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ON THE DYNAMICS OF THE McMULLEN FAMILY $R(z) = z^m + \lambda/z^\ell$

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Abstract. In this note we discuss the parameter plane and the dynamics of the rational family $R(z) = z^m + \lambda/z^\ell$, with $m \geq 2$, $\ell \geq 1$, and $0 < |\lambda| < \infty$.

1. Introduction

In my course on complex dynamical systems in the summer, 2005, the students had to discuss the dynamics of the well-known rational map

$$R(z) = z^2 + \lambda/z^3$$

due to McMullen [21], which has the following well-known property:

► *For $|\lambda| > 0$ small, the Julia set consists of uncountably many closed Jordan curves (actually quasi-circles) about the origin.*

Details of proof are worked out in Beardon's book [1], pp. 266–271. Of course the following question immediately arose:

► *What will happen if $|\lambda|$ is not small?*

It was, and is, the purpose of this paper to give an adequate, though by no means complete answer to this question. This project, however, has to bear in mind that a lot of detailed information is available in a series of recent papers and preprints–which I realised only in the winter of 2005 ([2], [3], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [26]), due to R. Devaney and collaborators, so that the picture to be drawn here has to be complementary to the picture drawn in the above papers; certain overlaps, however, are unavoidable.

In most (though not all) papers of this series the authors are concerned with the symmetric case $\ell = m$, thus they consider

$$R(z) = z^m + \lambda/z^m$$

rather than the general case

$$R(z) = z^m + \lambda/z^\ell,$$

with focus on the *Sierpiński curve Julia sets.* Due to the simple mapping properties of R (for example, circles about the origin are mapped onto ellipses with foci at the critical values $\pm 2\sqrt{\lambda}$, and from $R(\lambda^{1/m}/z) = R(z)$ it follows that the Julia set is symmetric with respect to the circle $|z| = |\lambda|^{1/2m}$), the symmetric case permits explicit computations and constructions which cannot be performed otherwise. Many

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of the proofs are based on these explicit expressions, and thus do not carry over to the general case. In section 5 of this paper it will become apparent that the case "d is a prime number" is extremely different from the other cases, this indicating that the general case needs more general proofs.

The purpose of this paper is to describe in detail the dynamics of the rational maps in the one-parameter family

$$R_{\mathtt{c}}(z) = z^m + \mathtt{c}^d/z^\ell,\ \mathtt{c} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \quad \ell \geq 1,\ m \geq 2,\ d = m + \ell, \tag{1}$$

and of the associated quasi-conjugate family

$$Q_{\mathtt{a}}(z) = \mathtt{a} z^m (1 + 1/z)^d,\ \mathtt{a} \in \mathbb{C}^*, \tag{2}$$

with emphasis on the hyperbolic *non-escape* case, to be described below, rather than the *Sierpiński case* dealt with more often than not in the papers cited above. For notation and background in complex dynamics, the reader is referred to Beardon [1], Milnor [24], and Steinmetz [27]. Throughout this paper the super-attracting basin at infinity shall be denoted $\mathcal{A}_\infty(\mathtt{c})$.

The rational map $R_{\mathtt{c}}$ has a critical point of order $\ell - 1$ at the origin, a critical point of order $m - 1$ at the point at infinity, and simple critical points at

$$\xi_j = \kappa \sigma^j \mathtt{c},\ 0 \leq j < d, \tag{3}$$

with $\kappa = \sqrt[d]{\ell/m}$ and $\sigma = e^{2\pi i/d}$. The critical points are not independent since $R_{\mathtt{c}}^n(\xi_j) = \sigma^{jm^n} R_{\mathtt{c}}^n(\kappa\mathtt{c})$, hence $|R_{\mathtt{c}}^n(\xi_j)| = |R_{\mathtt{c}}^n(\xi_0)|$, and the fate of the Julia set depends on whether or not the critical orbit $(R_{\mathtt{c}}(\kappa\mathtt{c}))$ escapes to infinity.

The rational functions $\mathtt{R}_n$, defined by $\mathtt{R}_n(\mathtt{c}) = R_{\mathtt{c}}^n(\kappa\mathtt{c})$, hence $\mathtt{R}_0(\mathtt{c}) = \kappa\mathtt{c}$,

$$\begin{aligned} \mathtt{R}_1(\mathtt{c}) &= \kappa^m \mathtt{c}^m (1 + \kappa^{-d}) = \mu \mathtt{c}^m, \\ \mathtt{R}_2(\mathtt{c}) &= \mu^m \mathtt{c}^{m^2} \big(1 + \mu^{-d}/\mathtt{c}^{(m-1)d}\big),\ \text{and} \\ \mathtt{R}_{n+1}(\mathtt{c}) &= \mathtt{R}_n(\mathtt{c})^m \big(1 + \mathtt{c}^d/\mathtt{R}_n(\mathtt{c})^d\big),\ n \geq 2, \end{aligned} \tag{4}$$

with $\deg \mathtt{R}_n = d^{n-1}(m-1)$ for $n \geq 2$ (and $1/\ell + 1/m < 1$), will be used to decompose the parameter plane $\mathbb{C}^*$ into the sets

$$\begin{aligned} \boldsymbol{\Omega}_0 &= \{\mathtt{c} : \mathtt{R}_0(\mathtt{c}) \in \mathcal{A}_\infty(\mathtt{c})\}, \\ \boldsymbol{\Omega}_n &= \{\mathtt{c} : \mathtt{R}_n(\mathtt{c}) \in \mathcal{A}_\infty(\mathtt{c})\} \setminus \boldsymbol{\Omega}_{n-1},\ n \in \mathbb{N},\ \text{and} \\ \boldsymbol{\Omega}_\infty &= \mathbb{C}^* \setminus \textstyle\bigcup_{n \in \mathbb{N}_0} \boldsymbol{\Omega}_n. \end{aligned} \tag{5}$$

Remark. The reason why we prefer to work in the parameter $\mathtt{c}$-plane rather than the $\lambda = \mathtt{c}^d$-plane is obvious—R_n is a function of $\mathtt{c}$, but not a function of λ! The true parameter, however, is neither λ nor $\mathtt{c}$, but $\mathtt{a} = \lambda^{m-1} = \mathtt{c}^{d(m-1)}$; see section 5. In the terminology of [22] $\partial\boldsymbol{\Omega}_\infty$ is the *bifurcation locus*, and its complement with respect to $\mathbb{C}^*$ is the J-*stable set*. The sequence $(\mathtt{R}_n)$ is normal in the J-stable set, and non-normal at any point of the bifurcation locus. In [7] the set $\boldsymbol{\Omega}_2$ is called the *McMullen domain*, $\boldsymbol{\Omega}_0$ is the *Cantor locus*, and $\bigcup_{3 \leq n < \infty} \boldsymbol{\Omega}_n$ may be called the *Sierpiński locus*.

This article is organised as follows. In section 2 we present an independent treatment of McMullen's example that may be used in any course on complex dynamics to discuss this example in moderate time. Section 3 contains a summary of results about the parameter plane and the dynamical plane of the family (1). Most of these results may be found in the series of papers mentioned in section 1, at least in the symmetric case $\ell = m$. In section 4 it is shown how the parameter plane is related to the dynamical planes of the rational functions $\mathtt{R}_n$ via Carathéodory's

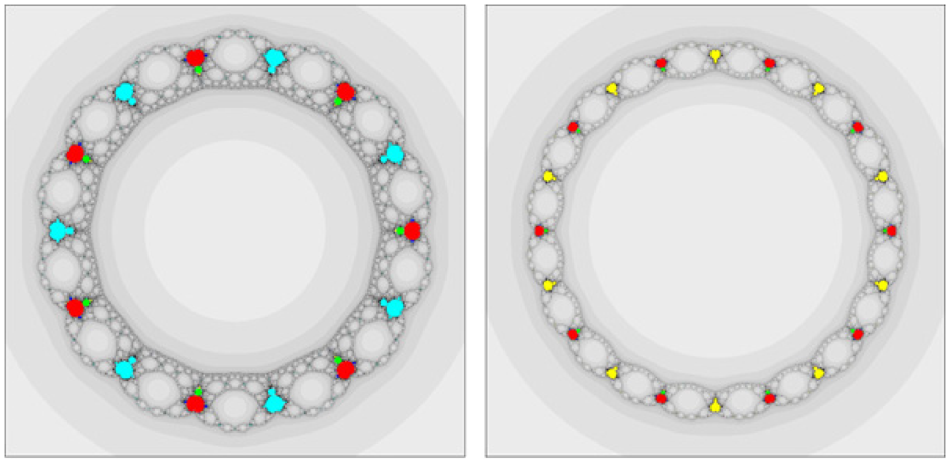

FIGURE 1. The parameter plane for the family $R_{\mathsf{c}}(z) = z^3 + \mathsf{c}^7/z^4$ (left) and $R_{\mathsf{c}}(z) = z^3 + \mathsf{c}^{10}/z^7$ (right). The coloured domains represent non-trivial hyperbolic components.

concept of kernel convergence. Section 5 is devoted to the study of the *non-trivial hyperbolic components* of the parameter plane of family (1), and to the dynamics of R_{c} in the non-escape case, while section 6 deals with the dynamics of the associated family (2) and its parameter plane. Finally, in section 7, we discuss the occurrence of copies of the classical Mandelbrot set in the parameter plane.

2. McMullen's example

We start with discussing McMullen's example in the following form:

Theorem A ([21])**.** *For $1/\ell + 1/m < 1$ and $|\mathsf{c}| > 0$ sufficiently small the Fatou set of R_{c} consists of the simply connected basin of attraction at infinity, its unique and simply connected pre-image $\mathcal{U}_0$ about $z = 0$, and the successive pre-images of $\mathcal{U}_0$. The pre-image of $\mathcal{U}_0$ is a conformal annulus $\mathfrak{A}$ separating 0 and ∞, whose pre-image consists of two conformal annuli $\mathfrak{A}_1$ and $\mathfrak{A}_2$ about the origin, each of which has two conformal $\mathfrak{A}_{jk}$ annuli as pre-images, etc. The Julia set consists of uncountably many closed Jordan curves (actually quasi-circles) about the origin, hence the Julia set is homeomorphic to $C \times S^1$ with $C \subset (0, \infty)$ a Cantor set.*

Remark. The restriction $1/\ell + 1/m < 1$ on ℓ and m in (1) is a necessary consequence of Grötzsch's inequality for moduli of nested conformal annuli, and will come out quite natural from the proof of Theorem A. For a different proof, see [11]. The proof presented here relies on the following Lemma, which will also be useful in a different context.

Lemma 1. *The set $C = \{z : |R_{\mathsf{c}}(z)| \le \mu|\mathsf{c}|^m\}$, with $\mu = \kappa^m(1 + \kappa^{-d})$ and $\kappa = (\ell/m)^{1/d}$, contains the zeros and the free critical points of R_{c}, and is bounded by two polar curves*

$$\Gamma_j : z = \mathsf{c}\, r_j(\theta)e^{i\theta},\ 0 \le \theta \le 2\pi,\ j = 1, 2,$$

with $r_j : [0, 2\pi] \to (0, \infty)$ continuous and independent of c.

