

CHECKING ATOMICITY OF CONFORMAL ENDING MEASURES FOR KLEINIAN GROUPS

KURT FALK, KATSUHIKO MATSUZAKI, AND BERND O. STRATMANN

ABSTRACT. In this paper we address questions of continuity and atomicity of conformal ending measures for arbitrary non-elementary Kleinian groups. We give sufficient conditions under which such ending measures are purely atomic. Moreover, we will show that if a conformal ending measure has an atom which is contained in the big horospherical limit set, then this atom has to be a parabolic fixed point. Also, we give detailed discussions of non-trivial examples for purely atomic as well as for non-atomic conformal ending measures.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The idea of a conformal ending measure was originally sketched briefly by Sullivan in [21] for a Kleinian group with parabolic elements. Subsequently, this construction was investigated more rigorously and in greater generality in [1]. The construction of a conformal ending measure is very similar to the well-known Patterson measure construction (see [16]). However, in [8] it was shown that there are cases in which conformal ending measures cannot be derived via Patterson's construction. The basic idea of a conformal ending measure is as follows (see Section 3 for a more detailed review of conformal ending measures). Let (z_n) be an *ending sequence*, that is, a sequence of points in the Poincaré ball model (\mathbb{D}, d) of the hyperbolic space which converges radially to some point $\zeta \in \mathbb{S} = \partial\mathbb{D}$ within some Dirichlet fundamental domain of a Kleinian group G . If ζ is a limit point of G , then the closure of the orbit $G(z_n)$ converges to the limit set $L(G)$, for n tending to infinity. Here, convergence is meant with respect to the Hausdorff metric on closed subsets of the closure $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}$ in the Euclidean topology. If $s \in \mathbb{R}^+$ is chosen such that the Poincaré series $\sum_{g \in G} j(g, z)^s$ of G at $z \in \mathbb{D}$ with exponent s converges, then there exist purely atomic s -conformal probability measures μ_n having their atoms in $G(z_n)$. Here, $j(g, z)$ denotes the conformal derivative of g at $z \in \overline{\mathbb{D}}$, that is, the uniquely determined positive number such that $g'(z)/j(g, z)$ is orthogonal, where $g'(z)$ is the Jacobian of g at z . For $\zeta \in \mathbb{S}$ we have $j(g, \zeta) = (1 - |g^{-1}(0)|^2)/|\zeta - g^{-1}(0)|^2$ (see Section 4 for further details). Consequently, weak limits of these measures μ_n give rise to s -conformal measures supported on $L(G)$, and these measures are called *s-conformal ending measures*. The aim of this paper is to address questions of continuity and atomicity for this

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type of ending measures. Throughout we will use the phrase *essential support* of a measure to denote any measurable set of full measure (not necessarily closed nor uniquely determined), and $\mathbf{1}_z$ denotes the Dirac delta at z . The following statement summarises Theorem 4.3, Lemma 4.4, Proposition 4.5 and Theorem 4.6.

Let (z_n) be a given ending sequence converging to $\zeta \in \mathbb{S}$, and assume that $j(h, \zeta) = 1$ for every element h of the stabiliser $\text{Stab}_G(\zeta)$ of ζ . Moreover, assume that the reduced horospherical Poincaré series $\sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s$ of G at ζ with exponent s converges. We then have that the essential support of the associated s -conformal ending measure μ is equal to $G(\zeta)$. In particular, μ is purely atomic and coincides with the measure μ_ζ given by

$$\mu_\zeta = \frac{\sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s \mathbf{1}_{g(\zeta)}}{\sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s}.$$

If ζ is an ordinary point of G or a bounded parabolic fixed point of G , then the reduced horospherical Poincaré series of G at ζ with exponent $s \in \mathbb{R}^+$ converges whenever the Poincaré series $\sum_{g \in G} j(g, z)^s$, $z \in \mathbb{D}$, converges.

This result provides us with typical examples of purely atomic conformal ending measures. Another class of examples is obtained by using a result of [1], stating that a conformal ending measure associated to a certain given end is essentially supported on the associated end limit set. We then have that if this end limit set is countable, then every ending measure essentially supported on it is necessarily purely atomic. Besides, this raises the question whether it is possible to have a countable end limit set associated to an end which does not originate from a parabolic fixed point. In fact, a result of Bishop [4], stating that a Kleinian group acting on three-dimensional hyperbolic space is geometrically finite if and only if the set of non-radial limit points is countable, strongly suggests that the case of parabolic cusps is the only case in which the end limit set of a given end is countable.

In Section 5 we first show that if a conformal ending measure for a Kleinian group G has an atom which is contained in the big horospherical limit set $L_H(G)$ (see Section 2 for the definition), then this atom has to be a parabolic fixed point. Note that above we gave a sufficient condition, in terms of the convergence of the reduced horospherical Poincaré series at ζ , under which a conformal ending measure is purely atomic. However, unless ζ is an ordinary point or ζ is a bounded parabolic fixed point, it is usually not an easy task to verify this convergence condition. Hence, it is desirable to find conditions which allow us to decide more easily whether a conformal ending measure has atoms or not. By employing a result of Tukia [22], which states that there always exists a measurable fundamental set for the action of a Kleinian group G on the complement of $L_H(G)$, in Section 5 we show that if G admits an s -conformal measure essentially supported on the dissipative part of the action of G on \mathbb{S} , then we always have that there exists a purely atomic s -conformal ending measure for G . This observation then gives rise to the following result (see Theorem 5.4). Here, $\delta(G)$ denotes the abzissa of convergence of the Poincaré series of G .

For $s \geq \delta(G)$, we have that every s -conformal ending measure for G supported on $L(G)$ is essentially supported on $L_H(G)$ if and only if every s -conformal ending measure for G supported on $L(G)$ can have atoms only at parabolic fixed points of G .

Eventually, in Section 6, we give various non-trivial examples of purely atomic and of non-atomic conformal ending measures. These include examples of infinitely generated Schottky groups whose conformal ending measures have atoms at Jørgensen points, and examples of non-atomic conformal ending measures on limit sets of normal subgroups of finitely generated Schottky groups. Moreover, we give the construction of a Kleinian group Γ which admits a purely atomic $\delta(\Gamma)$ -conformal ending measure at some Jørgensen point which is not a parabolic fixed point of Γ .

2. PRELIMINARIES

Hyperbolic geometry in the $(N + 1)$ -dimensional ball model (\mathbb{D}, d) , $N \geq 2$, and conformal geometry on the boundary sphere \mathbb{S} have the same automorphism group, namely, the group $\text{Con}(N)$ of conformal maps in \mathbb{S} which is isomorphic to the group of orientation-preserving hyperbolic isometries of \mathbb{D} . (For $N = 1$, $\text{Con}(1)$ is merely defined by the latter group.) Any transformation in $\text{Con}(N)$ gives rise to an isometry of \mathbb{D} , and vice versa. It is well known that this isomorphism arises naturally from the principle of Poincaré extension, based on the elementary observation that an $(N + 1)$ -dimensional hyperbolic half-space H in \mathbb{D} extends in a unique way to an N -dimensional ball $\overline{H} \cap \mathbb{S}$. A Kleinian group G is a discrete subgroup of $\text{Con}(N)$ and its limit set $L(G)$ is the derived set of an arbitrary G -orbit; that is, $L(G) = \overline{G(z)} \setminus G(z)$ for some arbitrary $z \in \mathbb{D}$. The limit set $L(G)$ is equal to the union of the set $L_r(G)$ of *radial limit points* (also called conical limit points) and the set $L_t(G) = L(G) \setminus L_r(G)$ of *transient limit points* (also called escaping limit points), where

$$L_r(G) = \left\{ \xi \in L(G) : \liminf_{T \rightarrow \infty} \Delta(\xi_T) < \infty \right\}.$$

