

## CROOKED PLANES

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ABSTRACT. Crooked planes are polyhedra used to construct fundamental polyhedra for discrete groups of Lorentz isometries acting properly on Minkowski (2+1)-space. These fundamental polyhedra are regions bounded by disjoint crooked planes. We develop criteria for the intersection of crooked planes and apply these criteria to proper discontinuity of discrete isometry groups.

### 1. INTRODUCTION

Let  $\Gamma$  be a group of isometries of Minkowski (2+1)-space  $\mathbb{E}$  acting freely and properly discontinuously on  $\mathbb{E}$ . The quotient space  $\mathbb{E}/\Gamma$  is complete flat Lorentz space-time. In 1983, Margulis constructed nonamenable subgroups of the Lorentz isometry group which act properly discontinuously on  $\mathbb{E}$  ([12],[13]), realizing a suggestion of Milnor [15]. That is, there exist complete flat Lorentz 3-manifolds  $M$  whose fundamental groups are nonamenable (that is, not virtually solvable) and these manifolds will be called *Margulis space-times*. By [7] the fundamental group of a Margulis space-time is a free group.

In [3], complete flat Lorentz manifolds with nonabelian free fundamental group were directly constructed using Poincaré's fundamental polyhedron technique ([3], [4]). These examples are diffeomorphic to solid handlebodies. Their fundamental groups are "affine Schottky groups" with fundamental domains derived from fundamental domains of Schottky subgroups of  $\mathbf{SO}^0(2, 1)$  acting on the hyperbolic plane  $\mathbf{H}^2$ . These fundamental polyhedra are bounded by certain disjoint polyhedral hypersurfaces in  $\mathbb{R}^3$ , called *crooked planes*.

Using crooked planes, it was shown in [4] that the underlying linear group of a free group of Lorentz isometries acting freely and properly discontinuously on Minkowski (2+1)-space can be any free discrete subgroup of  $\mathbf{SO}(2, 1)$ .

**Conjecture 1.** Every Margulis space-time has a fundamental polyhedron whose faces are crooked planes.

The classification of intersections of crooked planes is the beginning of our program to analyze the moduli space of Margulis space-times.

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## 2. MINKOWSKI 2 + 1-SPACE

Define  $\mathbb{R}^{2,1}$  to be the 3-dimensional real vector space with the indefinite inner product

$$\mathbb{B}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \cdot I_L \cdot \mathbf{v},$$

where  $\mathbf{u}$  and  $\mathbf{v}$  are written in terms of the usual basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  and  $I_L$  is the diagonal matrix with entries  $+1, +1, -1$ . Define the *Lorentzian cross-product* as

$$\mathbf{u} \boxtimes \mathbf{v} = I_L \cdot (\mathbf{u} \times \mathbf{v}),$$

where  $\times$  is the usual cross product. Then  $\mathbb{B}(\mathbf{x}, \mathbf{x} \boxtimes \mathbf{y}) = 0$ . Let  $\mathbf{O}(2, 1)$  be the full group of linear automorphisms of  $\mathbb{R}^{2,1}$ , with unimodular subgroup  $\mathbf{SO}(2, 1)$ , and identity component  $\mathbf{SO}^0(2, 1)$ .

For a vector  $\mathbf{v}$  denote its Euclidean norm by  $|\mathbf{v}|$ , the line containing  $\mathbf{v}$  by  $\mathbb{R}\mathbf{v}$ , and the ray containing  $\mathbf{v}$  by  $\mathbb{R}_+\mathbf{v}$ . A nonzero vector  $\mathbf{v} \in \mathbb{R}^{2,1}$  is called

- *spacelike* if  $\mathbb{B}(\mathbf{v}, \mathbf{v}) > 0$  (and *unit-spacelike* if  $\mathbb{B}(\mathbf{v}, \mathbf{v}) = 1$ );
- *null* (or *lightlike*) if  $\mathbb{B}(\mathbf{v}, \mathbf{v}) = 0$ ;
- *timelike* if  $\mathbb{B}(\mathbf{v}, \mathbf{v}) < 0$ ;

The *light cone*  $\mathbb{W}$  is the set of all *null* vectors and its upper nappe  $\mathbb{W}^+$  is called the *positive time-orientation*. Similarly  $\mathbb{U}$  is the set of *timelike* vectors with its upper connected component  $\mathbb{U}^+$  defining the positive time orientation.

