

## THE EHRHART POLYNOMIAL OF A LATTICE $n$ -SIMPLEX

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ABSTRACT. The problem of counting the number of lattice points inside a lattice polytope in  $\mathbb{R}^n$  has been studied from a variety of perspectives, including the recent work of Pommersheim and Kantor-Khovanskii using toric varieties and Cappell-Shaneson using Grothendieck-Riemann-Roch. Here we show that the Ehrhart polynomial of a lattice  $n$ -simplex has a simple analytical interpretation from the perspective of Fourier Analysis on the  $n$ -torus. We obtain closed forms in terms of cotangent expansions for the coefficients of the Ehrhart polynomial, that shed additional light on previous descriptions of the Ehrhart polynomial.

The number of lattice points inside a convex lattice polytope in  $\mathbb{R}^n$  (a polytope whose vertices have integer coordinates) has been studied intensively by combinatorialists, algebraic geometers, number theorists, Fourier analysts, and differential geometers. Algebraic geometers have shown that the Hilbert polynomials of toric varieties associated to convex lattice polytopes precisely describe the number of lattice points inside their dilates [3]. Number theorists have estimated lattice point counts inside symmetric bodies in  $\mathbb{R}^n$  to get bounds on ideal norms and class numbers of number fields. Fourier analysts have estimated the number of lattice points in simplices using Poisson summation (see Siegel's classic solution of the Minkowski problem [14] and Randol's estimates for lattice points inside dilates of general planar regions [13]). Differential geometers have also become interested in lattice point counts in polytopes in connection with the Dufree conjecture [18].

Let  $\mathbb{Z}^n$  denote the  $n$ -dimensional integer lattice in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be an  $n$ -dimensional lattice polytope in  $\mathbb{R}^n$ . Consider the function of an integer-valued variable  $t$  that describes the number of lattice points that lie inside the dilated polytope  $t\mathcal{P}$ :

$$L(\mathcal{P}, t) = \text{the cardinality of } \{t\mathcal{P}\} \cap \mathbb{Z}^n.$$

Ehrhart [4] inaugurated the systematic study of general properties of this function by proving that it is always a polynomial in  $t \in \mathbb{N}$ , and that in fact

$$L(\mathcal{P}, t) = \text{Vol}(\mathcal{P})t^n + \frac{1}{2}\text{Vol}(\partial\mathcal{P})t^{n-1} + \cdots + \chi(\mathcal{P})$$

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for closed polytopes  $\mathcal{P}$ . Here  $\chi(\mathcal{P})$  is the Euler characteristic of  $\mathcal{P}$ , and  $\text{Vol}(\partial\mathcal{P})$  is the surface area of  $\mathcal{P}$  normalized with respect to the sub-lattice on each face of  $\mathcal{P}$ . The Ehrhart-MacDonald reciprocity law  $L(\mathcal{P}^{int}, t) = (-1)^n L(\mathcal{P}^{clos}, -t)$  established that the study of the open polytope  $\mathcal{P}$  is essentially equivalent to the study of its closure.

The other coefficients of  $L(\mathcal{P}, t)$  remained a mystery, even for a general lattice 3-simplex, until the recent work of Pommersheim [12] in  $\mathbb{R}^3$ , Kantor and Khovanskii [6] in  $\mathbb{R}^4$ , and most recently Cappell and Shaneson [2] in  $\mathbb{R}^n$ . [12] and [6] used techniques from algebraic geometry related to the Todd classes of toric varieties to express these coefficients in terms of Dedekind sums and other cotangent expansions, and [2] used Grothendieck-Riemann-Roch for their work on convex lattice polytopes. The present paper introduces Fourier methods into the study of lattice polytopes, and some known recent results are shown to be easy corollaries of the main theorem. In the course of the page proofs Brion and Vergne have communicated to us that they recently and independently found another characterization of the Ehrhart polynomial using Fourier methods.

We compute the Ehrhart polynomial via Fourier integrals on the  $n$ -torus by first finding an explicit description of an associated generating function, defined by

$$(1) \quad \mathfrak{G}(\mathcal{P}, s) = \sum_{t=0}^{\infty} L(\mathcal{P}, t) e^{-2\pi st}.$$

A geometrical interpretation of this expression can be given that provides a direct link with Fourier analysis and the Poisson summation formula. The polytope  $\mathcal{P} \subset \mathbb{R}^n$  and its dilates can be regarded as parallel sections of a convex cone  $K \subset \mathbb{R}^{n+1}$  with vertex  $\{0\} \in \mathbb{R}^{n+1}$ , generated by  $\mathcal{P} \times \{1\}$ ; and the generating function can be regarded as the sum of the values of an exponentially decaying function taken over all integral lattice points lying in  $K$ . For a related construction see [1]. In order to investigate the general properties of such sums systematically it is convenient to introduce some notation and terminology. The polar cone associated to a convex cone  $K$  is the convex cone defined by

$$K^* = \{\eta \in \mathbb{R}^{n+1} : \langle x, \eta \rangle < 0, \forall x \in K\}.$$