Here, ξ_T denotes the unique point on the hyperbolic geodesic ray from the origin $0 \in \mathbb{D}$ to ξ for which $d(0, \xi_T) = T$, and $\Delta(z) = \inf\{d(z, g(0)) : g \in G\}$ denotes the hyperbolic distance of $z \in \mathbb{D}$ to the orbit $G(0)$. Important subsets of $L(G)$ are the set $L_{ur}(G)$ of *uniformly radial limit points* and the set $L_J(G)$ of *Jørgensen limit points*. These are given as follows (cf. [7] and [15]). $L_{ur}(G)$ is the set of $\xi \in L(G)$ for which $\limsup_{T \rightarrow \infty} \Delta(\xi_T) < \infty$, and $L_J(G)$ is the set of $\xi \in L(G)$ for which there exists a geodesic ray towards ξ which is fully contained in some Dirichlet fundamental domain of G . One easily verifies that $L_{ur}(G) \subset L_r(G)$ and $L_J(G) \subset L_t(G)$. Next, recall that an element of $\text{Con}(N)$ is called *parabolic* if it has precisely one fixed point in \mathbb{S} . If a Kleinian group G contains parabolic elements, then the fixed points of these elements are contained in $L_J(G)$. A horosphere at $\zeta \in \mathbb{S}$ can be represented by the level set $h_\zeta(c) = \{z \in \mathbb{D} : k(z, \zeta) = c\}$ of the Poisson kernel $k(z, \zeta) = (1 - |z|^2)/|\zeta - z|^2$, for some constant $c > 0$. Clearly, $h_\zeta(c)$ is an Euclidean sphere tangent to \mathbb{S} at ζ of radius equal to $1/(1+c)$. The associated horoball $H_\zeta(c)$ is the open ball bounded by this horosphere; that is,

$$H_\zeta(c) = \{z \in \mathbb{D} : k(z, \zeta) > c\}.$$

Let $\text{Stab}_G(\zeta)$ denote the stabiliser of $\zeta \in \overline{\mathbb{D}}$ in G consisting of all elements of G fixing ζ . Clearly, if ζ is the fixed point of a parabolic element of G , then we have that any horoball at ζ is invariant under $\text{Stab}_G(\zeta)$. Moreover, there is a constant $c > 0$ such that $g(H_\zeta(c)) \cap H_\zeta(c) = \emptyset$, for every $g \in G \setminus \text{Stab}_G(\zeta)$. A cusped region for ζ is given by the quotient $H_\zeta(c)/\text{Stab}_G(\zeta)$ which we assume to be embedded in the hyperbolic orbifold $M_G = \mathbb{D}/G$.

The *convex hull* $C(L(G))$ is the smallest closed convex subset of \mathbb{D} whose Euclidean closure in $\overline{\mathbb{D}}$ contains $L(G)$. Equivalently, $C(L(G))$ is the hyperbolic convex hull of the union of all geodesic lines with endpoints in $L(G)$. Clearly, the set $C(L(G))$ is invariant under G , and hence, $C_G = C(L(G))/G$ is a closed subset of $M_G = \mathbb{D}/G$. The set C_G is called the *convex core* of G or of M_G . A Kleinian group G is called *geometrically finite* if there exists $\varepsilon > 0$ such that the hyperbolic ε -neighbourhood of C_G has finite hyperbolic volume. We say that a parabolic fixed point ζ of G is *bounded* if the intersection of C_G and $h_\zeta(c)/\text{Stab}_G(\zeta)$ is compact. A theorem of Beardon and Maskit [3] asserts that G is geometrically finite if and only if the limit set $L(G)$ is equal to the union of $L_r(G)$ and the set of bounded parabolic fixed points.

Let us also recall the following well-known basic formulae (see e.g. [14]). Let $j(g, z)$ denote the *conformal derivative* of $g \in \text{Con}(N)$ at $z \in \overline{\mathbb{D}}$; that is,

$$j(g, z) = \frac{1 - |g^{-1}(0)|^2}{|z^* - g^{-1}(0)|^2 |z|^2},$$

where z^* denotes the image of z under the reflection at \mathbb{S} . An elementary calculation gives that if $\zeta \in \mathbb{S}$, then

$$j(g, \zeta) = \frac{1 - |g^{-1}(0)|^2}{|\zeta - g^{-1}(0)|^2},$$

and hence, $j(g, \zeta) = k(g^{-1}(0), \zeta)$. Also, if $z \in \mathbb{D}$, then one finds

$$j(g, z) = \frac{1 - |g(z)|^2}{1 - |z|^2}.$$

For $z_1, z_2 \in \mathbb{D}$, we define $d_\zeta(z_1, z_2)$ to be the *signed hyperbolic distance* between the horosphere at ζ through z_1 and the horosphere at ζ through z_2 , where ‘signed’ refers to that $d_\zeta(z_1, z_2)$ is positive if z_2 is contained in the horoball bounded by the horosphere at ζ through z_1 , and is negative otherwise. Clearly, $|d_\zeta(z_1, z_2)| \leq d(z_1, z_2)$. It is well known that the relation between this horospherical distance and the Poisson kernel is given by

$$d_\zeta(z, 0) = -\log k(z, \zeta).$$

The *Poincaré series* associated with a Kleinian group G is defined for $z \in \mathbb{D}$ and $s \in \mathbb{R}^+$ by

$$P(z, s) = \sum_{g \in G} j(g, z)^s.$$

Note that convergence and divergence of $P(z, s)$ does not depend on the choice of $z \in \mathbb{D}$. The *exponent of convergence* of G is then given by

$$\delta(G) = \inf\{s \in \mathbb{R}^+ : P(z, s) < \infty\}.$$

If $P(z, s)$ converges at $s = \delta(G)$; that is, if $P(z, \delta(G)) < \infty$, then G is said to be of *convergence type*. Otherwise, G is said to be of *divergence type*. Note that in [5] it was shown that if G is some arbitrary non-elementary Kleinian group, then the Hausdorff dimension of $L_r(G)$ and of $L_{ur}(G)$ are both equal to $\delta(G)$ (see also [17]).

Also, recall that a Borel probability measure μ on $\overline{\mathbb{D}}$ is called *s-conformal* for a Kleinian group G if, for each $g \in G$ and any measurable set A on $\overline{\mathbb{D}}$,

$$\mu(g(A)) = \int_A j(g, z)^s d\mu(z).$$

This condition is equivalent to that for each continuous function f on $\overline{\mathbb{D}}$ and for each $g \in G$, we have

$$\int_{\mathbb{D}} f(g^{-1}(z)) d\mu(z) = \int_{\mathbb{D}} f(z)j(g, z)^s d\mu(z).$$

For a Kleinian group G , the *big horospherical limit set* $L_H(G)$ consists of all points $\zeta \in L(G)$ with the property that there is a constant $c > 0$ such that the orbit $G(0)$ accumulates at ζ within the horoball $H_\zeta(c)$. It is easy to see that $L_r(G) \subset L_H(G)$ and that parabolic fixed points of G are contained in $L_H(G)$.

