

THE JULIA SETS OF BASIC UNICREMER POLYNOMIALS OF ARBITRARY DEGREE

ALEXANDER BLOKH AND LEX OVERSTEEGEN

ABSTRACT. Let P be a polynomial of degree d with a Cremer point p and no repelling or parabolic periodic bi-accessible points. We show that there are two types of such Julia sets J_P . The *red dwarf* J_P are nowhere connected im kleinen and such that the intersection of all impressions of external angles is a continuum containing p and the orbits of all critical images. The *solar* J_P are such that every angle with dense orbit has a degenerate impression disjoint from other impressions and J_P is connected im kleinen at its landing point. We study bi-accessible points and locally connected models of J_P and show that such sets J_P appear through polynomial-like maps for generic polynomials with Cremer points. Since known tools break down for $d > 2$ (if $d > 2$, it is not known if there are *small cycles* near p , while if $d = 2$, this result is due to Yoccoz), we introduce *wandering ray continua* in J_P and provide a new application of *Thurston laminations*.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

Consider a degree d polynomial map $P : \mathbb{C} \rightarrow \mathbb{C}$ of the complex plane \mathbb{C} ; denote the Julia set of P by $J_P = J$, the filled-in Julia set by $K_P = K$ and the basin of infinity by A_∞ . Assume that J is connected. Call an irrational neutral periodic point a *CS-point*. In his fundamental paper [K04] Kiwi showed that if P has no CS-points, then $P|_J$ is modeled by an *induced map on a topological (locally connected) Julia set* and J is locally connected at its preperiodic points. However, in the case when P has CS-points, similar results are not known, and there are few results about the topological structure of J .

We address this issue and consider *basic uniCremer* polynomials P defined as those with a Cremer periodic point and without repelling (parabolic) periodic bi-accessible points (by [GM93] and [K00], then the Cremer point is unique and fixed). We show that then for J either (a) all impressions are “big” (contain an invariant continuum) and J is nowhere *connected im kleinen* (topological notion close to local connectedness [Moo62]), or (b) there are plenty of degenerate impressions with dense orbits in J at each of which J is connected im kleinen. This dichotomy in a way takes further a standard dichotomy between non-locally connected and locally connected Julia sets.

Received by the editors May 8, 2008.

2000 *Mathematics Subject Classification*. Primary 37F10; Secondary 37F50, 37B45, 37C25, 54F15.

Key words and phrases. Complex dynamics, Julia set, Cremer fixed point.

The first author was partially supported by NSF grant DMS-0456748.

The second author was partially supported by NSF grant DMS-0405774.

©2009 American Mathematical Society
Reverts to public domain 28 years from publication

Similar results for $d = 2$ were obtained in [BO06a]. However in [BO06a] we strongly rely on [GMO99] and Yoccoz’s small cycles [Yoc95] none of which is known in general. The novelty of our present approach to polynomials of *arbitrary* degrees is that we use Thurston laminations (previously not used in this setting), and rely on our new fixed point result obtained in [BO08]. This together with a new argument leads to the generalization of the results of [BO06a] and to new results concerning bi-accessible points of a basic uniCremer Julia set J and the fact that any locally connected quotient spaces of J must be degenerate.

The study of basic uniCremer Julia sets is justified by two facts (see Section 2): (1) all polynomials with Cremer points but without critical preperiodic and parabolic points contain “basic uniCremer polynomial-like Julia sets” as subsets; (2) a polynomial with one critical point and a Cremer fixed point must be basic uniCremer.

Let us pass on to the precise statements. It is known that there exists a conformal isomorphism Ψ from the complement of the closure of the open unit disk \mathbb{D} onto the complement of K . Moreover, Ψ conjugates $z^d|_{\mathbb{C}\setminus\mathbb{D}}$ and $P|_{\mathbb{C}\setminus K}$. The Ψ -image R_α of the radial line of angle α in $\mathbb{C}\setminus\mathbb{D}$ is called an (*external*) *ray*. If J is locally connected, Ψ extends to a continuous function $\bar{\Psi}$ which semiconjugates $z^d|_{\mathbb{C}\setminus\mathbb{D}}$ and $P|_{A_\infty}$. Set $\psi = \bar{\Psi}|_{\mathbb{S}^1}$ and define an equivalence relation \sim_P on \mathbb{S}^1 by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$. The equivalence \sim_P is called the (*d*-invariant) *lamination (generated by P)*. The quotient space $\mathbb{S}^1 / \sim_P = J_{\sim_P}$ is homeomorphic to J and the map $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ induced by $z^d|_{\mathbb{S}^1} \equiv \sigma$ is topologically conjugate to $P|_J$. The set J_{\sim_P} (with the map f_{\sim_P}) is a model of $P|_J$, called the *topological Julia set*.

In his fundamental paper [K04] Kiwi extended this to polynomials P with connected Julia set and no irrational neutral periodic points (called *CS-points*) for which he obtained a *d*-invariant lamination \sim_P on \mathbb{S}^1 with $P|_J$ semi-conjugate to the induced map $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$ by a monotone map $m : J \rightarrow J_{\sim_P}$ (*monotone* means a map with connected point preimages). Kiwi also proved that for all periodic points $p \in J$ the set J is locally connected at p and $m^{-1} \circ m(p) = \{p\}$. However in some cases the results of [K04] do not apply. E.g., take a quadratic polynomial P with a *Cremer fixed point* (i.e. with a neutral non-linearizable fixed point $p \in J$ such that $P'(p) = e^{2\pi i\alpha}$ with α irrational). By Sullivan [Sul83], the Julia set J_P is not locally connected. Moreover, since P has a CS-point, Kiwi’s results from [K04] do not apply.

As mentioned above, we addressed this issue for quadratic polynomials with fixed Cremer point in [BO06a]. Yet the tools from [BO06a] have a limited, “quadratic”, nature. Most importantly, [BO06a] makes an essential use of [GMO99] where so-called *building blocks* were studied. However, the [GMO99] applies only in the quadratic case: it is based in particular upon the existence of *small cycles* near the Cremer point [Yoc95], and as of yet, it is not known for polynomials of higher degrees whether small cycles near Cremer points exist or not. Thus, the building blocks machinery breaks down for higher degrees and needs to be replaced.

Also, the main topological result in [BO06a] is Theorem 2.3 which says that *if K is a non-separating g^n -invariant plane continuum/point not containing a Cremer point of a polynomial g , $g^n|_K$ is a **homeomorphism** in an open neighborhood $V \supset K$, and some technical conditions are satisfied, then K is a point*. It sufficed to consider a homeomorphism in [BO06a] because if $d = 2$ (g is quadratic), then $p \notin K$ implies $c \notin K$ (the critical point c is recurrent and $p \in \omega(c)$ [Mn93]). If

$d > 2$, there are other critical points; hence, one needs to generalize Theorem 2.3 of [BO06a] to non-homeomorphisms.

Thus, to study polynomials of higher degrees with Cremer periodic points, new tools are needed. Here we suggest a different approach allowing us to extend the results of [BO06a] (by [SZ99] quadratic polynomials with a fixed Cremer point are basic uniCremer, so the present paper extends [BO06a]). Namely, we use Thurston laminations, the notion of a *wandering ray continuum*, and a new topological result from [BO08], and generalize [BO06a] to basic uniCremer polynomials of arbitrary degree. Also, we obtain such results as Theorem 4.3 (concerning certain wandering sets and bi-accessible points in basic uniCremer Julia sets) and Theorem 5.2 (which shows that *any* monotone map of J onto a locally connected continuum collapses J to a point).

To proceed we need a few definitions. Let Ψ be a conformal isomorphism from the complement of the closure of the open unit disk \mathbb{D} onto the complement of K . We identify points on the unit circle with their argument $\alpha \in [0, 2\pi)$ and call them *angles*. For each angle $\alpha \in \mathbb{S}^1$, the *impression of α* , denoted by $\text{Imp}(\alpha)$, is the set $\{w \in \mathbb{C} \mid \text{there exists a sequence } z_i \rightarrow \alpha \text{ such that } \lim \Psi(z_i) = w\}$. Then $\text{Imp}(\alpha)$ is a subcontinuum of J .

If the impression $\text{Imp}(\alpha)$ is disjoint from all other impressions, then the angle α is called *K-separate*. A continuum X is *connected im kleinen at a point x* [Moo62] provided for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C . Now we can state our main theorem which generalizes to basic uniCremer polynomials the results of [BO06a].

Theorem 5.10. *For a basic uniCremer polynomial P , the following facts are equivalent:*

- (1) *there is an impression not containing the Cremer point;*
- (2) *there is a degenerate impression;*
- (3) *the set Y of all K -separate angles with degenerate impressions contains all angles with dense orbits and a dense set of periodic angles, and the Julia set J is connected im kleinen at the landing points of the corresponding rays;*
- (4) *there is a point at which the Julia set is connected im kleinen.*

Basic uniCremer Julia sets with the properties from Theorem 5.10 are said to be *solar*. The remaining basic uniCremer Julia sets are called *red dwarf* Julia sets. They can be defined as basic uniCremer Julia sets such that all impressions contain p . These notions have been introduced in [BO06a] and were further studied in [BBCO07] where it was shown that solar Julia sets of degree two, with positive Lebesgue measure, exist. The following lemma describes red dwarf Julia sets and complements Theorem 5.10.

Lemma 5.3. *If J is a red dwarf Julia set, then the (non-empty) intersection of all impressions contains all forward images of all critical points, there exists $\varepsilon > 0$ such that the diameter of any impression is greater than ε , and there are no points at which J is connected im kleinen. Moreover, in this case no point of J is bi-accessible and p is not accessible from $\mathbb{C} \setminus J$.*

We also prove Theorem 4.3 which partially extends, to basic uniCremer polynomials, results of Schleicher and Zakeri [SZ99] related to the question of McMullen [McM94] as to whether a quadratic Julia set with a fixed Cremer point has bi-accessible points. Given a set B of angles, set $\text{Imp}(B) = \bigcup_{\beta \in B} \text{Imp}(\beta)$

$\Pi(B) = \bigcup_{\beta \in B} \Pi(\beta)$ ($\text{Imp}(\alpha)$ is the impression of an angle α and $\Pi(\alpha) = \overline{R_\alpha} \setminus R_\alpha$ is the *principal* set of an angle α). Call X a *ray continuum* if X is a continuum or a point and for some set of angles B we have $\Pi(B) \subset X \subset \text{Imp}(B)$. The maximal set B of angles such that $\Pi(B) \subset X \subset \text{Imp}(B)$ is said to be *connected* to X . A set is said to be *wandering* if all its images are pairwise disjoint.