Proof. We consider $f(r,\theta) = r^{2m}|1 + r^{-d}e^{-id\theta}|^2$ and note that

$$\min\{f(r,0) : 0 < r < \infty\} = f(\kappa, 0) = \mu^2.$$

Then for $d\theta \not\equiv 0 \mod 2\pi$ the equation $f(r,\theta) = \mu^2$ has exactly two solutions $r_1(\theta) < \kappa < r_2(\theta)$, while $r_1(\theta) = r_2(\theta) = \kappa$ for $d\theta \equiv 0 \mod 2\pi$. The second assertion is obvious, and the first one follows from $f(r,\theta) < f(r,0)$ for $d\theta \not\equiv 0 \mod 2\pi$, $\lim_{r\to 0} f(r,\theta) = \lim_{r\to\infty} f(r,\theta) = \infty$, and the fact that f is strictly decreasing in $0 < r < \rho_\theta$ and strictly increasing in $\rho_\theta < r < \infty$. To prove this we observe that, for θ fixed,

$$r^{2d-2m+1}\frac{d}{dr}f(r,\theta) = 2mr^{2d} + 2(m-\ell)\cos(d\theta)\, r^d - 2\ell.$$

The right-hand side, being a degree-two polynomial in r^d, has just one positive zero, hence $\frac{d}{dr}f(r,\theta)$ has exactly one zero at $r = \rho_\theta > 0$, this implying $\frac{d}{dr}f(r,\theta) < 0$ in $(0,\rho_\theta)$, and $\frac{d}{dr}f(r,\theta) > 0$ in (ρ_θ,∞). Finally writing $z = \mathsf{c}re^{i\theta}$, the inequality

$$|R_{\mathsf{c}}(z)| = |\mathsf{c}|^m r^m |1 + r^{-d}e^{-id\theta}| \le \mu|\mathsf{c}|^m$$

is equivalent to $r_1(\theta) \le r \le r_2(\theta)$. Obviously C contains the zeros of R_{c} and also the circle $|z| = \kappa|\mathsf{c}|$, *a fortiori* the free critical points. □

Proof of Theorem A. We assume *a priori* $0 < |\mathsf{c}| \le 1$, so that $|R_{\mathsf{c}}(z)| > 2$ for $|z| \ge 2$. It is then obvious that for $1/\ell + 1/m < 1$ (and only for pairs (ℓ, m) with this property) and $0 < |\mathsf{c}| < k_2 = k_2(\ell, m)$ sufficiently small (this could be made explicit) the following holds:

$$\mu|\mathsf{c}|^{m-1} < \min\{r_1(\theta) : 0 \le \theta \le 2\pi\},$$

$$\max\{r_2(\theta) : 0 \le \theta \le 2\pi\} < 2|\mathsf{c}|^{-1},$$

and

$$|R_{\mathsf{c}}(z)| > 2 \text{ for } 0 < |z| \le \mu|\mathsf{c}|^m.$$

The annulus

$$B = \{z : \rho_1 < |z| < \rho_2\}$$

with $\rho_1 = \mu|\mathsf{c}|^m$ and $\rho_2 = 2$ is backward invariant, since its complement is forward invariant. Actually B contains $R_{\mathsf{c}}^{-1}(\overline{B})$, and $R_{\mathsf{c}}^{-1}(B)$ consists of two conformal annuli $B_j \subset B$, B_j being bounded by the curve Γ_j and some closed curve within the conformal annulus bounded by Γ_j and the circle $|z| = \rho_j$, $j = 1, 2$. The Julia set $\mathcal{J}_{\mathsf{c}}$ is contained in $\overline{B}$. Since $\big(R_{\mathsf{c}}^{-n}(\overline{B})\big)_{n\in\mathbb{N}}$ is a nested sequence of closed sets, each separating 0 from ∞, the Julia set $\mathcal{J}_{\mathsf{c}} = \bigcap_{n\in\mathbb{N}} R_{\mathsf{c}}^{-n}(\overline{B})$ consists of uncountably many connected components

$$C_{j_1 j_2 \dots} = \bigcap_{n\in\mathbb{N}} \overline{B}_{j_1 j_2 \dots j_n}$$

separating 0 from ∞ (we have tacitly assumed that the connected components $B_{j_1 j_2 \dots j_n}$, $j_1, \dots, j_n \in \{1, 2\}$, of $R_{\mathsf{c}}^{-n}(B)$ are labelled in such a way that $B_{j_1 j_2 \dots j_n} \subset B_{j_1 j_2 \dots j_{n-1}}$). Thus the domains $\mathcal{A}_\infty(\mathsf{c})$ and $\mathcal{U}_0(\mathsf{c})$ are disjoint, hence simply connected as follows from the Riemann–Hurwitz formula, and $R_{\mathsf{c}}^{-1}(\mathcal{U}_0(\mathsf{c}))$ is a conformal annulus containing all zeros and critical points of R_{c} different from 0 and ∞. Otherwise $R_{\mathsf{c}}^{-1}(\mathcal{U}_0(\mathsf{c}))$ would consist of d domains, each containing a simple

FIGURE 2. The set $\{z : |R_{\mathtt{c}}(z)| \le \mu|\mathtt{c}|^m\}$ (black) for $R_{\mathtt{c}}(z) = z^2 + \mathtt{c}^5/z^3$, the annulus B, and the conformal annuli B_1 (green) and B_2 (blue).

zero and a critical point of $R_{\mathtt{c}}$, which would contradict $\deg R_{\mathtt{c}} = d$. The Riemann–Hurwitz formula, and the fact that there are no critical points left, then yield the assertion of Theorem A. $\square$

Remark. Actually the components of $\mathcal{J}_{\mathtt{c}}$ are closed Jordan curves about the origin, since the thickness of the components of $R_{\mathtt{c}}^{-n}(\overline{B})$ (the conformal modulus) tends to zero as $n \to \infty$, and even quasi-circles by hyperbolicity of $R_{\mathtt{c}}$. We note that, as $\mathtt{c} \to 0$, the equation $R_{\mathtt{c}}(z) = 2$, say, has m solutions $\sim \sqrt[m]{2}$ by Hurwitz' Theorem, and ℓ solutions $\sim \sqrt[\ell]{\mathtt{c}^d/2}$. Thus $R_{\mathtt{c}}$ has degree m in B_2 and degree ℓ in B_1, which implies $m \mod B_2 = \ell \mod B_1 = \mod B$, hence $1/\ell + 1/m < 1$ by Grötzsch's inequality.

3. The structure of the parameter plane

In the sequel we will collect some facts about the dynamical plane of the functions $\mathtt{R}_n$, $\mathtt{c} \in \bigcup_{n\in\mathbb{N}_0} \mathbf{\Omega}_n$, and the associated sets $\mathbf{\Omega}_n$ in the parameter plane. Much more detail can be found in the papers mentioned in the introduction.

Theorem B. *The sets $\mathbf{\Omega}_n$, $n \in \mathbb{N}_0$, are open.*

This follows from the λ-Lemma due to Mãñé, Sad and Sullivan [19] (see also [22]), since the Julia sets of $R_{\mathtt{c}}$ for $\mathtt{c}$ in different sets $\mathbf{\Omega}_n$ have different dynamical type, the latter being important for $n \ge 3$. We describe the different types without going into detail; the reasoning is straightforward. In all cases, the Fatou set consists solely of the basin $\mathcal{A}_\infty(\mathtt{c})$ and its successive pre-images.

- ► ($n = 0$) *The Fatou set is connected, and the Julia set is totally disconnected* (*a Cantor set*).
- ► ($n = 1$) *The basin $\mathcal{A}_\infty(\mathtt{c})$ is simply connected, while $\mathcal{U}_0(\mathtt{c})$ and all its pre-images are* $(d+1)$*-connected.* (It will be seen, however, that this description was purely hypothetical, since $\mathbf{\Omega}_1$ is actually empty.)

- ▶ ($n = 2$) *For* $1/\ell + 1/m < 1$ *the Julia set is disconnected without singleton components* (Theorem A), *while* $\mathbf{\Omega}_2 = \emptyset$ *for* $\ell = m = 2$ *or else* $\ell = 1$ *and* $m \geq 2$.
- ▶ ($n \geq 3$) *The domains* $\mathcal{A}_\infty(\mathtt{c})$ *and* $\mathcal{U}_0(\mathtt{c})$ *as well as the pre-images of* $\mathcal{U}_0(\mathtt{c})$ *of order* k*, denoted* $U_j^{[k]}$*, are simply connected. The critical points* ξ_l *are distributed over* d *Fatou components* $U_l^{[n-1]} = \sigma^l U_0^{[n-1]}$, $\sigma = e^{2\pi i/d}$, $0 \leq l < d$, *of order* $n-1$. *There are* $d/\gcd(m,d)$ *critical Fatou components of order* $n-2$*, each being the image of* $\gcd(m,d)$ *domains* $U_l^{[n-1]}$ *containing a critical point, and of* $d - 2\gcd(m,d)$ *domains free of critical points. The remaining Fatou components of order* $n-2$ *are non-critical, and there are* $d^k(d^{n-1}-d)$ *pre-images of order* $n-1+k$, $k \geq 0$. *The Julia set* $\mathcal{J}_\mathtt{c}$ *is connected, and even locally connected, actually a Sierpiński curve.*

Remark. The different cases ($n \leq 1$), ($n = 2$), and ($n \geq 3$) constitute the *escape trichotomy* (see [10]). Note, however, that escape time in [10] is measured for the critical *values* rather than the critical *points*. A *Sierpiński curve* is a compact, connected, locally connected, and nowhere dense subset of the plane such that the complementary components (Jordan domains) do not touch, i.e., any two distinct complementary components have disjoint closures.

Böttcher's function $\Phi_\mathtt{c}$ is the solution of *Böttcher's* functional equation

$$\Phi_\mathtt{c}(R_\mathtt{c}(z)) = \Phi_\mathtt{c}(z)^m;$$

it is uniquely determined by the normalisation $\Phi_\mathtt{c}(z)/z \to 1$ as $z \to \infty$. In our case even

$$\Phi_\mathtt{c}(z) = z + \frac{\mathtt{c}^d}{mz^{d-m-1}} + \ldots$$

holds at $z = \infty$, and $\Phi_\mathtt{c}$ is given by

$$\Phi_\mathtt{c}(z) = \lim_{n\to\infty} \sqrt[m^n]{R_\mathtt{c}^n(z)} \tag{6}$$

in a neighbourhood of $z = \infty$. If $\mathcal{A}_\infty(\mathtt{c})$ is simply connected, then (6) holds throughout $\mathcal{A}_\infty(\mathtt{c})$, and $\Phi_\mathtt{c}$ maps $\mathcal{A}_\infty(\mathtt{c})$ conformally onto $\Delta = \{w : |w| > 1\}$; in general, $\Phi_\mathtt{c}$ admits analytic continuation to any simply connected domain $D \subset \mathcal{A}_\infty(\mathtt{c})$ containing ∞, but none of the points 0 and ξ_j, $0 \leq j < d$. At any rate, if Ω_n denotes any connected component of $\mathbf{\Omega}_n$, then

$$\Xi_n(\mathtt{c}) = \Phi_\mathtt{c}(\mathtt{R}_n(\mathtt{c})) \tag{7}$$

is meromorphic in Ω_n, and by mimicking the Douady–Hubbard technique [18] (in case $n = 0$; the argument is much easier for $n \geq 1$) it is not hard to show that $|\Xi_n(\mathtt{c})| \to 1$ as $\mathtt{c} \to \partial\Omega_n \setminus \{0, \infty\}$. Since Ξ_n is non-constant, either Ω_n contains some pole of $\mathtt{R}_n$, hence a zero of $\mathtt{R}_{n-1}$, or else $\partial\Omega_n$ contains the origin/the point at infinity, and Ξ_n is a proper map of Ω_n onto $|w| > 1$. We will shortly indicate how the following results may be deduced from these properties of the functions Ξ_n.

Theorem C. *The set* $\mathbf{\Omega}_0$ *is connected, and* $\Omega_0 = \mathbf{\Omega}_0 \cup \{\infty\}$ *is a simply connected domain. It is mapped conformally onto* $\Delta = \{w : |w| > 1\}$ *by any branch of*

$$\Psi_0(\mathtt{c}) = \sqrt[m]{\Phi_\mathtt{c}(\mathtt{R}_1(\mathtt{c}))}.$$

The compact set $\mathbb{C} \setminus \mathbf{\Omega}_0$ *has logarithmic capacity* $\mu^{-1/m}$.

Theorem D. *The set* $\mathbf{\Omega}_1$ *is empty.*

Theorem E. *For $\ell = m = 2$, or else $\ell = 1$ and $m \geq 2$ arbitrary, the set $\mathbf{\Omega}_2$ is empty. For $1/\ell + 1/m < 1$ it is connected, and*

$$\Xi_2(\mathtt{c}) = \Phi_{\mathtt{c}}(\mathtt{R}_2(\mathtt{c})) = \lim_{k\to\infty} \sqrt[m^k]{\mathtt{R}_{k+2}(\mathtt{c})}$$

is a proper mapping $\Omega_2 = \mathbf{\Omega}_2 \cup \{0\} \to \Delta = \{w : |w| > 1\}$ of degree $p_2 = (m-1)d - m^2$. The McMullen domain Ω_2 is even simply connected, and is mapped conformally onto Δ by any branch of

$$\Psi_2(\mathtt{c}) = \sqrt[p_2]{\Xi_2(\mathtt{c})}.$$

Remark. Note that $p_2 \leq 0$ for $\ell = m = 2$ or else $\ell = 1$, $m \geq 2$, so this again shows that the condition $1/\ell + 1/m < 1$ is necessary for the occurrence of the McMullen phenomenon. The reader is referred to [7], where, for $\ell = m \geq 3$, it is shown in addition that Ω_0 is bounded by a Jordan curve. It is also shown there that for $|\lambda| < \sqrt[m-1]{1/4^m}$, that is, for $|\mathtt{c}| < \sqrt[m-1]{1/2}$, the boundary of $\mathcal{A}_\infty(\mathtt{c})$ is a quasi-circle. We will come back to that point in Theorem 8 in the general case.

