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ON EMBEDDING THE $1:1:2$ RESONANCE SPACE IN A POISSON MANIFOLD

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Abstract. The Hamiltonian actions of $\mathbf{S}^1$ on the symplectic manifold $\mathbb{R}^6$ in the $1:1:-2$ and $1:1:2$ resonances are studied. Associated to each action is a Hilbert basis of polynomials defining an embedding of the orbit space into a Euclidean space V and of the reduced orbit space $J^{-1}(0)/\mathbf{S}^1$ into a hyperplane V_J of V, where J is the quadratic momentum map for the action. The orbit space and the reduced orbit space are singular Poisson spaces with smooth structures determined by the invariant functions. It is shown that the Poisson structure on the orbit space, for both the $1:1:2$ and the $1:1:-2$ resonance, cannot be extended to V, and that the Poisson structure on the reduced orbit space $J^{-1}(0)/\mathbf{S}^1$ for the $1:1:-2$ resonance cannot be extended to the hyperplane V_J.

1. Introduction

In this paper we study certain singular Poisson spaces arising from Hamiltonian actions of Lie groups on symplectic manifolds. The singular Poisson spaces are embedded, using Hilbert maps, into Euclidean spaces and we prove that the singular Poisson structure cannot be extended to the Euclidean spaces. This disproves a conjecture raised by Cushman and Weinstein [12, 13].

For a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with an equivariant momentum map J, Marsden and Weinstein define in [5] the reduced orbit space M_μ, for a value μ in the dual of the Lie algebra of G. The space M_μ is the quotient space $J^{-1}(\mu)/G_\mu$ where G_μ is the isotropy group of μ with respect to the coadjoint action of G. For weakly regular values μ of J, if G_μ acts freely and properly on the manifold $J^{-1}(\mu)$, M_μ is a manifold and there is a unique symplectic structure on M_μ which lifts to $i_\mu^*\omega$ where i_μ is the inclusion map $i_\mu : J^{-1}(\mu) \to M$.

The space M/G is assigned a smooth structure $C^\infty(M/G)$ by the G-invariant functions on M and M/G inherits a Poisson bracket from M making M/G a Poisson variety. Even when $\mu \in J(M)$ is not a weakly regular value the reduced orbit space M_μ has a smooth structure defined by restricting functions in $C^\infty(M/G)$ to $J^{-1}(\mu)$. It is proved in [1] that the space M_μ inherits, by restriction, the structure of a Poisson variety from M/G. For a compact Lie group G Sjamaar and Lerman show in [12] that the reduced orbit space M_0 is a union of symplectic manifolds and moreover a stratified symplectic space.

If the Lie group G is compact and the action on M has only finitely many orbit types then by a theorem of Schwarz [11] there are functions $f_1, \ldots, f_n$ in

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$C^\infty(M/G)$ such that $F^* C^\infty(\mathbb{R}^n) = C^\infty(M/G)$ for $F = (f_1, \dots, f_n)$. This defines an embedding $M/G \to \mathbb{R}^n$ and in particular an embedding $M_\mu \to \mathbb{R}^n$. If a compact Lie group G acts orthogonally on a Euclidean space M then we can choose the invariants $f_1, \dots, f_n$ to be generators of the algebra of invariant polynomials on M, in which case the embedding F is called a Hilbert map for the action and $f_1, \dots, f_n$ are said to form a Hilbert basis for the algebra. It is known for many simple cases, see for example [12], that the Poisson structure on the image of M/G can be extended to all of $\mathbb{R}^n$. We will, however, outline a proof showing that for the $1 : 1 : -2$ resonance action and a Hilbert map corresponding to a minimal homogeneous Hilbert basis, described below, defining the embedding $M/G \to \mathbb{R}^{11}$ there exists no smooth Poisson structure on $\mathbb{R}^{11}$ extending the Poisson structure on the orbit space. Furthermore we show that the Poisson structure on the singular variety M_0 cannot be extended to the hyperplane in $\mathbb{R}^{11}$ induced by the momentum map for the action. The nonexistence of Poisson structures on $\mathbb{R}^{11}$ extending the Poisson structure on the orbit space of the $1 : 1 : 2$ resonance action is then established using results already obtained for the $1 : 1 : -2$ resonance.

