CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 14, Pages 184–189 (July 15, 2010)
S 1088-4173(2010)00211-0

A NOTE TO "MAPPINGS OF FINITE DISTORTION: FORMATION OF CUSPS II"

PEKKA KOSKELA AND JUHANI TAKKINEN

Abstract. We consider planar homeomorphisms $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that are of finite distortion and map the unit disk onto a specific cusp domain Ω_s. We study the relation between the degree s of the cusp and the integrability of the distortion function K_f by sharpening a previous result where K_f is assumed to be locally exponentially integrable.

1. Introduction

When $\Omega \subset \mathbb{R}^2$ is the image of the unit disk under a quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, i.e. a quasidisk, certain geometric properties hold for the boundary of Ω. Perhaps the simplest of them is the Ahlfors [1] three point property that can be formulated as follows: The boundary of a Jordan domain $\Omega \subset \mathbb{R}^2$ has the three point property if there exists a constant C such that for each pair of points P_1 and P_2 in the boundary of Ω we have $\min_{i=1,2} \operatorname{diam}(\gamma_i) \leq C|P_1 - P_2|$, where γ_1, γ_2 denote the components of $\partial\Omega \setminus \{P_1, P_2\}$. The three point property is also a characterizing property, so that every Jordan domain whose boundary satisfies it is the image of the unit disk under some quasiconformal mapping of the entire plane and thus a quasidisk.

One easily observes that the three point property rules out the existence of cusps in the boundary of Ω, i.e. points where the boundary curve is "pinched" to form a zero angle. When we move away from quasiconformality to more general homeomorphisms, by relaxing the boundedness of the distortion, we should expect to see cusps. Indeed, Haïssinsky showed in [2] that cusps can arise when the distortion is assumed to be (only) exponentially integrable. There are no characterizations available for the images of the disk in this generality.

In order to gain insight on the relationship between the geometry of the images of the disk under homeomorphism of integrable distortion and the degree of integrability we have [4, 9, 5] considered the model domains

$$\Omega_s = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\, |x_2| < x_1^{1+s}\} \cup B(x_s, r_s), \tag{1}$$

where $x_s = (s+2, 0)$ and $r_s = \sqrt{(s+1)^2 + 1}$, as images of the unit disk B under a planar homeomorphism of finite distortion. The purpose of this note is to sharpen the main result of [9] to the following statement.

Received by the editors April 17, 2010.
2010 *Mathematics Subject Classification.* Primary 30C62, 30C65.
Key words and phrases. Cusp, homeomorphism, mapping of finite distortion.
The first author was partially supported by the Academy of Finland grants nos. 120927 and 131477.

Theorem A. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of finite distortion for which $f(B) = \Omega_s$ for some fixed $s > 0$. Assume that $\exp(\lambda K_f) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\lambda > 0$. Then necessarily $\lambda \leq 2/s$. Conversely, for every $s > 0$ there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of finite distortion such that $f(B) = \Omega_s$ and $\exp(\lambda K_f) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for all $\lambda < 2/s$.*

Here the improvement to [9] is in the necessity part where the bound for λ is reduced from $4/s$ to $2/s$ so that it coincides with the bound from the existence part. Theorem A is analogous to the result of [4] which states that the corresponding bound for λ is $1/s$ in the case when f is, in addition, assumed or required to be quasiconformal on B.

2. Notation and definitions

The open disk of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is denoted $B(x, r)$ and in the case of the unit disk we omit the centre and the radius, writing $B := B(0, 1)$. The closure of a set $U \subset \mathbb{R}^2$ is denoted $\overline{U}$ and the boundary ∂U. The symbol Ω always refers to a domain, i.e. a connected and open subset of $\mathbb{R}^2$. We call a homeomorphism $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ a homeomorphism of finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ and

$$\|Df(x)\|^2 \leq K(x) J_f(x) \text{ a.e. in } \Omega, \tag{2}$$

for some measurable function $K(x) \geq 1$ that is finite almost everywhere. In the distortion inequality (2), $Df(x)$ is the formal differential of f at the point x and $J_f(x) := \det Df(x)$ is the Jacobian. The norm of $Df(x)$ is defined as

$$\|Df(x)\| := \max_{e \in \partial B} |Df(x)e|.$$

For a homeomorphism of finite distortion it is convenient to write K_f for the optimal distortion function. This is obtained by setting $K_f(x) = \|Df(x)\|^2 / J_f(x)$ when $Df(x)$ exists and $J_f(x) > 0$, and $K_f(x) = 1$ otherwise. The distortion of f is said to be exponentially integrable if $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}(\Omega)$, for some $\lambda > 0$. Note that if we assume $K_f(x)$ to be bounded, we recover the class of quasiconformal mappings (cf. [10, 6]).