Theorem 4.3. *Suppose that J is the Julia set of a basic uniCremer polynomial P of degree d . Suppose that $K' \subset J$ is a wandering ray continuum connected to a set of angles \mathcal{A}' with $|\mathcal{A}'| \geq 2$. Then there exists an n such that $|\sigma^n(\mathcal{A}')|$ is a singleton and for every $\theta \in \mathcal{A}'$, $\bigcup_{n \geq 0} \sigma^n(\theta)$ is not dense in \mathbb{S}^1 . In particular, K' is pre-critical, and if a point $x \in J$ is bi-accessible, then it is either precritical or preCremer.*

In Theorem 5.2 we show that Kiwi’s results [K04] do not apply to basic uniCremer polynomials for principal reasons: any monotone map of a basic uniCremer Julia set must collapse the whole set to a point.

Theorem 5.2. *Suppose that P is a basic uniCremer polynomial and $\varphi : J \rightarrow A$ is a monotone map of J onto a locally connected continuum A . Then A is a singleton.*

The paper is designed as follows. In Section 2 we motivate our work by showing that all polynomials with Cremer points, but without critical preperiodic and parabolic points, contain “basic uniCremer polynomial-like Julia sets” as subsets and that a polynomial with one critical point and a Cremer fixed point must be basic uniCremer. In Section 3 we develop the topological tools. In Section 4 we develop the laminational tools and use them to study some wandering continua in the Julia set of a basic uniCremer polynomial. Finally, in Section 5 we obtain our main results.

2. MOTIVATION

In this section we suggest additional motivation for our research and provide a criterion for a polynomial to be basic uniCremer. Applying renormalization ideas that allow one to find “little” Julia sets (of appropriate polynomial-like maps) inside “big” ones, we show that *if a polynomial P , with connected Julia set J , has no critical preperiodic and parabolic points, then a periodic Cremer point p of P belongs to a polynomial-like Julia set, contained in J , corresponding to a basic uniCremer polynomial*. Thus, basic uniCremer Julia sets appear in Julia sets of *almost all* polynomials with Cremer points as Julia sets of polynomial-like maps. Similar constructions, based upon [DH85b], can be found, e.g., in [Hub93], [Lev98], and [Sch04]. We also sketch the proof of the fact that if no critical point of a polynomial P with a fixed Cremer point p is separated from p by means of repelling periodic bi-accessible points and their preimages in J_P , then P is basic uniCremer. Finally, we show that a polynomial with unique critical point and fixed Cremer point is basic uniCremer.

Let p be a fixed Cremer point of a polynomial P with connected Julia set J . Denote the set of all periodic repelling bi-accessible points of P by $B' \neq \emptyset$. Consider the set B of *all* preimages of points of B' . The rays landing at any $x \in B$ partition \mathbb{C} into the wedges. Choose the closed wedge W_x containing p and denote its boundary rays by R_{α_x} and R_{β_x} . Connect α_x and β_x with a chord $\ell = \overline{\alpha_x \beta_x}$ inside \mathbb{D} and

consider the closure T_x of the part of $\mathbb{D} \setminus \ell$ which corresponds to the angles with rays inside W_x . In this case call ℓ an *admissible* chord, set $x = z_\ell$ and call $\text{Cut}_\ell = \overline{R_{\alpha_x} \cup R_{\beta_x}}$ an *admissible* cut. If $X \subset B$, then we denote by W_X the intersection of all wedges W_x , $x \in X$. Consider the intersection $W_B = W$ of wedges W_x and the intersection T of sets T_x over *all* points $x \in B$. For a chord $\ell = \overline{\alpha\beta} \subset \partial T$ let $I_\ell = (\alpha, \beta) \subset \mathbb{S}^1$ be the open arc disjoint from T ; call I_ℓ the *prime arc* of ℓ .

We show that $W \cap J_P$ is the connected Julia set of a polynomial-like map. It is easy to see that (1) $W \cap J_P$ is an invariant continuum, (2) T is a subcontinuum of \mathbb{D} , and (3) ∂T consists of points in $\partial T \cap \mathbb{S}^1$ (nowhere dense in \mathbb{S}^1) and a collection \mathcal{K} of pairwise disjoint chords (except perhaps for the endpoints). If we connect the σ -images of the endpoints of a chord ℓ from \mathcal{K} , we get a chord from \mathcal{K} understood as the “image” $\hat{\sigma}(\ell)$ of ℓ . Let \mathcal{H} be the family of all $\hat{\sigma}$ -periodic chords (in this sense) in \mathcal{K} . Then (a) \mathcal{H} is finite and *can be assumed to consist only of chords with fixed endpoints*, (b) there are no critical chords in \mathcal{K} , (c) all chords in \mathcal{K} eventually map onto chords of \mathcal{H} . In particular, \mathcal{H} is not empty.

Let $\mathcal{A} = \{A_1 = (\alpha_1, \beta_1), \dots, A_n = (\alpha_n, \beta_n)\}$ be *all* components of $\mathbb{S}^1 \setminus T$ longer than $\frac{1}{d}$ with $\ell_1 = \overline{\alpha_1\beta_1}, \dots, \ell_n = \overline{\alpha_n\beta_n}$. The family \mathcal{A} is non-empty, e.g. because for any chord $\ell \in \mathcal{H}$ the arc I_ℓ is longer than $\frac{1}{d}$. Moreover, by the construction there exists i and an arc $B_i \subset A_i$ with endpoints θ, γ such that B_i is longer than $\frac{1}{d}$ and $\overline{\theta\gamma}$ is admissible. Then the part of the plane separated from W by $\overline{R_\theta \cup R_\gamma}$ must contain a critical point. This implies one of our claims: *if no critical point is separated in the Julia set from a fixed Cremer point by a point of B , then the polynomial is basic uniCremer*.

On the other hand, it is easy to see that if p is a Cremer fixed point, then there must exist a critical point c which is not separated from p by points of B . Indeed, otherwise there is a set of points $X = \{x_1, \dots, x_m\}$ and a set of rays landing at them $Z = \{\tilde{R}'_1, \tilde{R}''_1, \dots, \tilde{R}'_m, \tilde{R}''_m\}$ such that the component M of $\mathbb{C} \setminus \bigcup_i (\tilde{R}'_i \cup \tilde{R}''_i)$ containing p contains no critical points. Choose an equipotential curve L intersecting the rays at points close to the appropriate points of X . Then construct small arcs connecting points of intersection of L with $\tilde{R}'_i, \tilde{R}''_i$ outside M . The thus constructed curve S encloses a simply connected region U whose closure contains no critical points. By [Per94] and [Per97], then the so-called *hedgehog* $H \subset J$ created inside U will reach out to the boundary of U . This implies that at least one point of X is a cutpoint of H contradicting [Per94] and [Per97]. Thus, *if a polynomial has a **unique** critical point and a **fixed** Cremer point, then it is basic uniCremer*; this gives another example of basic uniCremer polynomials.

Let us go back to the general case. For a critical point $c \notin W \cap J$ there are two angles $\gamma \neq \theta \notin W$ with $c \in \text{Imp}(\gamma) \cap \text{Imp}(\theta)$ and $\sigma(\gamma) = \sigma(\theta)$. Since c is cut off W by admissible cuts, the component A_{i_c} of $\mathbb{S}^1 \setminus T$ to which γ, θ belong, is well defined. Let ℓ_{i_c} be the leaf in \mathcal{K} whose endpoints coincide with the endpoints of A_{i_c} . For each i_c choose r_c as the smallest such number that $\sigma^{r_c}(\ell_{i_c}) \in \mathcal{H}$. Given $\bar{s} \in \mathcal{H}$, let $H_{\bar{s}}$ be the set of all critical points $c \notin W \cap J$ with $\hat{\sigma}^{r_c}(\ell_{i_c}) = \bar{s}$.

We construct the simple closed curve S needed to find a polynomial-like map. Let $\bar{s} = \overline{\theta\gamma} \in \mathcal{H}$ be admissible. Then $R_\theta = R'_1, R_\gamma = R''_1$ land at $z_{\bar{s}}$. Since $z_{\bar{s}}$ is a repelling fixed point, we can choose a very small neighborhood $U_{\bar{s}} = U_1$ of $z_{\bar{s}}$ in which points are repelled from $z_{\bar{s}}$. Moreover, by assumption, critical points of P are not mapped into repelling periodic points. Hence we can choose $U_{\bar{s}}$ so small that $P^{r_c}(c) \notin U_{\bar{s}}$ for all $c \in H_{\bar{s}}$. Do this for all admissible $\bar{s} \in \mathcal{H}$.

