Boltzmann Systems for Gas Mixtures in 1D Torus

By Kung-Chien Wu

Response for reviewer:

• In the abstract, the author says he develops ”an $L_\infty \times L_\infty^{\infty}$ analysis based on $L_\infty \times L_\infty^{\infty}$ analysis of the Boltzmann equation”. It seems to be a circular statement to me.

**Ans:** Yes, this statement is not clear. I will change it as ”An $L_\infty \times L_\infty^{\infty}$ analysis is developed to study this mass diffusion problem, which is based on the Boltzmann equation for the single species hard sphere collision...”.

• It would be desirable for readers if the author can provide a brief explanation on why $m_A/m_B$ factor is introduced in the definition of $M_A(\xi)$ in page 2.

**Ans:** First, we fix the Maxwellian state of the surrounding gas $B$ as the standard Gaussian, i.e.,

$$M_B(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right).$$

We want to choose $M_A$, the Maxwellian state of gas $A$, satisfies

$$Q^{AB}(M_A, M_B) = Q^{BA}(M_B, M_A) = 0,$$

this implies

$$M_A(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-m_A |\xi|^2}{m_B \cdot 2}\right).$$

This is why $m_A/m_B$ is introduced in the definition of $M_A(\xi)$. I explain this in the revised version.

• $g_{in}$ and $h_{in}$ suddenly pop up in no context in (4) and (5) of page 3.

**Ans:** I agree the reviewer’s point, this presentation in not good. I present this part after the introduction of the initial conditions. See page 3 of the new version.
In this paper, we consider both the total mass of gas $A$ and the perturbations in gas $B$ are sufficiently small, i.e., the state of gas $A$ is near vacuum and gas $B$ is near Maxwellian $M_B$, hence we can assume the initial density distribution functions $F_A(0, x, \xi)$ is near vacuum and $F_B(0, x, \xi)$ has the same total mass, momentum and energy as the Maxwellian $M_B$. I explain more in the new version. We always need those kind of assumptions to get exponential decay when we consider the "close to equilibrium" setting in the torus.

• The author should provide discussions on the ramification of the current result of the linearized problem on the related nonlinear problem.

**Ans:** I add the following sentences at the end of Section 1: When the linear estimate established, it is nature to consider the nonlinear problem (6) as our future project. Assuming that we have smallness assumption of the initial conditions, we expect that $(f_A, f_B)$ of (6) have time decay estimate $e^{-ae^2(1 + t)(1 + t)^{-1/2}\ln(1 + t)}$, the extra $\ln(1 + t)$ is due to the 1D spatial domain. However, there are some difficulties for this project. First, the collision invariants of $\Gamma_{AB}$ and $L_{BB}$ are different, this means that one can not estimate $f_B$ directly. Second, this problem is in 1D space, the nonlinear effect is much stronger than 3D case, hence we need more careful analysis for nonlinear interaction (see [5] for nonlinear analysis in 1D whole space case).

• Comparison with the previous results in the whole space are in order.

For example,

(1) In the whole space case, the spreading gas and the surrounding gas show different decay rates and configurations. The author should provide an explanation on why such difference cannot be captured in the torus case. Is it of intrinsic nature related to the physics of the system, or the author expects that the difference in the decay can be captured as in the whole space case by, for example, developing finer mathematical analysis in the future?

(2) Unlike the scale separation argument in the whole space case, the author applies the kinetic-remainder decomposition only on the short wave part. It would be helpful for readers if the author can explain the advantage of such approach over the conventional one.

**Answer of (1):** I add the following comparison with the whole space case and the torus case after Theorem 1 in the revised version:

Now, let us compare the results in the torus case (Theorem 1) and in the whole space case (constructed by Sotirov-Yu [2]).