2. The $1 : 1 : -2$ resonance

2.1. **Preliminaries.** Consider the space $\mathbb{R}^6$ as $\mathbb{C}^3$ and define an $\mathbf{S}^1$ action

$$\star : \mathbf{S}^1 \times \mathbb{R}^6 \to \mathbb{R}^6 \text{ by } \mathrm{z} \star (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3) = (\mathrm{z}\mathrm{u}_1, \mathrm{z}\mathrm{u}_2, \mathrm{z}^{-2}\mathrm{u}_3)$$

where the coordinates on $\mathbb{R}^6$ and $\mathbb{C}^3$ are related by $u_i = x_i + \imath y_i$ for $i = 1, 2, 3$. Also define coordinates $v_i = x_i - \imath y_i$ on $\mathbb{C}^3$ for $i = 1, 2, 3$, see [8].

The existence of a Hilbert basis for the invariant polynomials follows from a theorem of Hilbert [14], but for the action above it is easily established directly.

A minimal homogeneous generating set for the invariant polynomials in

$$\mathbb{R}[x, y] = \mathbb{R}[x_1, y_1, x_2, y_2, x_3, y_3]$$

is given by the polynomials $f_1, \dots, f_{11}$, i.e.,

$$\mathbb{R}[x, y]^{\mathbf{S}^1} = \mathbb{R}[f_1, \dots, f_{11}],$$

where

$$f_1 = x_1^2 + y_1^2,\ f_2 = x_2^2 + y_2^2,\ f_3 = x_3^2 + y_3^2,$$
$$f_4 + \imath f_5 = u_1 v_2,\ f_6 + \imath f_7 = u_1^2 u_3,\ f_8 + \imath f_9 = u_2^2 u_3 \text{ and}$$
$$f_{10} + \imath f_{11} = u_1 u_2 u_3.$$

A Hilbert map $F : \mathbb{R}^6 \to \mathbb{R}^{11}$ corresponding to the Hilbert basis $f_1, \dots, f_{11}$ is given by $F = (f_1, \dots, f_{11})$. We will refer to this particular choice of a Hilbert map for the $1 : 1 : -2$ action as the Hilbert map for the $1 : 1 : -2$ action in standard form.

2.2. **Complex coefficients.** On $\mathbb{R}^6$ the standard Poisson bivector field ϱ is given by

$$\varrho = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y_3}.$$

Denote by $M = \mathbb{R}^6$ and by $V = \mathbb{R}^{11}$ the spaces with linear coordinates

$$\{x_1, y_1, x_2, y_2, x_3, y_3\} \text{ and } \{l_1, l_2, l_3, R_1, I_1, \dots, R_4, I_4\}$$

respectively and in which $F : M \to V$ is given by the formula $F = (f_1, \dots, f_{11})$. Now let $M_{\mathbf{c}}$ and $V_{\mathbf{c}}$ be real vector spaces with linear coordinates

$$\{z_1, w_1, z_2, w_2, z_3, w_3\} \text{ and } \{l_1, l_2, l_3, Z_1, W_1, \dots, Z_4, W_4\}$$

respectively.

For the Euclidean space $E = \mathbb{R}^n$ denote by $\mathcal{X}^*(E)$ the C^∞-multivector fields on E, by $\mathcal{X}^*[E]$ the multivector fields on E with polynomial coefficients and by $\mathcal{X}^*[[E]]$ the multivector fields on E with formal coefficients.

Let T be the Taylor series operator

$$T : \mathcal{X}^*(V) \to \mathcal{X}^*[[V]]$$

replacing each coefficient with its Taylor series at 0.

Define a $\mathbb{C}$-algebra isomorphism

$$\kappa : \mathbb{C} \otimes \mathcal{X}^*[V] \to \mathbb{C} \otimes \mathcal{X}^*[V_{\mathbf{c}}]$$

by the formulas

$$l_s \mapsto l_s,\ R_t \mapsto \tfrac{1}{2}(Z_t + W_t) \text{ and } I_t \mapsto \tfrac{1}{2\imath}(Z_t - W_t),$$

$$\tfrac{\partial}{\partial l_s} \mapsto \tfrac{\partial}{\partial l_s},\ \tfrac{\partial}{\partial R_t} \mapsto \tfrac{\partial}{\partial Z_t} + \tfrac{\partial}{\partial W_t} \text{ and } \tfrac{\partial}{\partial I_t} \mapsto \imath(\tfrac{\partial}{\partial Z_t} - \tfrac{\partial}{\partial W_t})$$