For any nonzero vector  $\mathbf{v} \in \mathbb{R}^{2,1}$ , its *Lorentz-perpendicular plane* is

$$\mathcal{P}(\mathbf{v}) = \{\mathbf{u} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathbf{u}, \mathbf{v}) = 0\}$$

If  $\mathbf{v}$  is a spacelike vector,  $\mathcal{P}(\mathbf{v}) \cap \mathbb{W}$  is the union of 2 null lines. There exists a unique pair

$$\mathbf{x}^-(\mathbf{v}), \mathbf{x}^+(\mathbf{v}) \in \mathcal{P}(\mathbf{v}) \cap \mathbb{W}^+$$

such that both have Euclidean norm 1 and the ordered triple  $(\mathbf{v}, \mathbf{x}^-(\mathbf{v}), \mathbf{x}^+(\mathbf{v}))$  is a positively-oriented basis of  $\mathbb{R}^{2,1}$ .

For a lightlike  $\mathbf{w} \in \mathbb{W}^+$  with Euclidean norm 1, the plane  $\mathcal{P}(\mathbf{w})$  is the plane tangent to the light cone containing the line  $\mathbb{R}\mathbf{w}$ . The null line  $\mathbb{R}\mathbf{w}$  divides  $\mathcal{P}(\mathbf{w})$  into two half-planes with closures denoted  $\mathcal{P}^+(\mathbf{w})$  and  $\mathcal{P}^-(\mathbf{w})$ . Choose  $\mathcal{P}^+(\mathbf{w})$  so that  $\mathbf{v} \in \mathcal{P}^+(\mathbf{w})$  when  $\mathbf{w} = \mathbf{x}^+(\mathbf{v})$ . Choose  $\mathcal{P}^-(\mathbf{w})$  so that  $\mathbf{v} \in \mathcal{P}^-(\mathbf{w})$  when  $\mathbf{w} = \mathbf{x}^-(\mathbf{v})$ .

Let  $\mathbb{E}$  denote *Minkowski (2+1)-space*, the affine space corresponding to  $\mathbb{R}^{2,1}$ . Minkowski (2+1)-space is a simply connected complete flat Lorentzian manifold of dimension 2+1. For  $p, q \in \mathbb{E}$  the vector  $\mathbf{v} = q - p$  corresponds to the translation carrying  $p$  to  $q$ . We also write  $q = p + \mathbf{v}$ . Identify the vector space of translations of  $\mathbb{E}$  with the original inner product space  $\mathbb{R}^{2,1}$ . Denote the group of isometries of  $\mathbb{E}$  by

$$\mathbf{K} = \mathbf{O}(2, 1) \ltimes \mathbb{R}^{2,1},$$

and the homomorphism assigning to  $h \in \mathbf{K}$  its linear part by:

$$\mathbb{L} : \mathbf{H} \rightarrow \mathbf{O}(2, 1).$$

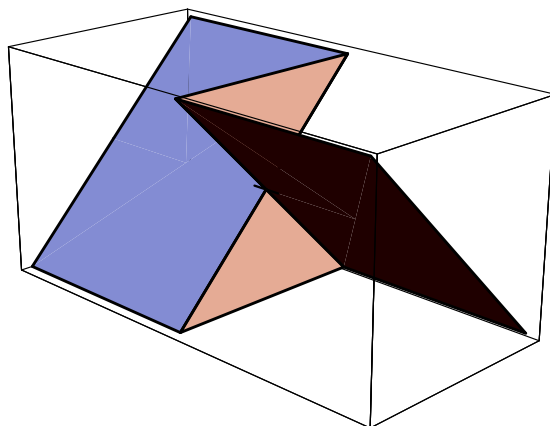


FIGURE 1. A crooked plane

Let  $\mathbf{H}$  and  $\mathbf{H}^0$  be the subgroups consisting of all  $h \in \mathbf{K}$  whose linear parts are  $\mathbb{L}(h) \in \mathbf{SO}(2, 1)$  and  $\mathbb{L}(h) \in \mathbf{SO}^0(2, 1)$ , respectively. For isometries  $h \in \mathbf{H}$  of  $\mathbb{E}$ , write  $h(\mathbf{x}) = g(\mathbf{x}) + \mathbf{v}$  where  $g = \mathbb{L}(h) \in \mathbf{SO}(2, 1)$ . The isometries  $g$  and  $h$  are called

- *hyperbolic* if  $g$  has 3 distinct real eigenvalues;
- *parabolic* if  $g$  has 1 as its only eigenvalue;
- *elliptic* if  $g$  has imaginary eigenvalues;

If  $\mathbf{v}$  is spacelike or timelike and  $l = p + \mathbb{R}\mathbf{v}$  is a line parallel to  $\mathbf{v}$ , then  $\iota_l$  denotes the inversion in line  $l$ . If  $\mathbf{v}$  is spacelike, then  $\iota_l \notin \mathbf{H}$ .