For each  $\eta \in K^*$  consider the convergent sum

$$\mathfrak{G}(K, \eta) = \sum_{x \in \mathbb{Z}^{n+1}} \chi_K(x) \exp(2\pi \langle x, \eta \rangle),$$

which can be regarded as a discrete, multi-variable Laplace transform of the characteristic function of  $K$ . We are interested in the behavior of the restriction of  $\mathfrak{G}(K, \eta)$  to the positive ray  $\eta = s\eta_0$  where  $s \in \mathbb{R}^+$  and  $\eta_0 = (0, 0, 0, \dots, -1) \in K^*$ , because  $\mathfrak{G}(K, s\eta_0) = \mathfrak{G}(\mathcal{P}, s)$  is the generating function of the Ehrhart polynomial of the simplex  $\mathcal{P}$  that generates  $K$ . We evaluate  $\mathfrak{G}(K, \eta)$  by first computing the (continuous) Fourier-Laplace transform of smoothed versions of the exponentially damped characteristic function of  $K$ , and then applying the Poisson summation formula to pass from the continuous to the discrete setting.

Once the generating function  $\mathfrak{G}(\mathcal{P}, s)$  has been determined, it is easy to recover the Ehrhart function itself. Precisely because  $L(\mathcal{P}, t)$  is a polynomial in  $t$ , the generating function is a rational function in  $e^{-2\pi s}$ . Its meromorphic continuation into the complex  $s$ -plane has a pole at  $s = 0$ , and the Ehrhart polynomial coefficients



**Lemma 1.** *For every complex frequency vector  $\zeta$  satisfying  $\text{Im } \zeta \in K^*$ , the Fourier-Laplace transform of the characteristic function of the cone  $K$  is a product of reciprocals of linear forms determined by the columns of the matrix  $M$  that generates  $K$ :*

$$\hat{\chi}_K(\zeta) = |\det M| \prod_{k=1}^{n+1} \frac{1}{\langle 2\pi i \zeta, M_k \rangle}.$$

□

Taking  $\zeta = (\xi_1, \xi_2, \dots, \xi_{n+1}) - i(0, 0, \dots, s) \in \mathbb{Z}^{n+1} + iK^*$  we see that Lemma 1 gives the explicit formula

$$(2) \quad \hat{f}_{s\eta_0}(\xi) = \frac{|\det M|}{(2\pi)^{n+1}} \prod_{k=1}^{n+1} \frac{1}{s + il_k(\xi)},$$

where the linear forms  $l_k(\xi) = \langle \xi, M_k \rangle$  are determined by the columns of  $M$ . Note that since  $M$  is a lower-triangular matrix, the linear form  $l_k(0, 0, \dots, \xi_k, \dots, \xi_{n+1})$  depends only on the last  $n+1-k$  coordinates of  $\xi$ .

We would like to use the Poisson summation formula to sum expression (2) over all integral frequency vectors  $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{Z}^{n+1}$ , but before doing so we multiply (2) by an additional damping function in the frequency domain that makes the resulting series absolutely summable. This is equivalent to mollifying the function  $f_\eta(x)$  by convolution. The mollifier will be taken to be an approximate identity  $\phi_\epsilon(x)$  constructed from a dilated version of  $f_{\eta_0}(x)$ , normalized to have mass one. This normalization condition is equivalent to requiring that  $\hat{\phi}_\epsilon(0) = 1$ . For each real  $\epsilon \neq 0$  we therefore multiply (2) by

$$\hat{\phi}_\epsilon(\xi) = \frac{\hat{f}_{\eta_0}(\epsilon\xi)}{\hat{f}_{\eta_0}(0)} = \prod_{k=1}^{n+1} \frac{1}{(1 + i\epsilon l_k(\xi))}$$

to get

$$(3) \quad (\widehat{f_{s\eta_0} * \phi_\epsilon})(\xi) = \frac{|\det M|}{(2\pi)^{n+1}} \prod_{k=1}^{n+1} \frac{1}{(s + il_k(\xi))(1 + i\epsilon l_k(\xi))}.$$

**Lemma 2.** *Suppose that  $\epsilon \neq 0$  and  $\eta \in K^*$ .*

(i) *The Poisson summation formula holds for  $f_\eta * \phi_\epsilon$  in the sense that the following two sums are both absolutely convergent representations of a common expression:*

$$\mathfrak{G}_\epsilon(K, \eta) := \sum_{x \in \mathbb{Z}^{n+1}} (f_\eta * \phi_\epsilon)(x) = \sum_{\xi \in \mathbb{Z}^{n+1}} \hat{f}_\eta(\xi) \hat{\phi}_\epsilon(\xi).$$