To prove Theorem C it suffices to show the following:

Lemma 2. *The set $\mathbf{\Omega}_0$ contains some disc $\Delta_0 = \{\mathtt{c} : |\mathtt{c}| > k_0\}$.*

Proof. For $|\mathtt{c}|$ sufficiently large, $|\mathtt{c}| \geq k_0$, say, the disc

$$D = \{z : |z| > \mu|\mathtt{c}|^m\}$$

is forward invariant under $R_{\mathtt{c}}$ (even $R_{\mathtt{c}}(\overline{D}) \subset D$), the domain

$$D_2 = \{z = re^{i\theta} : r > |\mathtt{c}|r_2(\theta),\ 0 \leq \theta \leq 2\pi\}$$

contains $\overline{D}$, and, by Lemma 1, is mapped by $R_{\mathtt{c}}$ onto D. Hence $\overline{D}_2 \subset \mathcal{A}_\infty(\mathtt{c})$, and since $\kappa\mathtt{c} \in \Gamma_2 \subset \overline{D}_2$, $\mathtt{c} \in \mathbf{\Omega}_0$ follows. □

Theorems C, D and E follow from Lemma 2, Theorem A, and the fact that $\mathtt{R}_n$, $n \leq 2$, has no poles in $\mathbb{C}^*$.

It was proved in [26] for $\ell = m = 2$, and in [9] for $\ell = m \geq 3$, that the connected components of $\mathbf{\Omega}_n$, called *Sierpiński holes*, are simply connected, and contain exactly one zero of $\mathtt{R}_{n-1}$. This may be re-written as follows.

Theorem F ([26], [9])**.** *For $\ell = m \geq 2$ and $n \geq 3$, the open set $\mathbf{\Omega}_n$ has $\deg \mathtt{R}_{n-1}$ connected components. Each component Ω_n is simply connected, contains exactly one (and always simple) zero of $\mathtt{R}_{n-1}$, and is mapped by*

$$\Xi_n(\mathtt{c}) = \Phi_{\mathtt{c}}(\mathtt{R}_n(\mathtt{c})) = \lim_{k\to\infty} \sqrt[m^k]{\mathtt{R}_{n+k}(\mathtt{c})}$$

properly onto $\Delta = \{w : |w| > 1\}$. Every branch of $\Xi_n^{1/\ell}$ maps Ω_n conformally onto Δ.

The proofs in [26] and [9] rely on very delicate computations and constructions, which only work for $\ell = m$. It is, nevertheless, quite natural to ask:

► *Does Theorem F remain true in the general case $1/\ell + 1/m < 1$?* [1]

[1] In the meanwhile the answer is known to be *yes*; see [28].

4. Kernel convergence

Let (D_n) be any sequence of domains, each containing some base point z_0. The *kernel* $\mathfrak{K}$ *of* (D_n) *with respect to* z_0 then is the union of all simply connected domains D, such that $z_0 \in D$ and $\overline{D} \subset D_n$ for $n \ge n_0(D)$. If no such D exists, we set $\mathfrak{K} = \{z_0\}$. The sequence (D_n) is said to *converge to* $\mathfrak{K}$ *in the sense of Carathéodory,* if $\mathfrak{K}$ is also the kernel of every sub-sequence (D_{n_k}). If all domains D_n are simply connected and if f_n denotes the conformal mapping of the unit disc $\mathbb{D}$ onto D_n, normalised by $f_n(0) = z_0$ and $f_n'(0) > 0$, then $D_n \to \mathfrak{K} \ne \{z_0\}$ in the sense of Carathéodory is equivalent to $f_n \to f$ locally uniformly, where f is the normalised conformal mapping $\mathbb{D} \to \mathfrak{K}$; for $\mathfrak{K} = \{z_0\}$ the sequence (f_n) tends to the constant z_0. For more details the reader is referred to Carathéodory [6]; the original definition for simply connected domains carries over to arbitrary domains D_n.

Every map $\mathtt{R}_n$, $n \in \mathbb{N}$, has a super-attracting fixed point at ∞, hence a super-attracting basin $\mathcal{B}_n$ and a corresponding Böttcher's function Θ_n, which is defined in some neighbourhood of infinity and satisfies

$$\Theta_n(\mathtt{R}_n(\mathtt{c})) = \mu^{m^{n-1}}\Theta_n(\mathtt{c})^{m^n}, \ \Theta_n \sim \mathtt{c} \text{ as } \mathtt{c} \to \infty.$$

It is easily shown that there exists some fixed disc $|\mathtt{c}| > \mathsf{P}$ contained in every $\mathcal{B}_n$; for a proof we just note that $|\mathtt{c}| > \mathsf{P} = \max\{1, \sqrt[m-1]{2/\mu}\}$ implies $|\mathtt{R}_n(\mathtt{c})| > 2|\mathtt{c}|$ for $n \ge 1$. Thus the *kernel* of the sequence $(\mathcal{B}_n)$ is a domain about ∞. If the domains $\mathcal{B}_n$ are simply connected, then Θ_n maps $\mathcal{B}_n$ conformally onto the disc $|w| > \sqrt[m^n-1]{\mu^{m^{n-1}}} \sim \sqrt[m]{\mu}$ as $n \to \infty$. In this case the statements

$$\Theta_n \to \mu^{-1/m}\Psi_0$$

and

$$\mathcal{B}_n \to \Omega_0 = \text{kernel of } (\mathcal{B}_n)$$

are equivalent.

Theorem 1. *The sequence* (Θ_n) *tends to* $\mu^{-1/m}\Psi_0$ (for the definition of Ψ_0 with normalisation $\Psi_0(\mathtt{c}) \sim \mu^{1/m}\mathtt{c}$ see the previous section), *locally uniformly in* Ω_0, *while the sequence* $(\mathcal{B}_n)$ *tends to its kernel* $\mathfrak{K}$ *with respect to* ∞, *and* $\mathfrak{K} = \Omega_0 = \mathbf{\Omega}_0 \cup \{\infty\}$.

A similar result holds for $\mathbf{\Omega}_2$, and also for the connected components of $\mathbf{\Omega}_n$, $n \ge 3$.

Theorem 2. *Let* $\mathtt{c}_n \in \mathbb{C}^*$, $n \ge 3$, *be a zero of* $\mathtt{R}_{n-1}$, *and let* $\mathcal{B}_k'$, $k \ge n$, *denote the Fatou component of* $\mathtt{R}_k$ *containing* $\mathtt{c}_n$ ($\mathcal{B}_k'$ *may coincide with* $\mathcal{B}_k$). *Then the sequence* $(\mathcal{B}_k')_{k\ge n}$ *tends to its kernel* $\mathfrak{K}$ *with respect to the point* $\mathtt{c}_n$, *and* $\mathfrak{K}$ *is the connected component of* $\mathbf{\Omega}_n$ *containing* $\mathtt{c}_n$. *The same is true if the McMullen domain* $\mathbf{\Omega}_2 \cup \{0\}$ *with* $n = 2$ *and* $\mathtt{c}_2 = 0$ *is considered.*

Remark. To my knowledge the phenomenon described in Theorems 1 and 2 was first observed by Busse in his doctoral thesis [5] in the case of the classical Mandelbrot set $\mathcal{M}$. For $P_{\mathtt{b}}(z) = \mathtt{b} + z^2$ the associated sequence $\mathsf{P}_n(\mathtt{b}) = P_{\mathtt{b}}^n(\mathtt{b})$ tends to ∞ exactly outside $\mathcal{M}$, and the sequence $(\mathcal{B}_n)$ of super-attracting basins of P_n tends to its kernel $\widehat{\mathbb{C}} \setminus \mathcal{M}$ in the sense of Carathéodory, while the sequence (Θ_n) of Böttcher functions tends to the conformal mapping $\widehat{\mathbb{C}} \setminus \mathcal{M} \to \Delta$. It is, however, not true that the domains $\mathcal{B}_n$ are simply connected, hence this result does not provide a new proof of the Douady–Hubbard Theorem [17] on the connectivity of the Mandelbrot set;

it is also not true that the sequence of filled-in Julia sets tends to the Mandelbrot set with respect to the Hausdorff metric, as one might suspect. In our case the corresponding question remains open: it is not known whether or not the Julia sets of the rational maps $\mathtt{R}_n$ are connected (at least for a sub-sequence).

Proof. The proof of Theorem 1 follows [5]. Böttcher's function Θ_n, the solution of Böttcher's functional equation

$$\Theta_n(\mathtt{R}_n(\mathtt{c})) = \mu^{m^{n-1}}\Theta_n(\mathtt{c})^{m^n},$$

is given by

$$\Theta_n(\mathtt{c}) = \lim_{k\to\infty} \sqrt[m^{nk}]{\mathtt{R}_n^k(\mathtt{c})/\mu^{\frac{m^{nk}-1}{m^n-1}m^{n-1}}} \sim \mathtt{c} \text{ as } \mathtt{c}\to\infty,$$

at least in $|\mathtt{c}| > \mathsf{P}$. Since $\mathtt{R}_n$ has no zeros and poles in $\Omega_0 \setminus \{\infty\}$, and $\mathtt{R}_n(\mathtt{c}) \to \infty$ in Ω_0, Θ_n is eventually defined in any simply connected domain D with $\overline{D} \subset \Omega_0$ and $\infty \in D$. Thus $D \subset \mathcal{B}_n$ for $n \geq n_0$, and the sequence of approximations $\Psi_n(\mathtt{c}) = \sqrt[m^n]{\mathtt{R}_n(\mathtt{c})}$ tend to Ψ_0, the normalised conformal mapping of Ω_0 onto Δ, uniformly in D. From $\mathtt{R}_n(\mathtt{c}) \to \infty$,

$$\mu^{m^{n-1}}\Theta_n(\mathtt{c})^{m^n} = \Theta_n(\mathtt{R}_n(\mathtt{c})) = \mathtt{R}_n(\mathtt{c}) + O(|\mathtt{R}_n(\mathtt{c})|^{-1}),$$

and $\Psi_n(\mathtt{c})^{m^n} = \mathtt{R}_n(\mathtt{c})$ then follows $\mu^{m^{n-1}}\Theta_n(\mathtt{c})^{m^n} - \Psi_n(\mathtt{c})^{m^n} \to 0$, *a fortiori* $\mu^{1/m}\Theta_n \to \Psi_0$, locally uniformly in Ω_0, and, in addition, $\Omega_0 \subset \mathfrak{K} =$ kernel of the sequence $(\mathcal{B}_n)$.

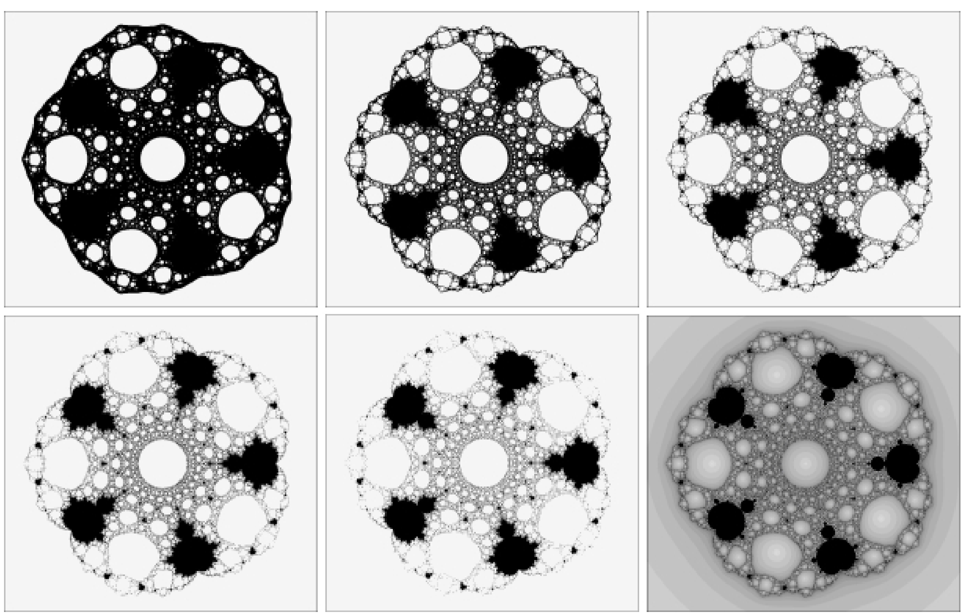

FIGURE 3. The dynamical planes for the rational maps $\mathtt{R}_n$, $n = 9, 12, 15, 18,$ and 21, and the parameter plane for the family $R_{\mathtt{c}}(z) = z^2 + \mathtt{c}^5/z^3$ (bottom right).

