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ON THE LENGTH SPECTRUM TEICHMÜLLER SPACES OF RIEMANN SURFACES OF INFINITE TYPE

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Abstract. On the Teichmüller space $T(R_0)$ of a hyperbolic Riemann surface R_0, we consider the length spectrum metric d_L, which measures the difference of hyperbolic structures of Riemann surfaces. It is known that if R_0 is of finite type, then d_L defines the same topology as that of Teichmüller metric d_T on $T(R_0)$. In 2003, H. Shiga extended the discussion to the Teichmüller spaces of Riemann surfaces of infinite type and proved that the two metrics define the same topology on $T(R_0)$ if R_0 satisfies some geometric condition. After that, Alessandrini-Liu-Papadopoulos-Su proved that for the Riemann surface satisfying Shiga's condition, the identity map between the two metric spaces is locally bi-Lipschitz.

In this paper, we extend their results; that is, we show that if R_0 has bounded geometry, then the identity map $(T(R_0), d_L) \to (T(R_0), d_T)$ is locally bi-Lipschitz.

1. Introduction

We say that a Riemann surface is hyperbolic if its universal cover is the Poincaré disk. For a hyperbolic Riemann surface R_0, consider a pair (R, f) of a Riemann surface R and a quasiconformal mapping f from R_0 to R. Such pairs (R_1, f_1) and (R_2, f_2) are called Teichmüller equivalent if there exists a conformal mapping $h : R_1 \to R_2$ that is homotopic to $f_2 \circ f_1^{-1}$, where the homotopy does not necessarily fix points of the ideal boundary ∂R_0. *The Teichmüller space* $T(R_0)$ of R_0 is the set of all Teichmüller equivalence classes.

$T(R_0)$ has a complete metric d_T called *the Teichmüller metric*. It is defined by

$$d_T([R_1, f_1], [R_2, f_2]) = \log \inf_f K(f),$$

where the infimum is taken over all quasiconformal mappings from R_1 to R_2 that are homotopic to $f_2 \circ f_1^{-1}$, and $K(f)$ is the maximal dilatation of f. This means that d_T measures the difference of complex structures of Riemann surfaces in $T(R_0)$.

We introduce another metric on $T(R_0)$. Let $\mathcal{C}(R_0)$ be the set of non-trivial and non-peripheral closed curves in R_0. We define *the length spectrum metric* d_L by

$$d_L([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \mathcal{C}(R_0)} \max \left\{ \frac{\ell_{R_1}([f_1(\alpha)])}{\ell_{R_2}([f_2(\alpha)])}, \frac{\ell_{R_2}([f_2(\alpha)])}{\ell_{R_1}([f_1(\alpha)])} \right\},$$

where $[f_i(\alpha)]$ is a geodesic which is freely homotopic to $f_i(\alpha)$, and $\ell_{R_i}(\cdot)$ is the hyperbolic length $(i = 1, 2)$. By definition, $d_L([R_1, f_1], [R_2, f_2]) = 0$ if and only if

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$\ell_{R_1}([f_1(\alpha)]) = \ell_{R_2}([f_2(\alpha)])$ for any $\alpha \in \mathcal{C}(R_0)$. Hence d_L measures the difference of hyperbolic structures of Riemann surfaces in $T(R_0)$.

Remark 1.1. Let $\mathcal{S}(R_0)$ be the set of non-trivial and non-peripheral simple closed curves in R_0. Thurston [16] showed that

$$d_L([R_1, f_1], [R_2, f_2]) = \log \sup_{\alpha \in \mathcal{S}(R_0)} \max \left\{ \frac{\ell_{R_1}([f_1(\alpha)])}{\ell_{R_2}([f_2(\alpha)])}, \frac{\ell_{R_2}([f_2(\alpha)])}{\ell_{R_1}([f_1(\alpha)])} \right\}.$$

In 1972, Sorvali [15] defined the metric d_L and showed the following.

Lemma 1.2 ([15])**.** *For any* $[R_1, f_1], [R_2, f_2] \in T(R_0)$*,*

$$d_L([R_1, f_1], [R_2, f_2]) \leq d_T([R_1, f_1], [R_2, f_2])$$

holds.

In 1986, Li [10] proved that d_T and d_L define the same topology on $T(R_0)$ if R_0 is a compact Riemann surface, and in 1999, Liu [11] showed that the same holds for any Riemann surface of finite type, i.e., compact surface from which at most finitely many points have been removed. In 2003, Shiga [14] showed that there exists a Riemann surface R_0 of infinite type such that d_T and d_L define different topologies on $T(R_0)$. After that, Liu-Sun-Wei [13], Kinjo [8], and Evren [6] gave new examples of Teichmüller spaces on which the two metrics define different topologies. Each Riemann surface they considered has a sequence of points whose injective radii either diverge or vanish. Here, the injective radius $\mathrm{inj}_r(x)$ of $x \in R_0$ is defined as follows:

$$\mathrm{inj}_r(x) = \sup\{r > 0 \mid \text{The } r\text{-neighborhood of } x \text{ is homeomorphic to a disk.}\}$$

First, we describe when d_T and d_L define the same topology on the Teichmüller space. In 2003, Shiga gave a sufficient condition ([14]) for the two metrics to define the same topology. (We call the assumption of the following theorem Shiga's condition.)

Theorem 1.3 ([14], Theorem 1.2)**.** *Let R_0 be a Riemann surface. Assume that there exists a pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$ satisfying the following conditions.*

(1) *Each connected component of ∂P_k $(k = 1, 2, 3...)$ is either a puncture or a simple closed geodesic of R_0.*
(2) *There exists a constant $M > 0$ such that if α is a boundary geodesic of some P_k, then*
$$0 < M^{-1} < \ell_{R_0}(\alpha) < M$$
holds.

Then d_T and d_L define the same topology on $T(R_0)$.

In our previous paper ([9], 2014), we gave a new sufficient condition. Before mentioning it, we shall define *bounded geometry* of Riemann surfaces. First, we define a cusp neighborhood in R_0 as a punctured disc bounded by a horocycle such that its area is one and the length of horocycle is one. Let R_0' be a subsurface of R_0 obtained by removing all cusp neighborhoods from R_0.