(ii) *The preceding expression,  $\mathfrak{G}_\epsilon(K, \eta)$ , possesses one-sided limits as  $\epsilon \rightarrow 0^\pm$ ; and these one-sided limits are the discrete Laplace transforms of the characteristic functions of the interior and closure of  $K$ :*

$$(ii.a) \quad \lim_{\epsilon \rightarrow 0^+} \mathfrak{G}_\epsilon(K, \eta) = \sum_{x \in \mathbb{Z}^{n+1}} \chi_{K^{int}}(x) \exp(2\pi \langle x, \eta \rangle) = \mathfrak{G}(K^{int}, \eta);$$

$$(ii.b) \quad \lim_{\epsilon \rightarrow 0^-} \mathfrak{G}_\epsilon(K, \eta) = \sum_{x \in \mathbb{Z}^{n+1}} \chi_{K^{clos}}(x) \exp(2\pi \langle x, \eta \rangle) = \mathfrak{G}(K^{clos}, \eta).$$

□

In view of Lemma 1 it is natural to investigate the sum  $\sum_{\xi \in \mathbb{Z}^{n+1}} \hat{f}_\eta(\xi) \hat{\phi}_\epsilon(\xi)$  using (3). To describe the main result let  $\mathcal{G}$  be the finite abelian group  $(\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})$ , where  $p_k = \prod_{i < k} a_{i,i}$  for  $k \leq n$  and  $p_{n+1} = p_n$ . For each group element  $r = (r_1, \dots, r_n) \in \mathcal{G}$ , let  $m_k(r) = l_k(0, r_1, \dots, r_n)$  denote the linear form that  $l_k(\xi) = \langle \xi, M_k \rangle$  induces on the group  $\mathcal{G}$ .

**Theorem.**

$$\sum_{t=0}^{\infty} L(\mathcal{P}^{int}, t) e^{-2\pi st} = \frac{1}{2^{n+1}} \frac{1}{|\mathcal{G}|} \sum_{r \in \mathcal{G}} \prod_{k=1}^{n+1} \left( 1 + \coth \frac{\pi}{p_k} (s + im_k(r)) \right);$$

$$1 + \sum_{t=1}^{\infty} L(\mathcal{P}^{clos}, t) e^{-2\pi st} = \frac{1}{2^{n+1}} \frac{1}{|\mathcal{G}|} \sum_{r \in \mathcal{G}} \prod_{k=1}^{n+1} \left( -1 + \coth \frac{\pi}{p_k} (s + im_k(r)) \right).$$

□

**Corollary.**  $L(\mathcal{P}^{int}, t)$  is a polynomial in  $t \in \mathbb{N}$ , and if  $L(\mathcal{P}^{int}, t) = \sum_{i=0}^n c_i t^i$ , then  $c_m$  is the coefficient of  $s^{-m-1}$  in the Laurent expansion at  $s = 0$  of

$$\frac{\pi^{m+1}}{m! 2^{n-m}} \frac{1}{|\mathcal{G}|} \sum_{r \in \mathcal{G}} \prod_{k=1}^{n+1} \left( 1 + \coth \frac{\pi}{p_k} (s + im_k(r)) \right).$$

□

We remark that for three-dimensional polytopes  $\mathcal{P}$ , the only coefficient of the Ehrhart polynomial  $L(\mathcal{P}, t) = c_3 t^3 + c_2 t^2 + c_1 t + 1$  that is difficult to determine is the codimension two coefficient  $c_1$  (see [12]). This coefficient can be found by solving for the coefficient of  $s^{-2}$  in the Laurent expansion of the product of cotangents given by the corollary. Consider for example the ‘‘Mordell-Pommersheim’’ Tetrahedron, whose vertices are  $(0, 0, 0)$ ,  $P = (a, 0, 0)$ ,  $Q = (0, b, 0)$ , and  $R = (0, 0, c)$ ; where  $a, b, c \in \mathbb{N}$  [8], [12]. There is no loss of generality in assuming that  $\gcd(a, b, c) = 1$ . Define  $A = \gcd(b, c)$ ,  $B = \gcd(a, c)$ ,  $C = \gcd(a, b)$ , and  $d = ABC$ .

We recall that the classical Dedekind sum  $\mathfrak{s}(q, p)$  can be defined for  $\gcd(p, q) = 1$  in terms of cotangent expansions [5] by

$$\mathfrak{s}(q, p) = \frac{1}{4p} \sum_{m=1}^{p-1} \cot \frac{\pi m}{p} \cot \frac{\pi m q}{p}.$$

Using the previous corollary, it is easy to recover Theorem 5 of [12]:

$$c_1 = \frac{1}{4} (A + B + C + a + b + c) + \frac{1}{12} \left( \frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} + \frac{d^2}{abc} \right) - A \mathfrak{s} \left( \frac{bc}{d}, \frac{aA}{d} \right) - B \mathfrak{s} \left( \frac{ac}{d}, \frac{bB}{d} \right) - C \mathfrak{s} \left( \frac{ab}{d}, \frac{cC}{d} \right).$$

As other corollaries of the main theorem, one quickly recovers the Ehrhart-MacDonald Reciprocity Law [4], [7], [16] and higher dimensional Dedekind sums appearing in Hirzebruch’s and Zagier’s work [5], [19].

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