and similarly define a $\mathbb{C}$-algebra isomorphism

$$\kappa' : \mathbb{C} \otimes \mathcal{X}^*[M] \to \mathbb{C} \otimes \mathcal{X}^*[M_{\mathbf{c}}]$$

by the formulas

$$x_t \mapsto \tfrac{1}{2}(z_t + w_t) \text{ and } y_t \mapsto \tfrac{1}{2\imath}(z_t - w_t),$$

$$\tfrac{\partial}{\partial x_t} \mapsto \tfrac{\partial}{\partial z_t} + \tfrac{\partial}{\partial w_t} \text{ and } \tfrac{\partial}{\partial y_t} \mapsto \imath(\tfrac{\partial}{\partial z_t} - \tfrac{\partial}{\partial w_t}).$$

Let

$$\tau : \mathbb{R}[V^*] \to \mathbb{R}[M^*]$$

be the algebra morphism given by

$$\tau(g) = g \circ F$$

and let $\tau_{\mathbf{c}}$ be the $\mathbb{C}$-algebra morphism induced by the diagram

$$\begin{array}{ccc} \mathbb{R}[V^*] & \xrightarrow{\tau} & \mathbb{R}[M^*] \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa'} \\ \mathbb{C}[V_{\mathbf{c}}^*] & \xrightarrow{\tau_{\mathbf{c}}} & \mathbb{C}[M_{\mathbf{c}}^*] \end{array}$$

and use $\tau_{\mathbf{c}}$ to define the complex counterpart of the Hilbert map

$$F_{\mathbf{c}} : \mathbb{C} \otimes M_{\mathbf{c}} \to \mathbb{C} \otimes V_{\mathbf{c}}$$

with complex polynomial coordinate functions defined by

$$\tau_{\mathbf{c}}(p) = p \circ F_{\mathbf{c}},$$

furthermore let $F_{\mathbf{c}*}$ be the $\mathbb{C}$-linear derivative of $F_{\mathbf{c}}$.

There is a one-to-one relationship between bivector fields $\pi \in \mathcal{X}^2[V]$ which are F-related to ϱ

$$F_* \circ \varrho = \pi \circ F$$

and bivector fields $\pi_{\mathbf{c}} \in \mathbb{C} \otimes \mathcal{X}^2[V_{\mathbf{c}}]$ which are $F_{\mathbf{c}}$-related to $\varrho_{\mathbf{c}}$

$$F_{\mathbf{c}*} \circ \varrho_{\mathbf{c}} = \pi_{\mathbf{c}} \circ F_{\mathbf{c}}$$

and satisfy $\kappa^{-1}(\pi_{\mathbf{c}}) \in \mathcal{X}^2[V]$. The relationship is given by $\pi = \kappa^{-1}(\pi_{\mathbf{c}})$.

Extend the Schouten-Nijenhuis [9, 10] bracket $[\![,]\!]$ by $\mathbb{C}$-bilinearity to the spaces $\mathbb{C} \otimes \mathcal{X}^*[V]$ and $\mathbb{C} \otimes \mathcal{X}^*[V_{\mathbf{c}}]$, then the identity

$$[\![\kappa(\mathrm{X}), \kappa(\mathrm{Y})]\!] = \kappa([\![\mathrm{X}, \mathrm{Y}]\!]) \text{ for } \mathrm{X}, \mathrm{Y} \in \mathcal{X}^*[\mathrm{V}]$$

follows easily.

Let

$$\varrho_{\mathbf{c}} = \kappa'(\varrho) = -2\imath \sum_n \frac{\partial}{\partial z_n} \wedge \frac{\partial}{\partial w_n}$$

then the commutative diagram below

$$\begin{array}{ccc} (\mathbb{R}[V^*], \pi) & \xrightarrow{\tau} & (\mathbb{R}[M^*], \varrho) \\ \Big\downarrow \kappa & & \Big\downarrow \kappa' \\ (\mathbb{C}[V_{\mathbf{c}}^*], \pi_{\mathbf{c}}) & \xrightarrow{\tau_{\mathbf{c}}} & (\mathbb{C}[M_{\mathbf{c}}^*], \varrho_{\mathbf{c}}) \end{array} \tag{1}$$

demonstrates the setting for finding a bivector field $\pi \in \mathcal{X}^2[V]$ which is F-related to ϱ.