Definition 1.4. A Riemann surface R_0 has bounded geometry if there exists a constant $M > 0$ satisfying the condition (BG): any (non-trivial and non-peripheral) closed geodesic has length greater than $1/M$ and every point x in R_0' has a closed curve based on x with the length less than M.

Note that a Riemann surface R_0 has bounded geometry if and only if there exists a constant $M > 0$ such that for any point $x \in R_0'$, $1/M < \mathrm{inj}_r(x) < M$ holds.

Now the new condition in our previous paper is the following:

Theorem 1.5 ([9], Corollary 1.5)**.** *Let R_0 be a Riemann surface with bounded geometry. Suppose R_0 has finite genus. Then d_T and d_L defines the same topology on $T(R_0)$.*

Since the Riemann surface of Theorem 1.5 is of finite genus, it does not extend Shiga's Theorem. However, 1.5 does not restrict his condition. There exist Riemann surfaces satisfying our condition but not satisfying his condition. See Section 2 of [8] or Section 2 of [9].

Next we consider Lipschitz continuity between the two metric spaces. Alessandrini-Liu-Papadopoulos-Su ([2]) proved the following.

Theorem 1.6 ([2], Theorem 1.4)**.** *Let R_0 be a Riemann surface satisfying Shiga's condition. Then the identity map $(T(R_0), d_L) \to (T(R_0), d_T)$ is locally bi-Lipschitz.*

In this paper, we extend the above three theorems 1.3, 1.5, and 1.6. Our main result is the following.

Theorem 1.7. *Let R_0 be a Riemann surface with bounded geometry. Then the identity map $(T(R_0), d_L) \to (T(R_0), d_T)$ is locally bi-Lipschitz.*

To make it easier to read this paper, we describe the outline of the proof of Theorem 1.7. For a Riemann surface R_0 with bounded geometry, let $M > 0$ be a constant satisfying the condition (BG) (in Definition 1.4). We would like to see that $\frac{1}{C} d_T(p,q) \le d_L(p,q) \le C d_T(p,q)$ for any $p, q \in B(p_0, r)$, where $p_0 = [R_0, id]$ is a basepoint of $T(R_0)$, $B(p_0, r) = \{p \in R_0 \mid d_L(p, p_0) < r\}$ is some neighborhood (with respect to d_L) of p_0 and $C = C(M, r) \ge 1$. By Sorvali's Lemma, $d_L(p,q) \le d_T(p,q)$ holds for any $p, q \in T(R_0)$, thus it is sufficient to show that $\frac{1}{C} d_T(p,q) \le d_L(p,q)$. In proving it, the following is a Key Lemma.

Lemma 1.8. *Let R_0 be a Riemann surface with bounded geometry. Then R_0 is decomposed into pairs of pants $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$, (where some component of ∂P_n may be a puncture) and hyperbolic right hexagons $\mathcal{H} = \{H_m\}_{m=1}^{\infty}$ satisfying the following conditions:*

1. *$\mathring{P}_n \cap \mathring{P}_k = \emptyset$ $(n \ne k)$, $\mathring{P}_n \cap \mathring{H}_m = \emptyset$, $\mathring{H}_m \cap \mathring{H}_l = \emptyset$ $(m \ne l)$ and $R_0 = \bigcup_{n=1}^{\infty} P_n \cup \bigcup_{m=1}^{\infty} H_m$. Here, every other edge of H_m $(m = 1, 2, \ldots)$ is contained in some ∂P_n.*
2. *The lengths of all closed geodesics in $\{\partial P_n\}_{n=1}^{\infty}$ are uniformly bounded from above and below.*
3. *The lengths of all edges of hexagons $\{H_m\}_{m=1}^{\infty}$ are uniformly bounded from above and below.*

If R_0 satisfies Shiga's condition, then R_0 can be decomposed by the above pairs of pants and hexagons. However, the converse is not true; that is, even if R_0 can be decomposed by those, it does not necessarily satisfy Shiga's condition (cf. [8], §2).

We prove this lemma in Section 2. And we prove Theorem 1.7 in Section 3 by taking the following steps.

Step 1. Take two arbitrary points $p_1 = [R_1, f_1], p_2 = [R_2, f_2] \in B(p_0, r)$. Decompose R_1 into pairs of pants $\mathcal{P}_1$ and right hexagons $\mathcal{H}_1$ by Lemma 1.8. Also, decompose R_2 into pairs of pants $\mathcal{P}_2$ corresponding to $\mathcal{P}_1$ and right hexagons $\mathcal{H}_2$ corresponding to $\mathcal{H}_1$.

Step 2. Construct a quasiconformal mapping $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $K(\varphi) < C(M, r)$, where $C(M, r) \to 1$ as $r \to 0$. We use Bishop's lemmas.

Step 3. Construct a quasiconformal mapping $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$ such that $K(\varphi) < C(M, r)$, where $C(M, r) \to 1$ as $r \to 0$. In constructing, take notice of the twist along any closed geodesic α of ∂P for any $P \in \mathcal{P}_1$. We use the fact that the twist is uniformly bounded since $d_L(p_1, p_2) < 2r$ and $\ell_{R_1}(\alpha)$ is uniformly bounded.

In these steps, we obtain a quasiconformal mapping $\varphi : R_1 \to R_2$ such that $K(\varphi) < C(M, r)$, where $C(M, r) \to 1$ as $r \to 0$. We can check that φ is homotopic to $f_2 \circ f_1^{-1}$. This means $d_T(p_1, p_2) < \log C(M, r)$, where $\log C(M, r) \to 0$ as $r \to 0$, thus $d_T(p_1, p_2) \leq C d_L(p_1, p_2)$.

2. Proof of the Key Lemma

In this section, we prove Lemma 1.8. First, we take disjoint simple closed geodesics all over R_0. Here, R_0' is a subsurface of R_0 obtained by removing all cusp neighborhoods from R_0.

