ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, August 1996

# GLOBAL SOLUTIONS OF THE EQUATIONS OF ELASTODYNAMICS FOR INCOMPRESSIBLE MATERIALS

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(Communicated by James Glimm)

Abstract. The equations of the dynamics of an elastic material are a non-linear hyperbolic system whose unknowns are functions of space and time. If the material is incompressible, the system has an additional pseudo-differential term. We prove that such a system has global (classical) solutions if the initial data are small. This contrasts with the case of compressible materials for which F. John has shown that such solutions may not exist even for arbitrarily small data.

## 1. Introduction

**Theorem.** *The initial value problem for the equations of motion of an incompressible hyperelastic homogeneous isotropic material has classical solutions for all time, if the initial displacement and velocity are small.*

The displacement and velocity must be small in the norm of equation (5.4) below, where $k$ must be at least 9 for the displacement and at least 8 for the velocity. This norm defines a Hilbert space which includes all functions which when differentiated up to $k$ times and multiplied by polynomials of degree no more than $k$ are in $L^2$. In particular it includes all smooth functions of compact support and even all rapidly decreasing functions. Thus the theorem applies to all small rapidly decreasing data.

The mathematical meaning of the physical properties of our material is described in Section 3, where the equations of motion are derived.

Our theorem was proved for the special case of neo-Hookean materials in [Eb], but the present more general result is important on two counts. From the standpoint of elasticity it is important because the neo-Hookean model is not appropriate in most cases. From the mathematical perspective it is significant because the non-linearities in the neo-Hookean case are due solely to the incompressibility, while in the more general case they are a consequence of the basic constitutive laws as well. Indeed, in our study of the neo-Hookean case we were able to rewrite the equations so that the non-linear terms were reduced to lower order. Here we are forced to proceed by a different method.

Received by the editors December 29, 1995.
1991 *Mathematics Subject Classification.* Primary 35L70, 35Q72, 73C50, 73D35.
*Key words and phrases.* Non-linear hyperbolic, elastodynamics, incompressible, global existence.
Partially supported by NSF grant DMS 9304403.

It is interesting to note that this more general theorem is false for compressible materials as is shown by the counter-example of F. John [Jo1]. Furthermore, our theorem does not apply to motions which are constant in any one dimension (a case in which shocks may appear) because the data for such motions are not finite in the norm of (5.4).

For small time, solutions of the above initial value problem are known [EbSa], as are solutions of corrresponding initial-boundary value problems [HrRe], [EbSi1], [EbSi2].

We thank Professors D. Christodoulou, S. Simanca, and S. Sogge for several very useful conversations.

## 2. Background

Over the past twenty years there have been a number of papers devoted to the initial value problem for non-linear wave equations with small initial data. Klainerman [Kla1] showed that solutions exist for all time if the number of space dimensions is at least four. John, on the other hand, [Jo2], [Jo3] found, in dimension three, examples with arbitrarily small data which develop a singularity in finite time. Then Klainerman [Kla3] discovered a condition on the non-linearities (Klainerman's null condition) which guarantees solutions in the large for the three-dimensional case as well.

These mathematical ideas have been applied in two areas, relativity theory and elastodynamics. In relativity theory one has Einstein equations which can be written as a system of non-linear wave equations on $\mathbf{R}^1 \times \mathbf{R}^3$. One can consider the initial value problem for this system by prescribing data on $0 \times \mathbf{R}^3$. If the data are zero, one gets the trivial solution which is Minkowski space.

Christodoulou and Klainerman [ChKl] studied this initial value problem and showed that it satisfies the null condition. Thus solutions to the problem with small data exist in the large and are small for all time. Hence Minkowski space is globally stable in the sense that small data give a solution which is globally close to that of Minkowski space.

We will pursue the second area of application — elastodynamics. This is a natural area for the following reasons. Many problems in elasticity involve small deformation. Such problems are sometimes modeled by the equations of elasticity linearized about the state of no deformation, the solutions of such equations being called "infinitesimal" elastic waves. A more accurate model uses the full non-linear equations (whose solutions might be called small elastic waves), but this model is only good for the length of time for which solutions exist. Thus it becomes important to compute this time. One also wants to know whether the small elastic waves remain small over long times and whether they are close to the infinitesimal elastic waves. Our techniques provide positive answers to all of these questions in the incompressible case.

Our equations do not satisfy Klainerman's null condition, but we still use techniques similar to his [Kla3] to prove existence in the large. In fact we construct an "energy" function $E(t)$ from $L^2$-norms of the displacement, velocity, and their derivatives and show that it obeys a certain differential inequality. Combining this inequality with Klainerman's inequality [Kla2], we can show that if $E(0)$ is sufficiently small, then it remains finite for all positive $t$, and from this it follows that our equations have global solutions.