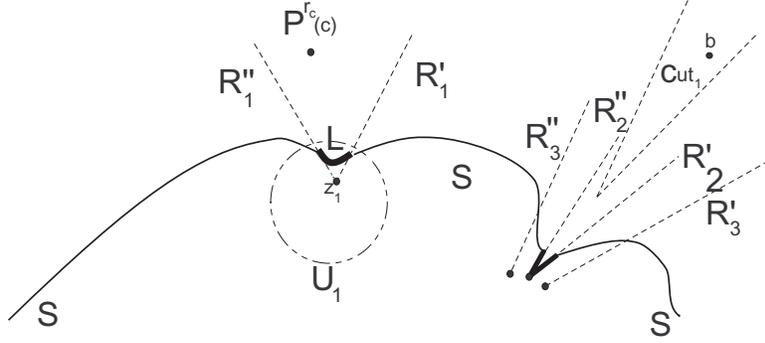


FIGURE 1. Construction of the curve S

Now consider $\bar{s} \in \mathcal{H}$ which is not admissible. For $b \in H_{\bar{s}}$ choose an admissible chord $\ell_b = \overline{\alpha\beta}$ with $R_\alpha = R'_2, R_\beta = R''_2$ very close to $\ell_{i_b} = \overline{\lambda\xi}$ with $R_\lambda = R'_3, R_\xi = R''_3$. We can choose the closeness so that $\text{Cut}_{\ell_b} = \text{Cut}_1$ separates b from W and then consider both $P^{rb}(\text{Cut}_1)$ (which by the construction is located outside W) and $\sigma^{rb}(\ell_b)$ (which by the construction is an admissible chord outside T). Then we choose an admissible chord $\ell'_{\bar{s}} = \overline{\alpha_{\bar{s}}\beta_{\bar{s}}}$ so close to \bar{s} that it separates all chords $\sigma^{rb}(\ell_b), b \in H_{\bar{s}}$ from T (this is possible since \bar{s} is not admissible and hence is a limit of admissible chords). Figure 1 illustrates this construction.

Then choose an equipotential Q which intersects all rays R_θ, R_γ taken over all admissible chords $\bar{s} = \overline{\theta\gamma} \in \mathcal{H}$ inside $U_{\bar{s}}$ at points $x_{\bar{s}}$ and $y_{\bar{s}}$. Connect $x_{\bar{s}}$ and $y_{\bar{s}}$ with a small curve $L_{\bar{s}} = L \subset U$ which is repelled away from $z_{\bar{s}}$ inside the corresponding wedge between R_θ and R_γ and replace the piece of the equipotential between R_θ and R_γ by L . Also, this equipotential intersects all rays $R_{\alpha_{\bar{s}}}, R_{\beta_{\bar{s}}}$ taken over non-admissible chords $\bar{s} \in \mathcal{H}$. In this case we replace the piece of the equipotential Q between the rays $R_{\alpha_{\bar{s}}}, R_{\beta_{\bar{s}}}$ by the union $L_{\bar{s}}$ of pieces of these rays between the point of their intersection with Q and their common landing point. Thus, the constructed curve S consists of pieces of the equipotential Q and either small curves $L_{\bar{s}}$ (taken over all admissible chords $\bar{s} \in \mathcal{H}$) and unions $L_{\bar{s}}$ of pieces of rays $R_{\alpha_{\bar{s}}}, R_{\beta_{\bar{s}}}$ leading to their common landing point (taken over all non-admissible chords $\bar{s} \in \mathcal{H}$).

Denote by H the Jordan disk enclosed by S . It follows from the construction that P maps the curve S outside \overline{H} . Then we can consider the pullback V of \overline{H} under P defined by $P(p) = p$. Clearly, $P : V \rightarrow H$ is a polynomial-like map. Moreover, the construction and standard arguments imply that the set of all points which never exit \overline{H} under P is exactly W : this can be observed on the disk model first and then transported to the dynamical plane using the construction, the fact that all periodic points are repelling, and the fact that so are all periodic angles on the circle. Thus, W is the filled-in Julia set of the polynomial-like map $P : V \rightarrow H$, and $J \cap W$ is its Julia set.

Denote by f the polynomial to which $P|_V$ corresponds. By [DH85b] there is a quasi-conformal conjugacy ψ between $P|_{J \cap W}$ and $f|_{J_f}$ (we will use terms “ P -ray”

and “ f -ray” in the obvious sense). Notice that J_f cannot have Fatou domains (or other Cremer points) because by [GM93] and [K00] all such sets are cut off W . Hence J_f is a non-separating continuum. It is easy to see that $q = \psi(p)$ is Cremer: there are arbitrarily small invariant subcontinua in $J \cup W$ containing p (hedgehogs [Per94] and [Per97]), hence the corresponding continua exist for f which excludes that q is repelling.

Suppose that f has a periodic repelling cutpoint $a = \psi(b)$ and set $\psi(p) = q$. Take m so that $f^m(a) = a$. Then $P^m(b) = b$. By the construction of W and T , there exists an P^m -invariant P -ray landing at b . Then its ψ -image will be an invariant (in a neighborhood of a) curve in the complement of J_f landing at a . This implies that all f -rays landing at a must be f^m -invariant too (there is no rotation around a in the f -plane). Denote by A_m all f^m -fixed points; similarly, we see that each such point (except perhaps for q) has an f^m -invariant f -ray landing at it. Consider the closure K of a component of $J_f \setminus \{A_m\}$ not containing q . Then by [GM93] (or by Corollary 3.1 applied to K) we see that there exists an f^m -fixed point in K which does not belong to A_m , a contradiction.

3. FIXED POINTS AND IMPRESSIONS

We begin by describing a corollary of [BO08] which serves as our major topological tool. Let $X \subset K$ be a non-separating continuum or a point such that:

- (1) Pairwise disjoint continua/points $E_1 \subset X, \dots, E_m \subset X$ and finite sets of angles $A_1 = \{\alpha_1^1, \dots, \alpha_{i_1}^1\}, \dots, A_m = \{\alpha_1^m, \dots, \alpha_{i_m}^m\}$ are given.
- (2) We have $\Pi(A_j) \subset E_j \subset \text{Imp}(A_j)$ (so the set $E_j \cup (\bigcup_{k=1}^{i_j} R_{\alpha_k^j}) = E'_j$ is closed and connected).
- (3) X intersects a unique component A of $\mathbb{C} \setminus \cup E'_j$ and X is equal to the union of all E_j and $K \cap A$.

We call such X a *general puzzle-piece* and call the continua E_i the *exit continua* of X . For each j , the set E'_j divides the plane into i_j open wedges; denote by W_j the one which contains $X \setminus E_i$. Also, let H_j be the chord connecting the angles whose rays form a part of the boundary of W_j , and let T_j be the component of $\mathbb{D} \setminus H_i$ taken from the side corresponding to X (so that angles whose rays have impressions in X belong to T_j). Then the intersection T of all sets T_j is a convex subset of \mathbb{D} whose boundary consists of chords H_j and some arcs of the circle. It follows that the impressions of all angles in $T \setminus \cup A_i$ are contained in X .

Corollary 3.1 ([BO08]). *Suppose that for a non-separating continuum X or a point, at least one of the following holds.*

- (1) X is a general puzzle-piece with exit continua E_1, \dots, E_m such that for every i either E_i is a fixed point or $P(E_i) \subset W_i$.
- (2) $X \subset J$ is an invariant continuum.

If all fixed points which belong to $X \cap J$ are repelling and all rays landing at them are fixed, then X is a repelling fixed point. Hence, if $X \subset K$ is a non-degenerate continuum which satisfies (1) or (2), then either X contains a non-repelling fixed point or X contains a repelling fixed point at which a non-fixed ray lands.

By Corollary 3.1 if P is a basic uniCremer polynomial, then a P^m -invariant continuum or a point $K \subset J$, not-containing the Cremer point p of P , is a point.

We now prove a few lemmas which can be of independent interest. Mostly they deal with topological properties of impressions. In this subsection X is a non-separating one-dimensional plane continuum. The assumption that X is non-separating and one-dimensional is not essential but simplifies the arguments (e.g., in this case each subcontinuum of X is also a non-separating, one-dimensional plane continuum and the intersection of any two subcontinua of X is connected). Speaking of *points* we mean points in the (dynamic) plane while *angles* mean arguments of external rays.

A *crosscut* C of K is an open arc $C \subset \mathbb{C} \setminus K$ whose closure is a closed arc with endpoints in K . By $Sh(C)$, the *shadow of* C , we denote the bounded component of $\mathbb{C} \setminus [K \cup C]$. Let R_α be an external ray and $\{C_n\}$ a family of crosscuts all of whom cross R_α essentially and are such that $\text{diam}(C_n) \rightarrow 0$. Then it is well known that the impression $\text{Imp}(\alpha) = \bigcap_n \overline{Sh(C_n)}$. A continuum K is said to be *decomposable* if there exist two *proper* subcontinua $A, B \subsetneq K$ such that $A \cup B = K$ and *indecomposable* otherwise. Theorem 3.2 holds for *all* polynomials.

Theorem 3.2 ([CMR05, Theorem 1.1]). *The Julia set of a polynomial P is indecomposable if and only if there exists an angle γ whose impression has non-empty interior in $J(P)$; in this case the impressions of all angles coincide with $J(P)$.*

From the topological standpoint, if the Julia set is indecomposable, then one cannot use impressions to further study its structure: representing $J(P)$ as the union of smaller more primitive continua is impossible in this case. Besides, if $J(P)$ is indecomposable, then the results of this paper are immediate. Thus, in the lemmas below we assume that no impression has interior in X .

By $\text{Ch}(A)$ we denote the convex hull of a planar set A . Lemma 3.3 studies how ray continua intersect. If two sets $A_1, A_2 \subset \mathbb{S}^1$ are such that $\text{Ch}(A_1) \cap \text{Ch}(A_2) = \emptyset$, then they are said to be *unlinked*.

Lemma 3.3. *Suppose that K_1, K_2 are disjoint ray continua connected to finite sets of angles A_1, A_2 respectively. Then A_1 and A_2 are unlinked.*

Proof. Since $K_1 \cap K_2 = \emptyset$, $A_1 \cap A_2 = \emptyset$. If A_1 and A_2 are not unlinked, then there exists $\alpha_1, \alpha_2 \in A_1$ such that A_2 separates α_1 and α_2 . This clearly implies that $K_1 \cap K_2 \neq \emptyset$, a contradiction. \square

Recall that a continuum X is *connected im kleinen at* x if for each open set U containing x there exists a connected set $C \subset U$ with x in the interior of C . A continuum X is *locally connected at* $x \in X$ provided for each neighborhood U of x there exists a connected and open set V such that $x \in V \subset U$. Observe that sometimes different terminology is used (see the discussion in [BO06a]).