It is known that a Kleinian group G acts *conservatively* on $L_H(G)$ with respect to any conformal measure μ for G (see Tukia [22]). This means that there are infinitely many elements $g \in G$ such that $\mu(A \cap g(A)) > 0$, for each measurable set $A \subset L_H(G)$ for which $\mu(A) > 0$. On the other hand, Tukia [22] also showed that G acts *dissipatively* on the complement $\mathbb{S} \setminus L_H(G)$; that is, there exists a measurable fundamental set $E \subset \mathbb{S} \setminus L_H(G)$ such that $G(\zeta) \cap E$ is a singleton, for each $\zeta \in \mathbb{S} \setminus L_H(G)$. These results provide significant generalisations of a theorem of Sullivan [20], who derived these assertions for the special case in which μ is equal to the N -dimensional Lebesgue measure m . Next, recall that the *big Dirichlet set* $D(G)$ of G is equal to the set of all points that lie in the boundary at infinity of some Dirichlet fundamental domain. Clearly, $D(G)$ can be written as a disjoint union as follows:

$$D(G) = \Omega(G) \cup L_J(G).$$

It is known that $D(G) \cup L_H(G) = \mathbb{S}$ and that $m(D(G) \cap L_H(G)) = 0$ (see [11]).

Finally, let us also recall from [1] that a connected open subset E of $M_G = \mathbb{D}/G$ is called an *end* of M_G , if ∂E is compact and non-empty and if E has non-compact closure in M_G . For a suitable index set I , let $\{E_i^0 : i \in I\}$ denote the set of all connected components of the lift of E to \mathbb{D} , and let G_i denote the stabiliser $\text{Stab}_G(E_i^0)$ of E_i^0 in G . The set E_i^0 is called an *end of G* , and G_i the corresponding *end group of G* . By construction, the G_i are subgroups of G which are all conjugate to each other. A point $\zeta \in \mathbb{S}$ is called an *endpoint of an end E_i^0 of G* if, whenever β is a hyperbolic ray towards ζ , there exists a subray β' of β that is contained in E_i^0 , for which the hyperbolic distance $d(z, \partial E_i^0)$ tends to infinity as z approaches ζ on β' . Note that the set of endpoints of E_i^0 has empty intersection with $L_r(G)$. If additionally $\zeta \in L(G_i)$, then ζ is called the *end limit point of E_i^0* . Note that the set of end limit points of E_i^0 is contained in $L(G_i) \setminus L_r(G)$. The union of all end limit points of all ends E_i^0 , $i \in I$, shall be referred to as the *end limit set of G associated with E* , or simply as the *end limit set of E* .

So far, ends have been subsets E of M_G (or of the lift to \mathbb{D} of such a set) such that E is a non-compact component of $M_G \setminus \partial E$, where the boundary ∂E of E in M_G is compact. Note that the definition of an end can also be given in terms of $\overline{M}_G = (\mathbb{D} \cup \Omega(G))/G$. Namely, in these terms, E is a non-compact component of $\overline{M}_G \setminus \partial E$, where the boundary ∂E taken in \overline{M}_G is compact. In [1] such a set E was called an ‘end with boundary’. Note that for ends with boundary one has by [1, Theorem 4.4] that there exists a non-trivial conformal measure supported by the end limit point set of an end with boundary. Moreover, [1, Proposition 4.5] then allows us to extend this conformal measure for an end group to a conformal measure for G . We shall make use of this observation in the proof of Proposition 6.2 of this paper. For further details on end limit sets and ends with boundary we refer to [1] and [8].

3. THE CONSTRUCTION OF CONFORMAL ENDING MEASURES

In order to recall the construction of an s -conformal ending measure, let F be a Dirichlet fundamental domain of some given Kleinian group G , and let \overline{F} be its closure in $\overline{\mathbb{D}}$. A sequence (z_n) of elements $z_n \in F$ is called an *ending sequence* tending to $\zeta \in \overline{F} \cap \mathbb{S}$, if each of the z_n lies within a bounded distance from some geodesic ray towards ζ contained in F , and if the sequence (z_n) tends to ζ with respect to the Euclidean metric. Let $s \in \mathbb{R}^+$ be chosen such that $P(z, s) < \infty$. For a given ending sequence (z_n) tending to $\zeta \in \overline{F} \cap \mathbb{S}$, we define, for each $n \in \mathbb{N}$, the point mass μ_n on the closed unit ball $\overline{\mathbb{D}}$ as follows:

$$\mu_n = \frac{1}{P(z_n, s)} \sum_{g \in G} j(g, z_n)^s \mathbf{1}_{g(z_n)}.$$

The Helly-Bray Theorem then ensures that, by passing to a subsequence if necessary, the sequence (μ_n) has a weak limit measure μ , and we will refer to this measure as the *s -conformal ending measure*. By construction, for each of the μ_n we have, for any $g \in G$ and for any continuous function f on $\overline{\mathbb{D}}$,

$$\begin{aligned} \int_{\overline{\mathbb{D}}} f(g^{-1}(z)) d\mu_n(z) &= \frac{1}{P(z_n, s)} \sum_{h \in G} f(g^{-1}h(z_n)) j(h, z_n)^s \\ &= \frac{1}{P(z_n, s)} \sum_{h \in G} f(h(z_n)) j(gh, z_n)^s \\ &= \frac{1}{P(z_n, s)} \sum_{h \in G} f(h(z_n)) j(g, h(z_n))^s j(h, z_n)^s \\ &= \int_{\overline{\mathbb{D}}} f(z) j(g, z)^s d\mu_n(z). \end{aligned}$$

Since (μ_n) converges weakly to μ , it therefore follows that μ satisfies the same s -conformal transformation rule. Therefore, we have, for all $g \in G$ and $A \subset \overline{\mathbb{D}}$ measurable,

$$\mu(g(A)) = \int_A j(g, z)^s d\mu.$$

Finally, let us remark that if $L_J(G)$ is non-empty and if μ is an s -conformal ending measure derived from an ending sequence (z_n) tending towards some $\zeta \in L_J(G)$, then the support of μ has to be equal to $L(G)$. Indeed, consider any open set $U \subset \overline{\mathbb{D}}$ whose closure \overline{U} is contained in $\overline{\mathbb{D}} \setminus L(G)$. The discontinuous action of G on $\overline{\mathbb{D}} \setminus L(G)$ immediately implies that there exist at most finitely many $g \in G$ such that $g(F) \cap U \neq \emptyset$. Furthermore, we have that $g(\overline{F}) \cap \overline{U}$ is compact in $\overline{\mathbb{D}} \setminus L(G)$, for each $g \in G$. Since the sequence (z_n) used in the definition of μ_n was chosen to be an ending sequence contained in F , it then follows that there exists $n_U \in \mathbb{N}$ such that $\mu_n(U) = 0$, for all $n \geq n_U$. This shows that $\mu(U) = 0$, and hence, μ is supported on $L(G)$.

4. CONVERGENCE AT INFINITY AND PURELY ATOMIC MEASURES

Throughout, let G be a Kleinian group. As already mentioned briefly in the introduction, extensions of the Poincaré series $P(z, s)$ to points in \mathbb{S} gives rise to what we call the *horospherical Poincaré series* of G . More precisely, the horospherical

Poincaré series $\mathcal{P}(\zeta, s)$ of G at $\zeta \in \mathbb{S}$ with exponent $s \in \mathbb{R}^+$ is given by

$$\mathcal{P}(\zeta, s) = \sum_{g \in G} j(g, \zeta)^s.$$

Note that Morosawa [13] also considered this type of horospherical Poincaré series and obtained basic results concerning atoms of conformal measures, some of which will be generalised in this paper. Also, we refer to [14] (Section 3.5) for some basic facts concerning the atomic part of a conformal measure.