It is interesting to contrast our study with that of John ([Jo1]). He shows that for compressible materials there are waves with arbitrarily small initial data which blow up (develop shocks) in finite time. Thus one has small waves which do not remain small and which diverge from infinitesimal waves with the same data.

## 3. Physical assumptions and derivation of equations

We consider a material that fills all of three-space and let $\eta(t) : \mathbf{R}^3 \to \mathbf{R}^3$ denote the map that takes each material point from its original position $x$ to the position $\eta(t)(x)$, which it reaches at time $t$. Thus the curve of maps $t \mapsto \eta(t)$ describes the entire motion. The velocity of this material point is $\dot{\eta}(t)(x)$ [1] and the kinetic energy of the motion at time $t$ is

$$K(\dot{\eta}(t)) = \frac{1}{2} \int_{\mathbf{R}^3} |\dot{\eta}(t)(x)|^2 dx. \tag{3.1}$$

The hypothesis that our material is homogeneous and hyperelastic means that the stresses on the material are derived from a potential energy function $W(D\eta^\tau D\eta)$ where $D\eta$ is the $3 \times 3$ matrix of spatial derivatives of $\eta(t) : \mathbf{R}^3 \to \mathbf{R}^3$ and $D\eta^\tau$ is its transpose. We shall denote $D\eta^\tau D\eta$ by $C$ below. [2] The additional hypothesis that the material is isotropic implies that $W$ depends only on the similarity class of $C$, or equivalently, that $W(C)$ is a symmetric function of the eigenvalues of $C$. We shall assume that $W$ is smooth. The final hypothesis, incompressibility, is expressed mathematically as the statement that $\det(D\eta)$, which we shall call $J(\eta)$, is identically one.

With these hypotheses we can derive the equations of motion by using Hamilton's principle. We denote by

$$V = \int_{\mathbf{R}^3} W(D\eta^\tau(t)(x) D\eta(t)(x)) dx \tag{3.2}$$

the potential energy of the configuration $\eta(t)$ and look for stationary curves $\eta(t)$ of the action:

$$\mathcal{L} = \int_0^T (K - V)\, dt. \tag{3.3}$$

We consider a variation $\eta(t,s)$ of $\eta(t) = \eta(t,0)$ where $\eta(0,s) = \eta(0)$ and $\eta(T,s) = \eta(T)$, and we require that

$$\frac{d}{ds}\mathcal{L}|_{s=0} = 0,$$

where

$$\mathcal{L}(s) = \int_0^T \left( K(\dot{\eta}(t,s)) - V(\eta(t,s)) \right) dt. \tag{3.4}$$

Computing $\frac{d}{ds}\mathcal{L}$ and performing the usual integration by parts we find that

$$\frac{d}{ds}\mathcal{L}|_{s=0} = -\int_0^T \int_{\mathbf{R}^3} \langle \ddot{\eta}(t), \sigma(t) \rangle - \langle \mathrm{Div}\, \frac{\partial W}{\partial D\eta}, \sigma(t) \rangle dt, \tag{3.5}$$

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[1] "˙" means time derivative.

[2] $C$ is commonly known as the Green deformation tensor or the right Cauchy-Green tensor [MaHu].

where $\sigma(t) = \partial_s \eta(t,s)|_{s=0}$, $\frac{\partial W}{\partial D\eta}$ is the matrix of derivatives of $W$ with respect to each $\frac{\partial \eta^i}{\partial x^j}$, and

$$\operatorname{Div} \frac{\partial W}{\partial D\eta} = \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \left( \frac{\partial W}{\partial \frac{\partial \eta^i}{\partial x^j}} \right). \tag{3.6}$$

Thus the requirement that $\eta(t)$ be stationary tells us that at any time $t$, $\ddot{\eta} - \operatorname{Div} \frac{\partial W}{\partial D\eta}$ is perpendicular in $L^2(\mathbf{R}^3, \mathbf{R}^3)$ to $\sigma(t)$.

One might think that $\sigma(t) : \mathbf{R}^3 \to \mathbf{R}^3$ is arbitrary so one gets the equation:

$$\ddot{\eta} - \operatorname{Div} \frac{\partial W}{\partial D\eta} = 0. \tag{3.7}$$

This is correct for compressible materials, but in our case the requirement $J(\eta) \equiv 1$ causes complications.

We note that

$$\eta(t,s) = \eta(t) + s\sigma(t) + o(s),$$

which we rewrite as

$$\eta(t,s) = (id + s\sigma(t) \circ \eta(t)^{-1}) \circ \eta(t) + o(s), \tag{3.8}$$

where $id : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ is identity.

