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THE SCHWARZIAN OPERATOR: SEQUENCES, FIXED POINTS AND N-CYCLES

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Abstract. Given a function $f(z)$ that is analytic in a domain D, we define the classical Schwarzian derivative $\{f,z\}$ of $f(z)$, and mention some of its most useful analytic properties. We explain how the process of iterating the Schwarzian operator produces a sequence of Schwarzian derivatives, and we illustrate this process with examples. Under a suitable restriction, these sequences become N-cycles of Schwarzian derivatives. Some properties of functions belonging to an N-cycle are listed. We conclude the article with a collection of related open problems.

1. Definitions

If $f(z)$ is an analytic, locally univalent (one-to-one) function which is defined in a domain D, then the *Schwarzian derivative* $\{f,z\}$ *of* $f(z)$ is defined there by

$$\{f,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

This definition makes sense, for if $f(z)$ is locally univalent in a neighborhood of z, then $f'(z) \neq 0$ there. It should be noted that the natural domains of $f(z)$ and $\{f,z\}$ are usually different [1, p. 184]. The *Schwarzian operator* $\mathcal{S}$ maps $f(z)$ to $\{f,z\}$.

An extremely important and useful property of the Schwarzian derivative is known as the *Composition Law.* If $f(z)$ is an analytic function and $z = g(\zeta)$ is another analytic function for which the composition $F(\zeta) = f(g(\zeta))$ is defined in some domain, then

$$\{F,\zeta\} = \{f, g(\zeta)\}\,[g'(\zeta)]^2 + \{g,\zeta\}.$$

In particular, if $f(z)$ is analytic in a circular domain centered at the origin, and $|\varepsilon| = 1$, then the composition $f_\varepsilon(z) = \bar{\varepsilon}\, f(\varepsilon z)$ is known as a *rotation* of $f(z)$, and an application of the Composition Law yields the relation $\{f_\varepsilon, z\} = \varepsilon^2\,\{f, \varepsilon z\}$.

In this paper, we investigate the behavior of sequences of Schwarzian derivatives produced by iteration in the following manner. Given a function $f(z)$ that is analytic on the domain D, we set $f_1(z) = f(z)$, and for $n \geq 1$, we set $f_{n+1}(z) = \mathcal{S} f_n(z)$. Each sequence element $f_n(z)$ has a natural domain D_n of definition, and these domains are usually different. If $f_{N+1}(z) = f_1(z)$ on the domain $D_{N+1} = D_1$, then the set $\{f_1(z), \ldots, f_N(z)\}$ will be called an *N-cycle of Schwarzian derivatives.*

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2. Examples of sequences of Schwarzian derivatives

The functions in each of the following four examples satisfy a functional equation of the form $\mathcal{S}f(z) = \beta\, f(\alpha\, z)$. Relations of this form allow for the direct iteration of $\mathcal{S}$.

2.1. Fixed points ($\beta = 1$ and $\alpha = 1$).

The functions

$$f(z; a, b) = \frac{-3\, a^2}{2(b + a\, z)^2}$$

are non-trivial fixed points of $\mathcal{S}$, since $\mathcal{S}f(z; a, b) = f(z; a, b)$ for any choice of $a \neq 0$ and b. Note that $f'(z; a, b) \neq 0$ if $z \neq -b/a$, and that $f(z; a, b)$ has one double pole at $-b/a$. If $b = 0$, then $f(z; a, 0) = -3/2z^2$.

2.2. 2-cycles ($\beta = 1$ and $\alpha = -1$).

We can also display a 2-cycle of Schwarzian derivatives. Let

$$f_1(z) = -\frac{4 + 2\, e^z + e^{2z}}{2(1 + e^z)^2} \qquad \text{and} \qquad f_2(z) = -\frac{1 + 2\, e^z + 4\, e^{2z}}{2(1 + e^z)^2}.$$

Straightforward calculations show that $\mathcal{S}f_1(z) = f_2(z) = f_1(-z)$ and that $\mathcal{S}f_2(z) = f_1(z) = f_2(-z)$. Thus, $\{f_1, f_2\}$ is a 2-cycle of Schwarzian derivatives. Since $\mathcal{S}^2 f_1(z) = f_1(z)$ and $\mathcal{S}^2 f_2(z) = f_2(z)$, both $f_1(z)$ and $f_2(z)$ can be viewed as fixed points of $\mathcal{S}^2$.