Let us first prove the following two elementary lemmata. These will turn out to be useful later.

Lemma 4.1. *If $\mathcal{P}(\zeta, s) < \infty$ for some $\zeta \in \mathbb{S}$, then $P(z, s) < \infty$ for every $z \in \mathbb{D}$. In particular, we hence have that if $\mathcal{P}(\zeta, s) < \infty$ for some $\zeta \in \mathbb{S}$, then $s \geq \delta(G)$. Moreover, if $P(z, s) < \infty$ for some (and hence for all) $z \in \mathbb{D}$, then $\mathcal{P}(\zeta, s) < \infty$ for each $\zeta \in \Omega(G)$.*

Proof. Using the identities for the conformal derivative $j(g, \cdot)$ stated in Section 2, one immediately verifies that for $\zeta \in \mathbb{S}$ and $z \in \mathbb{D}$ we have

$$j(g, \zeta) = \frac{|z^* - g^{-1}(0)|^2 |z|^2}{|\zeta - g^{-1}(0)|^2} j(g, z) \geq \frac{(1 - |z|)^2}{4} j(g, z).$$

This gives the first statement in the lemma. For the second statement note that if $\zeta \in \Omega(G)$, then $|\zeta - g^{-1}(0)|^2 \geq c$, for some constant $c > 0$ not depending on $g \in G$. Hence,

$$j(g, \zeta) \leq \frac{(1 + |z|)^2}{c} j(g, z),$$

which gives the second statement in the lemma. \square

Lemma 4.2. *Let (z_n) be a sequence in \mathbb{D} which converges to $\zeta \in \mathbb{S}$. Then the following hold:*

- (i) *We always have $\liminf_{n \rightarrow \infty} P(z_n, s) \geq \mathcal{P}(\zeta, s)$, and therefore, if $\mathcal{P}(\zeta, s)$ diverges, then $\lim_{n \rightarrow \infty} P(z_n, s) = \infty$.*
- (ii) *If $\zeta \in \Omega(G)$, then $\lim_{n \rightarrow \infty} P(z_n, s) = \mathcal{P}(\zeta, s)$.*
- (iii) *If (z_n) converges radially to $\zeta \in \mathbb{S}$, then $\lim_{n \rightarrow \infty} P(z_n, s) = \mathcal{P}(\zeta, s)$.*

Proof. The statement in (i) is an immediate consequence of Fatou's Lemma. For the remaining statements, we can assume without loss of generality that $\mathcal{P}(\zeta, s)$ converges. If $\zeta \in \Omega(G)$, then clearly $|\zeta - g^{-1}(0)|^2$ is uniformly bounded away from 0, for all $g \in G$. Therefore, since $|z_n^* - g^{-1}(0)|^2$ tends to $|\zeta - g^{-1}(0)|^2$ for n tending to infinity, there exists a constant $c_1 > 0$ such that, for each $g \in G$ and every $n \in \mathbb{N}$,

$$|z_n^* - g^{-1}(0)|^2 |z_n|^2 \geq c_1 |\zeta - g^{-1}(0)|^2.$$

Hence, $j(g, z_n)^s \leq c_1^{-s} j(g, \zeta)^s$. For the remaining assertion, assume that z_n converges radially to ζ , in the sense that each z_n is at most a uniformly bounded distance away from some geodesic ray ending at ζ . This immediately implies that the angle θ at ζ in the Euclidean triangle with vertices $g^{-1}(0)$, ζ and z_n^* is uniformly bounded away from zero. An application of the Euclidean sine rule then gives us that $|\zeta - g^{-1}(0)| \leq (1/\sin \theta) |z_n^* - g^{-1}(0)|$, for all $n \in \mathbb{N}$ and $g \in G$. Thus, there exists a constant $c_2 > 0$ such that, for each $g \in G$ and every $n \in \mathbb{N}$,

$$|z_n^* - g^{-1}(0)|^2 |z_n|^2 \geq c_2 |\zeta - g^{-1}(0)|^2.$$

This implies that $j(g, z_n)^s \leq c_2^{-s} j(g, \zeta)^s$. Since we have by definition that $j(g, z_n)$ tends to $j(g, \zeta)$ for n tending to infinity, the dominated convergence theorem now gives in the situation of (ii) and of (iii) that $P(z_n, s)$ tends to $\mathcal{P}(\zeta, s)$ for n tending to infinity. \square

The following theorem shows that convergence and divergence of the horospherical Poincaré series is crucial for finding out whether s -conformal ending measures have atoms or not.

Theorem 4.3. *Let (z_n) be a given ending sequence which tends to ζ , for some $\zeta \in D(G) = \Omega(G) \cup L_J(G)$ not fixed by any non-trivial element of G . Also, suppose that $\mathcal{P}(\zeta, s)$ converges for $s \geq \delta(G)$. We then have that the essential support of the associated s -conformal ending measure μ is equal to $G(\zeta)$. In particular, μ is purely atomic and coincides with*

$$\mu_\zeta = \frac{1}{\mathcal{P}(\zeta, s)} \sum_{g \in G} j(g, \zeta)^s \mathbf{1}_{g(\zeta)}.$$

Proof. Let $K = \{z_1, z_2, \dots\} \cup \{\zeta\}$ and consider

$$\mu_n = \frac{1}{P(z_n, s)} \sum_{g \in G} j(g, z_n)^s \mathbf{1}_{g(z_n)}.$$

Clearly K is a compact subset of $\overline{\mathbb{D}}$. Combining this with the observation that $\limsup_{n \rightarrow \infty} \mu_n(C) \leq \mu(C)$ for each $C \subset \overline{\mathbb{D}}$ compact, which is an immediate consequence of weak convergence, we obtain

$$\begin{aligned} 1 &\geq \mu(G(\zeta)) = \sum_{h \in G} \mu(h(K)) \geq \sum_{h \in G} \limsup_{n \rightarrow \infty} \mu_n(h(K)) \\ &= \sum_{h \in G} \limsup_{n \rightarrow \infty} \frac{j(h, z_n)^s}{P(z_n, s)}. \end{aligned}$$

By Lemma 4.2, we have

$$\lim_{n \rightarrow \infty} P(z_n, s) = \mathcal{P}(\zeta, s).$$

Since $\mathcal{P}(\zeta, s)$ converges by assumption, the last term in the above chain of inequalities is equal to 1. This shows that $\mu(G(\zeta)) = 1$, and thus the probability measure μ is essentially supported on the countable set $G(\zeta)$. \square

Note that the statement of Theorem 4.3 includes the case in which $\zeta \in \Omega(G)$. In this case the resulting measure $\mu_\zeta = \frac{1}{\mathcal{P}(\zeta, s)} \sum_{g \in G} j(g, \zeta)^s \mathbf{1}_{g(\zeta)}$ is a conformal measure in the sense we use here (see Section 2), although strictly speaking it is not supported on the limit set of G .