Lemma 2.1. *Let R_0 be a Riemann surface with bounded geometry and let $M > 0$ be a constant satisfying condition (BG) in Definition* 1.4. *Then there exists a family of pairwise disjoint and simple closed geodesics $\mathcal{G} := \{\alpha_n\}_{n=0}^{\infty}$ in R_0 satisfying following conditions.*

(1) $1/M < \ell_{R_0}(\alpha_n) < M$ *for any* $\alpha_n \in \mathcal{G}$.
(2) *There exists a constant $D = D(M) > 0$ such that for any $x \in R_0'$, $d_{R_0}(x, \alpha_n) < D$ holds for some $\alpha_n \in \mathcal{G}$, where $d_{R_0}(,)$ is hyperbolic distance on R_0.*

Proof. By the definition of bounded geometry, for any $x \in R_0'$, there exists a simple closed curve c_x based on x with $1/M < \ell_{R_0}(c_x) < M$. First, we take some constant $D = D(M) > 0$ as follows. For each $x \in R_0'$, put

$$d_x = \max_{c_x} \ d_{R_0}([c_x], x),$$

where c_x is taken over all simple closed curves based on x with the length $\in (1/M, M)$, and $[c_x]$ is a closed geodesic homotopic to c_x.

Claim 2.2. $d_x < 2\log(\sqrt{2}M) + \frac{1}{2}M$ holds for any $x \in R_0'$.

Proof. For an arbitrary point $x \in R_0'$, take an arbitrary simple closed curve c_x based on x with length $\in (1/M, M)$. If $[c_x] \cap c_x \neq \emptyset$, then $d_{R_0}([c_x], x) < \frac{1}{2}\ell_{R_0}(c_x) < \frac{1}{2}M$. If $[c_x] \cap c_x = \emptyset$, then there exists a cylinder C_x bounded by $[c_x]$ and c_x. Let $r_x > 0$ be the distance from $[c_x]$ to c_x. In the cylinder C_x, take an r_x-neighborhood of $[c_x]$ (Figure 1), i.e.,

$$C_x' = \{y \in C_x \mid d_{R_0}([c_x], y) < r_x\}.$$

Put $\zeta := \partial C_x' - [c_x]$, then

$$\ell_{R_0}(\zeta) = \ell_{R_0}([c_x]) \cdot \cosh r_x$$

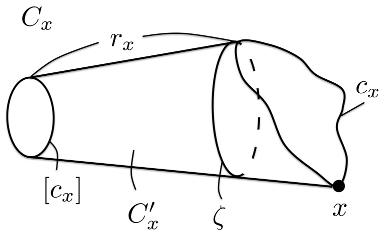

FIGURE 1. A cylinder C_x bounded by $[c_x]$ and c_x, and a subcylinder $C'_x \subset C_x$.

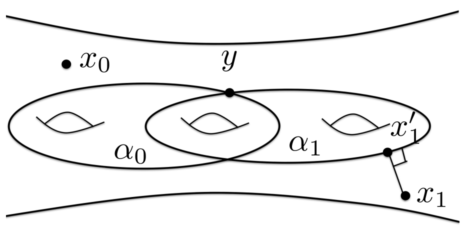

FIGURE 2. The case where $\alpha_0 \cap \alpha_1 \neq \emptyset$.

holds (cf. §4.1 or §5.2 of [4]). Since $\ell_{R_0}([c_x]) > 1/M$ and $\ell_{R_0}(\zeta) \leq \ell_{R_0}(c_x) < M$,

$$\cosh r_x = \frac{\ell_{R_0}(\zeta)}{\ell_{R_0}([c_x])} < M^2.$$

Since $e^{r_x}/2 < \cosh r_x$ holds, $e^{r_x} < 2M^2$, that is, $r_x < 2\log(\sqrt{2}M)$. Therefore $d_{R_0}([c_x], x) \leq r_x + d_{R_0}(\zeta, x) < r_x + 1/2 \cdot \ell_{R_0}(c_x) < 2\log(\sqrt{2}M) + 1/2M$. □

Put

$$d = \sup_{x \in R'_0} d_x.$$

Then $d = d(M)$ is a constant such that for any $x \in R'_0$, there exists a closed geodesic α_x satisfying $\ell_{R_0}(\alpha_x) < M$ and $d_{R_0}(x, \alpha_x) < d$. Indeed, for an arbitrary $x \in R'_0$, let c_x be an arbitrary simple closed curve based on x with $1/M < \ell_{R_0}(c_x) < M$. Then $\ell_{R_0}([c_x]) < M$ and $d_{R_0}([c_x], x) \leq d_x < d$. Put

$$D = 2\max\{M, d\}.$$

By using constants M, d, D, we shall take pairwise disjoint simple closed geodesics $\mathcal{G} = \{\alpha_n\}_{n=0}^{\infty}$. Let x_0 be an arbitrary point in R'_0 and take a closed geodesic α_0 with the length $< M$ and with $d_{R_0}(\alpha_0, x_0) < d$. Let U_0 be the D-neighborhood of α_0, i.e.,

$$U_0 := B(\alpha_0, D) = \{x \in R'_0 \mid d_{R_0}(\alpha_0, x) < D\}.$$

Now take a point $x_1 \in \partial U_0$. And for x_1, take a simple closed geodesic α_1 with $\ell_{R_0}(\alpha_1) < M$ and with $d_{R_0}(x_1, \alpha_1) < d$. Then:

Claim 2.3. $\alpha_0 \cap \alpha_1 = \emptyset$ and $U_0 \cap B(\alpha_1, D) \neq \emptyset$, where $B(\alpha_1, D)$ is the D-neighborhood of α_1 in R'_0.

Proof. Assume that $\alpha_0 \cap \alpha_1 \neq \emptyset$ and take a point $y \in \alpha_0 \cap \alpha_1$. Now let x'_1 be the point on α_1 that is closest to x_1. (See Figure 2.)

Since $y \in \alpha_1$,

$$\begin{aligned} d_{R_0}(y, x_1) &< d_{R_0}(y, x_1') + d_{R_0}(x_1', x_1) \\ &\leq \frac{1}{2}\ell_{R_0}(\alpha_1) + d_{R_0}(x_1', x_1) \\ &< \frac{1}{2}M + d \\ &< \frac{3}{2}\max\{M, d\}. \end{aligned}$$

On the other hand, $d_{R_0}(\alpha_0, x_1) < d_{R_0}(y, x_1)$ holds since $y \in \alpha_0$. Hence

$$d_{R_0}(\alpha_0, x_1) < \frac{3}{2}\max\{M, d\} < D.$$

This contradicts the assumption that $x_1 \in \partial U_0$, that is, $d_{R_0}(\alpha_0, x_1) = D$. Therefore $\alpha_0 \cap \alpha_1 = \emptyset$.