For the above diagram we have

$$[\![\pi, \pi]\!] = 0 \text{ if and only if } [\![\pi_{\mathbf{c}}, \pi_{\mathbf{c}}]\!] = 0$$

or equivalently [3, 9] that $\pi \in \mathcal{X}^2[V]$ is a Poisson bivector field if and only if $\pi_{\mathbf{c}} = \kappa(\pi)$ is a Poisson bivector field.

2.3. **Embeddings.** The $1 : 1 : -2$ resonance action above has momentum map J given by the quadratic polynomial

$$J = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_3^2 - 2y_3^2).$$

There exists a unique functional $\mathcal{J}$ in V^* satisfying

$$J = \mathcal{J} \circ F$$

and F maps the space $J^{-1}(0)$ into the hyperplane

$$V_J = \mathcal{J}^{-1}(0) \subset V.$$

The smooth structure on the reduced orbit space $J^{-1}(0)/\mathbf{S}^1$, obtained, see [1], by restricting the invariant functions on M to $J^{-1}(0)$, is described by

$$C^\infty(J^{-1}(0)/\mathbf{S}^1) = (\eta \circ \mathrm{F})^* \, \mathrm{C}^\infty(\mathrm{V_J})|_{\mathrm{J}^{-1}(0)}$$

where η is the projection

$$\eta : V \to V_J$$

onto V_J along the vector $\mathcal{J}^* = \frac{1}{2}(l_1^* + l_2^* - 2l_3^*)$.

Extending the Poisson structure on $M/\mathbf{S}^1$ to all of V is equivalent to finding a bivector field ζ in $\mathcal{X}^2(V)$ satisfying the conditions

1a) $[\![\zeta, \zeta]\!] = 0$ and

1b) $\zeta \circ F = F_* \circ \varrho$.

Extending the Poisson structure on $J^{-1}(0)/\mathbf{S}^1$ to all of V_J is equivalent to finding a bivector field ξ' in $\mathcal{X}^2(V_J)$ satisfying

2a) $[\xi', \xi'] = 0$ and

2b) $\xi' \circ (\eta \circ F)|_{J^{-1}(0)} = (\eta \circ F)_* \circ \varrho|_{J^{-1}(0)}$.

Further a Poisson bivector ξ' in $\mathcal{X}^2(V_J)$, i.e., $[\xi', \xi'] = 0$, extending the structure on $J^{-1}(0)/\mathbf{S}^1$ can be extended to a Poisson bivector ξ on all of V satisfying

3a) $\xi \circ \iota = \iota_* \circ \xi'$,

3b) $\xi \circ F|_{J^{-1}(0)} = F_* \circ \varrho|_{J^{-1}(0)}$,

3c) $[\xi, \mathcal{J}] = 0$, i.e., $\mathcal{J}$ is Casimir,

and where ι is the inclusion map $V_J \to V$.

The above Poisson bivector field ξ is obtained from ξ' by defining ξ to be constant along the fibres of the projection η.

Theorem 1. *Let $F : M \to V$ be the Hilbert map in standard form for the $1 : 1 : -2$ resonance action $\star : \mathbf{S}^1 \times \mathrm{M} \to \mathrm{M}$. Then the Poisson structure on the singular orbit space $M/\mathbf{S}^1$, embedded into V by F, cannot be extended to V. Furthermore the Poisson structure on the singular reduced orbit space $J^{-1}(0)/\mathbf{S}^1$ cannot be extended to V_J.*

Before proving the above theorem we establish the lemmas below.

Choose a bivector field π in $\mathcal{X}^2[V]$ such that

$$\pi \circ F = F_* \circ \varrho, \tag{2}$$

e.g. use diagram (1); the existence of a bivector field satisfying (2) follows from the definition of a Hilbert basis.

We can further assume that

$$[\pi, \mathcal{J}] = 0$$

and using the natural grading in $\mathcal{X}^2[V]$ write

$$\pi = \pi^1 + \pi^{2+}$$

where π^1 has coefficients of degree 1 and π^{2+} has coefficients of degrees 2 and higher.