Next we will define the two central tools for us – the modulus of a path family and the capacity. Let E and F be subsets of $\overline{\Omega}$. We denote by $\Gamma(E, F, \Omega)$ the path family consisting of all locally rectifiable paths joining E to F in Ω. A Borel function $\rho\colon \mathbb{R}^2 \to [0, \infty[$ is said to be admissible for $\Gamma(E, F, \Omega)$ if $\int_\gamma \rho\, ds \geq 1$ for all $\gamma \in \Gamma(E, F, \Omega)$. The modulus of a path family $\Gamma := \Gamma(E, F, \Omega)$ is defined as

$$\mathrm{mod}(\Gamma) := \inf \Big\{ \int_\Omega \rho^2(x)\, dx : \rho\colon \mathbb{R}^2 \to [0, \infty[\text{ is an admissible Borel function for } \Gamma \Big\}.$$

By $\mathrm{mod}_{K_f(x)}(\Gamma)$ we mean the $K_f(x)$-weighted modulus, where instead of $\int \rho^2(x)\, dx$ we take the infimum over $\int \rho^2(x) K_f(x)\, dx$.

Let E and F be disjoint compact sets in a domain Ω. Let ω be measurable with $0 \leq \omega(x) \leq 1$ almost everywhere. The ω-weighted capacity of the pair (F, E) with respect to Ω is defined to be

$$\operatorname{cap}_\omega(F,E;\Omega) := \inf\Big\{ \int_\Omega |\nabla u(x)|^2 \omega(x)\,dx : u \in C(\Omega)\cap W^{1,1}_{\mathrm{loc}}(\Omega),\\ u \le 0 \text{ on } F \text{ and } u \ge 1 \text{ on } E\Big\}.$$

3. Auxiliary results

We begin by introducing two lemmas that are slight modifications to the corresponding results in, [9, pp. 209–213].

Lemma 1. *Let $E \subset \overline{B}$ be a continuum such that $E \subset B(x_0, 1/6)$ for some $x_0 \in \partial B$ and $F := \overline{B}(0,1/4)$. Suppose that $v \in W^{1,1}(B)$ is continuous and satisfies: $v = 0$ on F and $\lim_{y\to x} v(y) \ge 1$ for every $x \in E$. If $L := \int_B \exp(\lambda K) < \infty$, for some measurable function $K(x) \ge 1$, then*

$$\int_B \frac{|\nabla v|^2}{K} \ge C\lambda\Big(\log\frac{\sqrt{4L/\pi}}{\operatorname{diam} E}\Big)^{-2}. \tag{3}$$

Proof. This lemma is basically the same as Lemma 3 in [9]. The only difference is that now $E \subset \overline{B}$ instead of just $E \subset B$. One can easily check that the proof of Lemma 3 in [9] goes through with just some minor modifications. □

Lemma 2. *Let Ω_s be a cusp domain as defined in (1) and $F \subset \Omega_s$ a compact set. Set $d := \min\{1, \operatorname{dist}(0,F)\}$ and let $0 < r < d/2$. If $E \subset \{(x_1,x_2) \in \overline{\Omega}_s : 0 \le x_1 \le r\}$, then there is a Lipschitz function u on Ω_s such that $u = 0$ on F, $\lim_{y\to x} u(y) = 1$ for every $x \in E$ and*

$$\int_{\Omega_s} |\nabla u|^2\,dx \le C(s,d) r^s. \tag{4}$$

Proof. This lemma is also a slight reformulation of a lemma from [9] (Lemma 1). The difference here is that now $E \subset \{(x_1,x_2) \in \overline{\Omega}_s : 0 \le x_1 \le r\}$ instead of $E \subset \{(x_1,0) \in \Omega_s : 0 < x_1 \le r\}$. The original proof from [9] goes through unmodified. □

Next we introduce a lemma that is in some sense a specifically crafted replacement for the general modulus of continuity result from [7] used in the proof of Theorem 3 on [9]. The extra information we have, i.e. $f(B) = \Omega_s$, can indeed be used to obtain a better diameter estimate for a certain subset of $f^{-1}(\partial\Omega_s)$ rather than by using the general modulus of continuity result.

