Theorem B), any point $z \in \widehat{W}$ belongs to exactly 1, 2 or 4 tiles. The points belonging to four tiles are the critical points of f.

Lemma 6. *Let p be any tile. Then there exists a homeomorphism $\phi_p : \mathbb{H}_+ \cup \mathbb{R} \to p$, such that $\phi(\mathbb{Z})$ is the set of critical points of f contained in p.*

Proof. We first claim that for any tile p,

- $f : p \to H_\pm \cup \mathbb{R}$ is a homeomorphism,
- the set Z of critical values of $f|p$ is discrete in $\mathbb{R}$, and
- Z accumulates on both $-\infty$ and $+\infty$.

Indeed, this has already been proved in Theorem 4 for the tiles $\overline{W}_\pm \cap \widehat{W}$: the set of critical values of $f|\partial W_\pm$ is $\{(-1/\lambda)^k \mid k \in \mathbb{N}\}$. Now, if p is any tile, we know that $f : \overset{\circ}{p} \to \mathbb{H}_\pm$ is an isomorphism (Lemma 5), and that there exists an n such that

$f_\lambda^{\circ n} : \overset{\circ}{p} \to W_\pm$ is an isomorphism that extends analytically in a neighborhood of p. Since

$$f = \left(-\frac{1}{\lambda}\right)^n f \circ f_\lambda^{\circ n},$$

we see that the set Z of critical values of $f|p$ is discrete and contains the set $\{(-1/\lambda)^k \mid k \geq n\}$. Hence it accumulates on both $-\infty$ and $+\infty$.

Next, observe that for any discrete set Z in $\mathbb{R}$ that accumulates both on $-\infty$ and $+\infty$, there exists a homeomorphism $h : \mathbb{H}_+ \cup \mathbb{R} \to \mathbb{H}_+ \cup \mathbb{R}$ mapping $\mathbb{Z}$ to Z. Then, the homeomorphism $\phi_p = f^{-1} \circ h : \mathbb{H}_+ \cup \mathbb{R} \to p$ provides the required homeomorphism. ∎

5.3. **Quadratic tilings.** We will first define a notion of a quadratic tiling of a simply connected open domain $U \subset \mathbb{C}$. We will then prove that given two quadratic tilings of simply connected domains U and U', there always exist a homeomorphism between U and U' mapping tiles to tiles.

Definition 11. A quadratic tiling of a simply connected open domain $U \subset \mathbb{C}$ is a set $\mathcal{P}$ of subsets of U such that any point $z \in U$ belongs to exactly 1, 2 or 4 tiles (in particular $U = \bigcup_{p \in \mathcal{P}} p$), and for each tile $p \in \mathcal{P}$ there exist a homeomorphism $\phi_p : \mathbb{H}_+ \cup \mathbb{R} \to p$ such that

- if $z \in \mathbb{H}_+$, then $\phi_p(z)$ belongs to exactly 1 tile,
- if $z \in \mathbb{Z}$, then $\phi_p(z)$ belongs to exactly 4 tiles,
- if $z \in \mathbb{R} \setminus \mathbb{Z}$, then $\phi_p(z)$ belongs to exactly 2 tiles, and
- when $z \in \mathbb{H}_+$ satisfies $|z| \to +\infty$, then $\phi_p(z)$ tends to a point $\phi_p(\infty)$ in the boundary of U.

We will say that $(U, \mathcal{P})$ is a tiled domain. The points belonging to four tiles will be called "corners", the closure of the connected components of the set of points belonging to exactly two tiles will be called "edges", and the point $\phi_p(\infty)$ will be called "the point at infinity in p".

Remark. This definition has been given so that the tiling $\mathcal{P}_{\widehat{W}}$ is a quadratic tiling of $\widehat{W}$.

Now, given a tiled domain $(U, \mathcal{P})$, a sub-tiling is a subset P of $\mathcal{P}$ such that the union of tiles $p \in P$ is connected and remains connected when the corners are removed. Let us say that $h : p \to p'$ is a tile homeomorphism if h is a homeomorphism between two tiles, respecting the boundary structure, i.e., sending corners and edges of p to corners and edges of p'. A sub-tiling homeomorphism between two sub-tilings P and P' is a homeomorphism

$$h : \bigcup_{p \in P} p \to \bigcup_{p' \in P'} p'$$

such that the restriction to any tile $p \in P$ is a tile homeomorphism. Finally, a tiling homeomorphism between two tiled domains $(U, \mathcal{P})$ and $(U', \mathcal{P}')$ is a sub-tiling homeomorphism between $\mathcal{P}$ and $\mathcal{P}'$.