### 3. CROOKED PLANES

Our strategy is to derive fundamental polyhedra for actions on Minkowski space from fundamental polygons for the corresponding actions on  $\mathbf{H}^2$ . A convenient model for  $\mathbf{H}^2$  is the hyperboloid defined by  $\mathbb{B}(\mathbf{v}, \mathbf{v}) = -1$ , which has an induced Riemannian metric of constant curvature -1. Let  $\mathbf{H}^2$  be the component of the hyperboloid lying in  $\mathbb{U}^+$ . Points of  $\mathbf{H}^2$  correspond to timelike lines and geodesics in  $\mathbf{H}^2$  to spacelike planes, that is to planes that are Lorentz perpendicular to spacelike vectors. Suppose that  $\mathbf{v} \in \mathbb{R}^{2,1}$  is spacelike. Then the plane  $\mathcal{P}(\mathbf{v}) \subset \mathbb{R}^{2,1}$  meets  $\mathbb{U}$  in timelike lines determining a geodesic in  $\mathbf{H}^2$ .

Two spacelike planes always intersect in a line. In particular, two ultraparallel geodesics in  $\mathbf{H}^2$  do not intersect but the spacelike planes corresponding to the geodesics intersect in a spacelike line. Any translations of these spacelike planes also intersect. We now introduce crooked planes as objects in  $\mathbb{E}$  that correspond to geodesics in  $\mathbf{H}^2$ .

Let  $p \in \mathbb{E}$  be a point and  $\mathbf{v} \in \mathbb{R}^{2,1}$  a spacelike vector. Define the *positively oriented crooked plane*  $\mathcal{C}(\mathbf{v}, p) \subset \mathbb{E}$  with *vertex*  $p$  and *direction vector*  $\mathbf{v}$  to be the union of two *wings*

$$\mathcal{W}^+(\mathbf{v}, p) = p + \mathcal{P}^+(\mathbf{x}^+(\mathbf{v})), \quad \mathcal{W}^-(\mathbf{v}, p) = p + \mathcal{P}^+(\mathbf{x}^-(\mathbf{v}))$$

and the *stem*

$$\mathcal{S}(\mathbf{v}, p) = p + \{ \mathbf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathbf{v}, \mathbf{x}) = 0, \mathbb{B}(\mathbf{x}, \mathbf{x}) \leq 0 \}.$$

Define the *negatively oriented crooked plane*  $\mathcal{K}(\mathbf{v}, p) \subset \mathbb{E}$  with *vertex*  $p$  and *direction vector*  $\mathbf{v}$  to be the union of two *wings*

$$\mathcal{M}^+(\mathbf{v}, p) = p + \mathcal{P}^-(\mathbf{x}^+(\mathbf{v})), \quad \mathcal{M}^-(\mathbf{v}, p) = p + \mathcal{P}^-(\mathbf{x}^-(\mathbf{v}))$$

and the stem  $\mathcal{S}(\mathbf{v}, p)$  defined above.

For the crooked plane  $\mathcal{C}(\mathbf{v}, p)$  (or  $\mathcal{K}(\mathbf{v}, p)$ ) there exists a unique spacelike line  $p + \mathbb{R}\mathbf{v}$  lying on it which will be called the *spine* of the crooked plane. Given a spacelike line  $\ell$  and a point  $p \in \ell$ , there is a unique positively (or negatively) oriented crooked plane with spine  $\ell$  and vertex  $p$ .