The elements of this 2-cycle have some interesting properties:

(1) The sum $f_1(z) + f_2(z)$ and product $f_1(z)\, f_2(z)$ of the cycle elements are even functions.
(2) The zeroes of $f_1(z)$ are located at $\ln 2 \pm 2\pi i/3 \pm 2n\pi i$, and the zeroes of $f_2(z)$ are symmetrically located at $-\ln 2 \pm 2\pi i/3 \pm 2n\pi i$. Furthermore, all of these zeroes are simple.
(3) The derivatives of the cycle elements,

$$f_1'(z) = \frac{3e^z}{(1 + e^z)^3} \qquad \text{and} \qquad f_2'(z) = -\frac{3e^{2z}}{(1 + e^z)^3},$$

do not vanish.
(4) Since the double poles of both $f_1(z)$ and $f_2(z)$ are located at $\pm(2n + 1)\pi i$, it follows that $D_1 = D_2$.

This example of a 2-cycle can be generalized. Let

$$g_1(z; a) = -\frac{4\, a^2 + 2\, a\, e^z + e^{2z}}{2(a + e^z)^2} \qquad \text{and} \qquad g_2(z; a) = -\frac{a^2 + 2\, a\, e^z + 4\, e^{2z}}{2(a + e^z)^2}.$$

Straightforward calculations show that $\mathcal{S}g_1(z; a) = g_2(z; a) = g_1(-z; \frac{1}{a})$ and that $\mathcal{S}g_2(z; a) = g_1(z; a) = g_2(-z; \frac{1}{a})$. Thus, $\{g_1, g_2\}$ is a 2-cycle of Schwarzian derivatives. Both $g_1(z; a)$ and $g_2(z; a)$ can be viewed as fixed points of $\mathcal{S}^2$. The cycle elements $g_1(z; a)$ and $g_2(z; a)$ are translations of $f_1(z)$ and $f_2(z)$, as

$$g_1(z; a) = f_1(z - \log a) \qquad \text{and} \qquad g_2(z; a) = f_2(z - \log a).$$

Consequently, the zeroes and poles of $g_1(z;a)$ and $g_2(z;a)$ are translations of the zeroes and poles of $f_1(z)$ and $f_2(z)$. Note that if $a=-1$, then both $g_1(z;-1)$ and $g_2(z;-1)$ have double poles at the origin.

2.3. **Rational functions of e^z with argument doubling ($\beta = 4$ and $\alpha = 2$).** If

$$h(z) = -\frac{1+10\,e^z + e^{2z}}{8(e^z-1)^2} = \frac{5+\cosh z}{8(1-\cosh z)},$$

then $\mathcal{S}h(z) = \{h,z\} = 4\,h(2z)$. Upon iteration, we obtain $\mathcal{S}^n h(z) = 4^n\,h(2^n\,z)$. Note that the algebraic form of $h(z)$ is similar to that of $f_1(z)$ and $f_2(z)$ in the second example above, but that the behavior of the resulting sequence is remarkably different. Note also that $h(z)$ also has a double pole at the origin.

2.4. **Elliptic functions with argument doubling ($\beta = -6$ and $\alpha = 2$).** The Weierstrass $\wp$-function is the elliptic function defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum ranges over all $\omega = n_1\omega_1 + n_2\omega_2$ except 0 [1, pp. 264–269]. It is well known that $\wp(z)$ satisfies the differential equation

$$(\wp'(z))^2 = 4\,\wp^3(z) - g_2\,\wp(z) - g_3$$

where

$$g_2 = 60 \sum_{\omega} \frac{1}{\omega^4} \qquad \text{and} \qquad g_3 = 140 \sum_{\omega} \frac{1}{\omega^6}.$$

If we differentiate this differential equation and then divide by $\wp'(z)$, we obtain

$$2\,\wp''(z) = 12\,\wp^2(z) - g_2.$$

If we differentiate this last equation and then divide by $\wp'(z)$, we obtain

$$\frac{\wp'''(z)}{\wp'(z)} = 12\,\wp(z).$$

It is also well known that $\wp(z)$ satisfies the identity

$$\frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = \wp(2z) + 2\wp(z).$$

By combining the last two equations, we have

$$\mathcal{S}\wp(z) = \{\wp, z\} = \frac{\wp'''(z)}{\wp'(z)} - \frac{3}{2}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = -6\,\wp(2z).$$

Upon iteration, we obtain $\mathcal{S}^n\wp(z) = -6\cdot 4^{n-1}\,\wp(2^n z)$.