Proposition 4. *Given any two tiled domains $(U, \mathcal{P})$ and $(U', \mathcal{P}')$ there exists a tiling homeomorphism between them.*

Proof. Let $\mathcal{S}$ be the set of sub-tiling homeomorphisms between sub-tilings $P \subset \mathcal{P}$ and sub-tilings $P' \subset \mathcal{P}'$. This set $\mathcal{S}$ is clearly non-empty, since for any tiles $p \in \mathcal{P}$

and $p' \in \mathcal{P}'$, there is always a tile homeomorphism between p and p'. Besides, $\mathcal{S}$ can be partially ordered as follows: a sub-tiling homeomorphism h_1 between P_1 and P_1' is larger than a sub-tiling homeomorphism h_2 between P_2 and P_2' if and only if

$$P_2 \subset P_1 \quad \text{and} \quad h_1 | \bigcup_{p \in P_2} p = h_2$$

(note that the restriction of h_1 to any sub-tiling $P \subset P_1$ is always a sub-tiling homeomorphism between P and its image). Observe that every totally ordered chain of sub-tiling homeomorphisms h_i, $i \in \mathcal{I}$, between sub-tilings P_i and P_i', has an upper bound which is the sub-tiling homeomorphism h between $\bigcup_{i \in \mathcal{I}} P_i$ and $\bigcup_{i \in \mathcal{I}} P_i'$, such that the restriction of h to P_i is equal to h_i. Hence, we can apply Zorn's lemma which shows that there exists a maximal sub-tiling homeomorphism h between two sub-tilings $P \subset \mathcal{P}$ and $P' \subset \mathcal{P}'$.

If $P = \mathcal{P}$, then the set $\bigcup_{p \in P} p$ contains no edge in its boundary. Since h is a homeomorphism, the same property holds for its image and $P' = \mathcal{P}'$. Hence, in this case the proof is completed. On the other hand, if $P \neq \mathcal{P}$, then the set $\bigcup_{p \in P} p$ contains an edge e in its boundary. We call $p_0 \notin P$ the tile that contains e in its boundary and e' the image $h(e)$. Since h is a homeomorphism, e' is in the boundary of $\bigcup_{p' \in P'} p'$. We call $p_0' \notin P'$ the tile that contains e' in its boundary. We will show that we can extend h to a sub-tiling homeomorphism between $P \cup \{p_0\}$ and $P' \cup \{p_0'\}$.

Let us first show that $\gamma = p_0 \cap \bigcup_{p \in P} p$ is a connected union of edges, and that $h(\gamma)$ is contained in p_0'. It is obviously a union of edges of p_0. Assume it is not connected. Then, there exists an edge e_1 such that the two connected components of $(\partial p_0 \setminus e_1) \cap U$ both intersect P. Moreover, let $p_1 \notin P$ be the tile which contains e_1 in its boundary. Then, $p_0 \cup p_1$ separates U in two connected components (because both tiles contain a point at infinity in their boundary), both intersecting P. This contradicts the fact that the sub-tiling of P is connected. To see that $h(\gamma)$ is contained in p_0', observe that for any two edges e_1 and e_2 in γ satisfying $e_1 \cap e_2 \neq \emptyset$, we have the following property: $h(e_1) \in p_0'$ implies that $h(e_2) \in p_0'$. Indeed, since $e_2 \in p_0$, it is a boundary edge of P, hence its image is a boundary edge of P'. Let $p_0'' \notin P'$ be the tile that contains $h(e_2)$. If we set $x = e_1 \cap e_2$, then $h(x) \in p_0''$. But, the same argument of connectivity as above shows that x is contained in three tiles of P, the fourth tile being p_0. Hence, there is only one tile which contains $h(x)$ and which is not contained in P', and $p_0'' = p_0'$.