Taking Jacobian determinants we get

$$J(\eta(t,s)) = \det(Id + sD(\sigma(t) \circ \eta(t)^{-1})) \circ \eta(t) + o(s), \tag{3.9}$$

where $Id$ means the identity matrix. Thus the requirement $J(\eta) \equiv 1$ tells us that

$$1 \equiv J(\eta(t,s)) \equiv \det(Id + sD(\sigma(t) \circ \eta(t)^{-1}) \circ \eta(t) + o(s) \tag{3.10}$$

and hence that

$$0 = \frac{\partial}{\partial s} J(\eta(t,s))|_{s=0} = \operatorname{tr}(D(\sigma(t) \circ \eta^{-1}(t))) \circ \eta(t), \tag{3.11}$$

where "tr" means trace of a matrix.

Equation (3.11) is a restriction on $\sigma(t)$, so we find that $\ddot{\eta} - \operatorname{Div} \frac{\partial W}{\partial D\eta}$ needs to be perpendicular only to all $\sigma(t)$ satisfying this restriction.

We note that the right side of (3.11) can be written as

$$(\operatorname{div}(\sigma(t) \circ \eta(t)^{-1})) \circ \eta(t),$$

where "div" means divergence. We will denote this by $\operatorname{div}_{\eta(t)} \sigma(t)$. In fact if $L$ is any differential operator and $\eta : \mathbf{R}^3 \to \mathbf{R}^3$ is a diffeomorphism, we will use $L_\eta f$ to mean $L(f \circ \eta^{-1}) \circ \eta$. We have need for such $L_\eta$ for the following reason: The functions which we study are usually given as functions of time $t$ and material coordinates $x$. However, in calculations concerning incompressibility, we must consider derivatives of such functions with respect to spatial coordinates $\eta(t)(x)$. If $L$ is an operator involving differentiations with respect to $x$, $L_\eta$ will have the same differentiations with respect to $\eta(x)$ and thus will be the desired operator.

To determine the significance of the restriction on $\sigma(t)$, we note that $L^2(\mathbf{R}^3, \mathbf{R}^3)$ can be decomposed as

$$L^2(\mathbf{R}^3, \mathbf{R}^3) = \operatorname{div}^{-1}(0) \oplus \nabla \mathcal{F}, \tag{3.12}$$

where the first summand is the space of divergence-free vector fields and the second is the space of gradients of functions on $\mathbf{R}^3$. The summands are clearly perpendicular with respect to $L^2$. A vector field $v$ is decomposed as $w + \nabla f$, where $f$ solves the equation

$$\Delta f = \operatorname{div} v \tag{3.13}$$

and $w$ is defined to be $v - \nabla f$.

There is of course a "sub-$\eta$" version of this decomposition — namely:

$$L^2(\mathbf{R}^3, \mathbf{R}^3) = {\operatorname{div}_\eta}^{-1}(0) \oplus \nabla_\eta \mathcal{F}. \tag{3.14}$$

If $J(\eta) \equiv 1$, this decomposition is also orthogonal with respect to $L^2$.

Thus to say that $\ddot{\eta} - \operatorname{Div} \frac{\partial W}{\partial D\eta}$ is orthogonal to all $\sigma$ satisfying $\operatorname{div}_\eta(\sigma) = 0$ is the same as saying:

$$\ddot{\eta} - \operatorname{Div} \frac{\partial W}{\partial D\eta} = \nabla_\eta q \tag{3.15}$$

for some function $q(t) : \mathbf{R}^3 \to \mathbf{R}$. Equation (3.15) with the accompanying condition $J(\eta(t)) \equiv 1$ is the equation of elastodynamics for incompressible materials.

## 4. Analysis of equations

Since we are interested in small deformations, we want to consider motions $\eta(t)$ which are close to $id : \mathbf{R}^3 \to \mathbf{R}^3$. For this reason it is natural for us to work with the displacement $u$ which we define to be $\eta - id$. Clearly $\dot{u} = \dot{\eta}$, $\ddot{u} = \ddot{\eta}$, $Du = D\eta - id$, and $D^2 u = D^2\eta$. Also we can write (3.15) as an equation in $u$.