The result in [2] basically gave full wave structures of the solutions in the
whole space:
(a) Outside the wave cone ($|x| > Mt$ for some $M > 0$): $g$ decay exponentially in both $t$ and $x$ and $h$ propagate along the Euler characteristic, i.e.,

$$\|g\|_{L^2_\xi} \leq e^{-(|x|+t)}$$

and

$$\|h\|_{L^2_\xi} \leq \sum_{j=1}^{3} (1+t)^{-1/2} e^{-\frac{(x-a_j t)^2}{c(t+1)}} + e^{-c(|x|+t)},$$

where $a_j$, $j = 1, 2, 3$ are wave speed of $h$.

(b) Inside the wave cone ($|x| \leq Mt$ for some $M > 0$): Both $g$ and $h$ propagate along the Euler characteristic, i.e.,

$$\|g\|_{L^2_\xi} \leq (1 + t)^{-1/2} e^{-\frac{|x|^2}{c(t+1)}} + e^{-c(|x|+t)}$$

and

$$\|h\|_{L^2_\xi} \leq \sum_{j=1}^{3} (1+t)^{-1/2} e^{-\frac{(x-a_j t)^2}{c(t+1)}} + e^{-t^{1/2}}.$$  

Here we remark that the wave speed of $g$ is 0.

However, in the torus case, we only focus on large time behavior, this is because the domain is bounded, the "outside wave cone" seems meaningless. Moreover, our estimate in the velocity variable $\xi$ is $L^\infty_{\xi,\beta}$ norm, which is better than $L^2_\xi$ estimate.

On the other hand, in the whole space case [1, 2], they got dissipative behavior of the type of the Navier-Stokes equation by using complex analysis method, but in torus case, the Fourier transform in space $x$ is replaced by Fourier series (it is discrete), it may difficult to introduce complex analysis method in discrete version.

If we only consider the large time behavior of $g$ and $h$ in the whole space, then the time decay rate is $(1 + t)^{-1/2}$, this coincides the large time behavior of the torus case. However, we have extra observation for very large time behavior of $g, h$ in torus case (exponential decay depends on the size of the domain).

**Answer of (2):** I modify the decomposition from single species of torus case in [3] but not whole space case. I think it is better to add the following explanation in the revised version (see Subsection 1.4).
Let us explain why we need this modification. If we apply the kinetic-remainder decomposition to the whole solution $g$ as in [3], then $g$ can be decomposed as the fluid part and the nonfluid part, where the fluid part is part of the long wave expansion, the nonfluid part is the sum of the kinetic part and the remainder part. However, when we estimate $h$, the fluid and nonfluid will intersect in the long wave expansion, but the nonfluid part of $g$ does not separate as the long wave nonfluid part and short wave nonfluid part. This is why we need to modify our method. Under the new decomposition in $g$, we will get fluid part, long wave nonfluid part and short wave nonfluid part, then it will be no problem to estimate the solution of $h$.

- It would be desirable to recall the definition of $E_D$ for readers convenience in the statement of Lemma 7 in page 7.
  
  **Ans:** I recall the definition of $E_D$ in the statement of Lemma 7 in page 7.

- I guess it would be better to present the correct form of each $I_{1j}(j=1;2;3)$ explicitly in line 4 from below in page 12.
  
  **Ans:** I present the correct form of each $I_{1j}(j=1;2;3)$ explicitly in the revised version.

- Some typos...
  (1) each parts should be each part in the first paragraph of page 5.
  (2) section X is scattered over the manuscript, which should be Section X.
  (3) nature in the line 4 from below in page 12 seems to be a typo.
  (4) There must be more. Please check the typographical errors.
  
  **Ans:** I correct the typos mentioned above. Moreover, I correct the following typos:
  (a) Lemma 8 in page 9, ”$h_0$” should be ”$h_{in}$”.
  (b) Page 15, line 2, ”$E_D = M_A$” should be ”$E_D = \sqrt{M_A}$”.
  (c) Page 15, line 7, ”integral” should be ”integrand”.
  (d) Page 17, line 8, ”$L_x^\infty L_{x,\beta}^\infty$” should be ”$L_x^\infty L_{\xi,\beta}^\infty$”.

**References**