Finally, given two tiles $p \in \mathcal{P}$ and $p' \in \mathcal{P}'$ and a homeomorphism h between a connected union γ of edges in the boundary of p, mapping each edge to an edge of p', there exists an extension of h to a tile homeomorphism $\tilde{h} : p \to p'$. This finishes the proof of the proposition.   ∎

5.4. **The tiling for a Blaschke product.** Let $B(z)$ be a Blaschke product

$$B(z) = z \frac{z - a}{1 - az}, \quad a \in]0, 1[.$$

Then B has a fixed point at 0 with multiplier $-a$. The basin of this critical point is $\mathbb{D}$ and contains only one simple critical point $\omega \in \mathbb{D}$. It is a classical result due to Fatou that if $\psi : \mathbb{D} \to \mathbb{C}$ is a linearizer of the Blaschke product

$$B(z) = z \frac{z - a}{1 - az}, \quad a \in]0, 1[,$$

then ψ is a ramified covering map, ramified at the inverse images of the critical point ω.

Figure 10 shows a tiling $\mathcal{P}_B$ associated to the Blaschke product B. The tiles are the closure in $\mathbb{D}$ of the connected components of $\mathbb{D} \setminus \psi^{-1}(\mathbb{R})$.
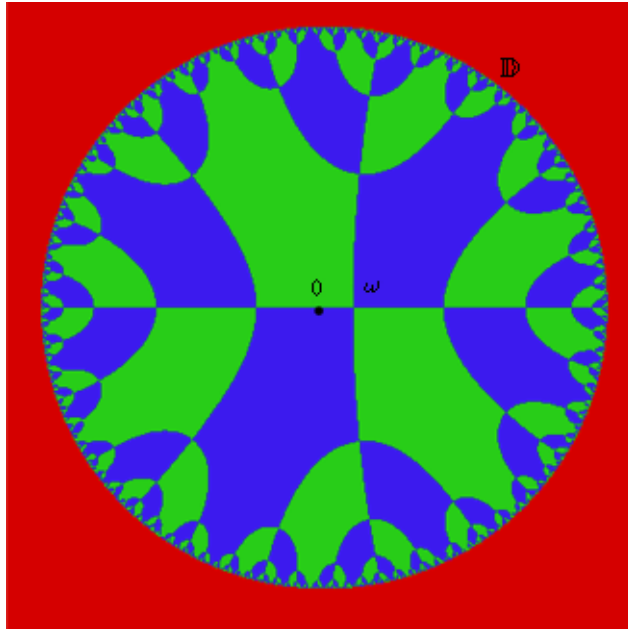

FIGURE 10. The linearizer ψ which is tangent to the identity at 0 maps blue regions to $\mathbb{H}_-$ and green regions to $\mathbb{H}_+$.

The critical points of ψ are the preimages by B of the critical point ω of B contained in $\mathbb{D}$. They are simple critical points. Hence, given a point $z \in \mathbb{D}$, we have three possibilities:

- $\psi(z) \in \mathbb{H}_\pm$ and z belongs to exactly one tile,
- z is a critical point of ψ and z belongs to exactly four tiles since ψ has only simple critical points,
- z is not a critical point of ψ and $\psi(z) \in \mathbb{R}$, and z belongs to exactly two tiles.

Besides, ψ maps each tile $p \in \mathcal{P}_B$ to $\mathbb{H}_\pm$ homeomorphically. The set of critical values of $\psi|p$ is of the form $\{\psi(\omega)/a^k\}_{k \geq k_0}$, for some $k_0 \geq 0$. Hence the image of the corners form a discrete set in $\mathbb{R}$ accumulating on both $+\infty$ and $-\infty$. We can then show, as in the case of the tiling $\mathcal{P}_{\widehat{W}}$, that the tiling $\mathcal{P}_B$ is a quadratic tiling. Hence, by Proposition 4 there exists a homeomorphism between $\mathbb{D}$ to $\widehat{W}$ mapping the tiles of $\mathcal{P}_B$ to the tiles of $\mathcal{P}_{\widehat{W}}$.