We also define the *positively oriented crooked half-space*  $\mathcal{H}(\mathbf{v}, p)$  and the *negatively oriented crooked half-space*  $\mathcal{N}(\mathbf{v}, p)$  to be the components of  $\mathbb{E} - \mathcal{C}(\mathbf{v}, p)$  and  $\mathbb{E} - \mathcal{K}(\mathbf{v}, p)$ , respectively, which contain

$$p + \{\mathbf{u} \in \mathbb{W}^+ \mid \mathbb{B}(\mathbf{v}, \mathbf{u}) > 0\}.$$

#### 4. INTERSECTIONS OF CROOKED PLANES

Since two nonparallel half planes of differing orientation meet in a ray, two oppositely oriented crooked planes always intersect:

**Theorem 2** ([9]). *Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be linearly independent spacelike vectors and let  $p_1, p_2 \in \mathbb{E}$ . Then  $\mathcal{C}(\mathbf{v}_1, p_1) \cap \mathcal{K}(\mathbf{v}_2, p_2)$  is nonempty.*

To examine the intersection of two crooked planes of the same orientation we introduce the following terminology: Vectors  $\mathbf{v}_1, \mathbf{v}_2$  are said to be *ultraparallel*, *asymptotic*, or *crossing* if  $\mathcal{P}(\mathbf{v}_1)$  and  $\mathcal{P}(\mathbf{v}_2)$  correspond to geodesics in  $\mathbf{H}^2$  which are ultraparallel, asymptotic, or crossing, respectively (compare [9]).

For crossing spacelike vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , the intersection of the stems  $\mathcal{S}(\mathbf{v}_1, p_1)$  and  $\mathcal{S}(\mathbf{v}_2, p_2)$  contains two rays. Since  $\mathcal{C}(\mathbf{v}_2, p_2)$  and  $\mathcal{K}(\mathbf{v}_2, p_2)$  share a stem,

**Theorem 3** ([9]). *Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be crossing spacelike vectors. Then the intersection of any two crooked planes with spines parallel to  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is nonempty.*

Now consider crooked planes of the same orientation whose spines are either ultraparallel or asymptotic. The following terminology will be useful since  $\mathcal{C}(\mathbf{v}, p) = \mathcal{C}(-\mathbf{v}, p)$  (and  $\mathcal{K}(\mathbf{v}, p) = \mathcal{K}(-\mathbf{v}, p)$ ). Spacelike vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are said to be *consistently oriented* if  $\mathbb{B}(\mathbf{v}_i, \mathbf{v}_j) \leq 0$  and  $\mathbb{B}(\mathbf{v}_i, \mathbf{x}^\pm(\mathbf{v}_j)) \leq 0$  for  $i \neq j$ . This means that the half-planes  $H(\mathbf{v}_i), H(\mathbf{v}_j) \in \mathbf{H}^2$  corresponding to  $\mathcal{H}(\mathbf{v}_i, p)$  and  $\mathcal{H}(\mathbf{v}_j, p)$  do not intersect.

**Theorem 4** ([9]). *Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be consistently oriented, ultraparallel, unit-spacelike vectors and  $p_1, p_2 \in \mathbb{E}$ . The positively oriented crooked planes  $\mathcal{C}(\mathbf{v}_1, p_1)$  and  $\mathcal{C}(\mathbf{v}_2, p_2)$  are disjoint if and only if*

$$\mathbb{B}(p_2 - p_1, \mathbf{v}_1 \boxtimes \mathbf{v}_2) > |\mathbb{B}(p_2 - p_1, \mathbf{v}_2)| + |\mathbb{B}(p_2 - p_1, \mathbf{v}_1)|. \quad (4.0.1)$$

*The negatively oriented crooked planes  $\mathcal{K}(\mathbf{v}_1, p_1)$  and  $\mathcal{K}(\mathbf{v}_2, p_2)$  are disjoint if and only if*

$$\mathbb{B}(p_2 - p_1, \mathbf{v}_1 \boxtimes \mathbf{v}_2) < -(|\mathbb{B}(p_2 - p_1, \mathbf{v}_2)| + |\mathbb{B}(p_2 - p_1, \mathbf{v}_1)|). \quad (4.0.2)$$