Lemma 3. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of finite distortion such that $\exp(\lambda K_f) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\lambda > 0$, and $f(B) = \Omega_s$. Let $E'_t = \{x \in \partial\Omega_s : |x| \le t\}$ and $E_t = f^{-1}(E'_t)$. Then for all $\varepsilon > 0$ there exists $t_0 > 0$ such that for some positive constants C and $\tilde{C}$,*

$$\operatorname{diam} E_t \ge C \exp\Big(\frac{-\tilde{C}}{(\operatorname{diam} E'_t)^{\frac{1+\varepsilon}{\lambda}}}\Big) \tag{5}$$

for all $0 < t < t_0$.

Proof. Set $F' = [-2,-1] \subset \mathbb{R}$, $F = f^{-1}(F')$ and fix $R > 0$ such that $R < \min\{1, \operatorname{dist}((1,0),F)\}$. As f is an homeomorphism, we find $t_0 > 0$ such that $\operatorname{diam} E_t < R/e^3$ for all $0 < t < t_0$. Now, for each such t, we may choose integers i_1 and i_2 such that $i_1 + 1 < i_2 < 0$ and

$$e^{i_1-1} < \operatorname{diam} E_t \le e^{i_1} < e^{i_2} \le R < e^{i_2+1}. \tag{6}$$

Denote $A_r := \partial B((1,0),r) \setminus \overline{B}$. A direct application of Lemma 1 from [4] shows that

$$\begin{aligned}(7)\quad &\operatorname{mod}_{K_f}(\Gamma(A_{e^{i_1}}, A_{e^{i_2}}, \mathbb{R}^2 \setminus \overline{B}))\\ &\qquad \le (\pi + e^{i_2})(1 + 8\log^{-1}(C_0/e^{2i_2}))\frac{2}{\lambda}\log^{-1}\Big(\log\frac{C_0}{e^{2i_1}}\Big/\log\frac{C_0}{e^{2i_2}}\Big)\end{aligned}$$

where C_0 is a positive constant. From (6) and (7) one readily obtains the upper bound

$$\begin{aligned}(8)\quad &\operatorname{mod}_K(\Gamma(E_t, F, \mathbb{R}^2 \setminus \overline{B})) \le \operatorname{mod}_K(\Gamma(A_{e^{i_1}}, A_{e^{i_2}}, \mathbb{R}^2 \setminus \overline{B}))\\ &\qquad \le (\pi + R)(1 + 8\log^{-1}(C_1/R^2))\frac{2}{\lambda}\log^{-1}\Big(C_2 \log\frac{C_3}{(\operatorname{diam} E_t)^2}\Big),\end{aligned}$$

where C_1, C_2 and C_3 are again some positive constants. As

$$\operatorname{mod}(\Gamma(E'_t, F', \mathbb{R}^2 \setminus \overline{\Omega}_s)) \le \operatorname{mod}_{K_f}(\Gamma(E_t, F, \mathbb{R}^2 \setminus \overline{B})) \tag{9}$$

(cf. [3]), it suffices to show that

$$\operatorname{mod}(\Gamma(E'_t, F', \mathbb{R}^2 \setminus \overline{\Omega}_s)) \ge 2\pi \log^{-1}(C_4/\operatorname{diam} E'_t), \tag{10}$$

because by combining (8), (9) and (10) we easily obtain

$$C_2 \log\frac{C_3}{(\operatorname{diam} E_t)^2} \le (C_4/\operatorname{diam} E'_t)^{(\pi+R)(1+8\log^{-1}(C_1/R^2))(\pi\lambda)^{-1}}$$

from which the claim (5) will follow by choosing R sufficiently small in the beginning of the proof and solving for $\operatorname{diam} E_t$.