6. TOPOLOGY OF THE BOUNDARY OF $\widehat{W}$

The domain $\widehat{W}$ is dense in $\mathbb{C}$, hence its boundary has empty interior. However, it seems to have a very complex structure. Studying its topology, its Hausdorff dimension, or its Lebesgue measure could be interesting. But we do not have many

tools at the moment. However, we will show the surprising result that there are bi-accessible points in the boundary of $\widehat{W}$.

First of all, remember that Theorem 4 tells that the boundary of ∂W_+ can be decomposed as follows.

Definition 12. The boundary ∂W_+ is the union of
- an arc γ_1 which is mapped by f to $\mathbb{R}_+$,
- an arc γ_2 which is mapped by f to $\mathbb{R}_-$, and
- a point x_1 such that $(x_1)^2 = F^{-1}(-i\infty)$.

Theorem C. *The point x_1/λ belongs to the boundary of $\widehat{W}$, and is accessible from $\widehat{W}$ with at least three accesses.*

Proof. We claim that $(x_1, \overline{x}_1)$ is a repelling cycle of period 2 for the map f_λ. Indeed, $f_\lambda^{\circ 2}$ is an automorphism of W_+ (see Lemma 1). Since there is an attracting fixed point in the boundary (the point x_0), there must also be a repelling fixed point on the boundary. But the only point in ∂W_+ which is not in the basin $\widehat{W}$ is the point x_1. Hence, x_1 is a repelling fixed point for $f_\lambda^{\circ 2}$. Since f_λ maps W_+ to W_-, x_1 cannot be a fixed point of f_λ. This shows that x_1 is a repelling periodic point of period 2 for f_λ. Since the map f_λ is a real map, the other point of the cycle is $\overline{x}_1$.

In particular, x_1 does not belong to the basin $\widehat{W}$, and it is in its boundary. Moreover, according to Lemma 2, $\lambda\widehat{W} \subset \widehat{W}$. Hence $\lambda^{-1}\partial\widehat{W} \subset \partial\widehat{W}$, so that $x_1/\lambda \in \partial\widehat{W}$.

The Feigenbaum map f maps γ_1 and γ_2 to $\mathbb{R}_+$ and $\mathbb{R}_-$, which are contained in $\widehat{W}$. Hence, f_λ maps $\gamma_1' = \gamma_1/\lambda$ and $\gamma_2' = \gamma_2/\lambda$ inside its basin of attraction $\widehat{W}$. Moreover, the imaginary axis is also contained in $\widehat{W}$. Let us call γ_3' its inverse image by $f_\lambda|W_+$. The arcs γ_1', γ_2' and γ_3' are in the basin of attraction of f_λ, i.e., in $\widehat{W}$ (see Figure 9), and they land at the same point x_1/λ which is "mapped" by f to $-i\infty$.

We must show that those arcs do not define the same access to x_1/λ. This condition is not difficult to check (see Figure 11). Those arcs cut W_+/λ in two pieces which are mapped by f_λ to the two quarter-planes contained in $\mathbb{H}_-$. Hence, each one of those pieces contains a point which is not contained $\widehat{W}$ (either $x_1 = f_\lambda^{-1}(\overline{x}_1)$ or $f_\lambda^{-1}(-x_1)$).
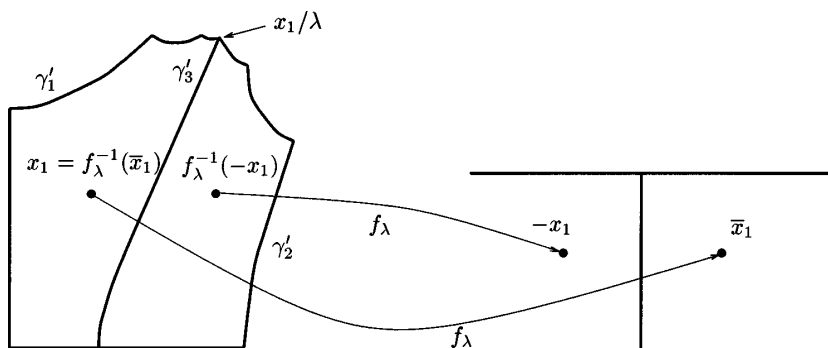

FIGURE 11. The arcs γ_1', γ_2' and γ_3'.