Next we consider the d-neighborhood $B(x_1, d)$ of x_1. Since x_1 a point in ∂U_0, $U_0 \cap B(x_1, d) \neq \emptyset$. For any $x \in U_0 \cap B(x_1, d)$,

$$d_{R_0}(x, \alpha_1) < d_{R_0}(x, x_1) + d_{R_0}(x_1, \alpha_1) < d + d \leq D.$$

Hence $x \in U_0$ and $x \in B(\alpha_1, D)$. Therefore $U_0 \cap B(\alpha_1, D) \neq \emptyset$. □

Next take an arbitrary point x_2 in $\partial U_0 - B(\alpha_1, D)$. Also take a simple closed geodesic α_2 such that $\ell_{R_0}(\alpha_2) < M$ and $d_{R_0}(x_2, \alpha_2) < d$. We can check that α_0, α_1 and α_2 are disjoint and $U_0 \cap B(\alpha_2, D) \neq \emptyset$ since $d_{R_0}(\alpha_0, x_2) = D$ and $d_{R_0}(\alpha_1, x_2) \geq D$. Continue to take a point in $\partial U_0 - \bigcup_i B(\alpha_i, D)$ and a geodesic until $\partial U_0 \subset \bigcup_{i=1}^{n(1)} B(\alpha_i, D)$.

After that, put

$$U_1 := U_0 \cup (\bigcup_{i=1}^{n(1)} B(\alpha_i, D)).$$

Next, for this, take a point $x_{n(1)+1} \in \partial U_1$, and take a simple closed geodesic $\alpha_{n(1)+1}$ such that $\ell_{R_0}(\alpha_{n(1)+1}) < M$ and $d_{R_0}(\alpha_{n(1)+1}, x_{n(1)+1}) < d$. Then $\alpha_0, \alpha_1, ..., \alpha_{n(1)}$ and $\alpha_{n(1)+1}$ are disjoint and $U_1 \cap B(\alpha_{n(1)+1}, D) \neq \emptyset$. Similarly, take points and closed geodesics $\alpha_{n(1)+2}, ..., \alpha_{n(2)}$ until $\partial U_1 \subset \bigcup_{i=n(1)+1}^{n(2)} B(\alpha_i, D)$. And put

$$U_2 := U_1 \cup (\bigcup_{i=n(1)+1}^{n(2)} B(\alpha_i, D)).$$

Continue to take a union U_L of the D-neighborhoods of geodesics such that $U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_L \subset R_0'$. Then, for any closed geodesic α_n taken in U_L, the length $< M$ holds, and for any $x \in U_L$, $d_{R_0}(x, \alpha_i) < D$ for some $\alpha_i \in \{\alpha_n\}_{n=0}^{n(L)}$ since $x \in B(\alpha_i, D)$ for some $\alpha_i \in \{\alpha_n\}_{n=0}^{n(L)}$. Put $U := \lim_{L\to\infty} U_L$, then $U = R_0'$. Indeed, $U \subset R_0'$ is trivial. Assume that $R_0' \not\subset U$. For any $x \in R_0' - U$, take the shortest geodesic segment s from x to α_0 (the first closed geodesic in U_0). Then there exists a point $y \in s \cap \partial U$. Since U is a union of neighborhoods $\{B(\alpha_i, D)\}$ of closed geodesics, there exists a number i such that $y \in \partial B(\alpha_i, D)$. However, for any i, $\partial B(\alpha_i, D)$ is covered by some neighborhoods $\{B(\alpha_j, D)\}$ by construction. This is a contradiction. Hence $R_0' \subset U$. Therefore, we obtain simple closed geodesics $\mathcal{G} = \{\alpha_n\}_{n=0}^{\infty}$ as desired. □

By using Lemma 2.1, we prove the Key Lemma.

Proof of Lemma 1.8. Let $\mathcal{G} = \{\alpha_n\}_{n=0}^{\infty}$ be the closed geodesics in Lemma 2.1. For each $\alpha_n \in \mathcal{G}$, take the following domain:

$$D_n := \{x \in R_0' \mid d_{R_0}(x, \alpha_n) < d_{R_0}(x, \alpha_\ell) \text{ for any } \ell \neq n\}.$$

We call D_n *a Voronoi domain of* α_n. Note that for any point $x \in \partial D_n$, $d_{R_0}(x, \alpha_n) \leq D$ holds by condition (2) in Lemma 2.1. Also, for any point $x \in \partial D_n$, there exists a Voronoi domain D_m of $\alpha_m (\neq \alpha_n)$ such that $x \in \partial D_m$ and $d_{R_0}(x, \alpha_n) = d_{R_0}(x, \alpha_m)$. Then $d_{R_0}(\alpha_n, \alpha_m) \leq 2D$.

We shall consider D_n and a domain $[D_n]$ obtained by straightening each component of ∂D_n for each n. (We assume that any component of ∂D_n is simple. If some component has self-intersection points, divide it into simple closed curves.) We describe how to take $[D_n]$: If $c \cap \partial D_n = \emptyset$ for any horocycle c bounding a cusp neighborhood of R_0, let $[D_n] \subset R_0'$ be a domain obtained by replacing each component σ_n of ∂D_n with the closed geodesic $[\sigma_n]$ homotopic to σ_n. Then note the following.

Claim 2.4. There exists a constant $E = E(M)$ such that for any $x \in [\sigma_n]$, $d_{R_0}(\alpha_n, x) \leq E$ $(n = 0, 1, 2...)$.

Proof. First, note that there exists a point $x_0 \in [\sigma_n]$ such that $d_{R_0}(\alpha_n, x_0) < 2D$ holds. Indeed, for σ_n, there exists a Voronoi domain D_m of $\alpha_m (\neq \alpha_n)$ such that $\sigma_n \cap \partial D_m \neq \emptyset$. Let s_{nm} be the shortest geodesic segment connecting α_n and α_m, then $\ell_{R_0}(s_{nm}) < 2D$. Any closed curve homotopic to σ_n intersects s_{nm}, therefore there exists a point $x_0 \in [\sigma_n] \cap s_{nm}$.

Next we take a constant E. In the unit disk $\mathbb{D}$, take an arbitrary segment α with the hyperbolic length M and let L be the length of the boundary $\partial B(\alpha, 2D)$ of the $2D$-neighborhood of α. We put $E := 2D + \frac{1}{2}L$.