Conversely, let D be any simply connected domain containing ∞ and such that $\overline{D} \subset \mathfrak{K}$. Then $D \subset \mathcal{B}_n$ for $n \geq n_0$ (actually one has to work with an arbitrary subsequence $\mathcal{B}_{n(\nu)}$), and the sequence $(\Theta_n)_{n \geq n_0}$ is normal in D, this following from the fact that any analytic continuation of Θ_n satisfies

$$|\Theta_n(\mathtt{c})| \geq \sqrt[m^n-1]{\mu^{-m^{n-1}}} \sim \mu^{-1/m} \text{ in } \mathcal{B}_n,$$

a fortiori in D. By Vitali's Theorem the sequence (Θ_n) converges to Θ, locally uniformly in D, with $|\Theta(\mathtt{c})| > \mu^{-1/m}$ in D and $\Theta(\mathtt{c}) = \mu^{-1/m}\Psi_0$ in $D \cap \Omega_0$. Since $|\Psi_0(\mathtt{c})| \to 1$ as $\mathtt{c} \to \partial\Omega_0$ this shows that D is a subset of Ω_0, hence $\mathfrak{K} \subset \Omega_0$. □

Proof. The proof of Theorem 2 runs along the same lines. All one has to know is that $\mathtt{R}_k$, $k \geq n > 2$, has only poles whose multiplicities are multiples of m^{k-n}, hence $\sqrt[m^{k-n}]{\mathtt{R}_k(\mathtt{c})}$ is analytic in any simply connected domain containing no zero of $\mathtt{R}_k$; the zeros of $\mathtt{R}_k$, however, do not belong to $\mathcal{B}'_k$. Details of proof are left to the reader. □

5. Hyperbolic components of $\mathbf{\Omega}^\circ_\infty$

As we have seen in section 3, the dynamics of $R_\mathtt{c}$ is not very exiting if every critical point escapes to infinity, although the Julia set is. On the other hand, it is obvious that the Fatou set of any rational function $R(z) = z^m Q(z^d)$, $2 \leq m < d$, is invariant under the symmetric group

$$\Sigma_d = \{\omega : \omega^d = 1\}$$

(we identify ω with the rotation $z \mapsto \omega z$), and the dynamics of $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ interacts with the dynamics of the map

$$\Sigma_d \to \Sigma_d,\ \omega \mapsto \omega^m,$$

in a non-trivial manner. This may be best observed for $R = R_\mathtt{c}$, with parameter $\mathtt{c} \in \mathbf{\Omega}_\infty$; choosing $\mathtt{c} \in \mathbf{\Omega}_\infty$ leads to exiting phenomena in particular if d is a prime number.

The set $\mathbf{\Omega}_\infty$ is compact, non-empty (the solutions of $\mathtt{R}_n(\mathtt{c}) = \mathtt{R}_0(\mathtt{c})$ belong to $\mathbf{\Omega}_\infty$), and invariant under the group $\Sigma_{d(m-1)}$. Also, any connected component Ω of $\mathbf{\Omega}^\circ_\infty$ is either invariant under $\Sigma_{d(m-1)}$ or else has $d(m-1)-1$ counterparts $e^{2\pi ij/d(m-1)}\Omega$.

A connected component H of $\mathbf{\Omega}^\circ_\infty$ is called *hyperbolic*, if $(R_\mathtt{c})_{\mathtt{c} \in H}$ is a hyperbolic family. Since for $\mathtt{c} \in H$ the critical points ξ_j, $0 \leq j < d$, do not escape to ∞, any such $R_\mathtt{c}$ has (super-) attracting cycles.

Remark. The connected components of $\mathbf{\Omega}_n$ are also hyperbolic, as is the unbounded complementary component of the classical Mandelbrot set. In [16] they are called *Sierpiński holes.*

- A cycle $\{U_0, \ldots, U_{p-1}\}$ of Fatou components of $R_\mathtt{c}$ is called *prime*, if $U_j \neq \omega U_k$ for every $\omega \in \Sigma_d$ and every pair (j,k), $j \neq k$.
- A hyperbolic component H is called *prime,* if one (and hence, by the Theorem of Mãné, Sad and Sullivan, every) function $R_\mathtt{c}$, $\mathtt{c} \in H$, has a prime attracting cycle.
- Every $\omega \in \Sigma_d \setminus \{1\}$ satisfying $\omega^{m^n} = \omega$ for some $n \in \mathbb{N}$ is called *periodic*. The smallest n with this property is called the *period* of ω, and $\{\omega, \omega^n, \ldots, \omega^{m^{n-1}}\}$ is a *periodic cycle* in Σ_d.

Remark. Obviously the definition of periodicity depends on the *pair* (m,d); $\omega = e^{2\pi ij/d} \neq 1$ is n-periodic if and only if $j(m^n-1) \equiv 0 \mod d$. If d is a prime number, this condition is equivalent to $m^n \equiv 1 \mod d$, independent of j. Since then $m^{d-1} \equiv 1 \mod d$ by Fermat's Little Theorem, there is a *universal* period $n = n(d,m) > 1$; it depends only on m and d, and divides $d-1$. This is also true if $\gcd(m,d) = 1$. For example, we have $n(5,2) = n(5,3) = 4$, $n(7,2) = n(7,4) = 3$, and $n(7,3) = n(7,5) = 6$. For $d = 2m$ and m odd there is only one periodic element ($\omega = -1$ with period 1), while for m even all elements of $\Sigma_{2m} \setminus \{1\}$ are pre-periodic. This shows that the dynamics of $R(z) = z^m + \lambda/z^m$ are rather tame.

We first describe the prime hyperbolic components.

Theorem 3. *Let R_{c} have a prime (super-) attracting cycle $\{U_0, \ldots, U_{p-1}\}$ of Fatou components, with multiplier λ. Then exactly one of these domains, U_0 say, contains a critical point, and the following is true:*

(i) *For any n-periodic $\omega \in \Sigma_d \setminus \{1\}$, the cycle $O^+(\omega U_0)$ has period $q = \operatorname{lcm}(n,p)$, multiplier $\lambda^{q/p}\omega^{m^q-1}$, and contains q/p critical points.*

(ii) *For non-periodic $\tilde\omega \in \Sigma_d$, the Fatou component $\tilde\omega U_j$ is pre-periodic. If $\omega = \tilde\omega^{m^r}$ is periodic (here $\omega = 1$ is included), then R_{c}^r maps $\tilde\omega U_j$ into the cycle $O^+(\omega U_j)$.*

(iii) *The Julia set is connected and locally connected.*

Remark. The prime cycle $\{U_0, \ldots, U_{p-1}\}$ is uniquely determined, except when $\Sigma_d \setminus \{1\}$ contains some n-periodic element ω with $n|p$; then the cycle $O^+(\omega U_0)$ is also prime.

Proof of Theorem 3. We write $\alpha = \{z_0, \ldots, z_{p-1}\}$ with $z_{j+1} = R_{\mathsf{c}}(z_j)$, and denote by U_j the Fatou component containing z_j. Since the free critical points are regularly distributed, the prime cycle $\{U_0, \ldots, U_{p-1}\}$ contains exactly one critical point, $\xi_0 \in U_0$, say. Also $R_{\mathsf{c}}^p : U_j \to U_j$ is a two-to-one map with one critical point, hence U_j is simply connected.

For every integer q and every $\omega \in \Sigma_d$ we have

$$R^q(\omega z_0) = \omega^{m^q} R^q(z_0) = \omega^{m^q} z_q.$$

Since α is prime, $\omega^{m^q} z_q = \omega z_0$ holds if and only if $q \equiv 0 \mod p$ and $m^q \equiv 1 \mod d$, hence

$$q \equiv 0 \mod p \quad \text{and} \quad q \equiv 0 \mod n$$

follow. Thus ωz_0 is $q = \operatorname{lcm}(n,p)$-periodic, and from $R_{\mathsf{c}}'(\omega z_0) = \omega^{m-1} R_{\mathsf{c}}'(z_0)$, hence $R_{\mathsf{c}}'(\omega^{m^j} z_j) = \omega^{m^{j+1}-m^j} R_{\mathsf{c}}'(z_j)$, follows that the cycle $\beta = O^+(\omega z_0)$ has multiplier $\lambda(\beta) = \lambda^{q/p}\omega^{m^q-1}$. The q-cycle

$$\{\omega U_0, \omega^m U_1, \ldots, \omega^{m^{q-1}} U_{q-1}\}$$

contains exactly q/p critical points; more precisely, $\omega^{m^j} U_j$, $0 \le j < q$, contains a (unique) critical point if and only if $j \equiv 0 \mod p$. This proves (i).

If $\tilde\omega$ is non-periodic and $\omega = \tilde\omega^{m^r}$ is periodic, but $\tilde\omega^{m^{r-1}}$ is not, then $R^r(\tilde\omega z_0)$ belongs to the periodic cycle $O^+(\omega z_0)$, but $R^{r-1}(\tilde\omega z_0)$ does not. Thus $\tilde\omega z_0$ is pre-periodic, which proves (ii).

To prove the final statement we note that any Fatou component contains at most one critical point. Since the domains U_j, $0 \le j < p$, and $\mathcal{A}_\infty(\mathsf{c})$ are simply connected, the Julia set is connected, and even locally connected, since R_{c} is hyperbolic. $\square$

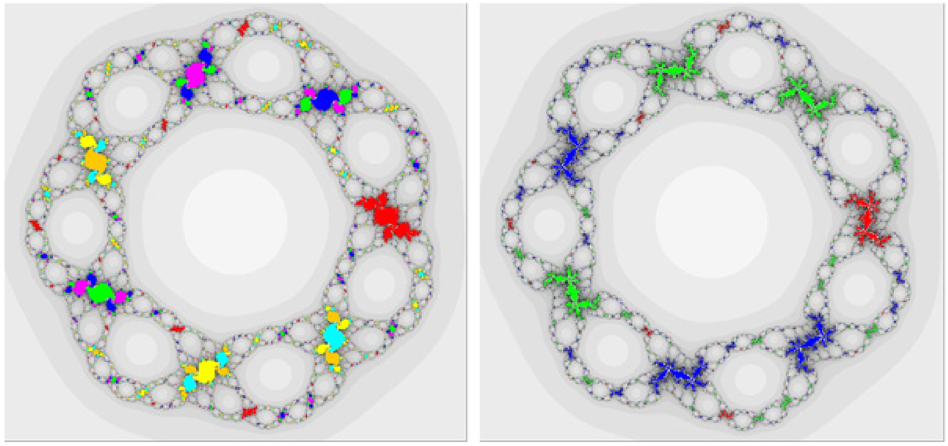

FIGURE 4. The dynamical plane for $R_{\mathsf{c}}(z) = z^4 + \mathsf{c}^7/z^3$, $\mathsf{c} = 0.78+0.03i$ (left) and $\mathsf{c} = 0.8025+0.0145i$ (right). In the first case, R_{c} has seven prime attracting 3-cycles ($p = n = 3$), only one being connected in the sense that it is embedded into a set homeomorphic to the filled-in Julia set of some degree-2 polynomial. In the second case, R_{c} has a prime connected attracting 5-cycle and two non-prime attracting 15-cycles ($p = 5$, $n = 3$, $q = \operatorname{lcm}(3, 5) = 15$).