Lemma 3.4 gives a sufficient condition for a continuum X to be connected im kleinen at some point x . It was proven in [BO06a], Theorem 3.5, so we give it here without a proof. The idea is to establish “short connections” among impressions which cut x off the rest of X and apply it to prove that X is connected im kleinen at some points. Recall that if $\text{Imp}(\theta)$ is disjoint from all other impressions, then we call θ a *K-separate angle*.

Lemma 3.4. *Suppose that θ is a K-separate angle and $\text{Imp}(\theta) = \{x\}$ is a singleton. Then arbitrarily close to θ there are angles $s < \theta < t$ such that $\text{Imp}(s) \cap \text{Imp}(t) \neq \emptyset$. Also, J is connected im kleinen at x .*

On the other hand, some conditions on X imply that it is nowhere connected im kleinen, or connected im kleinen at very few points. The next lemma combines Lemma 3.9 and Lemma 3.10 proven in [BO06a] and is given here without a proof.

Lemma 3.5. *The following holds.*

- (1) *If X is connected im kleinen at x , then for any ε there exists θ such that $\text{Imp}(\theta) \subset B(x, \varepsilon)$ (in particular, there are angles with impressions of arbitrarily small diameter).*
- (2) *Suppose that there exists $\delta > 0$ such that for each $\theta \in \mathbb{S}^1$, $\text{diam}(\text{Imp}(\theta)) > \delta$. Then X is nowhere connected im kleinen.*
- (3) *Suppose that the intersection Z of all impressions is not empty. Then the only case when X is connected im kleinen at a point is (possibly) when $Z = \{z\}$ is a singleton and X is connected im kleinen at z .*

4. WANDERING CONTINUA FOR UNICREMER POLYNOMIALS

In Section 4 we use Thurston’s invariant geometric laminations defined in [Thu85]. A *geometric lamination* is a compact set \mathcal{L} of chords in \mathbb{D} and points in \mathbb{S}^1 such that any two distinct chords can meet, at most, in an end-point (i.e., the intersection of any two distinct chords is either empty or a point in \mathbb{S}^1). We refer to a non-degenerate chord $\ell \in \mathcal{L}$ as a *leaf* and to a point in \mathcal{L} as a *degenerate leaf* (by “leaves” we mean non-degenerate leaves, and by “(degenerate) leaves” we mean both types of leaves). A degenerate leaf may be an endpoint of a leaf. If $\ell \cap \mathbb{S}^1 = \{a, b\}$ for a leaf ℓ , we write $\ell = ab$. We denote by \mathcal{L}^* the union of all leaves in \mathcal{L} . Then $\mathcal{L}^* \cup \mathbb{S}^1$ is a continuum. We can extend $\sigma : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ over \mathcal{L}^* by mapping $\ell = ab$ linearly onto the chord $\sigma(a)\sigma(b)$ and denoting this chord by $\sigma(\ell)$. A *gap* G is the closure of a complementary domain of $\mathbb{D} \setminus \mathcal{L}^*$. A geometric lamination \mathcal{L} is *d-invariant* if σ preserves gaps and leaves of \mathcal{L} in the following sense:

- (1) (Leaf invariance) For each leaf $\ell \in \mathcal{L}$, $\sigma(\ell)$ is a (degenerate) leaf in \mathcal{L} and there exist d pairwise disjoint leaves ℓ_1, \dots, ℓ_d in \mathcal{L} such that for each i , $\sigma(\ell_i) = \ell$.
- (2) (Gap invariance) For each gap G of \mathcal{L} , $\sigma(\text{Bd}(G))$ is a (degenerate) leaf or the boundary of a gap G' of \mathcal{L} . We denote by $\sigma(G)$ the (degenerate) leaf or the gap G' , respectively. If $\sigma(G) = G'$ is a gap, then we also require that $\sigma|_{\text{Bd}(G)} : \text{Bd}(G) \rightarrow \text{Bd}(G')$ be the composition of a positively oriented monotone map and a positively oriented covering map.

Wandering ray continua are defined in Section 1. One can iterate the set of angles \mathcal{A} connected to a wandering ray continuum; by Lemma 3.3, the sets $\sigma^n(\mathcal{A})$ are unlinked.

Observe that it may happen that K is contained in a larger wandering ray continuum K' . However the growth of the set of angles connected to wandering ray continua is limited as the following theorem shows (the result is due to Kiwi [K02], see also [BL02]).

Theorem 4.1 ([K02]). *A set of angles such that all its images under σ are unlinked consists of no more than 2^d angles.*

Theorem 4.2 shows that wandering ray continua connected to a non-trivial set of angles give rise to geometric laminations.

Theorem 4.2. *Suppose P is a polynomial of degree d with connected Julia set J which contains a wandering ray continuum K' and \mathcal{A}' is the set of angles connected to K' . Suppose that $|\mathcal{A}'| > 1$. Then there exists N such that the grand orbit of the ray continuum $K = P^N(K')$ is well defined and gives rise to a d -invariant geometric lamination $\mathcal{L}(K') = \mathcal{L}(K)$ such that \mathcal{A}' is contained in a leaf or a gap of $\mathcal{L}(K')$.*

Proof. First we show that any pullback A of any forward iterate of K' is tree-like. Indeed, A is wandering because so is K' . On the other hand, if A is not tree-like, then it contains the boundary of a Fatou domain, and by the Sullivan theorem [Sul85] cannot be wandering. Hence all pullbacks of images of K' are tree-like. Then by a theorem of J. Heath [Hea96] the map P on a pullback A is not one-to-one only if A contains a critical point of P .

Choose $K = P^N(K')$ so that the following holds. Since K' is wandering, then for each critical point c any image of K' may contain only one iteration of c . Choose a forward image K of K' so that for each critical point c , either some forward image of c belongs to K , or the orbit of c is disjoint from the orbit of K . In particular, $K, P(K), \dots$ do not contain critical points. Let us show that there are then d^m pairwise disjoint pullbacks of $P^n(K)$ of order $m \leq n$ none of which contains a critical point. Indeed, suppose that a pullback A of $P^n(K)$ of order m contains a critical point c . Since by the choice of K there exists $i > 0$ such that $P^i(c) \in K$ we then get that $P^n(K)$ contains both $P^{n+i}(c)$ and $P^m(c)$, a contradiction. By [Hea96] this implies that all powers of P restricted onto a pullback A of $P^n(K)$ of order $m \leq n$ are one-to-one.

This implies that the pullbacks of images of K are in fact well defined as sets: any two pullbacks A, B are either the same or disjoint. Indeed, suppose that A is a pullback of $P^n(K)$ of order m and B is a pullback of $P^r(K)$ of order s . Suppose that $A \cap B \neq \emptyset$ and then show that $A = B$. Choose a point $x \in A \cap B$ and consider several cases. For definiteness suppose that $m > s$. First let us show that $n - m = r - s$. Indeed, $P^m(x) \in P^n(K)$ and $P^s(x) \in P^r(K)$. Then the latter implies that $P^m(x) = P^{m-s}(P^s(x)) \in P^{m-s+r}(K)$. Hence $P^n(K)$ and $P^{m-s+r}(K)$ are not disjoint (both contain $P^m(x)$) and hence $n = m - s + r$ because K is wandering. Now, by the above, there is only one pullback of $P^m(K)$ containing $P^s(x)$, namely $P^r(K)$. Hence $P^s(A) = P^r(K)$ and A is the pullback of $P^r(K)$ of order s along the orbit $x, P(x), P^s(x)$. Since the same holds for B we conclude that $A = B$.

Choose a maximal set of angles \mathcal{A} connected to K and containing $\sigma^N(\mathcal{A}')$ (clearly, all angles from $\sigma^N(\mathcal{A}')$ are connected to $P^N(K)$). Denote a pullback of $P^n(K)$ by P of order m by $K(m, n, i)$ where different numbers i correspond to different pullbacks of the same order m of the same image $P^n(K)$ of K . In other words, $K(m, n, i)$ is the i -th component of the set $P^{-m}(P^n(K))$. To $K(m, n, i)$ we associate the set of angles $\Theta(m, n, i)$ which are all the angles from $\sigma^{-m}(\sigma^n(\mathcal{A}))$ connected to $K(m, n, i)$. The fact that all sets $K(m, n, i)$ in the grand orbit of K are pairwise disjoint and the definition of the set of angles connected to a ray continuum imply that the sets $\Theta(m, n, i)$ have the same property: any two such sets of angles either coincide or are disjoint.

If $G(m, n, i) = \text{Ch}(\Theta(m, n, i))$, then by Lemma 3.3 it follows that all the sets $G(m, n, i)$ are pairwise disjoint. By Theorem 4.1 all the sets $\Theta(m, n, i)$ are finite. Hence all $G(m, n, i)$ are pairwise disjoint finite gaps, leaves and points mapped onto each other by σ and its powers. Moreover, σ restricted on the sets $G(m, n, i)$

satisfies all the properties described in the definition of the lamination because this corresponds to the action of the map P on the plane. Hence $\bigcup_{m,n,i} G(m,n,i)$ is a σ -invariant non-compact lamination $\mathcal{L}(G)$. It follows easily that the closure of this non-compact lamination is a σ -invariant lamination $\mathcal{L}(K)$. Observe that by the construction \mathcal{A}' is contained in a leaf or a gap, and hence $\mathcal{L}(K)$ is non-trivial (because the cardinality of \mathcal{A} is more than 1). \square

We use geometric laminations in the proof of the following theorem.