Now we assume that there exists a point $x_1 \in [\sigma_n]$ such that $d_{R_0}(\alpha_n, x_1) > E$. By existence of x_0 and x_1, we can take some points $x_2, x_3 \in [\sigma_n]$ around x_1 such that $d_{R_0}(\alpha_n, x_i) = 2D$ $(i = 2, 3)$ and they are put on $[\sigma_n]$ in the order of x_2, x_1, x_3. Here, let S be the set surrounded by σ_n and $[\sigma_n]$, then S has genus zero and no horocycles (bounding cusp neighborhoods). In S, we can take an arc $c[x_2, x_3]$ such that its endpoints are x_2 and x_3, and $d_{R_0}(x, \alpha_n) = 2D$ for any $x \in c[x_2, x_3]$ since $d_{R_0}(y, \alpha_n) \leq D$ for any $y \in \sigma_n$. Then $\ell_{R_0}(c[x_2, x_3]) < L$ since $\ell_{R_0}(\alpha_n) < M$. On the other hand, let $g[x_i, x_j]$ be the geodesic segment on $[\sigma_n]$ such that its endpoints are x_i and x_j, and satisfies

$$\ell_{R_0}(g[x_2, x_3]) = \ell_{R_0}(g[x_2, x_1]) + \ell_{R_0}(g[x_1, x_3]).$$

Since $d_{R_0}(\alpha_n, x_1) > E = 2D + \frac{1}{2}L$ and $d_{R_0}(\alpha_n, x_i) = 2D$ $(i = 2, 3)$,

$$d_{R_0}(x_1, x_i) \geq d_{R_0}(\alpha_n, x_1) - d_{R_0}(\alpha_n, x_i) > 2D + \frac{1}{2}L - 2D = \frac{1}{2}L$$

$(i = 2, 3)$. Hence

$$\ell_{R_0}(g[x_2, x_3]) > \frac{1}{2}L + \frac{1}{2}L = L;$$

that is, $\ell_{R_0}(c[x_2, x_3]) < L < \ell_{R_0}(g[x_2, x_3])$. This is a contradiction. □

Next we consider the case where there exist a horocycle c bounding a cusp neighborhood and a domain D_n such that $c \cap \partial D_n \neq \emptyset$. We do not want to leave $R_0 - \cup_n [D_n]$ having punctures, so we take $[D_n]$ as follows: For a horocycle c, let

$\{D_i\}_{i\in I}$ be all Voronoi domains with $\partial D_i \cap c \neq \emptyset$ and take an arbitrary D_n in $\{D_i\}_{i\in I}$. For D_n, let σ_n be a component of ∂D_n with $c\cap\sigma_n \neq \emptyset$. Let c_n denote the intersection of c with the interior of D_n and put $\sigma'_n = ((\sigma_n \cap R'_0)\cup c) - c_n$. (See Figure 3.) And take a closed geodesic $[\sigma'_n]$ as a boundary component of $[D_n]$. On the other hand, for the other Voronoi domain D_m $(m \in I - \{n\})$, take a closed geodesic $[\sigma_m]$ homotopic to an original boundary component σ_m (i.e., a curve without adding or removing subhorocycles of c) as a boundary component of $[D_m]$. Note that for any $x \in [\sigma'_n]$ (or $x \in [\sigma_m]$), $d_{R_0}(\alpha_n, x) < E$ (or $d_{R_0}(\alpha_m, x) < E$) for the same constant $E = E(M)$ as in Claim 2.4 since $d_{R_0}(\alpha_n, \alpha_m) \leq 2D$.

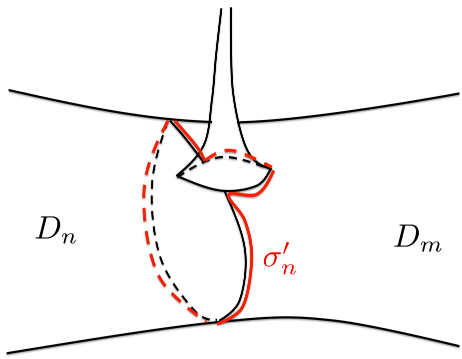

Figure 3. σ'_n

Now we consider pants decomposition in each $[D_n]$. First, let us check that genus g_n of $[D_n]$ and the number of connected components of $\partial[D_n]$ are uniformly bounded. For each $[D_n]$, let h_n be the number of horocycles bounding cusp neighborhoods and b_n be the number of closed geodesics of $\partial[D_n]$.

Claim 2.5. There exists a constant $A = A(M) > 0$ such that $g_n < A, h_n < A$ and $b_n < A$ for any $[D_n]$.

Proof. In the unit disk $\mathbb{D}$, consider an arbitrary geodesic segment α with the hyperbolic length M. And let A be the area of the E-neighborhood of α. Then (the area of $[D_n]$) $< A$, since $[D_n]$ is contained in the E-neighborhood of α_n.

Now let $[\hat{D_n}]$ be a surface obtained by adding original cusp neighborhoods of R_0 to all horocycles. Then, since a cusp neighborhood has area one, (the area of $[\hat{D_n}]$) = (the area of $[D_n]$) $+ h_n < A + h_n$.

On the other hand, $[\hat{D_n}]$ is of genus g_n and with h_n punctures and b_n boundary geodesics, so, by the theorem of Gauss-Bonnet,

$$\text{(The area of } [\hat{D_n}]) = 2\pi(2g_n - 2 + h_n + b_n) < A + h_n.$$

Therefore

$$4\pi g_n + (2\pi - 1)h_n + 2\pi b_n < A + 4\pi.$$

Put $A := A + 4\pi$, then $g_n < A, h_n < A$ and $b_n < A$ hold. $\square$

Also, we check the following:

Claim 2.6. There exists a constant $B = B(M) > 0$ such that $\ell_{R_0}([\sigma_n]) < B$ for any $[D_n]$ and any component $[\sigma_n]$ of $\partial[D_n]$.

Proof. In the unit disk $\mathbb{D}$, take an arbitrary segment α with the hyperbolic length M and let K be the length of the boundary $\partial B(\alpha, E)$ of the E-neighborhood of α. Put $B := 2(K+E)$.