Then π^1 is uniquely determined by (2) and is already a Poisson bivector,

$$[\pi^1, \pi^1] = 0.$$

Using the Poisson bivector π^1 we define the π^1-cohomology coboundary operator, see [3], on $\mathcal{X}^*(V)$ by

$$\delta = -[\pi^1, \cdot].$$

Also define $\pi_{\mathbf{c}}, \pi^1_{\mathbf{c}}$ and $\pi^{2+}_{\mathbf{c}}$ in $\mathbb{C} \otimes \mathcal{X}^2[V_{\mathbf{c}}]$ as the images of π, π^1 and π^{2+} under κ respectively, and define $\delta_{\mathbf{c}}$ to be the coboundary operator on $\mathbb{C} \otimes \mathcal{X}^*(V_{\mathbf{c}})$ given by

$$\delta_{\mathbf{c}} = -[\pi^1_{\mathbf{c}}, \cdot].$$

Furthermore define Γ to be the ideal in $\mathbb{C}[V^*_{\mathbf{c}}]$ generated by the terms

$$\mathcal{J},\ l_3,\ Z_1,\ W_1,\ Z_2,\ W_2,\ Z_3,\ W_3,\ Z_4,\ W_4^2$$

and all the monomials of degree 3 and higher.

For $X \in \mathbb{C} \otimes \mathcal{X}^3[V_{\mathbf{c}}]$ denote the coefficient of $\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_2} \wedge \frac{\partial}{\partial W_4}$ in X by X_ϕ, i.e.,

$$X_\phi = X(dZ_2 \wedge dW_2 \wedge dW_4),$$

and for $Y \in \mathbb{C} \otimes \mathcal{X}^2[V_{\mathbf{c}}]$ define $\widetilde{Y}$ by

$$\widetilde{Y} = Y_{Z_2W_2}\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_2} + Y_{Z_2W_4}\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_4} + Y_{W_2W_4}\frac{\partial}{\partial W_2} \wedge \frac{\partial}{\partial W_4}$$

where $Y_{Z_2W_2}$ is the coefficient of $\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_2}$ in Y.

Lemma 1. *Let $\epsilon_{\mathbf{c}}$ be a bivector in $\mathbb{C} \otimes \mathcal{X}^2[V_{\mathbf{c}}]$. Then $\delta_{\mathbf{c}}(\epsilon_{\mathbf{c}})_\phi \equiv -[\varphi_{\mathbf{c}}, \widetilde{\epsilon_{\mathbf{c}}}]_\phi \pmod{\Gamma}$ where $\varphi_{\mathbf{c}}$ is the bivector field given by the formula*

$$\begin{aligned}\varphi_{\mathbf{c}} \;&= -2\imath(W_4\tfrac{\partial}{\partial l_1} \wedge \tfrac{\partial}{\partial W_4} + W_4\tfrac{\partial}{\partial l_2} \wedge \tfrac{\partial}{\partial W_4} + W_4\tfrac{\partial}{\partial l_3} \wedge \tfrac{\partial}{\partial W_4}\\ &+(l_2 - l_1)\tfrac{\partial}{\partial Z_1} \wedge \tfrac{\partial}{\partial W_1} + 2W_4\tfrac{\partial}{\partial Z_1} \wedge \tfrac{\partial}{\partial W_2} + 2W_4\tfrac{\partial}{\partial W_1} \wedge \tfrac{\partial}{\partial W_3}).\end{aligned}$$

Proof: All the coefficients of the basis vectors $\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_2}$, $\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_4}$ and $\frac{\partial}{\partial W_2} \wedge \frac{\partial}{\partial W_4}$ in $\pi^1_{\mathbf{c}}$ and $\varphi_{\mathbf{c}}$ are zero and the bivector field $\pi^1_{\mathbf{c}} - \varphi_{\mathbf{c}}$ is in the ideal generated by $\Gamma \cdot \mathcal{X}^2[V_{\mathbf{c}}]$. From this and the definition of the Schouten-Nijenhuis bracket it follows that $\delta_{\mathbf{c}}(\epsilon_{\mathbf{c}})_\phi \equiv -[\varphi_{\mathbf{c}}, \widetilde{\epsilon_{\mathbf{c}}}]_\phi \pmod{\Gamma}$. □

One can choose $\pi^{2+}_{\mathbf{c}}$ to be homogeneous of degree 2 and an example of $\widetilde{\pi^{2+}_{\mathbf{c}}}$ is given by

$$\widetilde{\pi^{2+}_{\mathbf{c}}} = -2\imath((l_1^2 + 4l_1l_3)\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_2} + (l_1Z_1 + 2l_3Z_1)\frac{\partial}{\partial Z_2} \wedge \frac{\partial}{\partial W_4}).$$