3. Two theorems concerning N-cycles

In this section, we illuminate the structure of N-cycles.

Theorem 1. *Suppose that $\{f, z\} = f(\alpha\, z)$ in a circular, N-fold symmetric domain D centered at the origin. If N is a positive integer and $\alpha = e^{2\pi i/N}$, then the set*

$$\left\{\alpha^{2n-2}\, f(\alpha^n\, z)\right\}_{n=1}^{N}$$

constitutes an N-cycle of Schwarzian derivatives. The elements of this N-cycle are simply rotations of $f(z)$ multiplied by a suitable power of α.

Proof. We first observe that if $\mathcal{S} f(z) = f(\alpha\, z)$, then

$$\mathcal{S}^2 f(z) = \{f, \alpha\, z\} \cdot \alpha^2 = \alpha^2\, f(\alpha^2\, z)$$

by the Composition Law. By induction, if $\mathcal{S}^n f(z) = \alpha^{2n-2}\, f(\alpha^n\, z)$, then

$$\mathcal{S}^{n+1} f(z) = \alpha^{2n}\, \{f, \alpha^n\, z\} = \alpha^{2n}\, f(\alpha^{n+1}\, z).$$

Now, if $\alpha = e^{2\pi i/N}$, so that $\alpha^N = 1$, then $\mathcal{S}^{N+1} f(z) = \mathcal{S}^1 f(z) = f(\alpha\, z)$. Thus, the given set constitutes an N-cycle of Schwarzian derivatives, as claimed.

Note that $\mathcal{S}^N f(z) = f(z)/\alpha^2$ belongs to the N-cycle. With $|\alpha| = 1$, each element of the N-cycle is a multiple of a rotation of $f(z)$, since $\alpha^{2n-2}\, f(\alpha^n\, z) = \alpha^{3n-2}\, \bar{\alpha}^n\, f(\alpha^n\, z) = \alpha^{3n-2}\, f_{\alpha^n}(z)$. □

The elements of any N-cycle, as described in this theorem, have some very interesting properties:

(1) It is not necessary to apply the Schwarzian operator $N-1$ times to $f(\alpha z)$ in order to compute the rest of the cycle elements; it is only necessary to rotate $f(z)$ sufficiently.

(2) Each element of the N-cycle can be viewed as a fixed point of $\mathcal{S}^N$, or equivalently, as a solution of the nonlinear ordinary differential equation $\mathcal{S}^N f(z) = f(z)$.

(3) If

$$F_N(z) = \sum_{n=1}^{N} \alpha^{2n-2}\, f(\alpha^n z),$$

denotes the sum of the elements of an N-cycle, then $F_N(z) = \alpha^2\, F_N(\alpha z)$. Thus, $F_N(z)$ displays N-fold symmetry if and only if $N = 2$.

(4) If

$$G_N(z) = \prod_{n=1}^{N} \alpha^{2n-2}\, f(\alpha^n z),$$

denotes the product of the elements of an N-cycle, then $G_N(z) = G_N(\alpha z)$. Thus, $G_N(z)$ is N-fold symmetric for all $N \geq 1$.

(5) The zeroes and poles of any element of the N-cycle are exactly the rotations of the zeroes and poles of $f(z)$ or any other cycle element. Thus, every cycle element has the same number of zeroes and poles as any other cycle element.