Now, for $[\sigma_n]$ and α_n, take the shortest geodesic segment s connecting them, and let $x \in [\sigma_n]$ be the endpoint of s, and let ℓ be the length of s. Then $\ell \leq E$. Let y be the point on $[\sigma_n]$ such that $d_{R_0}(x,y) = \frac{1}{2}\ell_{R_0}([\sigma_n])$, and let $\frac{1}{2}[\sigma_n]$ be the half of $[\sigma_n]$ with endpoints x and y. Note that $\frac{1}{2}[\sigma_n]$ is the shortest geodesic segment between x and y since $[D_n]$ is convex by the definition. Also, take the shortest geodesic segment s' from y to α_n. Note that $\ell \leq \ell_{R_0}(s') \leq E$. Now, on s', take the point z such that $d_{R_0}(z, \alpha_n) = \ell$ and take the shortest geodesic segment s_1 from x to z. And let s_2 be the shortest geodesic segment from z to y. Then

$$\ell_{R_0}(\frac{1}{2}[\sigma_n]) \leq \ell_{R_0}(s_1) + \ell_{R_0}(s_2)$$

holds. Since $\ell_{R_0}(s_2) < E$ and we see that $\ell_{R_0}(s_1) < K$, hence $\frac{1}{2}\ell_{R_0}([\sigma_n]) = \ell_{R_0}(\frac{1}{2}[\sigma_n]) < K + E = \frac{1}{2}B$. $\square$

From Claims 2.5 and 2.6 and Bers' constant proved by Farb and Margalit ([7], §12.4.2) or Buser ([4], §5.2), we obtain the following:

Claim 2.7. There exists a constant $F = F(M) > 0$ and pairs of pants $\{P_k\}$ such that $[\hat{D_n}] = \bigcup_k P_k$, and for any closed geodesic γ_i in ∂P_k, $\ell_{R_0}(\gamma_i) < F$ $(i = 1, ..., 3g_n - 3 + h_n + b_n)$ for each $[\hat{D_n}]$.

Let $\mathcal{P} = \{P_k\}_{k=0}^{\infty}$ be a family of all pairs of pants in all domains $\{[\hat{D_n}]\}_{n=0}^{\infty}$. Finally we take hexagons in $R_0 - \bigcup_{k=0}^{\infty} P_k$. Let $\mathcal{D} = \{D_n\}_{n=0}^{\infty}$ be all Voronoi domains and take an arbitrary domain $D_0 \in \mathcal{D}$. Let $D_1 \in \mathcal{D}$ be a domain with $\partial D_0 \cap \partial D_1 \neq \emptyset$.

Case I: Suppose $\partial D_2 \cap (\partial D_0 \cap \partial D_1) = \emptyset$ for any $D_2 \in \mathcal{D}$.

By construction, there exist two pairs of pants $P_0 \subset [D_0]$ and $P_1 \subset [D_1]$ with $\partial P_0 \cap \partial P_1 \neq \emptyset$, that is, $D_0 \cup D_1$ is connected, hence we do not take a hexagon. (See Figure 4.)

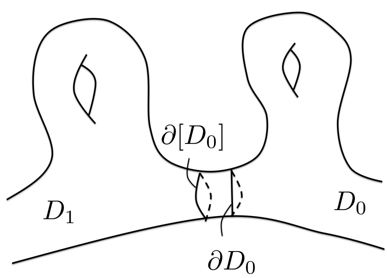

FIGURE 4. Case I

Case II: Suppose $\partial D_2 \cap (\partial D_0 \cap \partial D_1) \neq \emptyset$ for some $D_2 \in \mathcal{D}$.

Note that $\partial D_2 \cap (\partial D_0 \cap \partial D_1)$ consists of a point or finite points. Take a point $v \in \partial D_2 \cap (\partial D_0 \cap \partial D_1)$ and let $\{D_0, D_1, ..., D_\ell\} \subset \mathcal{D}$ be a set of all domains each of whose boundary contains v and which is in counterclockwise direction around v. Connect v to $\partial[D_i]$ with the shortest segment s_i for each $i = 0, 1, ..., \ell$. (See Figure 5.) Regard $s_i \cdot s_{i+1}$ as an arc connecting $\partial[D_i]$ and $\partial[D_{i+1}]$, and take the shortest geodesic

$[s_i \cdot s_{i+1}]$ homotopic to $s_i \cdot s_{i+1}$ for each i, where the homotopy glides endpoints on each closed geodesic. Then we get a $2(\ell+1)$-sided right polygon. If $\ell \geq 3$, take the geodesics $[s_0 \cdot s_2], ..., [s_0 \cdot s_{\ell-1}]$. Then we get $(\ell-1)$ right hexagons. The length of each edge is bounded by $2E$ since $\ell_{R_0}(s_i) < \max\{d_{R_0}(\partial D_i, \alpha_i), d_{R_0}(\partial[D_i], \alpha_i)\} \leq E$ for each i.

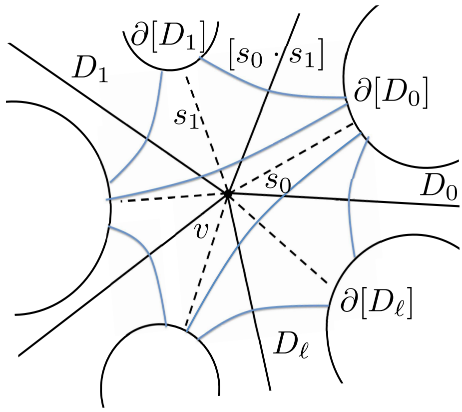

Figure 5. Right hexagons around v

Take hexagons for each intersection of boundaries of $\{D_n\}$. Then a family of hexagons $\{H_m\}_{m=1}^{\infty}$ we desire is obtained. □

3. Proof of Theorem 1.7

Before starting to prove Theorem 1.7, we introduce Bishop's lemmas on quasiconformal mappings.

Lemma 3.1 (Bishop [3], Lemma 3.1)**.** *Let $T_1, T_2 \subset \mathbb{D}$ be two hyperbolic triangles with sides (a_1, b_1, c_1) and (a_2, b_2, c_2), respectively. Suppose all their angles are bounded below by $\theta > 0$ and*

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$

Then there is a constant $C = C(\theta, A) > 0$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi : T_1 \to T_2$ such that φ maps each vertex to the corresponding vertex and φ is affine on each edge of T_1.