Call an element ρ in $\mathcal{X}^*[V]$ a relation if $\rho \circ F = 0$ and define

$$\Phi^*[V] = \{\rho' \in \mathcal{X}^*[V] : \rho' \circ F = 0\}.$$

Lemma 2. *Let $\zeta \in \mathcal{X}^2[V]$ be a bivector of the form*

$$\zeta = \pi + \rho + \mathcal{J}\gamma$$

for a relation ρ in $\Phi^2[V]$, a bivector $\gamma \in \mathcal{X}^2[V]$ and such that at least one of the two conditions below are satisfied

i) *$[\zeta, \mathcal{J}]$ contains no terms of degree 2,*
ii) *γ contains no constant terms.*

Then $[\zeta, \zeta]$ has nonzero terms of degree 2.

Proof: Let $\zeta_{\mathbf{c}} = \kappa(\zeta)$. Using Lemma 1 we calculate that an ϵ in $\Phi^2[V] + \mathcal{J}\mathcal{X}^2[V]$ satisfies the formula $\delta_{\mathbf{c}}(\kappa(\epsilon))_\phi \equiv 0 \pmod{\Gamma}$.
Furthermore the formula $[\pi^1_{\mathbf{c}}, \pi^{2+}_{\mathbf{c}}]_\phi \not\equiv 0 \pmod{\Gamma}$ holds.
Now we conclude that if either of conditions i) or ii) holds then

$$[\zeta_{\mathbf{c}}, \zeta_{\mathbf{c}}]_\phi \equiv 2[\pi^1_{\mathbf{c}}, \pi^{2+}_{\mathbf{c}}]_\phi \pmod{\Gamma}$$

and the lemma follows. □

Lemma 3. *Let $\Delta \in \mathcal{X}^*(V)$ satisfy $\Delta \circ F|_{J^{-1}(0)} = 0$. Then there exists a formal multivector field $\gamma \in \mathcal{X}^*[[V]]$ such that $T(\Delta) - \mathcal{J}\gamma$ is a formal relation, i.e.,*

$$T(\Delta) - \mathcal{J} \cdot \gamma \in \Phi^*[[V]] = \{\rho \in \mathcal{X}^*[[V]] : \rho \circ F = 0\}.$$

This lemma is proved by using the Malgrange-Mather division theorem [4, 6].□

Outline of the proof of Theorem 1: Let $\zeta \in \mathcal{X}^2(V)$ be a bivector field such that ζ and ϱ are F-related, i.e., $\zeta \circ F = F_* \circ \varrho$.

Now write $T(\zeta) - \pi = \rho' + {\rho'}^{3+}$ where $\rho' \in \Phi^2[V]$ has polynomial coefficients and such that ${\rho'}^{3+} \in \Phi^2[[V]]$ contains only terms of degrees greater than two.

Now since $\pi + \rho'$ satisfies condition ii) of Lemma 2, deduce that the Taylor series of $[\zeta, \zeta]$ has nonzero terms of degree 2. Hence ζ is not Poisson.

Assume now that there exists a Poisson bivector field ξ' in $\mathcal{X}^2(V_J)$ extending the Poisson structure of $J^{-1}(0)/\mathbf{S}^1$ to all of V_J.

By 3a, 3b and 3c we extend ξ' to a Poisson bivector field ξ on V in such a way that ξ extends the Poisson structure of $J^{-1}(0)/\mathbf{S}^1$ in V and has $\mathcal{J}$ for a Casimir function.

Use Lemma 3 to write

$$T(\xi) = \pi + \rho + \mathcal{J}\gamma + \rho^{3+} + \mathcal{J}\gamma^{2+}$$

where as before ρ is in $\Phi^2[V]$ and ρ^{3+} is in $\Phi^2[[V]]$ and has only coefficients of degrees greater than two, γ is in $\mathcal{X}^2[V]$ and γ^{2+} is a formal bivector field with coefficients of degrees greater than one. Conclude, since $\pi + \rho + \mathcal{J}\gamma$ satisfies condition i) of Lemma 2, that the Taylor series of $[\xi, \xi]$ has nonzero terms of degree 2. Thus contradicting the existence of the Poisson bivector field ξ'.□

3. The $1:1:2$ resonance

3.1. **Preliminaries.** To analyze the relationship between the $n_1 : \cdots : n_k$ resonance and the $\sigma_1 n_1 : \cdots : \sigma_k n_k$ resonance, $\sigma_i \in \{1, -1\}$ for $i = 1, \ldots, k$, we define an automorphism, see [2], ψ^σ on the space $\mathbb{C} \otimes \mathcal{X}^*[V]$. Here we will define ψ^σ for the $1 : 1 : -2$ resonance and the $\sigma = (1, 1, -1)$ case.