Next we consider the case in which ζ is a parabolic fixed point of G . In this situation we have that the horospherical Poincaré series $\mathcal{P}(\zeta, s)$ at ζ diverges, for every $s \in \mathbb{R}$. This follows, since $j(g, \zeta) = e^{-d_\zeta(g^{-1}(0), 0)}$ is constant for infinitely many $g \in G$. In this situation we then consider the *reduced horospherical Poincaré series*

$$\mathcal{P}_{\text{red}}(\zeta, s) = \sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s,$$

and if this series converges, then we have that the associated s -conformal measure

$$\mu_\zeta = \frac{1}{\mathcal{P}_{\text{red}}(\zeta, s)} \sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s \mathbf{1}_{g(\zeta)}$$

is purely atomic. Note that μ_ζ is indeed an s -conformal measure for G , as a computation similar to the one in Section 3 readily shows. For the case in which ζ is a bounded parabolic fixed point, the following lemma shows that if the Poincaré series converges, then the reduced horospherical Poincaré series converges as well. Therefore, if $s > \delta(G)$ or if $s = \delta(G)$ for G of convergence type, then we always have that there exists a purely atomic s -conformal measure essentially supported on $G(\zeta)$.

Lemma 4.4. *For ζ a bounded parabolic fixed point of a Kleinian group G , we have that if $P(z, s)$ converges for $s \geq \delta(G)$, then so does $\mathcal{P}_{\text{red}}(\zeta, s)$.*

Proof. Let $c > 0$ be chosen such that $H_\zeta(c)/\text{Stab}_G(\zeta)$ is a cusped region in M_G . Without loss of generality, we may assume $0 \in h_\zeta(c) \cap C(L(G))$. Since ζ is a bounded parabolic fixed point, there exists a constant $b > 0$ such that $d(0, (\text{Stab}_G(\zeta))(z)) \leq b$, for each $z \in h_\zeta(c) \cap C(L(G))$. Then choose representatives $\gamma \in G$ of the cosets in $G/\text{Stab}_G(\zeta)$ such that

$$d(0, \gamma(0)) = \min\{d(0, g(0)) : g \in \gamma \text{Stab}_G(\zeta)\}.$$

With this choice, we then have, for each of these coset representatives γ ,

$$e^{-d_\zeta(\gamma^{-1}(0), 0)} \leq e^b e^{-d(\gamma^{-1}(0), 0)}.$$

Since $j(g, \zeta) = e^{-d_\zeta(\gamma^{-1}(0), 0)}$ for every $g \in \gamma \text{Stab}_G(\zeta)$, it follows that

$$\mathcal{P}_{\text{red}}(\zeta, s) = \sum_{\gamma \in G/\text{Stab}_G(\zeta)} e^{-sd_\zeta(\gamma^{-1}(0), 0)} \leq e^{sb} \sum_{\gamma \in G/\text{Stab}_G(\zeta)} e^{-sd(\gamma^{-1}(0), 0)} \leq e^{sb} P(0, s).$$

This finishes the proof of the lemma. □

We now look at the more general situation in which there exists a purely atomic s -conformal measure essentially supported on $G(\zeta)$ for some point $\zeta \in \mathbb{S}$. Note that the measures considered in the following proposition are conformal measures which are not necessarily ending measures as introduced in Section 3.

Proposition 4.5. *Let G be a Kleinian group, and let $\zeta \in \mathbb{S}$ and $s \in \mathbb{R}^+$ be fixed. Then there exists a purely atomic s -conformal measure μ_ζ essentially supported on $G(\zeta)$ if and only if*

$$j(h, \zeta) = 1 \text{ for each } h \in \text{Stab}_G(\zeta), \text{ and } \mathcal{P}_{\text{red}}(\zeta, s) < \infty.$$

Proof. Without loss of generality we may assume that G has no elliptic elements. This follows mainly since $j(h, \zeta) = 1$ for any fixed point ζ of an elliptic element $h \in G$, and since $\text{Stab}_G(\zeta)$ has a subgroup of finite index with no elliptic elements. The details are left to the reader.

If $j(h, \zeta) = 1$, for each $h \in \text{Stab}_G(\zeta)$, and if the reduced horospherical Poincaré series converges, then μ_ζ is a purely atomic conformal measure. Here, μ_ζ is defined as prior to Lemma 4.4. This gives one direction of the equivalence. For the reverse direction, assume that there exists a purely atomic conformal measure μ essentially supported on $G(\zeta)$. Let us split the argument into the two cases: $\text{Stab}_G(\zeta)$ is trivial and $\text{Stab}_G(\zeta)$ is non-trivial. If $\text{Stab}_G(\zeta)$ is trivial, then the s -conformality

of μ implies that $\mu(\{g\zeta\}) = j(g, \zeta)^s \mu(\{\zeta\})$. Thus, $\mu = \sum_{g \in G} j(g, \zeta)^s \mu(\{\zeta\}) \mathbf{1}_{g(\zeta)}$, and therefore, since μ is a probability measure,

$$\sum_{g \in G} j(g, \zeta)^s = 1/\mu(\{\zeta\}) < \infty.$$

If $\text{Stab}_G(\zeta)$ is non-trivial, then ζ is either a parabolic or a loxodromic fixed point. If ζ is a loxodromic fixed point of G , then a conformal measure cannot have an atom at ζ . This follows, since $j(h, \zeta) \neq 1$ for each loxodromic element $h \in G$ fixing ζ . Therefore, ζ has to be a parabolic fixed point. In this situation we have that $j(h, \zeta) = 1$, for each $h \in \text{Stab}_G(\zeta)$. Arguing in a similar way as in the trivial stabiliser case, it follows that the reduced horospherical Poincaré series converges. \square

Combining Theorem 4.3 and Proposition 4.5, the statement of Theorem 4.3 can be extended to the case in which $\text{Stab}_G(\zeta)$ is non-trivial. The proof is straightforward and left to the reader.

Theorem 4.6. *For a Kleinian group G , let (z_n) be an ending sequence converging to $\zeta \in D(G)$, and let μ be the associated s -conformal ending measure. Then μ has an atom at ζ if and only if*

$$j(h, \zeta) = 1 \text{ for each } h \in \text{Stab}_G(\zeta), \text{ and } \mathcal{P}_{red}(\zeta, s) < \infty.$$

In case the latter is satisfied, we then in particular have that μ coincides with the purely atomic s -conformal measure

$$\mu_\zeta = \frac{1}{\mathcal{P}_{red}(\zeta, s)} \sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s \mathbf{1}_{g(\zeta)},$$

which is essentially supported on $G(\zeta)$.

Let us also briefly mention another way to produce purely atomic conformal ending measures. For this, let us assume that the end limit set associated to some end of a hyperbolic manifold $M_G = \mathbb{D}/G$ is countable, for some Kleinian group G . Here, the notions of end and end limit set are adopted from [1] (see also Section 2). Theorem 4.4 in [1] then ensures that the type of ending measure constructed in the previous section is essentially supported on the corresponding end limit set, and hence, this measure is necessarily purely atomic. Recall the basic definitions concerning ends of hyperbolic manifolds given in Section 2, in particular the notion of end limit set of G associated with some end E of M_G .

Proposition 4.7. *Let G be a Kleinian group such that M_G has an end E . Furthermore, assume that the end limit set associated with E is countable. Then every conformal ending measure obtained from any ending sequence converging to any end limit point of any end E is purely atomic.*

Bishop [4] improved a classical result of Beardon and Maskit [3], by showing that a Kleinian group G acting on hyperbolic 3-space is geometrically finite if and only if $L(G) \setminus L_r(G)$ is countable (possibly empty). It is not difficult to see that this result continues to hold for Fuchsian groups acting on hyperbolic 2-space. Then, a combination of this result of Bishop and Proposition 4.7 strongly supports the following conjecture. (Note that this conjecture is certainly true in two and three dimensions.)