In order to prove (10) we denote the open upper and lower half-planes H_+ and H_-, respectively, and set $E'_{t\pm} := E'_t \cap H_\pm$. Consider the path families $\Gamma := \Gamma(E'_t, F', \mathbb{R}^2 \setminus \overline{\Omega}_s)$ and $\Gamma_\pm := \Gamma(E'_{t\pm}, F', H_\pm \setminus \overline{\Omega}_s)$. Notice that $\Gamma_+ \cup \Gamma_- \subset \Gamma$, the path families Γ_+ and Γ_- are separate, and that the situation is symmetric with respect to Γ_+ and Γ_-. Therefore it follows from the basic properties of the modulus (cf. [10]), that

$$\operatorname{mod}(\Gamma) \ge \operatorname{mod}(\Gamma_+) + \operatorname{mod}(\Gamma_-) = 2\operatorname{mod}(\Gamma_+).$$

Let $V = \{(x_1, x_2) \in \mathbb{R}^2 : s + 2 \le x_1, 0 \le x_2 \le \sqrt{(s+1)^2 + 1}\}$ and set $G = H_+ \setminus (\overline{\Omega}_s \cup V)$ so that now $G \subset H_+ \setminus \overline{\Omega}_s$ and $E'_{t+} \subset \partial G$ (cf. Figure 1). Denote $\Gamma_G := \Gamma(E'_{t+}, F', G)$. It follows that

$$\operatorname{mod}(\Gamma_+) \ge \operatorname{mod}(\Gamma_G).$$

Next, we pick conformal mapping $h\colon H_+ \to G$ and two auxiliary Möbius transformations $m_1, m_2\colon \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ such that $m_1(B) = H_+$ and that $m_2(G)$ is some bounded domain. As the boundary of $m_2(G)$ is a Dini-smooth Jordan curve (because ∂G is) we may apply Theorem 3.5 from [8] to the conformal mapping $g\colon B \to m_2(G)$, $g(x) = m_2 \circ h \circ m_1(x)$ to see that its derivative g' has a continuous extension to $\overline{B}$ and that $g'(x) \ne 0$ for all $x \in \overline{B}$. Thus we obtain an extension $\overline{g}\colon \overline{B} \to \overline{m_2(G)}$

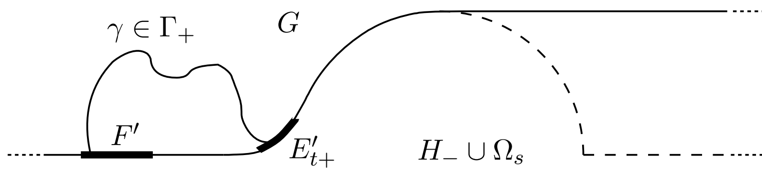

FIGURE 1. The setting in the proof of estimate (10)

for g that is not only homeomorphic but also Lipschitz. As m_1^{-1} and m_2^{-1} are also Möbius transformations, we see that $m_2^{-1} \circ \overline{g} \circ m_1^{-1}$ defines a bijective Lipschitz extension of h to $\overline{H}_+$. For simplicity, we also denote this extension by h.

The preimages $h^{-1}(F')$ and $h^{-1}(E'_{t+})$ are now two separate bounded intervals on $\mathbb{R}$ such that $\operatorname{diam} h^{-1}(F')$ and $\operatorname{dist}(h^{-1}(F'), h^{-1}(E'_{t+}))$ are independent of t. As h is Lipschitz and bijective, we have for some $L > 1$ that

$$\operatorname{diam} E'_{t+} = \operatorname{diam} h(h^{-1}(E'_{t+})) \leq L \operatorname{diam} h^{-1}(E'_{t+}).$$

Now, if $h^{-1}(\Gamma_+) := \Gamma(h^{-1}(F'), h^{-1}(E'_{t+}), H_+)$, then

$$\operatorname{mod}(h^{-1}(\Gamma_+)) \geq \pi \log^{-1}(C/\operatorname{diam} h^{-1}(E'_{t+})),$$

where C is some constant depending on $\operatorname{dist}(h^{-1}(F'), h^{-1}(E'_{t+}))$ and $\operatorname{diam} h^{-1}(F')$. By applying the (conformal) invariance of the modulus and noticing that $\operatorname{diam} E'_t = \operatorname{diam} E'_{t+}$, we obtain by combining the previous estimates that

$$\begin{aligned} \operatorname{mod}(\Gamma) \geq 2\operatorname{mod}(\Gamma_G) &= 2\operatorname{mod}(h^{-1}(\Gamma_+)) \\ &\geq 2\pi \log^{-1}(C/\operatorname{diam} h^{-1}(E'_{t+})) \geq 2\pi \log^{-1}(\hat{C}/\operatorname{diam} E'_t), \end{aligned}$$

which proves the claim (10). □

4. MAIN PROOF

Here we show how one can improve Theorem 3 on [9] by using Lemma 3. In fact, we give the following improved necessity part for Theorem A, stated as Theorem 1.