Lemma 3.2 (Bishop [3], Corollary 3.3)**.** *Let $H, H' \subset \mathbb{D}$ be two hyperbolic hexagons with sides $(a_1, \ldots, a_6)$ and $(b_1, \ldots, b_6)$, respectively. Suppose $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$ are bounded above by a constant B and below by $1/B$. Also assume that three alternating angles of H and the corresponding angles of H' are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. If $\varepsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a constant $C = C(\theta, B) > 0$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi : H \to H'$ such that φ maps each vertex to the corresponding vertex and φ is affine on each edge of H.*

Let us start the proof of Theorem 1.7. For a sufficiently small number $r > 0$ and a basepoint $p_0 = [R_0, id]$, put $B(p_0, r) = \{p \in T(R_0) \mid d_L(p_0, p) < r\}$. For any $p_1, p_2 \in B(p_0, r)$, we show that $\frac{1}{C} d_T(p_1, p_2) \leq d_L(p_1, p_2)$. Put $p_1 := [R_1, f_1]$, $p_2 := [R_2, f_2]$ and $g := f_2 \circ f_1^{-1}$.

Since R_0 has bounded geometry for a constant M, R_1 has bounded geometry for some constant M_1 depending on M and r. Hence R_1 can be decomposed into pairs of pants $\mathcal{P}_1$ and right hexagons $\mathcal{H}_1$ by Lemma 1.8. First, let us consider the corresponding decomposition in R_2. Note that $d_L(p_1, p_2) < 2r$, thus

$$\left| \log \frac{\ell_{R_1}([\alpha])}{\ell_{R_2}([g(\alpha)])} \right| < 2r$$

holds for any $\alpha \in \mathcal{C}(R_1)$. For an arbitrary hexagon $H \in \mathcal{H}_1$, let c_1, c_2, c_3 be connected components of boundaries of three pairs of pants $P_1, P_2, P_3 \in \mathcal{P}_1$ around H. Let $e_1, ..., e_6$ be the edges of H (in counterclockwise direction), where e_2 is the segment connecting c_1 and c_2. (See Figure 6.) Take a closed curve $c_{12} := c_1 \cdot e_2 \cdot c_2 \cdot e_2^{-1}$ in R_1. For $g(c_{12})$ in R_2, take a closed geodesic $[g(c_{12})]$ and consider a pair of pants with boundary $\{[g(c_1)], [g(c_2)], [g(c_{12})]\}$. Let e_2' be the geodesic orthogonal to $[g(c_1)]$ and $[g(c_2)]$ in the pants. Since the lengths of e_2 and e_2' are determined by the lengths of $\{c_1, c_2, c_{12}\}$ and $\{[g(c_1)], [g(c_2)], [g(c_{12})]\}$, respectively,

$$\frac{1}{A} < \frac{\ell_{R_1}(e_2)}{\ell_{R_2}(e_2')} < A$$

holds, where A is a constant depending on r and M. Similarly, take segments e_4', e_6' for e_4, e_6, respectively. Let H' be a right hexagon with edges e_2', e_4', e_6' and subarcs e_1', e_3', e_5' of $[g(c_1)]$, $[g(c_2)]$, $[g(c_3)]$ in R_2. Then

$$\frac{1}{B} < \frac{\ell_{R_1}(e_i)}{\ell_{R_2}(e_i')} < B$$

$(i = 1, 3, 5)$ holds, where B is a constant depending on r and M, since a right hexagon is determined by the lengths of three alternating edges. For each hexagon in R_1, take a hexagon in R_2 in the above way; then we have a similar family of hexagons $\mathcal{H}_2 \subset R_2$. And we have a quasiconformal map φ from $\mathcal{H}_1$ to $\mathcal{H}_2$ by Lemma 3.2.

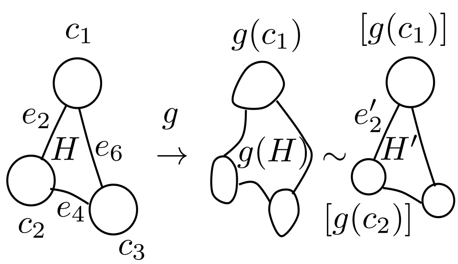

FIGURE 6. $H \in \mathcal{H}_1$ and $H' \in \mathcal{H}_2$.

Next we consider the quasiconformal map on $\mathcal{P}_1$. Let P_n be an arbitrary pair of pants in $\mathcal{P}_1$ and $\{\alpha_1, \alpha_2, \alpha_3\}$ be boundary components of P_n. (If some α_i is a puncture, replace it with a horocycle.) We consider triangulation of P_n. First, take a subset $X_j = \{x_i\}$ of α_j $(j = 1, 2, 3)$ consisting of vertices of hexagons $\{H_m\}$ with $\alpha_j \cap H_m \neq \emptyset$. If there exists a closed geodesic α_j such that $\alpha_j \cap H_m = \emptyset$ for any $H_m \in \mathcal{H}_1$, then $X_j = \emptyset$. Also, decompose P_n into two symmetric right hexagons

$\{h_1, h_2\}$ and let $Y = \{y_1, ..., y_6\}$ (counterclockwise direction) be a subset of ∂P_n consisting of the vertices of $\{h_1, h_2\}$, where the segment $[y_1, y_2]$ are connecting α_3 and α_1, and $[y_2, y_3] \subset \alpha_1$. (See Figure 7.) Now let w be a midpoint of $[y_1, y_2]$ and put $Z := X_1 \cup X_2 \cup X_3 \cup Y$. Connect w to each point of Z by the geodesic segment $[w, x_i]$ or $[w, y_k]$; then we obtain triangulation of P_n.

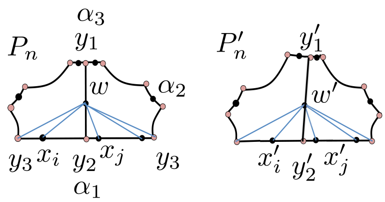

FIGURE 7. $Z \subset \partial P_n$ and $Z' \subset \partial P'_n$.