Conjecture. *The end limit set of an end is countable if and only if it consists of finitely many orbits of parabolic fixed points.*

Remark 4.8. Note that if the latter conjecture holds, then this would imply that the class of examples mentioned prior to Lemma 4.4 represents the only class of purely atomic conformal ending measures emerging from Proposition 4.7.

5. CONFORMAL ENDING MEASURES AND THE BIG HOROSPHERICAL LIMIT SET

In this section we show that if ζ is not fixed by any parabolic element of a Kleinian group G and if ζ is an atom of some conformal ending measure for G , then ζ lies in the dissipative part of the action of G on \mathbb{S} . Moreover, we will see that if the dissipative part of $L(G)$ has positive mass for some conformal measure, then there exists an atomic conformal ending measure for G .

Recall that a Kleinian group G is said to act *dissipatively*, on some measurable G -invariant set A , if there exists a measurable set $E \subset A$ such that $E \cap G(\zeta)$ is a singleton, for each $\zeta \in A$. Note that the dissipative action is defined independently of conformal measures. Let $L_H(G)$ be the big horospherical limit set of G and consider the complement $\mathbb{S} \setminus L_H(G)$, which is clearly G -invariant and contained in the big Dirichlet set $D(G) = \Omega(G) \cup L_J(G)$. In [22], Tukia showed that G acts dissipatively on $\mathbb{S} \setminus L_H(G)$. Moreover, in [22] it was also shown that if G acts dissipatively on a G -invariant set A , then A is contained in $\mathbb{S} \setminus L_H(G)$ up to parabolic fixed points of G and up to sets of measure zero, for any conformal measure for G . In this sense, we can say that $\mathbb{S} \setminus L_H(G)$ is the maximal subset of \mathbb{S} on which G acts dissipatively, and therefore we can say that $\mathbb{S} \setminus L_H(G)$ represents the dissipative part of the action of G on \mathbb{S} .

Proposition 5.1. *For a Kleinian group G we have that if $\zeta \in L_H(G)$ is not a parabolic fixed point of G , then ζ is not an atom of μ , for any s -conformal measure μ for G .*

Proof. If $\zeta \in L_H(G)$, then there is a sequence (g_n^{-1}) of elements $g_n^{-1} \in G$ such that $(g_n^{-1}(0))$ converges to ζ within a horoball $H_\zeta(c)$ for some $c > 0$. In this case, we have $j(g_n, \zeta) > c$, for all $n \in \mathbb{N}$. Combining this with Proposition 4.5, it follows that ζ can be an atom only if $\text{Stab}_G(\zeta)$ is infinite and if $j(h, \zeta) = 1$, for every $h \in \text{Stab}_G(\zeta)$. However, this is only possible if ζ is a parabolic fixed point of G . \square

Remark 5.2. Note that the intersection $L_J(G) \cap L_H(G)$ can contain a point ζ which is not a parabolic fixed point. In this case, the horospherical Poincaré series $\mathcal{P}(\zeta, s)$ at ζ diverges for every exponent $s \in \mathbb{R}^+$. Indeed, if it would converge, then Theorem 4.3 guarantees the existence of a purely atomic s -conformal ending measure μ with an atom at ζ , contradicting Proposition 5.1. An example of such a point $\zeta \in L_J(G) \cap L_H(G)$ can be given by taking a normal subgroup G of some Kleinian group Γ . By choosing a suitable hyperbolic element $\gamma \in \Gamma$ with $G \cap \langle \gamma \rangle = \{\text{id.}\}$, we can guarantee on the one hand that the fixed point ζ of γ is contained in the boundary at infinity of some Dirichlet fundamental domain of G . On the other hand, by a result of [10] we have that $\zeta \in L_r(\Gamma) \subset L_H(G)$.

The following proposition shows that if there exists a conformal measure which gives positive mass to the dissipative part, then there also exists a purely atomic conformal ending measure. This result relies on Tukia’s results mentioned above.

Proposition 5.3. *If there exists an s -conformal measure μ for G such that $\mu(A \setminus L_H(G)) > 0$, for some G -invariant measurable set A , then there exists a purely atomic s -conformal ending measure ν for G essentially supported on A such that each atom of ν is not fixed by any parabolic element of G .*

Proof. Let A be G -invariant and measurable such that $\mu(A \setminus L_H(G)) > 0$. By [22], we then have that there exists a measurable set $E \subset A \setminus L_H(G)$ such that $G(\zeta)$ has exactly one point in E , for each $\zeta \in A \setminus L_H(G)$. Using $\mu(A \setminus L_H(G)) > 0$, it follows that $\mu(E) > 0$. Define a measurable subset E_n , for each positive integer n , by

$$E_n = \{\zeta \in E : \#\text{Stab}_G(\zeta) \leq n\}.$$

Remark that non-elliptic fixed points belong to $L_H(G)$. Since $E = \bigcup_{n=1}^\infty E_n$ and $\mu(E) > 0$, there exists some n such that $\mu(E_n) > 0$. Hence,

$$1 \geq \mu(A \setminus L_H(G)) \geq \mu\left(\bigcup_{g \in G} g(E_n)\right) \geq \frac{1}{n} \sum_{g \in G} \mu(g(E_n)).$$

Since μ is an s -conformal measure for G , the final term in the latter chain of inequalities is equal to

$$\frac{1}{n} \sum_{g \in G} \int_{E_n} j(g, \zeta)^s d\mu(\zeta) = \frac{1}{n} \int_{E_n} \mathcal{P}(\zeta, s) d\mu(\zeta).$$

This implies that $\mathcal{P}(\zeta, s)$ converges, for μ -almost every $\zeta \in E_n$. In particular, there exists some $\xi \in E_n$ such that $\mathcal{P}_{\text{red}}(\xi, s) < \infty$. Since $\xi \in D(G)$, Theorem 4.6 shows that the measure

$$\nu = \frac{1}{\mathcal{P}_{\text{red}}(\xi, s)} \sum_{g \in G/\text{Stab}_G(\xi)} j(g, \xi)^s \mathbf{1}_{g(\xi)}$$

is a purely atomic s -conformal ending measure. In particular, since $\text{Stab}_G(\xi)$ is finite, we have that ξ is not fixed by any parabolic element of G . \square

We now summarize the two previous propositions in the following theorem.

Theorem 5.4. *For a Kleinian group G and for $s \geq \delta(G)$, the following are equivalent.*

- (i) *Every s -conformal ending measure for G supported on $L(G)$ is essentially supported on $L_H(G)$.*
- (ii) *Every s -conformal ending measure for G supported on $L(G)$ can have atoms only at parabolic fixed points of G .*

Proof. Let $\zeta \in L(G)$ be given such that ζ is not a parabolic fixed point of G . Suppose that there exists a purely atomic s -conformal ending measure μ for G essentially supported on $G(\zeta)$. By Proposition 5.1, we then have that ζ does not belong to $L_H(G)$, and hence, μ is essentially supported on $L(G) \setminus L_H(G)$. This shows that (i) implies (ii).

For the reverse implication, suppose that there is an s -conformal ending measure μ for G such that $\mu(L(G) \setminus L_H(G)) > 0$. By applying Proposition 5.3 to the situation where $A = L(G)$, it follows that there exists a purely atomic s -conformal ending measure for G supported on $L(G)$ whose atoms are not parabolic fixed points of G . \square

6. EXAMPLES OF CONFORMAL ENDING MEASURES

In this section we discuss three non-trivial examples of purely atomic, as well as non-atomic conformal ending measures. The first example considers infinitely generated Schottky groups which admit purely atomic conformal ending measures with atoms at Jørgensen points. The second example looks at normal subgroups of finitely generated Schottky groups. We show that for groups of this type there can be an uncountable family of mutually singular non-atomic conformal ending measures. The third example considers a Kleinian group Γ which admits a purely atomic $\delta(\Gamma)$ -conformal ending measure at a Jørgensen point which is not a parabolic fixed point.