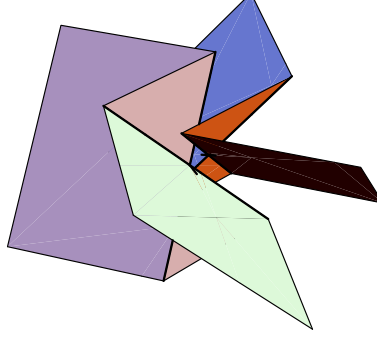


FIGURE 2. Disjoint ultraparallel crooked planes

These conditions follow from explicit description of the intersections of the components of crooked planes. We rephrase them in terms of the displacement vector  $w = p_2 - p_1$  between their vertices. Fix spacelike vectors  $v_1$  and  $v_2$  and a point  $p \in \mathbb{E}$ . Consider crooked planes with vertex  $p_1 = p$  and  $p_2 = p + w$ . For any  $w \in \mathbb{R}^{2,1}$ , the stems  $\mathcal{S}(v_1, p)$  and  $\mathcal{S}(v_2, p + w)$  are disjoint if and only if

$$|\mathbb{B}(p_2 - p_1, v_1 \boxtimes v_2)| > |\mathbb{B}(p_2 - p_1, v_2)| + |\mathbb{B}(p_2 - p_1, v_1)|.$$

The set

$$A = \{w \in \mathbb{R}^{2,1} \mid \mathcal{S}(v_1, p) \cap \mathcal{S}(v_2, p + w) = \emptyset\}$$

is a disjoint union of the two open pyramids

$$A_+ = \{w \in \mathbb{R}^{2,1} \mid \mathcal{C}(v_1, p) \cap \mathcal{C}(v_2, p + w) = \emptyset\}$$

and

$$A_- = \{w \in \mathbb{R}^{2,1} \mid \mathcal{K}(v_1, p) \cap \mathcal{K}(v_2, p + w) = \emptyset\}$$

**Theorem 5** ([9]). *Let  $v_1$  and  $v_2$  be consistently oriented, asymptotic, unit-spacelike vectors. Let  $\mathcal{C}_i = \mathcal{C}(v_i, p_i)$  and  $\mathcal{K}_i = \mathcal{K}(v_i, p_i)$  for  $i = 1, 2$ . The positively oriented crooked planes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are disjoint if and only if*

$$\left\{ \begin{array}{l} \mathbf{x}^-(v_1) = \mathbf{x}^+(v_2), \\ \mathbb{B}(p_2 - p_1, v_1) < 0, \\ \mathbb{B}(p_2 - p_1, v_2) < 0, \\ \mathbb{B}(p_2 - p_1, \mathbf{x}^+(v_1) \boxtimes \mathbf{x}^-(v_2)) > 0; \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \mathbf{x}^+(v_1) = \mathbf{x}^-(v_2), \\ \mathbb{B}(p_2 - p_1, v_1) > 0, \\ \mathbb{B}(p_2 - p_1, v_2) > 0, \\ \mathbb{B}(p_2 - p_1, \mathbf{x}^-(v_1) \boxtimes \mathbf{x}^+(v_2)) > 0. \end{array} \right\}$$

*The negatively oriented crooked planes  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are disjoint if and only if*

$$\left\{ \begin{array}{l} \mathbf{x}^-(v_1) = \mathbf{x}^+(v_2), \\ \mathbb{B}(p_2 - p_1, v_1) > 0, \\ \mathbb{B}(p_2 - p_1, v_2) > 0, \\ \mathbb{B}(p_2 - p_1, \mathbf{x}^+(v_1) \boxtimes \mathbf{x}^-(v_2)) < 0; \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \mathbf{x}^+(v_1) = \mathbf{x}^-(v_2), \\ \mathbb{B}(p_2 - p_1, v_1) < 0, \\ \mathbb{B}(p_2 - p_1, v_2) < 0, \\ \mathbb{B}(p_2 - p_1, \mathbf{x}^-(v_1) \boxtimes \mathbf{x}^+(v_2)) < 0. \end{array} \right\}$$

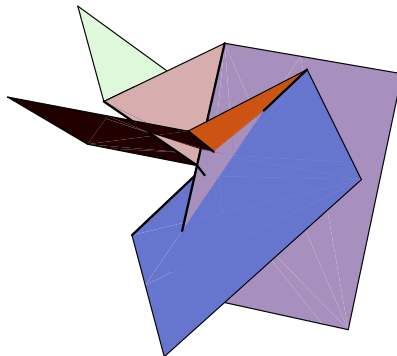


FIGURE 3. Intersecting ultraparallel crooked planes

A pair of positively oriented ultraparallel crooked planes which are disjoint are shown in Figure 2. A pair of negatively oriented ultraparallel crooked planes with the same stems and translation vectors as in Figure 2 are shown in Figure 3 and have a nonempty intersection. These two figures illustrate the following more general fact.