For the $1 : 1 : 2$ resonance the Hilbert map $F^\sigma : M \to V$ in standard form, by definition here, is obtained by interchanging u_3 and v_3 in the definitions of the invariants $f_1, \ldots, f_{11}$.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) = (1, 1, -1)$ and define

$$\omega^\sigma : \{l_1, l_2, l_3, Z_1, W_1, \ldots, Z_4, W_4\} \to \{1, -1, \imath, -\imath\}$$

by the formula

$$\omega^\sigma_\gamma = \imath^{\deg_{z_3} \tau_{\mathbf{c}}(\gamma)} \cdot \imath^{\deg_{w_3} \tau_{\mathbf{c}}(\gamma)}.$$

Now, using ω^σ, define the automorphism

$$\psi^\sigma_{\mathbf{c}} : \mathbb{C} \otimes \mathcal{X}^*[V_{\mathbf{c}}] \to \mathbb{C} \otimes \mathcal{X}^*[V_{\mathbf{c}}]$$

by the formulas

$$l_s \mapsto \omega^\sigma_{l_s} l_s,\ Z_t \mapsto \omega^\sigma_{Z_t} Z_t \text{ and } W_t \mapsto \omega^\sigma_{W_t} W_t,$$

$$\frac{\partial}{\partial l_s} \mapsto \frac{1}{\omega^\sigma_{l_s}} \frac{\partial}{\partial l_s},\ \frac{\partial}{\partial Z_t} \mapsto \frac{1}{\omega^\sigma_{Z_t}} \frac{\partial}{\partial Z_t} \text{ and } \frac{\partial}{\partial W_t} \mapsto \frac{1}{\omega^\sigma_{W_t}} \frac{\partial}{\partial W_t}$$

and also define

$$\psi^\sigma : \mathbb{C} \otimes \mathcal{X}^*[V] \to \mathbb{C} \otimes \mathcal{X}^*[V] \text{ by } \psi^\sigma = \kappa^{-1} \circ \psi^\sigma_{\mathbf{c}} \circ \kappa.$$

The map ψ^σ is a Schouten-Nijenhuis morphism defining a cochain map between the complexes

$$(\mathbb{C} \otimes \mathcal{X}^*[V], \delta) \xrightarrow{\psi^\sigma} (\mathbb{C} \otimes \mathcal{X}^*[V], \delta^\sigma)$$

where $\delta = -[\pi^1, \cdot]$ and $\delta^\sigma = -[\psi^\sigma(\pi^1), \cdot]$.

Lemma 4. *Let π' be a bivector field in $\mathcal{X}^2[V]$. The bivectors π' and ϱ are F-related if and only if the bivectors $\psi^\sigma(\pi')$ and ϱ are F^σ related.*

Proof: [2].□

Thus ψ^σ allows us to apply results for the $1 : 1 : -2$ resonance to the $1 : 1 : 2$ resonance case.

3.2. **Embeddings.**

Theorem 2. *Let $F^\sigma : M \to V$ be the Hilbert map in standard form for the* $1 : 1 : 2$ *resonance action* $\star : \mathbf{S}^1 \times \mathrm{M} \to \mathrm{M}$. *Then the Poisson structure on the singular orbit space $M/\mathbf{S}^1$, embedded into V by F^σ, cannot be extended to V.*

Proof: Follows from applying the morphism ψ^σ to results for the $1 : 1 : -2$ resonance. □

4. Generalizations

In order to analyze which circle actions induce Poisson embeddings of the orbit spaces one should study the semigroup structures obtained from the invariant polynomials. The semigroup structure arises from complexifying the phase space and the ambient Euclidean space. A proper framework for Poisson structures and maps for these semigroups would be helpful in resolving the general case. The author is currently working in this direction.

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