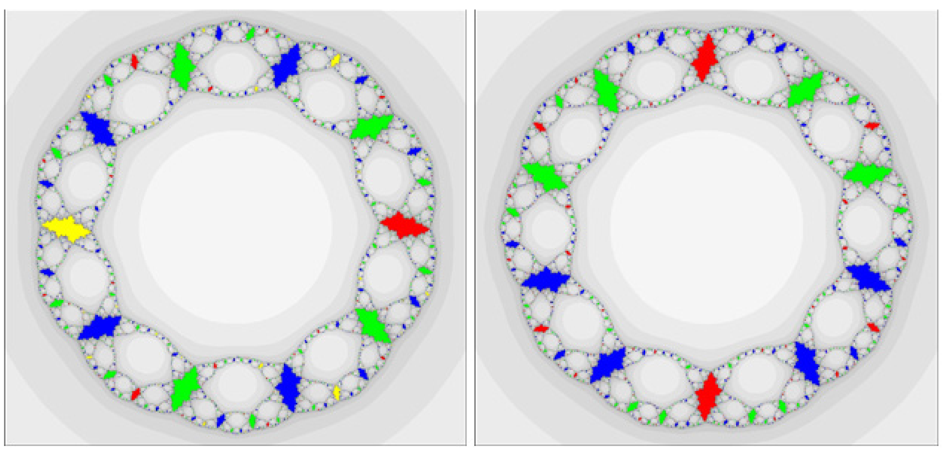

FIGURE 5. The dynamical plane for $R_{\mathsf{c}}(z) = z^3 + \mathsf{c}^{10}/z^7$, $\mathsf{c} = 0.75 + 0.01i$ (left) and $\mathsf{c} = -0.01 + 0.75i$ (right). In the first case, R_{c} has two fixed domains and four non-prime 4-cycles ($p = 1$, $n = 4$, $q = 4$ in the notion of Theorem 4). In the second case, R_{c} has a non-prime 2-cycle and two non-prime 4-cycles ($p = 2$, $n = 4$, $q = 4$). The Julia sets are rotations of each other, but dynamically they are quite different.

Remark. The fact that hyperbolic rational functions with connected Julia set have even locally connected Julia set, provided there exists a *completely invariant* fixed domain (or 2-cycle), is due to Fatou. The general result for so-called *geometrically* *finite* rational maps, even *without completely invariant* Fatou component, was first proved by Tan Lei and Y.-Ch. Yin [29], and, independently, by Mattler in his doctoral thesis [20].

At the beginning of this work it seemed to me a minor problem (to be postponed to the end) to prove that prime attracting cycles always exist. After several ineffective attempts, computer experiments evidently showed that prime attracting cycles do not exist in general, so this possibility has to be discussed separately. I also learned from these experiments that there are different types of prime attracting cycles; see Figure 6.

Theorem 4. *Let $\beta = \{u_0, \ldots, u_{q-1}\}$ be a non-prime (super-) attracting cycle of R_{c}, with corresponding cycle $\{U_0, \ldots, U_{q-1}\}$ of Fatou components. Then there exists some n-periodic $\omega \in \Sigma_d \backslash \{1\}$ and some integer p dividing q, such that $R_{\mathsf{c}}^p(U_0) = \omega U_0$ and $q = \operatorname{lcm}(n, p)$. The cycle $\{U_0, \ldots, U_{q-1}\}$ contains q/p critical points. The Julia set is connected, and even locally connected.*

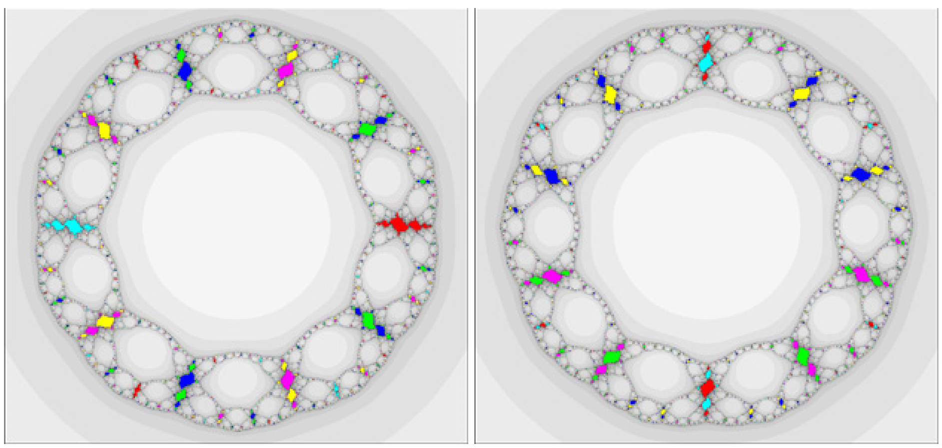

FIGURE 6. The dynamical plane for $R_{\mathsf{c}}(z) = z^3 + \mathsf{c}^{10}/z^7$, $\mathsf{c} = 0.735 + 0.005i$ (left) and $\mathsf{c} = -0.005 + 0.735i$ (right). In the first case, R_{c} has two prime 2-cycles and four non-prime 4-cycles; the 2-cycles are connected in the sense that they are embedded in the filled-in Julia set of some degree-2 polynomial-like mapping. In the second case, R_{c} has two prime but non-connected 2-cycles and four non-prime 4-cycles. The Julia sets are rotations of each other, but dynamically they are quite different.

Proof of Theorem 4. The map R_{c} is *quasi-conjugate* to

$$\tilde{R}_{\mathsf{c}}(z) = z^m(1 + \mathsf{c}^d/z)^d \tag{8}$$

via $R_{\mathsf{c}}(z)^d = \tilde{R}_{\mathsf{c}}(z^d)$. Since the map $\tilde{R}_{\mathsf{c}}$ has only one free critical point $\kappa^d \mathsf{c}^d$, it has a unique (super-) attracting cycle $\alpha = \{v_0, \ldots, v_{p-1}\}$ with corresponding Fatou components

$$V_0, V_1, \ldots, V_{p-1},$$

and $z \mapsto z^d$ maps β onto α. Then to each domain V_ρ, $0 \le \rho < p$, there correspond d domains $U_\rho^{[j]} = e^{2\pi i j/d} U_\rho^{[0]}$, $0 \le j < d$, which are mapped by $z \mapsto z^d$ onto V_ρ.

Hence, if we assume $U_0 = U_0^{[j_0]}$, the orbit

$$V_0 \xrightarrow{\tilde{R}_{\mathsf{c}}} V_1 \xrightarrow{\tilde{R}_{\mathsf{c}}} V_2 \xrightarrow{\tilde{R}_{\mathsf{c}}} \cdots \xrightarrow{\tilde{R}_{\mathsf{c}}} V_p = V_0$$

corresponds to

$$U_0 \xrightarrow{R_{\mathsf{c}}} U_1^{[j_1]} \xrightarrow{R_{\mathsf{c}}} U_2^{[j_2]} \xrightarrow{R_{\mathsf{c}}} \cdots \xrightarrow{R_{\mathsf{c}}} U_p^{[j_p]} = U_0^{[j_p]} = \omega U_0$$

for some $\omega \in \Sigma_d$. Since β is assumed to be non-prime, $\omega \neq 1$ is n-periodic and β has period $q = \mathrm{lcm}(n, p)$. Thus the proof worked as if α were a prime attracting cycle for R_{c}. The cycle $\{U_0, \ldots, U_{q-1}\}$ contains q/p critical points. The domains V_ρ are simply connected, and so are the domains $U_\rho^{[j]}$, which together with simple connectivity of $\mathcal{A}_\infty(\mathsf{c})$ implies that the Julia set is connected, and again even locally connected. □

Theorem 5. *If d is a prime number, then every hyperbolic component is prime, i.e., every R_{c} with c in some hyperbolic component has a prime attracting cycle.*

Proof. There is some universal period $n = n(m, d)$ with $2 \leq n \leq d-1$ and $n|(d-1)$. By analysing the proof of Theorem 4, one may conclude that there exists some unique $p \in \mathbb{N}$, such that every non-prime (super-) attracting cycle of R_{c} has period $q = \mathrm{lcm}(n, p)$ and contains $k = q/p > 1$ critical points. Since $k|n$ and thus $k|(d-1)$, there is at least one critical point not belonging to some q-cycle; it thus belongs to some prime attracting cycle of period p. □

Remark. If d is a prime number, then every $\omega \in \Sigma_d \setminus \{1\}$ is $n(m, d)$-periodic, and all critical points belong to attracting or super-attracting cycles, but only one of these cycles is prime, except when $n(m, d)|p$; in this case all (super-) attracting cycles are prime.

In a certain sense the symmetric case $\ell = m$ is just the contrary: the map $\omega \mapsto \omega^m$ has no periodic element in $\Sigma_{2m} \setminus \{1\}$ if m is even, and exactly one, namely $\omega = -1$ with period 1, if m is odd. By Theorem 4 every finite attracting cycle is prime, and the dynamics of

$$R_{\mathsf{c}}(z) = z^m + \mathsf{c}^{2m}/z^m \tag{9}$$

is thus very tame. From this observation easily follows Theorem 6 below. The different situations (m odd resp. even) are illustrated in Figure 7.

Theorem 6. *Any attracting cycle $\{U_0, \ldots, U_{p-1}\}$ of R_{c}, given by* (9), *is prime. For m even, there exists no other attracting cycle, while in case m is odd, R_{c} has a second attracting* (*and also prime*) *cycle $\{-U_0, \ldots, -U_{p-1}\}$.*

Remark. Theorems 3, 4, 5, and 6 also hold if *attracting* cycle is replaced with *parabolic* cycle or *Siegel* cycle, and *Fatou component* with *Leau flower* or *Siegel disc*, respectively. The statements about the critical points, of course, have to be modified suitably in the case of Siegel discs.

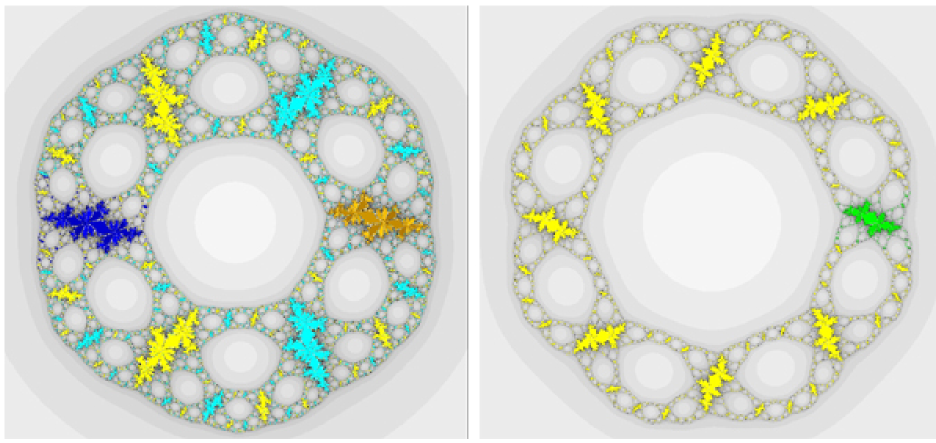

FIGURE 7. The dynamical plane for $R_{\mathtt{c}}(z) = z^3 + \mathtt{c}^6/z^3$, $\mathtt{c} = 0.655 + 0.027i$ (left) and $R_{\mathtt{c}}(z) = z^4 + \mathtt{c}^8/z^4$, $\mathtt{c} = 0.7775 + 0.018i$ (right). In the first case, $R_{\mathtt{c}}$ has two attracting and prime 9-cycles; they are connected in the sense that they are embedded in the filled-in Julia set (brown and blue, respectively) of some degree-2 polynomial-like mapping. In the second case, there is only one attracting cycle; it is prime, has period 5, and is embedded in the filled-in Julia set (green) of some degree-2 polynomial-like mapping.

6. Polynomial-like mappings

Let D_0 and D be simply connected domains bounded by piecewise smooth Jordan curves and such that $\overline{D}_0 \subset D$, and let $f : D_0 \to D$ be a proper mapping of degree $\deg f = k > 1$. Then, following Douady and Hubbard [18], $(f; D_0, D)$ is called a *polynomial-like mapping.* The compact set

$$\mathcal{K}_f = \bigcap_{n\in\mathbb{N}} f^{-n}(\overline{D}_0)$$

is called the *filled-in Julia set* of f. It is connected if and only if the critical orbit of f is (defined and) contained in D_0. Every polynomial-like mapping is *hybrid equivalent* to some polynomial with $\deg P = \deg f$, i.e., there exists some quasi-conformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ such that the conjugation $\phi \circ f = P \circ \phi$ holds on D_0; ϕ is analytic in the interior of $\mathcal{K}_f$. If $\mathcal{K}_f$ is connected, then P is uniquely determined up to analytic conjugation. Every polynomial P with degree $k > 1$ may be regarded as a polynomial-like mapping $P : P^{-1}(\Delta_r) \to \Delta_r = \{z : |z| < r\}$ for $r > 0$ sufficiently large, with $\mathcal{J}_P = \partial\mathcal{K}_P$.