Theorem 1. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism of finite distortion with $\exp(\lambda K_f) \in L^1(2B)$ for some $\lambda > 0$. If $f(B) = \Omega_s$, then $\lambda \leq 2/s$.*

Proof. The proof of this theorem goes through more or less the same way as the proof of Theorem 3 in [9]. As we basically only replace the use of the modulus of continuity with our more specific Lemma 3, we therefore do not reproduce the complete argument here, but instead state only the key points of difference and ask the reader to have [9] in hand.

Let $\varepsilon > 0$. Define $E'_t = \{x \in \partial\Omega_s : |x| \leq t\}$, $F = \overline{B}(0, 1/4)$ and set $E_t = f^{-1}(E'_t)$, $F' = f(F)$. From Lemma 3 we obtain $t_0 > 0$ such that for some positive constants C and $\tilde{C}$,

$$\operatorname{diam} E_t \geq C \exp\Big(\frac{-\tilde{C}}{(\operatorname{diam} E'_t)^{\frac{1+\varepsilon}{\lambda}}}\Big), \tag{11}$$

for all $0 < t < t_0$. As f is an homeomorphism we may assume, by making t_0 smaller if necessary, that $E_t \subset B(f^{-1}(0), 1/6)$ for all $0 < t < t_0$ and $\operatorname{diam} E'_{t_0} < \operatorname{dist}(0, F')$.

So far the only real difference here to the arguments of [9], is that in [9] we have $E'_t \subset \Omega_s$, but now $E'_t \subset \partial\Omega_s$ (likewise $E_t \subset \partial B$ instead of $E_t \subset B$). However, this does not pose a problem as the lemmas that are used in the proof of Theorem 3 in [9] (Lemma 1 and Lemma 3), can easily be adjusted to this case and their results remain the same (also Lemma 2 that is used on the proof of Lemma 3). If fact, we have already reformulated the required lemmas as Lemma 1 and Lemma 2.

Thus, by following the arguments presented in the proof of Theorem 3 in [9], we arrive at the capacity estimate

$$C_1\Big(\log\frac{C_2}{\operatorname{diam} E_t}\Big)^{-2} \leq \operatorname{cap}_{1/K}(F, E_t; B) \leq C_3(\operatorname{diam} E_t')^s, \tag{12}$$

where C_1, C_2 and C_3 are some positive constants. Next, by combining (11) and (12) we obtain that for all $0 < t < t_0$,

$$C_4(\operatorname{diam} E_t')^{\frac{2+2\varepsilon}{\lambda}} \leq C_3(\operatorname{diam} E_t')^s.$$

From this it follows by letting $t \to 0$ (and thus $\operatorname{diam} E_t' \to 0$), that for all $\varepsilon > 0$ we must have $(2+2\varepsilon)/\lambda \geq s$. This proves the claim. □

References

1. L.V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301. MR0154978 (27:4921)
2. P. Haïssinsky, *Chirurgie parabolique*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 2, 195–198. MR1645124 (99i:58127)
3. P. Koskela and J. Onninen, *Mappings of finite distortion: Capacity and modulus inequalities*, J. Reine Angew. Math. **599** (2006), 1–26. MR2279096 (2007k:30035)
4. P. Koskela and J. Takkinen, *Mappings of finite distortion: formation of cusps*, Publ. Mat. **51** (2007), no. 1, 223–242. MR2307153 (2008e:30026)
5. ———, *Mappings of finite distortion: Formation of cusps III*, Acta. Math. Sin. (English Series) **26** (2010), no. 5, 817–824.
6. O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, second ed., Springer-Verlag, New York, 1973. MR0344463 (49:9202)
7. J. Onninen and X. Zhong, *A note on mappings of finite distortion: the sharp modulus of continuity*, Michigan Math. J. **53** (2005), no. 2, 329–335. MR2152704 (2006c:30025)
8. Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 299, Springer-Verlag, Berlin, 1992. MR1217706 (95b:30008)
9. J. Takkinen, *Mappings of finite distortion: Formation of cusps II*, Conform. Geom. Dyn. **11** (2007), 207–218. MR2354095 (2009f:30055)
10. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, vol. 229, Springer-Verlag, New York, 1971. MR0454009 (56:12260)

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, FI-40014 University of Jyväskylä, Finland

E-mail address: pekka.j.koskela@jyu.fi

Linnantie 8 C 21, 40800 Vaajakoski, Finland

E-mail address: juhani.takkinen@kolumbus.fi