(6) No cycle element has a pole of order $m \geq 3$. Suppose that $f(z)$ has a pole of order $m \geq 2$ at $z = z_0$. Then

$$\mathcal{S}^N f(z) = f(z)/\alpha^2 = \frac{g(z)}{(z - z_0)^m},$$

where $g(z)$ is analytic at $z = z_0$ and $g(z_0) \neq 0$ [1, p. 184]. By direct computation, we obtain the representation

$$\begin{aligned}\mathcal{S}^{N+1} f(z) &= \frac{1}{(z-z_0)^2}\left[\frac{(1-m^2)}{2} + \frac{(m^2-1)\, g'(z_0)}{m g(z_0)}(z-z_0) + \cdots\right] \\ &= f(\alpha\, z).\end{aligned}$$

From this representation, it follows that

$$f(z) = \frac{\alpha^2}{(z-\alpha z_0)^2}\left[\frac{(1-m^2)}{2} + \frac{(m^2-1)\, g'(z_0)}{\alpha\, m g(z_0)}(z-\alpha z_0) + \cdots\right].$$

To summarize, we have shown that if $f(z)$ has a pole of order $m \geq 2$ at $z = z_0$, then $f(z)$ also has a pole of order 2 at $z = \alpha z_0$. By repeating this argument $N-1$ more times, we conclude that $f(z)$ has symmetrically placed poles of order 2 at $z = \alpha^n z_0$ for $n = 1, \ldots, N$. In particular, since $z = \alpha^N z_0 = z_0$, it must be the case that $m = 2$.

(7) The derivative of every cycle element does not vanish. Suppose on the contrary that $f'(z_0) = 0$. Since $\mathcal{S}^N f(z) = f(z)/\alpha^2$, it follows from the definition of the Schwarzian derivative that $\mathcal{S}^{N+1} f(z) = f(\alpha z)$ has a pole of order 2 at $z = z_0$. By repeating the argument given in the last item, we conclude that $f(z)$ also has a pole of order 2 at $z = z_0$, contrary to assumption.

(8) No cycle element has a zero of order $m \geq 2$. Since the proof is entirely similar to the argument above, it is omitted here.

It is easy to state another theorem that characterizes N-cycles.

Theorem 2. *Suppose that $\{f, z\} = f(z - \log a)$ in some horizontal strip domain D. If N is a positive integer and $a = e^{2\pi i/N}$, then the set*

$$\{f(z - n \log a)\}_{n=1}^{N}$$

constitutes an N-cycle of Schwarzian derivatives.

Proof. If $\mathcal{S} f(z) = f(z - \log a)$, then $\mathcal{S}^2 f(z) = f(z - 2 \log a)$ by the Composition Law. Inductively, $\mathcal{S}^n f(z) = f(z - n \log a)$ for all n. With $a^N = 1$, we have $N \log a = 0$, so that $\mathcal{S}^{N+1} f(z) = f(z - \log a) = \mathcal{S} f(z)$. Thus, the given set constitutes an N-cycle of Schwarzian derivatives, as claimed. □

4. Open problems

Some characteristics of cycle elements for all N have been noted above, but their precise form for $N \geq 3$ is an open problem. There are two obvious approaches here.

On the one hand, one could solve the nonlinear ordinary differential equation $\mathcal{S}^N f(z) = f(z)$ to determine all N cycle elements at once. After the denominators are cleared in this equation, it becomes a polynomial equation in $f(z)$ and its derivatives whose order and degree depends upon N.

On the other hand, one could solve the Schwarzian operator equation $\mathcal{S} f(z) = \{f, z\} = \beta\, f(\alpha z)$ with $\beta = 1$ and $\alpha = e^{2\pi i/N}$ to determine the one cycle element, and then, according to Theorem 1, rotate it $N-1$ times to determine the other cycle elements.

Given the need to determine the elements of all N-cycles for $N \geq 3$ and the four interesting examples displayed in §2, it is of interest to solve the Schwarzian operator equation $\{f, z\} = \beta\, f(\alpha z)$ for all complex values of α and β.

It is easy to generalize this problem in many directions. For example, if $P(z)$ denotes an arbitrary polynomial of degree N, do functions $f(z)$ exist for which the equation $\{f, z\} = P(f(\alpha z))$ holds on some domain?

References

1. Lars V. Ahlfors, *Complex Analysis*, McGraw-Hill Book Company, New York, 1966. MR510197 (80c:30001)

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