Theorem 4.3. *Suppose that J is the Julia set of a basic uniCremer polynomial P of degree d . Suppose that $K' \subset J$ is a wandering ray continuum connected to a set of angles \mathcal{A}' with $|\mathcal{A}'| \geq 2$. Then there exists an n such that $|\sigma^n(\mathcal{A}')|$ is a singleton and for every $\theta \in \mathcal{A}'$, $\bigcup_{n \geq 0} \sigma^n(\theta)$ is not dense in \mathbb{S}^1 . In particular, K' is pre-critical, and if a point $x \in J$ is bi-accessible, then it is either precritical or preCremer.*

Proof. By Theorem 4.2, there exists a d -invariant lamination $\mathcal{L}(K')$ such that \mathcal{A}' is contained in a gap or leaf of $\mathcal{L}(K')$. Let p be the fixed Cremer point of P . We can replace K' by K constructed in Theorem 4.2 and replace \mathcal{A}' by the set of all angles \mathcal{A} connected to K . It suffices to prove the theorem for K and \mathcal{A} . By way of contradiction, suppose that for some $m > 1$ and every n we have $|\sigma^n(\mathcal{A})| = m$. Denote the grand orbit of K under P by $\Gamma(K)$; denote the grand orbit of \mathcal{A} under σ (whose closure is the geometric lamination $\mathcal{L}(K)$) by $\Theta(\mathcal{A})$. It is easy to see that there are no critical leaves on the boundaries of sets from $\Theta(K)$ (a set $T \in \Gamma(K)$ may cover a critical point of P , but then the corresponding set from $\Theta(\mathcal{A})$ will be mapped onto its image in a k -to-1 fashion so that one can draw critical chords inside its convex hull, but not on its boundary).

The idea of the proof is to show that in the situation of the theorem one can always find a continuum $X \subset J$ to which Corollary 3.1 applies leading to a contradiction with the definition of a uniCremer polynomial. To do so, we analyze the dynamical behavior of K and show that at some moment an image of K will be mapped “farther away” from p by the appropriate power of P . To implement this plan we need the following construction.

For each element D of $\Gamma(K)$, the union of the rays of angles from the appropriate element of $\Theta(\mathcal{A})$ and D cut the plane into open wedges one of which contains p . Denote by W_D the union of this wedge, its two boundary rays and D (clearly, W_D is closed). Simultaneously consider the corresponding leaf ℓ_D on the boundary of the gap/leaf of $\Theta(K)$ corresponding to D and the closure H_D of the component of $\mathbb{D} \setminus \ell_D$ corresponding to angles whose rays are contained in W_D . Then consider the intersection W of sets W_D and the intersection H of sets H_D over all elements D in the grand orbit of K .

By the construction H is a convex subset of \mathbb{D} whose boundary consists of leaves of $\mathcal{L}(K)$ and a nowhere dense subset $H \cap \mathbb{S}^1$ of \mathbb{S}^1 . Since by the construction no convex hull of a set of $\Theta(\mathcal{A})$ crosses H , either H is a point in \mathbb{S}^1 , or H is a gap or a leaf of $\mathcal{L}(K)$. Denote by $A(p)$ the set of all angles whose impressions contain p . Then it follows that $A(p) \subset H \cap \mathbb{S}^1 = H'$. Clearly, $\sigma(A(p)) \subset A(p)$ and hence $\sigma(H') \cap H' \supset \sigma(A(p))$.

Consider first the case when $H = H'$ is a point. Then by the above $H = H' = A(p)$ is a fixed point of σ . Choose an element T of $\Theta(\mathcal{A})$ close to H and consider

the corresponding element Q of $\Gamma(K)$. Then there is a unique open wedge N - one of the components of $\mathbb{C} \setminus [Q \cup \bigcup_{\alpha \in T} R_\alpha]$ - into which Q is mapped by P . Consider the union X of Q with the intersection of J and N . It is easy to check that Corollary 3.1 applies to X (with N being its unique exit continuum). This implies that X contains a P -fixed point a with several rays landing at a which rotate under P (i.e., a is a bi-accessible fixed point), a contradiction with the definition of basic uniCremer polynomials.

Suppose next that H is a gap or a leaf. Observe that there are no critical leaves on the boundary of H . Indeed, if $\ell \in \partial(H)$ is a critical leaf, then it cannot belong to an element from $\Theta(\mathcal{A})$, hence it is a limit of such elements. By the properties of laminations this is impossible in the case when $\sigma(H)$ is not a point because then the images of these elements of $\Theta(\mathcal{A})$ will cross the image of H . Now, if $\sigma(H)$ is a point, this point must coincide with $A(p)$ and is fixed. In this case the images of elements of $\Theta(\mathcal{A})$ which approximate ℓ will cross H itself, again a contradiction.

Let us show (by way of contradiction) that $\sigma(H') \subset H'$. By the above $\sigma(H')$ cannot be a point. Then the fact that $\sigma(H') \not\subset H'$ implies that H' and $\sigma(H')$ have one or two points in common. Suppose they have only one point a in common. Then $a \in \sigma(A(p))$ and $\sigma(a) = a$. For geometric reasons there is then a leaf $\ell \in \partial H$ and there is a leaf $\ell' \in \partial H \setminus \partial(\sigma(H))$ which have the point a in common. Elements of $\Theta(\mathcal{A})$ are disjoint from these leaves because they are wandering; on the other hand, by the construction of $\mathcal{L}(K)$ elements of $\Theta(\mathcal{A})$ must converge to ℓ and ℓ' . Since for each set $D \in \Gamma(K)$ we have $a \in H_D$ (because $p \in \text{Imp}(a)$), hence by the construction *both* leaves ℓ, ℓ' must belong to H , a contradiction.

Suppose that $\sigma(H')$ and H' have two points in common. Since by the assumption $\sigma(H') \not\subset H'$, this means that both H and $\sigma(H)$ are gaps with a common boundary leaf ℓ . Thus, ℓ is isolated in $\mathcal{L}(K)$ and lies on the boundary of a wandering set from $\Theta(\mathcal{A})$. On the other hand, $\sigma(H) \cap H \supset \sigma(A(p))\sigma^2(A(p))$ which means that no endpoint of ℓ can be wandering, a contradiction. Thus, $\sigma(H') \subset H'$.

Since by the above there are no critical leaves in ∂H , there must exist a periodic leaf $\ell \in \partial H$. This is clear if H is a leaf. Otherwise, leaves from ∂H map onto one another and never collapse. Moreover, for each $\ell \in \partial H$, let M_ℓ be the arc with the same endpoints as ℓ and disjoint from H . Then for distinct leaves from ∂H their arcs are disjoint, and if the length of the arc M_ℓ is smaller than $\frac{1}{a+1}$, then the length of the arc $M_{\sigma(\ell)}$ is greater than that of M_ℓ . Since there cannot be infinitely many such arcs with bounded from below length, there exists a periodic leaf $\ell \in \partial H$. Suppose that its period is m .

Choose an element T of $\Theta(\mathcal{A})$ close to ℓ and consider the corresponding element Q of $\Gamma(K)$. Then there is a unique open wedge N - one of the components of $\mathbb{C} \setminus [Q \cup \bigcup_{\alpha \in T} R_\alpha]$ - into which Q is mapped by P^m . Consider the union X of Q with the intersection of J and N . It is easy to check that Corollary 3.1 applies to X (with N being its unique exit continuum) and P^m . This implies that X contains a P^m -fixed point a with several landing rays which rotate under P , a contradiction with the definition of basic uniCremer polynomials.

Finally, if the σ -orbit of \mathcal{A} is dense in \mathbb{S}^1 , then clearly we will be able to find $m > n$, such that $\sigma^m(K)$ is separated from p by K united with the rays of the angles from \mathcal{A} . Then as before by Corollary 3.1 we get a contradiction with the definition of a uniCremer polynomial. \square

Observe that even if the set $\sigma^n(\mathcal{A})$ is a point for some n , the construction of the sets W and H goes through and shows that in the basic uniCremer case the dynamical behavior of K (and \mathcal{A}) is very specific.

5. MAIN RESULTS

First we establish some facts which may be of independent interest and serve as an additional motivation for us. As was explained in the Introduction, Kiwi's results [K04] do not apply to uniCremer polynomials. Still, one could hope to model (topologically) a basic uniCremer Julia set J by monotonically mapping J onto a locally connected continuum. It turns out that this is impossible. In the quadratic case we proved in [BO06b] that a monotone map of J onto a locally connected continuum collapses J to a point. However the proofs in [BO06b] rely also upon results of [GMO99] not known for basic uniCremer polynomials. Using a new argument we fill this gap and prove Theorem 5.2.

Let us state some results of [BO06b]. An *unshielded* continuum $K \subset \mathbb{C}$ is a continuum which coincides with the boundary of the unbounded component of $\mathbb{C} \setminus K$. Given an external (conformal) ray R , a crosscut C is said to be *R -transversal* if \overline{C} intersects \overline{R} only once and the intersection is transverse and contained in $C \cap R$; if $t \in R$, then by C_t we always denote an R -transversal crosscut such that $C_t \cap \overline{R} = \{t\}$. Given an external ray R , we define the (*induced*) *order* on R so that $x <_R y$ ($x, y \in R$) if and only if the point x is “closer to K **on the ray** R than y ”. Given an external ray R , we call a family of R -transversal crosscuts C_t , $t \in R$ an *R -defining* family of crosscuts if for each $t \in R$ there exists an R -transversal crosscut C_t such that $\text{diam}(C_t) \rightarrow 0$ as $t \rightarrow K$ and $\text{Sh}(C_t) \subset \text{Sh}(C_s)$ if $t <_R s$.

Lemma 5.1 ([BO06b, Lemma 2.1]). *Let K be an unshielded continuum and R be an external ray to K . Then there exists an R -defining family of R -transversal crosscuts C_t , $t \in R$.*

Now we can prove Theorem 5.2.

Theorem 5.2. *Suppose that P is a basic uniCremer polynomial and $\varphi : J \rightarrow A$ is a monotone map of J onto a locally connected continuum A or a point. Then A is a singleton.*

Proof. By way of contradiction, suppose that $\varphi : J \rightarrow A$ is a monotone map onto a locally connected non-degenerate continuum A . Define the map Φ on the complex plane \mathbb{C} so that it identifies precisely *fibers* (point-preimages) of φ and does not identify any points outside J . Since the decomposition of \mathbb{C} into fibers of φ and points of $\mathbb{C} \setminus J$ is upper-semicontinuous, the map Φ is continuous. Since J and hence all its subcontinua are non-separating, then by the Moore Theorem [Moo62] the map Φ maps \mathbb{C} onto \mathbb{C} , and so $\Phi(J) = \varphi(J) = A$ is a *dendrite* (locally connected continuum containing no simple closed curve). External (conformal) rays R_α in the J -plane are then mapped into continuous pairwise disjoint curves $\Phi(R_\alpha)$ in the A -plane called below *A -rays*. Clearly, if $R_\alpha = R$ lands, then so does $\Phi(R)$. Let us show that $\Phi(R)$ *always* lands, and in fact the impression $\text{Imp}(\alpha)$ maps under φ to the landing point of the A -ray $\Phi(R)$. By Lemma 5.1 there exists an R -defining family of crosscuts C_t . Since Φ is continuous, then $\text{diam}(\Phi(C_t)) \rightarrow 0$ as $t \rightarrow J$. Consider two cases.