**Corollary 6** ([9]). *Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be spacelike vectors. Either  $\mathcal{C}(\mathbf{v}_1, p_1) \cap \mathcal{C}(\mathbf{v}_2, p_2)$  or  $\mathcal{K}(\mathbf{v}_1, p_1) \cap \mathcal{K}(\mathbf{v}_2, p_2)$  is nonempty.*

## 5. APPLICATIONS

**5.1. Groups generated by inversions.** For any spacelike line  $l$  and  $p \in l$ , the inversion  $\iota_l$  preserves  $\mathcal{C}(\mathbf{v}, p)$  leaving each wing invariant, permuting the two components of  $\mathcal{W}^+(\mathbf{v}, p)$  and  $\mathcal{W}^-(\mathbf{v}, p)$  and the two components of  $\mathcal{S}(\mathbf{v}, p) - \{p\}$ . Further,  $\iota_l(\mathcal{H}(\mathbf{v}, p)) = \mathcal{H}(-\mathbf{v}, p)$  so that  $E - \mathcal{H}(\pm\mathbf{v}, p)$  are fundamental domains for the action of the group generated by the inversion in  $l$ .

Alternatively,  $\iota_l$  also preserves  $\mathcal{K}(\mathbf{v}, p)$  and  $E - \mathcal{N}(\pm\mathbf{v}, p)$  are fundamental domains for the action of the group generated by the inversion in  $l$ .

Identify  $\mathbb{R}^{2,1}$  with  $\mathbb{E}$  via a choice of origin  $o \in \mathbb{E}$ . For  $l = p + \mathbb{R}\mathbf{v}$ , let  $\ell = \mathbb{R}\mathbf{v} \subset \mathbb{R}^{2,1}$  be the line parallel to  $l$  through the origin. The inversion  $\iota_\ell$  of  $\mathbb{R}^{2,1}$  through  $\ell$  equals the reflection of  $\mathbf{H}^2$  through the geodesic

$$S(l) = \mathcal{P}(\mathbf{v}) \cap \mathbf{H}^2 = \mathcal{C}(\mathbf{v}, o) \cap \mathbf{H}^2 = \mathcal{K}(\mathbf{v}, o) \cap \mathbf{H}^2.$$

Let  $\Delta \subset \mathbf{H}^2$  be a convex  $n$ -gon whose pairs of adjacent sides  $S_1, \dots, S_n$  are ultraparallel.  $\iota_{\ell_i}$  is the reflection in  $S_i$  and let  $\Gamma \subset \mathbf{SO}(2, 1)$  be the group generated by  $\iota_{\ell_1}, \dots, \iota_{\ell_n}$ . Then  $\Gamma$  acts properly and discretely on  $\mathbf{H}^2$  with fundamental domain  $\Delta$ . Choose  $\mathbf{v}_1, \dots, \mathbf{v}_n$  to be consistently oriented so that

$$\Delta = \mathbf{H}^2 - (\cup_{i=1}^n H(\mathbf{v}_i))$$

An *affine deformation* of  $\Gamma$  is a collection  $\mathbf{l} = (\mathbf{l}_1, \dots, \mathbf{l}_n)$  of oriented spacelike lines such that each  $l_i$  has direction vector  $\mathbf{v}_i$ . Such an affine deformation determines

an affine representation of  $\Gamma$  whose image  $\Gamma_1$  is generated by inversions  $\iota_j$  in  $l_j$ . Lines  $l \subset \mathbb{E}$  with direction vector  $\mathbf{v}$  are parametrized by a point  $p \in l$  which is determined up to translation by  $\mathbb{R}\mathbf{v}$ . Affine deformations of  $\Gamma$  comprise the product

$$\mathbb{E}/\mathbb{R}\mathbf{v}_1 \times \cdots \times \mathbb{E}/\mathbb{R}\mathbf{v}_n$$

of the quotient affine spaces  $\mathbb{E}/\mathbb{R}\mathbf{v}_j$ .

A fundamental domain for the action of  $\Gamma_1$  is built from disjoint crooked planes  $\mathcal{C}_i = \mathcal{C}(\mathbf{v}_i, q_i)$  with spine  $l_i$  which bound a common region. Since a crooked plane with given spine  $l$  is determined by its vertex, the family  $(\mathcal{C}_1, \dots, \mathcal{C}_n)$  is determined by the  $n$ -tuple  $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in \mathbf{I}_1 \times \cdots \times \mathbf{I}_n$ .