To construct domains D_0 and D for

$$f(z) = Q_{\mathtt{a}}(z) = \mathtt{a}z^m(1 + 1/z)^d, \ \mathtt{a} \in \mathbb{C}^*,$$

with critical points -1, 0, and ∞, and *free* critical point $\varpi = \ell/m$, let D be any simply connected domain containing the critical value

$$\upsilon = Q_{\mathtt{a}}(\varpi) = \varpi^m(1 + 1/\varpi)^d\mathtt{a},$$

but none of the critical values 0 and ∞. Then $Q_{\mathtt{a}}^{-1}(D)$ consists of $d-1$ simply connected components D_j, $0 \le j \le d-2$, such that D_0 contains the critical point ϖ, and $Q_{\mathtt{a}} : D_0 \to D$ is a proper mapping of degree two, while $Q_{\mathtt{a}} : D_j \to D$, $1 \le j \le d-2$, is a conformal mapping. Thus $Q_{\mathtt{a}} : D_0 \to D$ is a polynomial-like mapping of degree two, and hence is hybrid equivalent to some quadratic polynomial $P_{\mathtt{b}}(z) = z^2 + \mathtt{b}$, if one can choose D in such a way that $\overline{D}_0 \subset D$ holds. This, of course, is the crucial point. The construction of D is illustrated in Figure 8; in the symmetric case $\ell = m$ it is much more explicit (see [9]).

Lemma 3. *To any* $\mathtt{a} \in \mathbb{C}^*$ *there exist simply connected and piecewise smooth Jordan domains* D *and* D_0*, such that* $Q_{\mathtt{a}} : D_0 \to D$ *is a polynomial-like mapping of degree two.*

Remark. One may choose D independent of $\mathtt{a}$, provided $\mathtt{a}$ varies in a compact subset of $\mathbb{C} \setminus (-\infty, 0]$.

Proof. We first assume that $\mathtt{a}$ is not real and negative. We start with the slit-plane

$$D^* = \{z : |\arg z| < \pi\}$$

(this idea was supported by computer experiments, see Figure 8); it contains the critical value $\upsilon = Q_{\mathtt{a}}(\varpi) = \varpi^m(1+1/\varpi)^d\mathtt{a}$. The pre-image of the negative real axis under $Q_{\mathtt{a}}$ consists of d open Jordan arcs, denoted γ_j; they start at $z=-1$ with equally distributed tangent directions. While m of these arcs, with labels $1 \le j \le m$, say, tend to infinity (note that $Q_{\mathtt{a}}(z) \sim \mathtt{a}z^m$ as $z \to \infty$), the arcs labelled by $m+1 \le j \le d$ land at $z=0$ (since $Q_{\mathtt{a}}(z) \sim \mathtt{a}z^{-\ell}$ as $z \to 0$).

The arcs γ_j, $m+1 < j < d$, are contained in the bounded domain D_b^* with boundary curve $\gamma_{m+1} \cup \gamma_d \cup \{-1, 0\}$; D_b^* also contains part of $(-1, 0)$. The curve $\gamma_1 \cup \gamma_m \cup \{-1, \infty\}$ divides the sphere into two domains, one of which, denoted D_u^*, contains the arcs γ_j, $1 < j < m$ (hence is unbounded), and part of $(-\infty, -1)$; the other one contains D_b^*. By D_0^* we denote the component of $Q_{\mathtt{a}}^{-1}(D^*)$ which is bounded by $\gamma_1 \cup \gamma_m \cup \gamma_{m+1} \cup \gamma_d \cup \{-1, 0, \infty\}$; it contains part of $(0, \infty)$.

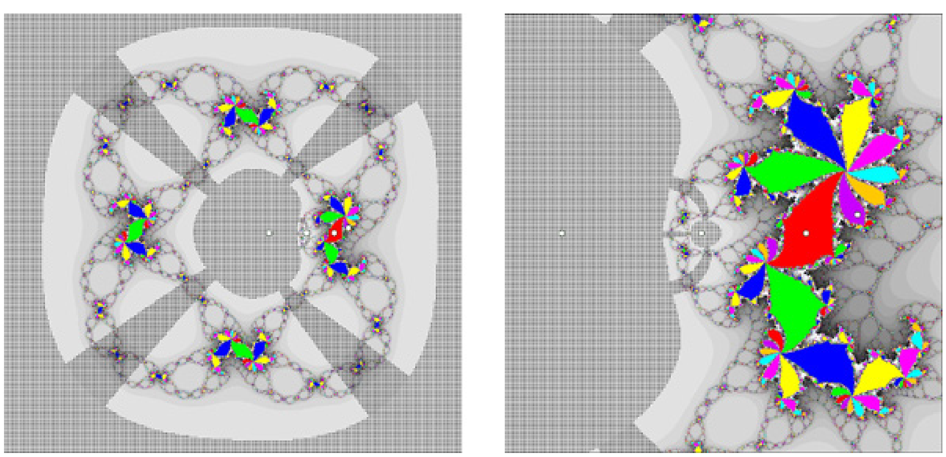

FIGURE 8. The pre-image of $D = \{z : |\arg z| < \frac{5}{6}\pi\} \setminus Q_{\mathtt{a}}^{-1}(\overline{\Delta}_{10})$ (left, and more detailed right) under $Q_{\mathtt{a}}(z) = \mathtt{a}z^4(1+1/z)^7$, $\mathtt{a} = 0.0094 + 0.0011i$.

Since $\arg Q_{\mathtt{a}}(z) = \arg \mathtt{a}$ for $z > 0$ and $\mathtt{a} \notin (-\infty, 0)$, it follows from the intermediate value theorem that the whole positive real axis, and thus the critical point $z = \varpi$, belong to D_0^*. For $z < -1$ we obtain $\arg Q_{\mathtt{a}}(z) = \arg \mathtt{a} + m\pi \pmod{2\pi}$, and again from the intermediate value theorem it follows that either $(-\infty, -1) \subset D_u^*$ or else $Q_{\mathtt{a}}(-\infty, -1) = (-\infty, 0)$; the latter occurs if and only if m is odd and $\mathtt{a} > 0$. Similarly we obtain $(-1, 0) \subset D_b^*$ or else $Q_{\mathtt{a}}(-1, 0) = (-\infty, 0)$. Putting things together, it follows that $D_0^* \subset D^*$ for $\mathtt{a} \notin (-\infty, 0)$. To complete the construction we replace D^* with

$$D = \{z : |\arg z| < \pi - \epsilon\} \setminus Q_{\mathtt{a}}^{-1}(\overline{\Delta}_r), \; \epsilon > 0,$$

where $\Delta_r = \{z : |z| > r\}$ is any disc satisfying $Q_{\mathtt{a}}(\overline{\Delta}_r) \subset \Delta_r$. Then D and D_0, the connected component of $Q_{\mathtt{a}}^{-1}(D)$ which is contained in D_0^* (and which contains $z = \varpi$ for $\epsilon > 0$ sufficiently small), are simply connected and bounded by piecewise smooth Jordan curves. For $\epsilon > 0$ sufficiently small D_0 is compactly contained in D, and $Q_{\mathtt{a}} : D_0 \to D$ is a polynomial-like mapping of degree two.

Finally, for $\mathtt{a}$ negative we replace some short interval $(\upsilon - \sigma, \upsilon + \sigma)$ about the critical value $\upsilon = Q_{\mathtt{a}}(\varpi)$ with a semi-circle centred at υ to obtain a modified domain D^*. The construction of D and D_0 then runs along the same lines as in the first part of the proof. □

Remark. We note that the arcs γ_j are trajectories of the autonomous system $\dot{z} = Q_{\mathtt{a}}(z)/Q'_{\mathtt{a}}(z) = \frac{z(z+1)}{m(z-\varpi)}$, whose phase portrait coincides with that of

$$\dot{z} = z(z+1)(\bar{z} - \varpi), \; \varpi = \ell/m > 0. \tag{10}$$

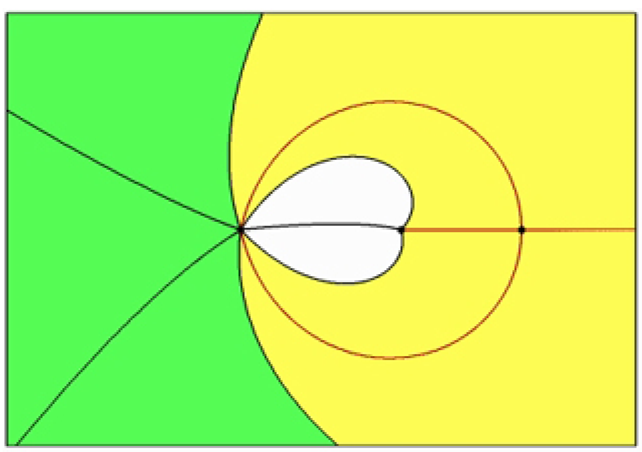

FIGURE 9. Phase portrait for (10) with $\ell = 3$ and $m = 4$. The plane is divided into three domains: the component D_0^* (yellow) of $Q_{\mathtt{a}}^{-1}(\mathbb{C} \setminus (-\infty, 0])$ containing the critical point $z = 3/4$ and the stable and unstable manifold (red) of the saddle point $z = 3/4$; the unbounded domain D_u^* (green) containing the negative real axis and the arcs γ_2 and γ_3; the bounded domain D_b^* (white) containing γ_6 and the interval $(-1, 0)$.

It is easily seen that (10) has an unstable node (star) at $z=-1$, stable nodes (stars) at $z=0$ and $z=\infty$ (the latter may be seen by substituting $z=1/\zeta$, hence $\dot\zeta=-\frac{\zeta}{|\zeta|^2}(\zeta+1)(1-\varpi\bar\zeta)$), and a saddle point at $z=\varpi$, see Figure 9. The stable manifold of the saddle point consists of two trajectories $\gamma^{\pm}$, orthogonal to the real axis at $z=\varpi$, and having tangent directions $\arg(z+1)=\pm\theta(\varpi)$ at $z=-1$. The closed Jordan curve

$$\Gamma=\{\varpi,-1\}\cup\gamma^-\cup\gamma^+$$

is a *separatrix*, its interior and exterior are attracted by $z=0$ and $z=\infty$, respectively. Given any value α there exist trajectories with tangent direction $\arg(z+1)=\alpha+2j\pi/d$, $0\le j<d$, at $z=-1$, and for $d\alpha\not\equiv\pm d\theta(\varpi) \mod 2\pi$, ℓ of these trajectories land at $z=0$, of course inside Γ, while the others tend to infinity. The same problem as in the proof of Lemma 3 occurs if $\alpha=\pm\theta(\varpi)$, this corresponding to $\mathtt{a}\in(-\infty,0)$, in which case we have d pre-image domains. By the way we find $\theta(\varpi)=\pi\ell/d$.

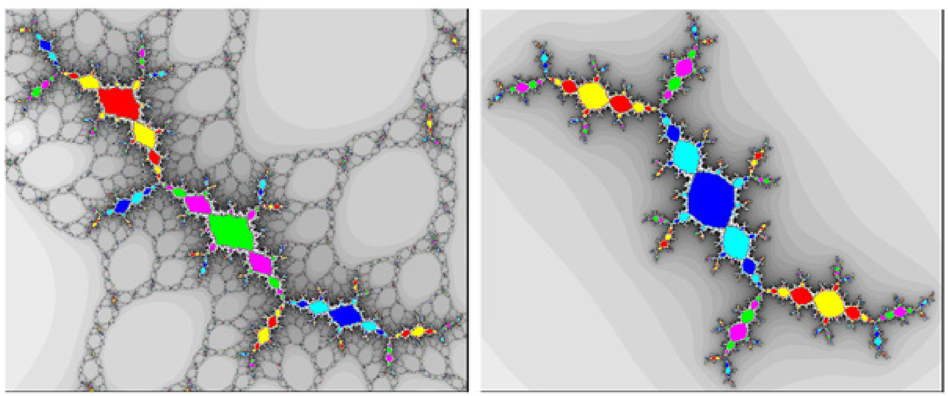

FIGURE 10. Detail of the dynamical plane for $\tilde Q_{\mathtt{a}}(z)=z^4(1+\mathtt{a}/z)^7$, $\mathtt{a}=0.783+0.013i$ (left), and for a corresponding polynomial $P_{\mathtt{b}}(z)=z^2+\mathtt{b}$, $\mathtt{b}=-0.11+0.8625$ (right); note that $\tilde Q_{\mathtt{a}}$ is conjugate to $Q_{\mathtt{a}^3}(z)=\mathtt{a}^3z^4(1+1/z)^7$. The dynamical plane of $\tilde Q_{\mathtt{a}}$ contains infinitely many copies of a set homeomorphic to the filled-in Julia set for $P_{\mathtt{b}}$. The picture is not quite correct, since no efforts were made to determine the true parameter $\mathtt{b}=\mathtt{b}(\mathtt{a})$, which is uniquely determined. However, $\mathtt{b}$ is chosen in the hyperbolic component of the classical Mandelbrot set containing also $\mathtt{b}(\mathtt{a})$.