Suppose that $\Phi(C_t)$ is an arc for all $t \in R$, and hence a crosscut of A . Since A is locally connected, then by Carathéodory theory $\Phi(C_t)$ converges to a unique

point $x \in A$ which implies that $\Phi(R)$ lands. Also, by Carathéodory theory the Φ -images of the closures of the shadows $\text{Sh}(C_t)$ converge to the same point x which belongs to them all. Since the intersection of the closures of the shadows $\text{Sh}(C_t)$ is the impression $\text{Imp}(\alpha)$ of α , then $\varphi(\text{Imp}(\alpha)) = \{x\}$ as desired. Otherwise there exists $t \in R$ such that $\Phi(C_t)$ is a simple closed curve. This can only happen if the endpoints of C_t map under φ to the same point, say, z . It follows that $\Phi(C_s)$ is a simple closed curve for all $s <_R t$, and these curves all contain z . Thus, $\Phi(R)$ lands at z . Moreover, in this case the shadows $\text{Sh}(C_s)$ are mapped inside the simple closed curves $\Phi(C_s)$, $s <_R t$ which, as before, implies that $\varphi(\text{Imp}(\alpha)) = \{z\}$.

Since impressions are upper-semicontinuous, the family of A -rays is continuous, and the map ψ associating to every angle α the landing point of the A -ray $\Phi(R_\alpha)$ is a continuous map of \mathbb{S}^1 onto A . Define the *valence* of a point $y \in A$ as the number of components of the set $A \setminus \{y\}$ if it is finite and infinity otherwise. Let \mathcal{B}_y be the set of A -rays landing at y . Then the number of components of $\mathbb{S}^1 \setminus \mathcal{B}_y$ equals the valence of y . Indeed, if (α, β) is a component of $\mathbb{S}^1 \setminus \mathcal{B}_y$, then the A -rays of angles in (α, β) land in a component of $A \setminus \{y\}$ and for distinct components of $\mathbb{S}^1 \setminus \mathcal{B}_y$ we get distinct components of $A \setminus \{y\}$. In fact there is exactly one component of $A \setminus \{y\}$ contained in the appropriate wedge in the plane formed by the A -rays $\Phi(R_\alpha)$ and $\Phi(R_\beta)$. Indeed, otherwise choose angles $\gamma, \theta \in (\alpha, \beta)$ so that their A -rays $\Phi(R_\gamma)$ and $\Phi(R_\theta)$ land at points x, z from distinct components of $A \setminus \{y\}$. Then the path $\psi([\gamma, \theta])$ connects points x and z inside A and hence must pass through y . On the other hand, by the construction $y \notin \psi([\gamma, \theta])$, a contradiction.

We show that except for a countable set of points there are no more than two A -rays landing at $y \in A$. Let Q' be the set of all points $y \in A$ with finite valence for which there are infinitely many A -rays landing at y . Then by the previous paragraph, \mathcal{B}_y has a non-empty interior for any point $y \in Q'$, and so Q' is countable. On the other hand, by Theorem 10.23 of [Nad92] the set Q'' of all branch points of A is countable (a *branch point* is a point of valence greater than 2). Hence, for any point $y \in A \setminus (Q' \cup Q'')$ such that its valence is greater than 1 (such points are called *cutpoints*) exactly two A -rays land at y .

Choose a point $y \in A \setminus (Q' \cup Q'')$ using a bit of dynamics. Let H be the union of grand orbits of p and all critical points of P and set $Q''' = \varphi(H)$. Then $\hat{Q} = Q' \cup Q'' \cup Q'''$ is countable. Choose $y \in A \setminus \hat{Q}$ to be a cutpoint of A . Then exactly two A -rays land at y . Moreover, let $\varphi^{-1}(y) = K$; then by the choice of y , forward images of K avoid p and critical points of P . Since impressions are mapped by φ into points, there are exactly two angles α, β with $K = \text{Imp}(\alpha) \cup \text{Imp}(\beta)$ and the impressions of other angles are disjoint from K .

By Theorem 4.3 there are integers $0 \leq l < m$ such that $P^l(K)$ and $P^m(K)$ intersect. Since forward images of K avoid critical points, $\sigma^r(\alpha) \neq \sigma^r(\beta)$ for any r and hence both $P^l(K) = \text{Imp}(\sigma^l(\alpha)) \cup \text{Imp}(\sigma^l(\beta))$ and $P^m(K) = \text{Imp}(\sigma^m(\alpha)) \cup \text{Imp}(\sigma^m(\beta))$ are unions of impressions of two distinct angles. Now, if the pair of angles $\sigma^l(\alpha), \sigma^l(\beta)$ maps into itself by σ^{m-l} , then $P^l(K)$ is a P^{m-l} -fixed continuum not containing p , hence by Corollary 3.1 it is a repelling periodic point at which two rays land, a contradiction. Hence there exists an angle $\gamma \in \{\sigma^m(\alpha), \sigma^m(\beta)\} \setminus \{\sigma^l(\alpha), \sigma^l(\beta)\}$ such that $\text{Imp}(\gamma)$ non-disjoint from the $P^l(K)$. If we now pull $P^l(K)$ back to K we will get an angle $\gamma' \notin \{\alpha, \beta\}$ whose impression is non-disjoint from K , a contradiction. \square

Since we assume that J is decomposable, all impressions are proper and have empty interior. In particular no countable union of impressions coincides with J . We begin by studying *red dwarf* Julia sets, i.e. uniCremer Julia sets such that impressions of all angles contain the Cremer point p .

Lemma 5.3. *If J is a red dwarf Julia set, then (1) the intersection K of all impressions contains all forward images of all critical points, (2) there exists $\varepsilon > 0$ such that the diameter of any impression is greater than ε , (3) there are no points at which J is connected im kleinen, and (4) no point of J is biaccessible and p is not accessible from $\mathbb{C} \setminus J$.*

Proof. Let us show that for any angle α its impression $\text{Imp}(\alpha)$ contains all critical images. Indeed, otherwise there exists a critical point c such that $P(c) \notin \text{Imp}(\alpha)$ which implies that $c \notin K$. Choose a curve T starting at $P(c)$, going to infinity and bypassing $\text{Imp}(\alpha)$. Then the pullback of T containing c cuts the plane into at least two components each of which contains at least one pullback of $\text{Imp}(\alpha)$, a contradiction. Hence K contains all critical values, and since K is forward invariant, K contains all forward images of all critical points.

Clearly, (1) implies (2) (by [Mn93] there is a recurrent critical point whose forward orbit avoids p). By Lemma 3.5(2) J is nowhere connected im kleinen. Moreover, no point $x \in J$ is biaccessible from $\mathbb{C} \setminus J$. Indeed, if $x \neq p$ is biaccessible, then one of the two half-planes into which x and rays landing at x cut the plane will not contain p , hence all rays contained in that half-plane will not contain p in their impressions, a contradiction. On the other hand, if $x = p$, we can take a preimage of p and get the same contradiction. Hence p cannot be accessible from $\mathbb{C} \setminus J$ because if it is, then the corresponding ray is not fixed (periodic rays cannot land on p by Douady and Hubbard [DH85a]) and hence this ray and its image show that p is biaccessible, a contradiction. \square

Lemma 5.4 follows from Corollary 3.1.

Lemma 5.4. *If $K \subset J$ is a P^n -invariant continuum or singleton not containing p , then K is a singleton. In particular, if γ is a periodic angle and $p \notin \text{Imp}(\gamma)$, then $\text{Imp}(\gamma)$ is a singleton.*

Proof. All P^n -fixed points in K are repelling, and by the definition of a basic uniCremer polynomial at each of them, exactly one P^n -fixed ray lands. Hence by Corollary 3.1, K is a singleton. \square

For $x \in J$ let $A(x)$ be the set of all angles whose impressions contain x , and let $B(x)$ be the union of these impressions. Then $A(x)$ and $B(x)$ are closed sets. In Lemma 5.5 we study these sets for a periodic x .

Lemma 5.5. *If x is periodic, then one of the following holds:*

- (1) $x \notin B(p)$, then $\{x\} = B(x) = \text{Imp}(\theta)$ for a periodic K -separate angle θ , $A(x) = \{\theta\}$, and x is a repelling periodic point;
- (2) $x \in B(p)$, then $B(x)$ is non-degenerate and no angle $\theta \in A(x)$ is K -separate. In particular, no angle $\theta \in A(p)$ is K -separate, and $B(p)$ is non-degenerate.

Proof. Consider first the case when $x = p$. We need to show that then the case (2) holds. First let us show that $B(p)$ is non-degenerate. Suppose that $A(p)$ is infinite. Since $A(p)$ is invariant, it follows from a well-known result from the topological

dynamics of locally expanding maps that $\sigma|_{A(p)}$ is not one-to-one. Hence by [Hea96], $B(p)$ contains a critical point and cannot coincide with p . Suppose that $A(p)$ is finite. Then $A(p)$ contains periodic angles, hence $B(p)$ contains periodic points distinct from p and hence again $B(p)$ is not degenerate.