Second, we consider triangulation of pairs of pants in R_2. Let $P'_n \in \mathcal{P}_2$ be a pair of pants with boundary $\{[g(\alpha_1)], [g(\alpha_2)], [g(\alpha_3)]\}$. Let $X'_j = \{x'_i\}$ $(j = 1, 2, 3)$ be a subset of $[g(\alpha_j)]$ such that each point x'_i is the vertex of $H'_m \in \mathcal{H}_2$ corresponding to $x_i \in R_1$. If $X_j = \emptyset$, then $X'_j = \emptyset$. Also, we take points Y' corresponding Y as follows. If $X'_j \neq \emptyset$, let y'_k be the point satisfying $\ell_{R_1}[x_i, y_k]/\ell_{R_1}[x_i, x_j] = \ell_{R_2}[x'_i, y'_k]/\ell_{R_2}[x'_i, x'_j]$, where $[\cdot, \star]$ is a segment with endpoints $\cdot$ and $\star$, and $[x_i, x_j], [x'_i, x'_j]$ are the smallest segments including y_k and y'_k, respectively. If $X'_j = \emptyset$, let y'_k be the point on $[g(\alpha_j)]$ which is one of vertices of two symmetric right hexagons $\{h'_1, h'_2\}$ decomposing P'_n. As above, we assume that the segment $[y'_1, y'_2]$ are connecting $[g(\alpha_3)]$ and $[g(\alpha_1)]$, and $[y'_2, y'_3] \subset [g(\alpha_1)]$. Let w' be a midpoint of $[y'_1, y'_2]$ and put $Z' := X'_1 \cup X'_2 \cup X'_3 \cup Y'$. Connect w' to each point of Z' by the geodesic segment $[w', x'_i]$ or $[w', y'_k]$; then we obtain triangulation of P'_n. Note the lengths of the corresponding sides of triangles.

Claim 3.3. There exists a constant $D = D(r, M)$ such that for any i and k,

$$\frac{1}{D} < \frac{\ell_{R_1}([w, x_i])}{\ell_{R_2}([w', x'_i])} < D \text{ and } \frac{1}{D} < \frac{\ell_{R_1}([w, y_k])}{\ell_{R_2}([w', y'_k])} < D$$

hold.

Proof. For P_n, take the nearest pair of pants $P_\ell \in \mathcal{P}_1 - \{P_n\}$ to α_1 such that $\partial P_n \cap \partial P_\ell = \emptyset$. Also take the shortest arc β connecting α_1 and a component α_ℓ of ∂P_ℓ, and let P be a pair of pants obtained by α_1, α_ℓ and $\alpha_1 \cdot \beta \cdot \alpha_\ell \cdot \beta^{-1}$. (See Figure 8.) Put $S_1 := P \cup P_n \cup P_\ell (\subset R_1)$, then S_1 is a Riemann subsurface satisfying Shiga's condition. Let $\hat{S}_1$ be a Nielsen extension of S_1 and consider the Teichmüller space $T(\hat{S}_1)$ and the Fenchel-Nielsen coordinate $(\ell_{\hat{S}_1}(\alpha), \theta_{\hat{S}_1}(\alpha))$ for the pants decomposition $S_1 = P \cup P_n \cup P_l$. Now put $S_2 := g(S_1) (\subset R_2)$ and let $\hat{S}_2$ be a Nielsen extension of S_2. Since

$$\left| \log \frac{\ell_{\hat{S}_1}(\alpha)}{\ell_{\hat{S}_2}([g(\alpha)])} \right| < 2r$$

holds for any $\alpha \in \mathcal{C}(\hat{S}_1)$,

$$\left| \theta_{\hat{S}_1}(\alpha_1) - \theta_{\hat{S}_2}([g(\alpha_1)]) \right| < E \cdot 2r$$

holds for the twist parameters $\theta_{\hat{S_1}}(\alpha_1)$ and $\theta_{\hat{S_2}}([g(\alpha_1)])$, where $E = E(r, M)$ is a constant depending only on r and M (cf. Proposition 3.3 of [2] or Lemma 4.1 of [14]). Hence, for some constant $D = D(r, M)$, the statement is true. $\square$

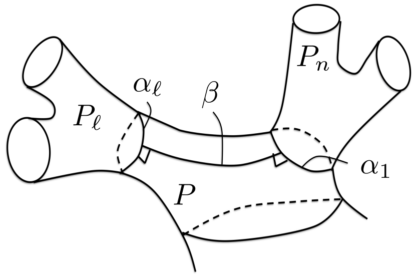

FIGURE 8. $S_1 = P \cup P_n \cup P_\ell$

Similarly, the angles of the triangles in P_n are bounded below by some constant $\theta(r, M) > 0$ since the lengths of sides of triangles are bounded above by some constant depending on r and M. Now triangulate all pairs of pants in $\mathcal{P}_1$ in the above way. For each triangle in each pair of pants, we define the quasiconformal mapping given by Lemma 3.1. Then we obtain a quasiconformal map $\varphi : \mathcal{P}_1 \to \mathcal{P}_2$. Put $\varepsilon := \max\{\log A, \log B, \log D\}$. Then we obtain a quasiconformal map $\varphi : R_1 \to R_2$ such that $K(\varphi) = 1 + \varepsilon \cdot F$, where F is a constant depending r, M and ε. Lastly, we show the following.

Claim 3.4. g is homotopic to φ.

Proof. It is sufficient to show that for any simple closed curve c in R_1, $g(c)$ and $\varphi(c)$ are homotopic (cf. [5], Lemma 4). Let X be a finite type subsurface of R_1 that is closed (as a subset), contains c, and can be written as a union of triangles appearing in the given triangulation of R_1. By construction and the fact that X has finite topology, there is a homotopy $H : R_1 \times I \to R_2$ such that $H(x, 0) = g(x)$ and $H|_{T\times\{1\}} = \varphi|_T$ for every triangle T in X. In particular, we see that $\varphi^{-1} \circ H : R_1 \times I \to R_1$ is a homotopy between $\varphi^{-1} \circ g$ and a map restricting to the identity on each triangle in X. In particular, for every simple closed curve $d \subset X$, we have $\varphi^{-1} \circ g(d)$ is homotopic to d. (See the Alexander Method of [7].) Thus $g(c)$ and $\varphi(c)$ are homotopic. $\square$

Thus $d_T(p, p') < C d_L(p, p')$ for some constant $C = C(r, M)$. $\square$

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