Example 1. Atomic measures at Jørgensen points of infinitely generated Schottky groups.

We consider a particular class of infinitely generated classical Schottky groups (see also [18]). In order to define these groups, let us fix some increasing function $\phi : \mathbb{N} \rightarrow \mathbb{R}^+$ such that ϕ is compatible with the following inductive construction of the set $\mathcal{C} = \{C_n^\epsilon : \epsilon \in \{+, -\}, n \in \mathbb{N}\}$ of pairwise disjoint open N -dimensional discs C_n^\pm in \mathbb{S} with radii r_n^\pm . For ease of exposition, let us assume that $r_n := r_n^+ = r_n^- < 1$, for all $n \in \mathbb{N}$. With $C_n^{\pm, \phi} (\supset C_n^\pm)$ denoting the disc concentric to C_n^\pm of radius $\phi(n)r_n$, we now choose the elements in \mathcal{C} by way of induction as follows. Let C_1^+ and C_1^- be chosen such that $C_1^{+, \phi} \cap C_1^{-, \phi} = \emptyset$. For the inductive step, assume that $C_1^+, C_1^-, \dots, C_n^+, C_n^-$ have been constructed for some $n \in \mathbb{N}$. Then choose C_{n+1}^+ and C_{n+1}^- such that

$$C_{n+1}^{+, \phi} \cap C_{n+1}^{-, \phi} = \emptyset \text{ and } C_{n+1}^{\epsilon, \phi} \cap C_k^{\epsilon, \phi} = \emptyset \text{ for all } k \in \{1, \dots, n\}, \epsilon \in \{+, -\}.$$

With these discs at hand, we then define hyperbolic transformations $g_n^\epsilon \in \text{Con}(N)$, for each $n \in \mathbb{N}$ and $\epsilon \in \{+, -\}$, such that $g_n^- = (g_n^+)^{-1}$ and

$$g_n^+ (\text{Ext}(C_n^+)) = \text{Int}(C_n^-) \text{ and } g_n^- (\text{Ext}(C_n^-)) = \text{Int}(C_n^+).$$

Here, Int and Ext denote respectively the interior and exterior in the topology of \mathbb{S} . Then let G^ϕ be the group generated by the set

$$G_1^\phi = \{g_n^\epsilon : \epsilon \in \{+, -\}, n \in \mathbb{N}\}.$$

One easily verifies that $\bigcap_{C_n^\epsilon \in \mathcal{C}} \text{Ext}(C_n^\epsilon)$ is a fundamental domain for the action of G^ϕ on \mathbb{S} (cf. [9], Proposition VIII.A.4), which shows in particular that G^ϕ is a Kleinian group. Note that the set $L_J(G^\phi)$ of Jørgensen limit points contains the accumulation points of $\{C_n^\epsilon \in \mathcal{C}\}$. Then let $L'_J(G^\phi)$ denote the subset of $L_J(G^\phi)$ given by

$$L'_J(G^\phi) = L_J(G^\phi) \cap \bigcap_{\epsilon \in \{+, -\}, n \in \mathbb{N}} \text{Ext}(C_n^{\epsilon, \phi}).$$

Clearly, by construction we have that $L'_J(G^\phi) \neq \emptyset$.

Proposition 6.1. *Let $s \in \mathbb{R}^+$ be given, and let $\phi_s : \mathbb{N} \rightarrow [2, \infty)$ be an increasing function such that $\sum_{n \in \mathbb{N}} (4/\phi_s(n))^{2s} < 1/2$. We then have that the horospherical Poincaré series for G^{ϕ_s} at ζ converges at the exponent s , for each $\zeta \in L'_J(G^{\phi_s})$. Moreover, the essential support of the associated s -conformal ending measure μ_ζ is equal to $G^{\phi_s}(\zeta)$.*

Proof. First note that for each $\zeta \in \text{Ext}(C_n^{\epsilon, \phi_s})$, $g_n^\epsilon \in G_1^{\phi_s}$ and $\epsilon \in \{+, -\}$, we have that

$$|\zeta - (g_n^\epsilon)^{-1}(0)| \geq \phi_s(n)r_n - r_n = (\phi_s(n) - 1)r_n.$$

Using this together with $\phi_s(n) \geq 2$ and the fact that $1 - |g_n^\epsilon(0)| \leq 2r_n^2$, it follows, for each $\zeta \in \text{Ext}(C_n^{\epsilon, \phi_s})$,

$$j(g_n^\epsilon, \zeta) = \frac{1 - |g_n^\epsilon(0)|^2}{|(g_n^\epsilon)^{-1}(0) - \zeta|^2} \leq \frac{2(1 - |g_n^\epsilon(0)|)}{(\phi_s(n) - 1)^2 r_n^2} \leq \frac{4}{(\phi_s(n) - 1)^2} \leq \left(\frac{4}{\phi_s(n)}\right)^2.$$

Then note that every element $\gamma \in G^{\phi_s}$ is uniquely represented by a reduced word $g_{n_1}^{\epsilon_1} \cdots g_{n_k}^{\epsilon_k}$ of elements in $G_1^{\phi_s}$. As usual, k will be referred to as the word length of γ . With $G_k^{\phi_s}$ denoting the set of elements of G^{ϕ_s} of word length $k \in \mathbb{N}$, we then obtain for $\zeta \in L'_J(G^{\phi_s})$, using the chain rule,

$$\begin{aligned} \sum_{\gamma \in G^{\phi_s} \setminus \{\text{id.}\}} j(\gamma, \zeta)^s &= \sum_{k \in \mathbb{N}} \sum_{\gamma \in G_k^{\phi_s}} j(g_{n_1}^{\epsilon_1}(\gamma) \cdots g_{n_k}^{\epsilon_k}(\gamma), \zeta)^s \\ &= \sum_{k \in \mathbb{N}} \sum_{\gamma \in G_k^{\phi_s}} \prod_{i=1}^k j(g_{n_i}^{\epsilon_i}(\gamma), g_{n_{i+1}}^{\epsilon_{i+1}}(\gamma) \cdots g_{n_k}^{\epsilon_k}(\gamma)(\zeta))^s \\ &\leq \sum_{k \in \mathbb{N}} \sum_{\gamma \in G_k^{\phi_s}} \prod_{i=1}^k \left(\frac{4}{\phi_s(n_i(\gamma))}\right)^{2s} \\ &\leq \sum_{k \in \mathbb{N}} \left[2 \sum_{n \in \mathbb{N}} \left(\frac{4}{\phi_s(n)}\right)^{2s} \right]^k < \infty. \end{aligned}$$

This shows that the horospherical Poincaré series for G^{ϕ_s} at ζ converges at the exponent s . Applying Theorem 4.3 to the associated s -conformal ending measure μ_ζ , finishes the proof of the proposition. \square

Example 2. Non-atomic conformal ending measures for normal subgroups of Schottky groups.