Let us show that no angle $\theta \in A(p)$ is K-separate. We may assume that $A(p) = \{\theta\}$ consists of just one angle and need to show that θ is not K-separate. Clearly, $\sigma(\theta) = \theta$. Denote the landing point of R_θ by x , and show that there is a critical point $c \in B(p) = \text{Imp}(\theta)$. Indeed, otherwise choose a neighborhood U of $B(p)$ such that no critical points belong to \overline{U} , consider the set of all points never exiting \overline{U} , and then the component K of this set containing p . Such sets are called *hedgehogs* (see papers by Perez-Marco [Per94] and [Per97]) and have a lot of important properties. In particular, by [Per94] and [Per97], K cannot contain a periodic point other than p , a contradiction (clearly, $B(p) \subset K$ and $B(p)$ contains x). Hence $c \in B(p)$ for some critical point c . This implies that for some integer i , $1 \leq i \leq d-1$ we have $c \in \text{Imp}(\theta + i/d)$ and hence $B(p) = \text{Imp}(\theta)$ is not disjoint from the impression of another angle and θ is not K-separate as desired.

Suppose now that $x \notin B(p)$ is a periodic point of period m . Then $p \notin B(x)$ and hence by Lemma 5.4, $B(x)$ is degenerate. By the above quoted topological result this implies that $A(x)$ is finite and therefore, by the assumptions on P , $A(x) = \{\theta\}$ where θ is periodic and K-separate. Now assume that $x \in B(p)$. Then there is an angle $\alpha \in A(p)$ with $p, x \in \text{Imp}(\alpha)$ and hence $p \in B(x)$ and $B(x)$ is not degenerate. Now, if there is more than one angle in $A(x)$, then all such angles are not K-separate and we are done. If however there is only one angle in $A(x)$, then this angle is $\alpha = \theta$ which belongs to $A(p)$ and by the previous paragraph is not K-separate either. \square

Lemma 5.5 implies a few facts: e.g., if θ is a K-separate periodic angle, then $\text{Imp}(\theta)$ is a repelling periodic point. Lemma 5.6 shows that in some cases there are lots of such angles. For $F \subset \mathbb{S}^1$ denote by $\text{Imp}(F)$ the set $\bigcup_{\theta \in F} \text{Imp}(\theta)$. Let E be the set of all K-separate periodic angles; by Lemma 5.5 each angle in E has degenerate impression.

Lemma 5.6. *Suppose $p \notin \text{Imp}(\theta)$ for some angle θ . Then (1) $B(p)$ is a nowhere dense subset of J , (2) the set E is dense in \mathbb{S}^1 , (3) the set $\text{Imp}(E)$ is dense in J , (4) for a closed set of angles $F \neq \mathbb{S}^1$ the set $\text{Imp}(F)$ is a proper subset of J , (5) J is connected im kleinen at every point $y \in \text{Imp}(E)$, (6) in any arc $W \subset \mathbb{S}^1$ there are two angles whose impressions meet.*

Proof. To prove the first claim it is enough to show that $B(p) \neq J$. Suppose otherwise. Since by our standing assumption J is decomposable, then no finite union of impressions can coincide with J , and $A(p)$ is infinite. Then in any open arc V there are angles whose impressions meet. Indeed, we can find a big integer N such that $\sigma^{-N}(A(p)) \cap V$ is infinite. Each angle from $\sigma^{-N}(A(p))$ contains a P^N -preimage of p in its impression. Since there are only finitely many P^N -preimages of p , then there are two angles $\gamma, \gamma' \in \sigma^{-N}(A(p)) \cap V$ which contain the same preimage of p and therefore meet as desired.

Denote by U an open arc containing θ such that for any angle in U its impression does not contain p (then by Lemma 5.4 periodic angles from U have degenerate impressions). Assume that $(\gamma, \gamma') \subset U$ is an arc such that the impressions of γ, γ' meet. The union $Z = R(\gamma) \cup \text{Imp}(\gamma) \cup \text{Imp}(\gamma') \cup R(\gamma')$ cuts the plane into two half-planes H and G ; assume for the sake of definiteness that $p \in G$. Since no

finite union of impressions coincides with J , then the union of impressions of angles from (γ, γ') is not contained in Z . Hence there exists a point $h \in J \cap H$. Then the impression of an angle from (γ', γ) cannot contain h because Z separates this angle's ray from h . On the other hand, the impression of an angle from $[\gamma, \gamma']$ cannot contain p either. Hence no impression contains h and p simultaneously, implying that $B(p) \neq J$.

By Lemma 5.5, E is the set of all periodic angles such that landing points of their rays do not belong to $B(p)$. By the upper semi-continuity of impressions, E is open in the set of all periodic angles. Since E is invariant, E is dense in \mathbb{S}^1 which proves (2). We claim that $\text{Imp}(E)$ is dense in J . Indeed, otherwise there exists an open set $U \subset J$ disjoint from $\text{Imp}(E)$. Since periodic points are dense in J , they are dense in U , and by the definition of E all these periodic points belong to $B(p)$. Hence $B(p)$ has non-empty interior, a contradiction with (1). Thus, $\text{Imp}(E)$ is dense in J which proves (3). The claim (2) implies (4). By Lemma 3.4 the rest of the lemma follows. \square

So far the results of this section use mainly topological tools. This changes in the lemmas below where we rely upon both continuum theory and dynamics. Our aim is to prove that the angles with orbits dense in \mathbb{S}^1 have degenerate impressions. Problems of this kind are often related to the dynamics of critical points. The result obtained in Lemma 5.8 enables us to apply some standard tools and seems to be interesting by itself. However first we need a simple lemma.

Lemma 5.7. *Suppose that there exists an angle θ whose impression does not contain p . Then the following statements are equivalent:*

- (1) *An angle α has a dense orbit in \mathbb{S}^1 .*
- (2) *Any point of $\text{Imp}(\alpha)$ has a dense orbit in J .*
- (3) *There exists a point in $\text{Imp}(\alpha)$ which has a dense orbit in J .*

Proof. Suppose that α has a dense orbit in \mathbb{S}^1 and choose $x \in \text{Imp}(\alpha)$. Let $y \in J$ and $\varepsilon > 0$. By Lemma 5.6 there exists $\gamma \in \mathbb{S}^1$ such that $\text{Imp}(\gamma) = \{z\}$ and $d(z, y) < \varepsilon/2$. Since the orbit of α is dense in \mathbb{S}^1 and impressions are upper semi-continuous, there exists $n > 0$ such that $\sigma^n(\alpha)$ is so close to γ that $\text{Imp}(\sigma^n(\alpha)) \subset B(z, \varepsilon/2)$, and hence $d(P^n(x), y) < \varepsilon$. Therefore, $\omega(x) = J$ and (1) implies (2). Clearly, (2) implies (3). Now, suppose that (3) holds. Then $\text{Imp}(\omega(\alpha)) = J$ which by Lemma 5.6(4) implies that $\omega(\alpha) = \mathbb{S}^1$ as desired. \square

Now we show that if there exists an angle θ whose impression does not contain p , then no critical point has a dense orbit in J .

Lemma 5.8. *Suppose that there exists an angle θ whose impression does not contain p . Suppose that α is an angle with a dense orbit in \mathbb{S}^1 . Then α is K -separate and $\text{Imp}(\alpha)$ does not contain a critical point of P . In particular, no critical point can have a dense orbit in J .*

Proof. The proof consists of several steps. Suppose that there are finitely many angles $\alpha = \alpha_0, \alpha_1, \dots, \alpha_k$ such that the union I of their impressions is a continuum. Let us show that then I has to be wandering (all its iterates are pairwise disjoint). By way of contradiction and without loss of generality we may assume that $P(I) \cap I \neq \emptyset$. Consider the set Q of all angles whose impressions are not disjoint from $B(p)$ and the set $\text{Imp}(Q)$. Then $Q \neq \mathbb{S}^1$ by Lemma 5.5. Hence by Lemma 5.6(4),

$\text{Imp}(Q) \neq J$. Let us show that we may assume that *all* images of I intersect $\text{Imp}(Q)$.

By Lemma 5.5 the set Q contains two angles with non-disjoint impressions. Denote these angles γ and β . Then the rays of these angles together with their impressions cut the plane into two components H and G so that the periodic K-separate angles from (γ, β) have point-impressions in H and the periodic K-separate angles from (β, γ) have point-impressions in G . Choose periodic K-separate angles $\theta' \in (\gamma, \beta)$ and $\theta'' \in (\beta, \gamma)$, and then choose $k < l$ so that $\sigma^k(\alpha) \in (\gamma, \beta)$ is very close to θ' and $\sigma^l(\alpha) \in (\beta, \gamma)$ is very close to θ'' . Then the continuum $Z = P^k(I) \cup P^{k+1}(I) \cup \dots \cup P^l(I)$ connects points from H to points from G . Hence by Lemma 3.3, $Z \cap \text{Imp}(Q) \neq \emptyset$. Thus, from some time on the images of I are non-disjoint from $\text{Imp}(Q)$, and we may assume that in fact $I \cap \text{Imp}(Q) \neq \emptyset$ and hence *all* images of I intersect $\text{Imp}(Q)$.

Just like the set $B(p)$ was constructed as the union of all impressions non-disjoint from p , and the set $\text{Imp}(Q)$ was constructed as the union of all the impressions non-disjoint from $B(p)$, this process can be continued for $k+1$ more steps resulting into a union of impressions which we will denote by T . Clearly, T is closed; moreover, since no point of $\text{Imp}(E)$ can belong to T , then T is a proper subset of J . Since α has a dense orbit, by Lemma 5.7 the orbit of any point of $\text{Imp}(\alpha)$ is dense in J . On the other hand, by the previous paragraph the orbit of $\text{Imp}(\alpha)$ is contained in T and $T \neq J$, a contradiction. So, the assumption that I is not wandering leads to a contradiction which implies that I is wandering. Then observe that by Theorem 4.1 $k \leq 2^d$.