Theorem 7. *For* $\mathtt{a}=\mathtt{c}^{d(m-1)}\in\mathbb{C}^*$ *the quasi-conjugate*

$$Q_{\mathtt{a}}(z)=\mathtt{a}z^m(1+1/z)^d:D_0\to D$$

of $R_{\mathtt{c}}$ *is hybrid equivalent to some polynomial* $P_{\mathtt{b}}(z)=z^2+\mathtt{b}$.

Remark. The fact whether or not the filled-in Julia set of $Q_{\mathtt{a}}:D_0\to D$ is a continuum (versus totally disconnected) is independent of the pair (D_0,D). In fact, if $Q_{\mathtt{a}}:D_0'\to D'$ is also a degree-two polynomial-like mapping, and if D'' is the connected component of $D\cap D'$ containing the critical value $Q_{\mathtt{a}}(\varpi)$, and D_0'' is the connected component of $Q_{\mathtt{a}}^{-1}(D'')$ containing the critical point $z=\varpi$, then the corresponding parts $\mathcal{K}_{Q_{\mathtt{a}}} \cap D_0''$ and $\mathcal{K}'_{Q_{\mathtt{a}}} \cap D_0''$ of the filled-in Julia sets agree with the filled-in Julia set of $Q_{\mathtt{a}} : D_0'' \to D''$.

We note that $\mathtt{c} \in \mathbf{\Omega}_\infty$ (correspondingly $\mathtt{a} \in \tilde{\mathbf{\Omega}}_\infty$) does not necessarily imply that $\mathtt{b}$ belongs to the classical Mandelbrot set $\mathcal{M}_2$. If, however, even $\mathtt{b} \in \mathcal{M}_2^\circ$, then all domains belonging to the bounded attracting cycles of $R_{\mathtt{c}}$ and $Q_{\mathtt{a}}$, respectively, and also their pre-images of any order, are bounded by quasi-circles. We will show that this holds always in the hyperbolic case.

If $R_{\mathtt{c}}$ is hyperbolic and $\mathcal{A}_\infty(c)$ is simply connected, then the boundary of $\mathcal{A}_\infty(c)$ is locally connected, hence a closed curve. Computer experiments indicate that it is even a Jordan curve. If so, this is a non-trivial fact. Nevertheless it is true for $|\mathtt{c}|$ not too large, Devaney [7] has shown that, for $\ell = m$ and $|\mathtt{c}^{2m}| = |\lambda| < \sqrt[m-1]{4^{-m}}$, $\partial\mathcal{A}_\infty(c)$ is a quasi-circle. This is also true in the general case for $|\mathtt{c}| < \sqrt[m-1]{\kappa/\mu}$ (see Theorem 8 below), and computer experiments indicate that, for d not too large, the disc $|\mathtt{c}| < \sqrt[m-1]{\kappa/\mu}$ covers part of the bifurcation locus $\partial\mathbf{\Omega}_\infty$.

Theorem 8. *For $0 < |\mathtt{c}| < \sqrt[m-1]{\kappa/\mu}$, the boundary of $\mathcal{A}_\infty(\mathtt{c})$ is a quasi-circle.*

Proof. We consider the unbounded component D of $\{z : |R_{\mathtt{c}}(z)| > \mu|\mathtt{c}|^m\}$, with $\mu = \kappa^m(1+\kappa^{-d})$ and $\kappa = (\ell/m)^{1/d}$. By Lemma 1, D is bounded by the closed polar curve $\Gamma_2 : z = \mathtt{c}r_2(\theta)e^{i\theta}$, $0 \le \theta \le 2\pi$, and contains no finite critical point of $R_{\mathtt{c}}$. For

$$\mu|\mathtt{c}|^m < |\mathtt{c}|\min\{r_2(\theta) : 0 \le \theta \le 2\pi\} = \kappa|\mathtt{c}|, \tag{11}$$

D is compactly contained in $\Delta = \{z : |z| > \mu|\mathtt{c}|^m\}$, and hence the conjugate

$$\tilde{R}(z) = 1/R_{\mathtt{c}}(1/z)$$

is a polynomial-like mapping $\tilde{R} : 1/D \to 1/\Delta$ with $\deg \tilde{R} = m$. Since

$$\tilde{R}(z) = z^m + \cdots \text{ at } z = 0,$$

and since $1/D$ contains no non-zero critical point, $\tilde{R}$ is hybrid equivalent to $z \mapsto z^m$,

$$\tilde{R}(\tilde{\psi}(z)) = \tilde{\psi}(z^m),$$

and has filled-in Julia set $\mathcal{K}_{\tilde{R}} = 1/\overline{\mathcal{A}_\infty(\mathtt{c})}$. Reversing the conjugation $z \mapsto 1/z$ we find that the solution of Böttcher's inverse functional equation

$$R_{\mathtt{c}}(\Psi_{\mathtt{c}}(z)) = \Psi_{\mathtt{c}}(z^m)$$

admits a quasi-conformal extension $\psi(z) = 1/\tilde{\psi}(1/z)$ to $\hat{\mathbb{C}}$, at least as long as (11) holds, and this is true for $|\mathtt{c}|^{m-1} < \kappa/\mu$. □

7. Copies of the Mandelbrot set

Copies of the classical Mandelbrot are closed sets homeomorphic to $\mathcal{M}_2$. They occur in many parameter families, in [23] it is explained why. In [8] and [9], the existence of small copies (baby and satellite Mandelbrot sets) is discussed for the symmetric family $R(z) = z^m + \lambda/z^m$, in much more detail than seems to be possible in the general case.

We prefer to consider the family $Q_{\mathtt{a}}(z) = \mathtt{a}z^m(1+1/z)^d$, $d = m+\ell$, with free critical point $\varpi = \ell/m$, and associated sequence $\mathtt{Q}_n(\mathtt{a}) = Q^n_{\mathtt{a}}(\varpi)$, hence

$$\begin{aligned} \mathtt{Q}_1(\mathtt{a}) &= \mu\mathtt{a}, \\ \mathtt{Q}_2(\mathtt{a}) &= \frac{(\mu\mathtt{a}+1)^d}{\mu^\ell\mathtt{a}^{\ell-1}}, \text{ and} \\ \mathtt{Q}_{n+1}(\mathtt{a}) &= \mathtt{a}\frac{(\mathtt{Q}_n(\mathtt{a})+1)^d}{\mathtt{Q}_n(\mathtt{a})^\ell}, \ n \geq 2. \end{aligned}$$

In the terminology of [23] the boundary $\tilde{B}$ of the set $\tilde{\mathbf{\Omega}}_\infty$ (the latter is just the image of $\mathbf{\Omega}_\infty$ under the covering map $\mathtt{c} \mapsto \mathtt{a} = \mathtt{c}^{d(m-1)}$, and may be written as $\tilde{\mathbf{\Omega}}_\infty = \{\mathtt{a} : 1/K \leq |\mathtt{Q}_n(\mathtt{a})| \leq K, \ n \in \mathbb{N}\}$ for some $K > 1$) denotes the *bifurcation locus*, and the critical point ϖ is called *active* for $\mathtt{a} \in \tilde{B}$. The boundaries of the copies of $\mathcal{M}_2$ are dense in $\tilde{B}$.

Rational functions $Q_{\mathtt{a}}$ with super-attracting n-cycles correspond to parameters $\mathtt{a}$ satisfying $\mathtt{Q}_n(\mathtt{a}) = \varpi$, but $\mathtt{Q}_j(\mathtt{a}) \neq \varpi$ for $j < n$. A compact set $\mathcal{M}_2^{[n]} \subset \tilde{\mathbf{\Omega}}_\infty$ homeomorphic to $\mathcal{M}_2$ is called a *copy of order* n, if its *main cardioid* (which is the image of the main cardioid of $\mathcal{M}_2$) contains a solution of $\mathtt{Q}_n(\mathtt{a}) = \varpi$, and $\mathtt{a}$ is called the *centre* of $\mathcal{M}_2^{[n]}$; the corresponding map $Q_{\mathtt{a}}$ then has a super-attracting n-cycle. Any rational map $Q_{\mathtt{a}}$ with $\mathtt{a}$ in some hyperbolic component of $\mathcal{M}_2^{[n]}$ has a (super-)attracting cycles whose period is a multiple of n.

Since $\mathtt{Q}_1(a) = \varpi$ has only one solution $\mathtt{a} = \mathtt{a}_0 = \varpi^{-m+1}(1+\varpi^{-1})^{-d}$, there is at most one copy $\mathcal{M}_2^{[1]}$ of order one, and by Theorem 9 below there is exactly one.

Theorem 9. *There exists exactly one copy $\mathcal{M}_2^{[1]}$ of $\mathcal{M}_2$ of order one; it is contained in*

$$A = \{\mathtt{a} : \rho_0 < |\mathtt{a}| < \rho_1, \ |\arg \mathtt{a}| < \pi - \delta\},$$

with $\rho_0 > 0$ and $\delta > 0$ sufficiently small, and ρ_1 sufficiently large.

Proof. Let A be any simply connected domain in the parameter plane containing $\mathtt{a}_0 = \varpi^{-m}(1+\varpi^{-d})^{-1}$. Then A contains some copy $\mathcal{M}_2^{[1]}$, if

$$\upsilon = Q_{\mathtt{a}}(\varpi) \notin D_0 \text{ for } \mathtt{a} \in \partial A, \tag{12}$$

see the interpretation of the Douady–Hubbard Theorem in [4]; the notation D, D_0 refers to Lemma 3. The meaning of (12) is that A is surrounded by polynomial-like mappings $Q_{\mathtt{a}} : D_0(\mathtt{a}) \to D(\mathtt{a})$ with *totally disconnected* filled-in Julia sets.

To check these conditions in our case, we first consider the circles $|\mathtt{a}| = \rho_j$, $j = 0, 1$, and note that the construction in Lemma 3 contains a parameter $r > 0$, subject only to the condition that $|z| \geq r$ implies $|Q_{\mathtt{a}}(z)| > r$. Thus $z \in D_0$ implies $|Q^2_{\mathtt{a}}(z)| < r$, and, in particular,

$$|Q^2_{\mathtt{a}}(\varpi)| = |Q_{\mathtt{a}}(\nu\mathtt{a})| < r, \qquad \nu = \varpi^m(1+\varpi^{-d})^d. \tag{13}$$

Now

$$Q^2_{\mathtt{a}}(\varpi) \sim \nu^m\mathtt{a}^{m+1} \text{ and } Q^2_{\mathtt{a}}(Q_{\mathtt{a}}(\varpi)) \sim \nu^{m^2}\mathtt{a}^{1+m+m^2} \text{ as } |\mathtt{a}| = \rho_1 \to \infty$$

imply $\upsilon = Q_{\mathtt{a}}(\varpi) \in D^* \setminus D_0^*$, hence $\upsilon \in D \setminus D_0$, if we set $r = \rho_1^{m+2}$ with ρ_1 sufficiently large. On the other hand, we have

$$Q^2_{\mathtt{a}}(\varpi) \sim \nu^{m-d}\mathtt{a}^{m+1-d} \text{ and } Q^3_{\mathtt{a}}(\varpi) \sim \nu^{m^2-md}\mathtt{a}^{1+m+m^2-md} \text{ as } |\mathtt{a}| \to 0.$$

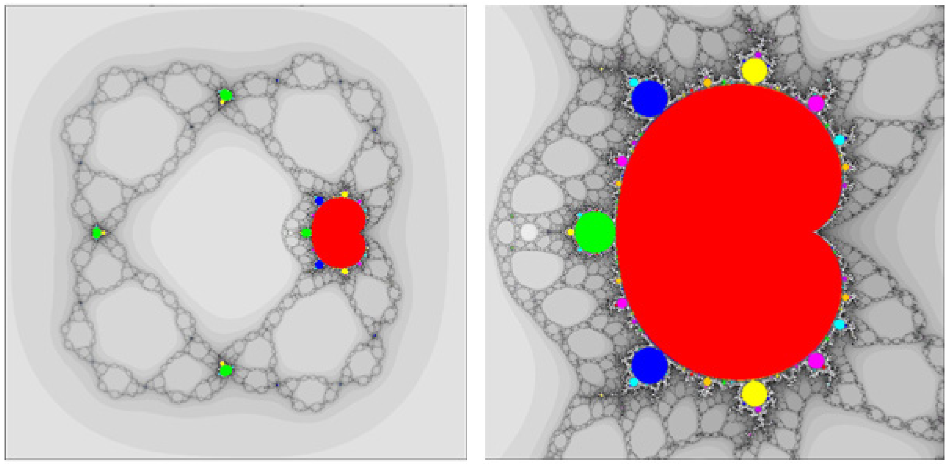

FIGURE 11. The parameter plane for $Q_{\mathtt{a}}(z) = \mathtt{a}z^3(1+1/z)^7$ (left), and a detail showing $\mathcal{M}_2^{[1]}$ (right). The coloured domains represent non-trivial hyperbolic components.