■

7. LOCAL CONNECTIVITY OF $K(f)$

Definition 13. We define puzzles $\mathcal{P}_n$, $n \in \mathbb{N}$ in the following way:
- the puzzle pieces of the puzzle $\mathcal{P}_0$ of depth 0 are the puzzle pieces P_x defined in Definition 10, and
- the puzzle pieces of the puzzle $\mathcal{P}_n$ of depth n are the pieces $\lambda^n P$, $P \in \mathcal{P}_0$.

Figure 12 shows some pieces of the puzzles of depth 1, 2, and 3. We have colored the puzzle pieces of depth 2 in blue and green and the puzzle pieces of depth 3 in brown and yellow. A lot of pieces are missing, but drawing them takes a bit of time. That is why the region covered by the puzzle pieces seem to be disconnected, which is not the case: remember that the tiles cover a dense open subset $\widehat{W} \subset \mathbb{C}$ which is connected and simply connected (see Figure 9).
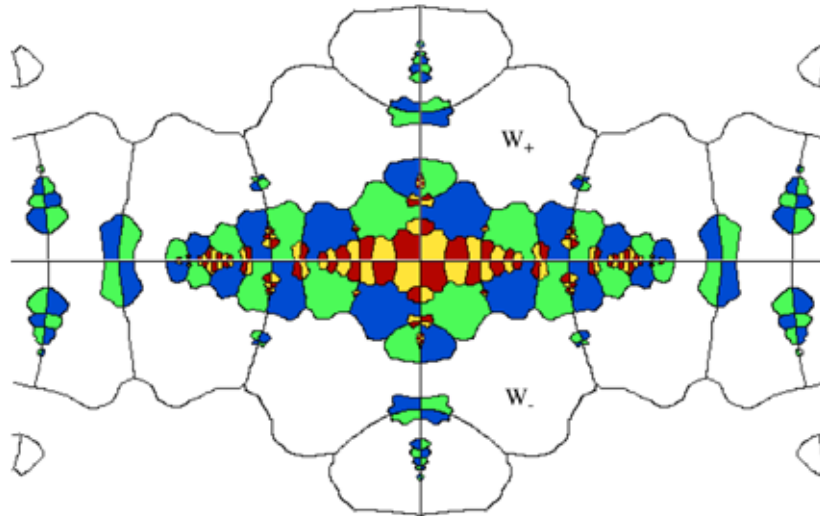

FIGURE 12. Some pieces of the puzzles $\mathcal{P}_i$, $i = 1, 2, 3$.

Since the closure of the puzzle pieces of depth n cover the set $\lambda^n \widehat{W}$, and since $K(f) \subset \lambda^n \widehat{W}$ for every $n \geq 0$ (see Lemma 3), we see that the closure of the puzzle pieces of depth n cover the Julia set $K(f)$. We will first show that the intersection of the closure of a puzzle piece with $K(f)$ is always connected (Lemma 7), and then we will prove that the puzzle of depth n subdivides the puzzle of depth $n - 1$ (Lemma 8).

Lemma 7. *If P is a puzzle piece, then for all $n \geq 0$, the intersection $\lambda^n \overline{P} \cap K(f)$ is connected.*

Proof. Given a puzzle piece P, there exists an integer $k \geq 0$ such that $f_\lambda^k(P) = W_\pm$. We will call this k the age of the puzzle piece, and we will prove this lemma by induction on the age of the puzzle piece P.

For $k = 0$, $\lambda^n \overline{W}_\pm \cap K(f)$ is connected, because $f^{\circ 2^n}(\lambda^n \overline{W}_\pm)$ is a half-plane, which cuts $K(f)$ in two parts along the real axis, and because $K(f)$ is totally invariant by the polynomial-like map $f : W \to \mathbb{C}_\lambda$.

If P is a puzzle piece of age $k + 1 \geq 1$, we first notice that $K(f) \subset W$. Hence,

- either $\lambda^n P \cap W = \emptyset$, and $\lambda^n P \cap K(f) = \emptyset$ is connected,
- or $\lambda^n P \subset W$.