Let us now show that α is K-separate. Indeed, otherwise let $\alpha' \neq \alpha$ be such that $\text{Imp}(\alpha) \cap \text{Imp}(\alpha') \neq \emptyset$. Then by the above, the maximal finite collection of angles $\alpha_0 = \alpha, \alpha_1 = \alpha', \dots, \alpha_k$ such that the union I of their impressions is connected, consists of $k+1 > 1$ angles, and is such that I is wandering. By maximality the impressions of other angles are disjoint from I . However the orbit of α is dense which contradicts Theorem 4.3. Hence α is K-separate as desired. To complete the proof it remains to notice that if a critical point c belongs to $\text{Imp}(\alpha)$, then because locally around c the map P is not one-to-one, there exists an angle α' such that $\sigma(\alpha) = \sigma(\alpha')$ and $c \in \text{Imp}(\alpha')$ implying that α is not K-separate. Hence $\text{Imp}(\alpha)$ does not contain critical points. Since this holds for any angle with dense orbit, we conclude that by Lemma 5.7 no critical point can have a dense orbit in J . \square

Lemma 5.9 completes a series of claims made in Lemma 5.7 and Lemma 5.8.

Lemma 5.9. *Suppose that there exists an angle θ whose impression does not contain p . Let α be an angle such that the σ -orbit of α is dense in \mathbb{S}^1 . Then $\text{Imp}(\alpha)$ is a singleton, and, moreover, the angle α is K-separate.*

Proof. By Mañé [Mn93] the closure B' of the union of orbits of all critical points of P contains p . By Lemma 5.8, B' is nowhere dense in J . By Lemma 5.6 there is an angle $\gamma \in E$ such that the singleton $\text{Imp}(\gamma)$ is not contained in B' . By the upper semi-continuity of impressions we can find an arc U around γ so that the union of impressions of angles from \overline{U} is positively distant from B' . By Lemma 3.4 we can find two angles $\tau' < \gamma < \tau''$ in U (the order is considered within U) such that $\text{Imp}(\tau') \cap \text{Imp}(\tau'') \neq \emptyset$. Set $U' = (\tau', \tau'')$.

Consider the two connected open components of $\mathbb{C} \setminus R_{\tau'} \cup \text{Imp}(\tau') \cup \text{Imp}(\tau'') \cup R_{\tau''}$; let V be the component containing rays of angles from U' . Then there are points

of J in V . Indeed, otherwise $\text{Imp}(\tau') \cup \text{Imp}(\tau'')$ contains the impressions of all angles from U' which yields that a forward σ -image of τ' or τ'' will coincide with J implying by Theorem 3.2 that J is indecomposable, a contradiction with the standing assumption. Let us prove that α is K -separate and its impression is a point. Indeed, by the previous paragraph, V is positively distant from B' . Since V is simply connected, we can find two Jordan disks $W' \supset \overline{W''} \supset J \cap V$ which are both positively distant from B' . Therefore all pull-backs of W' and W'' are univalent. By Mañé [Mn93] this implies that the diameter of the pull-backs of W'' converge to 0 as the power of the map approaches infinity. Observe that as σ -images of α approach γ , the corresponding P -images of $\text{Imp}(\alpha)$ get closer and closer to $\text{Imp}(\gamma)$ (because of the upper semi-continuity of impressions) and thus we may assume that infinitely many P -images of $\text{Imp}(\alpha)$ are contained in W'' . Pulling W'' back along the orbit of $\text{Imp}(\alpha)$ for longer and longer time we see that the diameter of $\text{Imp}(\alpha)$ cannot be positive, and hence $\text{Imp}(\alpha) = \{y\}$ is a point as claimed. Moreover, by Lemma 5.8 the angle α is K -separate. This completes the proof. \square

We can now prove our main theorem.

Theorem 5.10. *For a uniCremer polynomial P the following facts are equivalent:*

- (1) *there is an impression not containing the Cremer point;*
- (2) *there is a degenerate impression;*
- (3) *the set Y of all K -separate angles with degenerate impressions contains all angles with dense orbits and a dense set of periodic angles, and the Julia set J is connected im kleinen at landing points of the corresponding rays;*
- (4) *there is a point at which the Julia set is connected im kleinen.*

Proof. Let us prove that (1) implies (2). Indeed, suppose that there is an angle not containing p in its impression. Then by Lemma 5.6 there exist angles with degenerate impressions.

We show that (2) implies (3). Indeed, let $\text{Imp}(\alpha)$ be a point. Then so are the impressions of the angles $\alpha + 1/d, \dots, \alpha + (d-1)/d$. At least one of them is not p , so we may assume that $\text{Imp}(\alpha) \neq \{p\}$. Then by Lemma 5.6 the set E is dense in \mathbb{S}^1 . Let us now consider an angle β whose σ -orbit is dense in \mathbb{S}^1 . By Lemma 5.9, $\text{Imp}(\beta)$ is a point and β is K -separate. By Lemma 3.4, J is connected im kleinen at the landing points of the rays with arguments either from E , or from the set of angles with dense orbits in \mathbb{S}^1 . This shows that indeed (2) implies (3).

Clearly, (3) implies (4). It remains to show that (4) implies (1). Indeed, if all impressions contain p , then by Lemma 5.3, J is nowhere connected im kleinen, a contradiction. The proof is complete. \square

REFERENCES

- [BBCO07] A. Blokh, X. Buff, A. Chéritat and L. Oversteegen, *The solar Julia sets of basic quadratic Cremer polynomials*, preprint 2007.
- [BO06a] A. Blokh and L. Oversteegen, *The Julia sets of quadratic Cremer polynomials*, *Topology and its Applications*, **153** (2006), pp. 3038–3050. MR2248408 (2007j:37070)
- [BO06b] A. Blokh and L. Oversteegen, *Monotone images of Cremer Julia sets*, preprint (2006). MR2248408 (2007j:37070)

- [BO08] A. Blokh and L. Oversteegen, *Fixed points in non-invariant plane continua*, preprint, arXiv:0805.1069 (2008).
- [BL02] A. Blokh and G. Levin, *An inequality for laminations, Julia sets and ‘growing trees’*, Erg. Th. and Dyn. Sys., **22** (2002), pp. 63–97. MR1889565 (2003i:37045)
- [CMR05] D. Childers, J. Mayer and J. Rogers, Jr., *Indecomposable continua and the Julia sets of polynomials, II*, Topology and its Applications **153** (2006), 1593–1602. MR2216123 (2007c:37058)
- [DH85a] A. Douady and J. H. Hubbard, *Étude dynamique des polynômes complexes I, II*, Publications Mathématiques d’Orsay **84-02** (1984), **85-04** (1985).
- [DH85b] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 2, 287–343. MR816367 (87f:58083)
- [GM93] L. Goldberg and J. Milnor, *Fixed points of polynomial maps, Part II. Fixed point portraits*, Ann. Sci. Ec. Norm. Sup., 4^e série **26** (1993), pp. 51–98. MR1209913 (95d:58107)
- [GMO99] J. Grispolakis, J. Mayer, and L. Oversteegen, *Building Blocks for Julia sets*, Trans. Amer. Math. Soc. **351** (1999), 1203–1225. MR1615975 (99g:30030)
- [Hea96] J. Heath, *Each locally one-to-one map from a continuum onto a tree-like continuum is a homeomorphism*, Proc. Amer. Math. Soc., **124** (1996), 2571–2573. MR1371127 (97c:54016)
- [Hub93] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz*, in Topological methods in Modern Mathematics, Publish or Perish (1993), 467–511. MR1215974 (94c:58172)
- [K00] J. Kiwi, *Non-accessible critical points of Cremer polynomials*, Erg. Theory and Dyn. Sys. **20**, (2000), pp. 1391–1403. MR1786720 (2001i:37067)
- [K02] J. Kiwi, *Wandering orbit portraits*, Trans. Amer. Math. Soc. **254** (2002), pp. 1473–1485. MR1873015 (2002h:37070)
- [K04] J. Kiwi, *Real laminations and the topological dynamics of complex polynomials*, Advances in Math. **184** (2004), no. 2, pp. 207–267. MR2054016 (2005b:37094)
- [Lev98] G. Levin, *On backward stability of holomorphic dynamical systems*, Fund. Math. **158**, 97–107. MR1656942 (99j:58171)
- [Mn93] R. Mañé, *On a theorem of Fatou*, Bol. Soc. Bras. Mat. **24** (1993), 1–11. MR1224298 (94g:58188)
- [McM94] C. T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematical Studies **135**, Princeton University Press, Princeton, NJ (1994). MR1312365 (96b:58097)
- [Moo62] R. L. Moore, *Foundations of point set theory. Revised edition*, AMS Colloquium Publications **13** (1962), AMS, Providence, R.I. MR0150722 (27:709)
- [Nad92] S. Nadler, Jr., *Continuum theory. An introduction*. Monographs and Textbooks in Pure and Applied Mathematics **158**, Marcel Dekker, Inc., New York (1992). MR1192552 (93m:54002)
- [Per94] R. Perez-Marco, *Topology of Julia sets and hedgehogs*, Publications Mathématiques d’Orsay **94-48** (1994).
- [Per97] R. Perez-Marco, *Fixed points and circle maps*, Acta Math. **179** (1997), pp. 243–294. MR1607557 (99a:58130)
- [Sch04] D. Schleicher, *On fibers and local connectivity of Mandelbrot and Multibrot sets*, in Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 1, Proc. Sympos. Pure Math., **72**, Part 1, Amer. Math. Soc., Providence, RI (2004), 477–517. MR2112117 (2006b:37088)
- [SZ99] D. Schleicher and S. Zakeri, *On biaccessible points in the Julia set of a Cremer quadratic polynomial*, Proc. Amer. Math. Soc. **128** (1999), pp. 933–937. MR1637424 (2000e:37057)
- [Sul83] D. Sullivan, *Conformal dynamical systems*, Springer Lecture Notes in Mathematics, **1007** (1983), 725–752. MR730296 (85m:58112)
- [Sul85] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I, solution of the Fatou-Julia problem on wandering domains*, Ann. Math., **122** (1985), pp. 401–418. MR819553 (87i:58103)

- [Thu85] W. P. Thurston, *On the geometry and dynamics of iterated rational maps*, Preprint, 1985.
- [Yoc95] J. C. Yoccoz, *Petits diviseurs en dimension 1*, Asterisque **231** (1995).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, ALABAMA 35294-1170

E-mail address: `ablokh@math.uab.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, ALABAMA 35294-1170

E-mail address: `overstee@math.uab.edu`