**Theorem 7** ([9]). *Let  $\Gamma \subset \mathbf{SO}(2, 1)$  be as above with consistently oriented  $\mathbf{v}_i$ 's parallel to the  $l_i$ 's. An affine deformation  $\Gamma_1$  of  $\Gamma$  is properly discontinuous if there exist  $\mathbf{q} \in \mathbf{I}_1 \times \cdots \times \mathbf{I}_n$  such that either (but not both) of the following are true.*

- $\mathcal{C}(q_1, \mathbf{v}_1), \dots, \mathcal{C}(q_n, \mathbf{v}_n)$  are disjoint and bound a common region

$$\mathbb{E} - (\cup_{i=1}^n \mathcal{H}(q_i, \mathbf{v}_i)),$$

*which is a fundamental domain for the action.*

- $\mathcal{K}(q_1, \mathbf{v}_1), \dots, \mathcal{K}(q_n, \mathbf{v}_n)$  are disjoint and bound a common region

$$\mathbb{E} - (\cup_{i=1}^n \mathcal{N}(q_i, \mathbf{v}_i)),$$

*which is a fundamental domain for the action.*

These results follow directly from Theorem 3.5 of [4]. Conversely:

**Conjecture 8.** Let  $\Gamma \subset \mathbf{SO}(2, 1)$  be as above. An affine deformation  $\Gamma_1$  of  $\Gamma$  is properly discontinuous if and only if there exist  $\mathbf{q} \in \mathbf{I}_1 \times \cdots \times \mathbf{I}_n$  such that either (but not both) of the following are true.

1.  $\mathcal{C}(q_1, \mathbf{v}_1), \dots, \mathcal{C}(q_n, \mathbf{v}_n)$  are disjoint and bound a common region.
2.  $\mathcal{K}(q_1, \mathbf{v}_1), \dots, \mathcal{K}(q_n, \mathbf{v}_n)$  are disjoint and bound a common region.

**5.2. Groups generated by hyperbolic transformations.** All  $g \in \mathbf{SO}(2, 1)$  have 1 as an eigenvalue. For hyperbolic  $g$ , we define  $\mathbf{x}^0(g)$  to be the unique fixed unit-spacelike vector such that  $\mathbf{x}^-(\mathbf{x}^0(g))$  is an eigenvector with a corresponding eigenvalue  $< 1$ , and  $\mathbf{x}^+(\mathbf{x}^0(g))$  is an eigenvector with a corresponding eigenvalue  $> 1$ .

For hyperbolic  $h(\mathbf{x}) = g(\mathbf{x}) + \mathbf{v}$ , Margulis ([12],[13]) defined an invariant  $\alpha(h) = \mathbb{B}(\mathbf{v}, \mathbf{x}^0(g))$ .  $h$  admits a fixed point if and only if  $\alpha(h) = 0$ . Further, Margulis has shown ([12],[13], see also [7]) that

**Theorem 9** (Margulis). *If  $h_1$  and  $h_2$  are hyperbolic Lorentz transformations such that  $\alpha(h_1)$  and  $\alpha(h_2)$  have opposite signs then any group containing  $h_1$  and  $h_2$  does not act properly on  $\mathbb{R}^{2,1}$ .*

If there exist crooked planes  $\mathcal{C}(\mathbf{v}_1, p_1)$  and  $\mathcal{C}(\mathbf{v}_2, p_2)$  such that  $h(\mathcal{C}(\mathbf{v}_1, p_1)) = \mathcal{C}(\mathbf{v}_2, p_2)$  and  $\mathbf{v}_1$  and  $\mathbf{x}^0(g)$  cross (equivalently,  $\mathbf{v}_2$  and  $\mathbf{x}^0(g)$  cross) then the region bounded by  $\mathcal{C}(\mathbf{v}_1, p_1)$  and  $\mathcal{C}(\mathbf{v}_2, p_2)$  is a fundamental domain for the action of  $\langle h \rangle$  on  $\mathbb{R}^{2,1}$ . The orientation of the crooked planes is related to  $\alpha(h)$  by the following corollary of Theorems 4 and 5:

**Corollary 10.** *Let  $h \in \mathbf{H}$  be hyperbolic. If  $\alpha(h) < 0$  ( $\alpha(h) > 0$ ) then there is no fundamental domain of the action of the group generated by  $h$  on  $\mathbb{E}$  bounded by disjoint positively (negatively) oriented crooked planes.*

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