On the other hand, $f_\lambda(P) = Q$ is a puzzle piece of age k, and by inductive hypothesis, every $\lambda^n \overline{Q} \cap K(f)$, $n \in \mathbb{N}$, is connected. Besides,

$$f(\lambda P) = \left(-\frac{1}{\lambda}\right)^{n-1} f^{\circ 2^{n-1}}(\lambda^{n-1} \lambda P).$$

Hence,

$$f^{\circ 2^{n-1}}(\lambda^n P) = (-\lambda)^{n-1} f_\lambda(P) = (-\lambda)^{n-1} Q.$$

Since $K(f)$ is totally invariant by $f|W$, we can conclude that $\lambda^n \overline{P} \cap K(f)$ is also connected when $\lambda^n P \subset W$. ∎

We can now show that the puzzle $\mathcal{P}_{n+1}$ subdivides the puzzle $\mathcal{P}_n$. In other words, we will show the following lemma.

Lemma 8 (see Figure 12). *Every puzzle piece $\lambda P \in \mathcal{P}_1$ is contained in a puzzle piece $Q \in \mathcal{P}_0$.*

Proof. Let z be a point in λP and let Q be the puzzle piece which contains z. Using Lemma 5, we see that $f(Q) = \mathbb{H}_\pm$. Since $f(\lambda P) = f_\lambda(P)$ is a puzzle piece (by definition), and since puzzle pieces do not intersect the real line (they are the connected components of $\widehat{W} \setminus f^{-1}(\mathbb{R})$, and $\mathbb{R} \subset f^{-1}(\mathbb{R})$), we know that

$$f(\lambda P) \subset \mathbb{H}_\pm = f(Q),$$

and as $f: Q \to \mathbb{H}_\pm$ is an isomorphism, we can deduce that

$$\lambda P \subset Q. \quad \square$$

This construction makes it possible to recover the following result due to Hu and Jiang [HJ].

Theorem 5 (see Figure 13). *The Julia set $K(f)$ is locally connected at the critical point.*

Up to now, we can only prove local connectivity at the critical point 0:

- $0 \in \lambda^n W \subset \lambda^{n-1} W$, and
- $\lambda^n \overline{W} \cap K(f)$ is connected.

One can use classical arguments of recurrence to spread around this local connectivity. However, those arguments do not exhibit a nice basis of connected neighborhoods. We would like to prove that if we consider the nested sequence of puzzle pieces $P_n(z) \in \mathcal{P}_n$ which contain a point $z \in K(f)$, then the diameter of the piece $P_n(z)$ decreases geometrically with n. We just need to show the following conjecture.

Conjecture 1. *There exists a constant $\varepsilon > 0$ such that if $P \in \mathcal{P}_0$ and $Q \in \mathcal{P}_0$ are puzzle pieces and $\lambda P \subset Q$, then*

$$\frac{\mathrm{diam}(Q)}{\mathrm{diam}(\lambda P)} \geq 1 + \varepsilon.$$

We think it is possible to show this conjecture using some classical arguments of moduli of annuli. Moreover, we think that there is a stronger result.
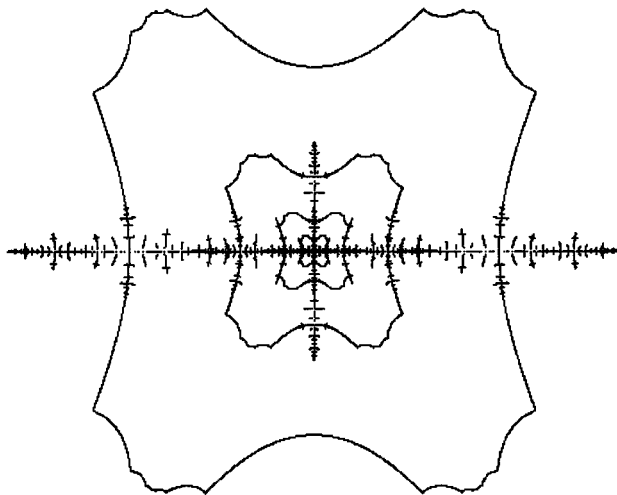
FIGURE 13. Local connectivity of the Feigenbaum Julia set at the critical point.

Conjecture 2. *There exists a constant $K \in \mathbb{R}$ such that the diameter of every puzzle piece $P \in \mathcal{P}_0$ is bounded by K:*

$$\mathrm{diam}(P) \leq K.$$

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