Since $1+m+m^2-md < m+1-d$ (equivalent to $m^2 < d(m-1)$) we may conclude that the critical value $v = \nu\mathtt{a}$ belongs to $D^* \setminus D_0^*$, if we set $r = \rho_0^{m-d+1/2}$ (note that $m-d+1/2 < 0$), now with ρ_0 sufficiently small.

We finally consider $\mathtt{a} = -|\mathtt{a}|e^{\pm i\delta}$. Then $v = \nu\mathtt{a} = -\nu|\mathtt{a}|e^{\pm i\delta} \in D_u^*$, hence $v \in D^* \setminus D_0^*$ for $|\mathtt{a}|$ fixed and $\delta > 0$ sufficiently small. If $|\mathtt{a}|$ is restricted to $\rho_0 \le |\mathtt{a}| \le \rho_1$, then δ may be chosen independent of $\mathtt{a}$.

□

Remark. Since $\mathtt{a} = \mathtt{c}^{d(m-1)}$ and $\lambda = \mathtt{c}^d$, thus $\mathtt{a} = \lambda^{m-1}$, Theorem 9 says that the λ-plane contains $m-1$ regularly distributed copies $\mathcal{M}_2^{[1]}$; see [8] and [9] for $R(z) = z^m + \lambda/z^m$.

Theorem 10. *Any non-trivial hyperbolic component of $\tilde{\mathbf{\Omega}}_\infty$ is contained in some copy $\mathcal{M}_2^{[n]}$.*

Proof. Let H be a non-trivial hyperbolic component (in the parameter $\mathtt{a}$-plane of the family $(Q_{\mathtt{a}})$). Then, for some positive integer k, H contains some solution $\mathtt{a} = \mathtt{a}_0$ of $\mathtt{Q}_k(\mathtt{a}) = \varpi$ not solving any equation $\mathtt{Q}_j(\mathtt{a}) = \varpi$, $j < k$; also, every $Q_{\mathtt{a}}$, $\mathtt{a} \in H$, has an attracting k-cycle. By Lemma 3 there exist domains D and D_0, such that $Q_{\mathtt{a}_0} : D_0 \to D$ is a polynomial-like mapping of degree two. It may, however, happen that $Q_{\mathtt{a}_0}(\varpi) \notin D_0$, so that the filled-in Julia set is totally disconnected; this always happens to polynomials $P_{\mathtt{b}}(z) = \mathtt{b} + z^2$ in satellites of the classical Mandelbrot set.

For $l = k$ there exists a component $\tilde{D}_0$ of $Q_{\mathtt{a}_0}^{-l}(D_0)$ which contains $Q_{\mathtt{a}_0}^k(\varpi)$, and this may happen also for some $l < k$ with $l|k$; we choose the smallest l with this property, and refer the reader to Figure 12, where this situation is illustrated for $k = 6$, $l = 2$, and $k = 6$, $l = 3$ (see also Figure 12). Thus the critical orbit $(Q_{\mathtt{a}_0}^{l\sigma}(\varpi))_{\sigma\in\mathbb{N}}$ remains in $\tilde{D}_0$, and the *first return map* $Q_{\mathtt{a}_0}^l : \tilde{D}_0 \to D_0$ is polynomial-like of degree two, but now has connected filled-in Julia set, and thus is hybrid equivalent to some polynomial $P_{\mathtt{b}_0}(z) = z^2 + \mathtt{b}_0$ with $\mathtt{b}_0 \in \tilde{H} \subset \mathcal{M}_2^\circ$. The polynomial

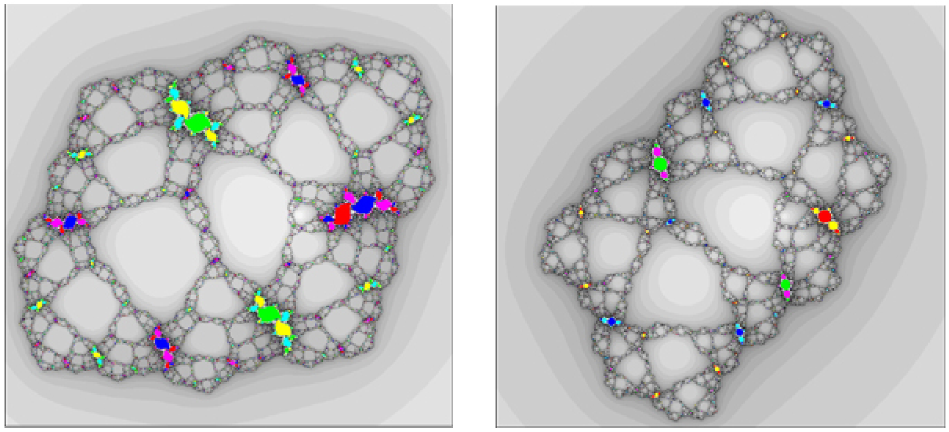

FIGURE 12. Two rational functions $Q_{\mathtt{a}}(z) = \mathtt{a}z^2(1+1/z)^5$ with attracting 6-cycles of different type. In the first case, $\mathtt{a} = -0.1 + 0.108i \in \mathcal{M}_2^{[2]}$ (left), the cycle consists of two parts, each looking like a 3-cycle, and $Q_{\mathtt{a}}^2$ is hybrid equivalent (in some neighbourhood of $z = \varpi = 3/2$) to some polynomial $P_{\mathtt{b}}$. In the second case, $\mathtt{a} = 0.047 + 0.148i \in \mathcal{M}_2^{[3]}$ (right), the cycle consists of three parts, each looking like a 2-cycle, and this time $Q_{\mathtt{a}}^3$ is hybrid equivalent to some polynomial $P_{\mathtt{b}}$. In both cases the domains are mapped as follows: cyan → magenta → yellow → blue → green → red

.

$P_{\mathtt{b}_0}$ has a super-attracting (k/l)-cycle, and H is homeomorphic to the hyperbolic component $\tilde{H} \subset \mathcal{M}_2^\circ$, hence is a connected component of some copy $\mathcal{M}_2^{[n]}$ (actually $n = l$). □

Remark. For $1/\ell + 1/m < 1$ the rational map $\mathtt{Q}_n$ has degree d^{n-1}, hence there are at most d^{n-1} different roots of the equation $\mathtt{Q}_n(\mathtt{a}) = \varpi$ in $\mathbb{C}^*$. Since $\mathcal{M}_2^{[j]}$, $j < n$, absorbs also roots of the equation $\mathtt{Q}_n(\mathtt{a}) = \varpi$, there exist less than d^{n-1} copies $\mathcal{M}_2^{[n]}$ in the $\mathtt{a}$-plane. In [9] it was shown that, again for the family $R(z) = z^m + \lambda/z^m$, there exist at least $(m-2)m^{n-1} + 1$ copies $\mathcal{M}_2^{[n]}$ in the λ-plane, thus $m^{n-1} - \frac{m^{n-1}-1}{m-1} = m^{n-1} - (m^{n-2} + m^{n-3} + \cdots + m + 1)$ copies $\mathcal{M}_2^{[n]}$ in the $\mathtt{a}$-plane. One may thus pose the following problem:

▶ *For $1/\ell + 1/m < 1$, determine (a non-trivial lower bound for) the number of copies $\mathcal{M}_2^{[n]}$ in the $\mathtt{a}$-plane.*

Final remark. The parameter plane of the family (1) consists of

- the bifurcation locus $\partial\mathbf{\Omega}_\infty$,
- the Cantor locus $\mathbf{\Omega}_0$,
- the McMullen domain $\mathbf{\Omega}_2$ (non-empty only for $1/\ell + 1/m < 1$),
- the Sierpiński holes, and
- the components of the interior of the copies $\mathcal{M}_2^{[n]}$.

On combination with Theorem 10 the construction of the domains D and D_0 will explain what happens in the dynamical plane of $R_{\mathtt{c}}$ for $\mathtt{c}$ in some non-trivial

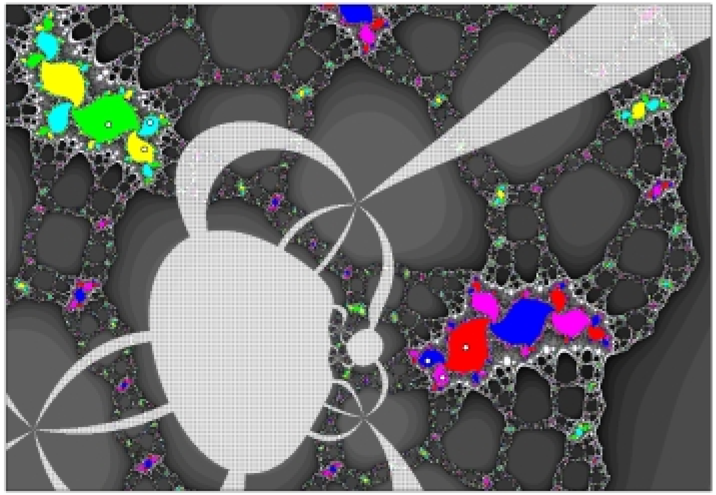

FIGURE 13. Construction of some domain U_0 such that $Q_{\mathtt{a}}^2 : U_0 \to Q_{\mathtt{a}}^2(U_0)$, $\mathtt{a} = -0.1 + 0.108i \in \mathcal{M}_2^{[2]}$, is a polynomial-like mapping of degree two with attracting 3-cycle (red/blue/magenta), while the rational map $Q_{\mathtt{a}}$ has a attracting 6-cycle (red/green/blue/yellow/magenta/cyan).

hyperbolic component; see Theorems 3 and 4. We remind the reader that $R_{\mathtt{c}}$ is quasi-conjugate to $Q_{\mathtt{a}}$ via

$$R_{\mathtt{c}}(z)^d = \tilde{R}_{\mathtt{c}}(z^d) \text{ and } \tilde{R}_{\mathtt{c}}(\mathtt{c}^d z)/\mathtt{c}^d = Q_{\mathtt{a}}(z), \text{ with } \mathtt{a} = \mathtt{c}^{d(m-1)}.$$

Suppose that $\mathtt{c}$ is in some non-trivial hyperbolic component. Then $Q_{\mathtt{a}}$ has an attracting p-cycle $\{V_0, \ldots, V_{p-1}\}$, which, by the proof of Lemma 3 may be assumed to belong to the slit plane $S : |\arg z| < \pi$. The branches ϕ_j of $z \mapsto \mathtt{c}^{m-1} z^{1/d}$ map S onto sectors

$$S_j : |\arg z - (m-1)\arg \mathtt{c} - 2\pi j/d| < \pi/d, \ 0 \le j < d,$$

and each domain V_k is mapped onto $\phi_j(V_k) = U_k^{[j]} \subset S_j$. Although one might suspect that the domains $U_0^{[j]}, \ldots, U_{p-1}^{[j]}$ form an attracting cycle for $R_{\mathtt{c}}$, this is not true in general. In many cases, to say the least, these domains belong to different (and even longer) attracting cycles which are distributed over several sectors. This happens to all non-prime attracting cycles, but may also happen to prime cycles; see Figures 4, 5, and 6. In particular, if $\mathtt{a}$ belongs to $\mathcal{M}_2^{[1]}$, then $Q_{\mathtt{a}}$ is hybrid equivalent to a quadratic polynomial $P_{\mathtt{b}}(z) = z^2 + \mathtt{b}$ with $\mathtt{b} = \mathtt{b}(\mathtt{a})$ in some hyperbolic component of $\mathcal{M}_2$. Although the cycle $\{V_0, \ldots, V_{p-1}\}$ is connected in the sense that it is embedded into a compact connected set (the filled-in Julia set of $Q_{\mathtt{a}} : D_0 \to D$) homeomorphic to the filled-in Julia set of $P_{\mathtt{b}}$, this need not be the case for the attracting cycles of $R_{\mathtt{c}}$.

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