Here, we consider a class of infinitely generated Kleinian groups which has already been studied in [7]. Let $G = G_1 * G_2$ denote the Schottky group obtained as the free product of the two finitely generated non-elementary Schottky groups G_1 and G_2 . Assume that $\delta(G) > N/2$, and that G_1 and G_2 have Dirichlet fundamental domains F_1 and F_2 at $0 \in \mathbb{D}$ such that $(\mathbb{D} \setminus F_1) \cap (\mathbb{D} \setminus F_2) = \emptyset$. Then G is isomorphic to the semidirect product $\Gamma \rtimes G_2$, where Γ denotes the kernel of the canonical group homomorphism from G onto G_2 . By the splitting lemma for groups, this is equivalent to the existence of a short exact sequence of group homomorphisms

$$1 \rightarrow \Gamma \rightarrow G \rightarrow G_2 \rightarrow 1.$$

Then note that, since Γ is normal in G , we have that $L(G) = L(\Gamma)$. Moreover, by a result of Brooks [6], we have that $\delta(\Gamma) < \delta(G)$. Finally, note that by construction we have that $L(G_2) \subset L_J(\Gamma)$.

Proposition 6.2. For G, G_1, G_2 and Γ given as above, the following hold:

- (i) Γ is of convergence type.

- (ii) Any $\delta(\Gamma)$ -conformal measure for Γ is atomless. In particular, for each $\zeta \in L_J(\Gamma)$, the $\delta(\Gamma)$ -conformal ending measure μ_ζ for Γ derived from some ending sequence towards ζ has no atoms.
- (iii) If μ_{ζ_1} and μ_{ζ_2} are two $\delta(\Gamma)$ -conformal ending measures for Γ , for distinct $\zeta_1, \zeta_2 \in L(G_2)$, then μ_{ζ_1} and μ_{ζ_2} are mutually singular.

Proof. For the proof of (i), note that Corollary 4.3 in [12] states that if Γ' is a Kleinian group of divergence type which is normal in some other Kleinian group G' , then we have that $\delta(\Gamma') = \delta(G')$ and that G' is of $\delta(G')$ -divergence type. Therefore, assuming that Γ is of $\delta(\Gamma)$ -divergence type would imply that $\delta(\Gamma) = \delta(G)$, and this is clearly not the case. This finishes the proof of (i).

For the proof of (ii), note that Theorem 6 in [10] states that if Γ' is a non-trivial normal subgroup of a Kleinian group G' , then $L_r(G') \subset L_H(\Gamma')$. Hence for our situation here, where Γ is normal in G and where G is a Schottky group, it follows that $L(\Gamma) = L(G) = L_r(G) \subset L_H(\Gamma)$ and hence, we can deduce $L(\Gamma) = L_H(\Gamma)$. An application of Proposition 5.1 in Section 5 then gives that any $\delta(\Gamma)$ -conformal measure for Γ is atomless (note that we have by construction that Γ does not have parabolic elements). This completes the proof of (ii).

Finally, in order to prove (iii), let $\zeta_1, \zeta_2 \in L(G_2)$ be given such that $\zeta_1 \neq \zeta_2$. Since G_2 is a finitely generated Schottky group, it is not difficult to see that there are hyperplanes in \mathbb{D} , denoted H_1 and H_2 , which are at the boundary of some image of the Dirichlet fundamental domain F_2 of G_2 , such that the following holds. First, H_1 separates \mathbb{D} into half-spaces one of which has ζ_1 at infinity, and the other contains H_2 and has ζ_2 at infinity; secondly, the analogous condition holds for H_2 . By construction of Γ , it now follows that H_1 and H_2 , when projected to $\overline{M}_\Gamma = (\mathbb{D} \cup \Omega(\Gamma))/\Gamma$, separate distinct ‘ends with boundary’ as introduced in [1], Subsection 4.3. Since Theorem 4.4 in [1] holds for such ends with boundary as well, we can conclude that the measures μ_{ζ_1} and μ_{ζ_2} have distinct essential supports and thus are mutually singular. \square

Example 3. Atomic $\delta(\Gamma)$ -conformal ending measures at limit points which are not fixed points.

We construct a Kleinian group Γ for which there is a purely atomic $\delta(\Gamma)$ -conformal ending measure at a Jørgensen point of Γ which is not a parabolic fixed point. As in Example 2, let $G = G_1 * G_2$ such that we have an exact sequence of group homomorphisms

$$1 \rightarrow \Gamma \rightarrow G \rightarrow G_2 \rightarrow 1.$$

This time G_2 denotes a parabolic Abelian subgroup of $\text{Con}(N)$ with fixed point equal to ζ , and $G_1 < \text{Con}(N)$ is a non-elementary Kleinian group with $\Omega(G_1) \neq \emptyset$. As in the previous example, let us assume that G_1 and G_2 have Dirichlet fundamental domains F_1 and F_2 at $0 \in \mathbb{D}$ such that $(\mathbb{D} \setminus F_1) \cap (\mathbb{D} \setminus F_2) = \emptyset$. Moreover, we assume that $\delta(G) > N/2$, and that G is of convergence type. Clearly, the latter condition is always satisfied, for instance, if we choose G_1 such that $\delta(G_1) = N$. Then note that, since G/Γ is isomorphic to the amenable group G_2 , an application of a result of [6] gives us that $\delta(\Gamma) = \delta(G)$. Also, note that ζ is a bounded parabolic fixed point of G , whereas ζ is a Jørgensen limit point of Γ which is not fixed by any non-trivial element of Γ .

Proposition 6.3. *For G, G_1, G_2, Γ and ζ given as above, we have that the $\delta(\Gamma)$ -conformal ending measure μ_ζ for Γ is essentially supported on $\Gamma(\zeta)$. In particular, we hence have that μ_ζ is purely atomic.*

Proof. By Lemma 4.4, we have that the reduced horospherical Poincaré series

$$\mathcal{P}_{\text{red}}(\zeta, s) = \sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^s$$

for G converges for s equal to the exponent $\delta(G)$ since we assume that G is of convergence type. Recall that $\delta(G) = \delta(\Gamma)$, and that $\text{Stab}_G(\zeta) = G_2$. We now show that the horospherical Poincaré series $\sum_{\gamma \in \Gamma} j(\gamma, \zeta)^{\delta(\Gamma)}$ for Γ at the exponent $\delta(\Gamma)$ converges. Indeed, since $G = \Gamma \rtimes G_2$, we can take Γ as a system of representatives of $G/\text{Stab}_G(\zeta)$. With this choice, we then have

$$\sum_{g \in G/\text{Stab}_G(\zeta)} j(g, \zeta)^{\delta(G)} = \sum_{\gamma \in \Gamma} j(\gamma, \zeta)^{\delta(\Gamma)}.$$

Therefore, the horospherical Poincaré series for Γ at the exponent $\delta(\Gamma)$ converges, and hence we can apply Theorem 4.3, which then gives the existence of a purely atomic $\delta(\Gamma)$ -conformal ending measure for Γ which is essentially supported on $\Gamma(\zeta)$. \square

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FACHBEREICH 3 - MATHEMATIK UND INFORMATIK, UNIVERSITÄT BREMEN, BIBLIOTHEKSTR. 1,
D-28359 BREMEN, GERMANY

E-mail address: `khf@math.uni-bremen.de`

DEPARTMENT OF MATHEMATICS, SCHOOL OF EDUCATION, WASEDA UNIVERSITY, SHINJUKU,
TOKYO 169-8050, JAPAN

E-mail address: `matsuzak@waseda.jp`

FACHBEREICH 3 - MATHEMATIK UND INFORMATIK, UNIVERSITÄT BREMEN, BIBLIOTHEKSTR. 1,
D-28359 BREMEN, GERMANY

E-mail address: `bos@math.uni-bremen.de`