Furthermore, since $W(C)$ is a symmetric function of the eigenvalues of $C$, it depends only on their elementary symmetric functions, $\operatorname{tr}(C)$, $\operatorname{tr}(C^2)$, and $\operatorname{tr}(C^3)$. But since $J(\eta) \equiv 1$, we know that $\det(C) \equiv 1$, and since the symmetric functions obey the equation:

$$6 \det(C) = \operatorname{tr}(C)^3 - 3\operatorname{tr}(C)\operatorname{tr}(C^2) + 2\operatorname{tr}(C^3),$$

we find that $\operatorname{tr}(C^3)$ is a function of $\operatorname{tr}(C)$, and $\operatorname{tr}(C^2)$. Thus we can consider $W(C)$ as a function of the two variables $\operatorname{tr}(C)$ and $\operatorname{tr}(C^2)$.

We now proceed to find $\operatorname{Div} \frac{\partial W}{\partial D\eta}$ in terms of $u$. Direct computation shows that $\frac{\partial \operatorname{tr}(C)}{\partial D\eta} = 2D\eta$ and $\frac{\partial \operatorname{tr}(C^2)}{\partial D\eta} = 4D\eta D\eta^\tau D\eta$. Thus letting $a = \operatorname{tr}(C)$ and $b = \operatorname{tr}(C^2)$ we find that

$$\frac{\partial W}{\partial D\eta} = 2\partial_a W D\eta + 4\partial_b W D\eta D\eta^\tau D\eta. \tag{4.1}$$

We apply Div to this expression and keep in mind that since $J(\eta) \equiv 1$,

$$1 + \operatorname{div} u + Q(Du) = 1, \tag{4.2}$$

where $Q(Du)$ is a polynomial in $Du$ of order 2.[3] Then doing a lengthly but routine computation, we find that [4]

$$\left(\operatorname{Div} \frac{\partial W}{\partial D\eta}\right)^i = c_0 \Delta u^i + c_1 \nabla^i \operatorname{div}(u) + c_2 \nabla^i \mid Du \mid^2 + c_3(\partial_j u^k + \partial_k u^j)\partial_j \partial_k u^i$$
$$+ c_4(\partial_i u^j + \partial_j u^i)\Delta u^j + Q(Du)^{ij}_{k\ell} \partial_k \partial_\ell u^j,$$

[3] Throughout the paper $Q$ will stand for any function of order at least two; that is, any function whose value and first derivatives vanish at the origin.

[4] Repeated indices are summed throughout.

where $c_0, \dots, c_4$ are positive constants [5] and each $Q(Du)^{ij}_{k\ell}$ again stands for a function in $Du$ of order at least two. For convenience we scale units so that $c_0 = 1$.

Thus we derive:

$$\begin{aligned}\ddot{u}^i &= \Delta u^i + c_1 \nabla^i \mathrm{div}\,(u) + c_2 \nabla^i \mid Du \mid^2 + c_3(\partial_j u^k + \partial_k u^j)\partial_j \partial_k u^i \\ &+ c_4(\partial_i u^j + \partial_j u^i)\Delta u^j + Q(Du)^{ij}_{k\ell}\partial_k \partial_\ell u^j + \nabla^i_\eta q.\end{aligned}$$

However, since $\mathrm{div}\, u = Q(Du)$, this simplifies to

$$\begin{aligned}\ddot{u}^i &= \Delta u^i + \nabla^i Q(Du) + c_3(\partial_j u^k + \partial_k u^j)\partial_j \partial_k u^i \\ &+ c_4(\partial_i u^j + \partial_j u^i)\Delta u^j + Q(Du)^{ij}_{k\ell}\partial_k \partial_\ell u^j + \nabla^i_\eta q.\end{aligned} \tag{4.3}$$

Now we must investigate $\nabla_\eta q$, which is the second summand of $\ddot{\eta} - \mathrm{Div}\, \frac{\partial W}{\partial D\eta}$ in the decomposition (3.13). $q$ satisfies:

$$\Delta_\eta q = \mathrm{div}_\eta \left( \ddot{\eta} - \mathrm{Div}\, \frac{\partial W}{\partial D\eta} \right).$$

Taking a time derivative of $J(\eta) \equiv 1$, we find that

$$\mathrm{div}_\eta\, \dot{\eta} = 0. \tag{4.4}$$

Taking a second time derivative we get

$$\mathrm{div}_\eta\, \ddot{\eta} = [v \cdot \nabla, \mathrm{div}\,]_\eta \dot{\eta},$$

where $v$ is defined as $\dot{\eta} \circ \eta^{-1}$ and $[\, ,\, ]$ means commutator. Thus $q$ is the solution of

$$\Delta_\eta q = \left( [v \cdot \nabla, \mathrm{div}\,]_\eta \dot{\eta} - \mathrm{div}_\eta\, \mathrm{Div}\, \frac{\partial W}{\partial D\eta} \right),$$

so $q$ depends only on $\eta$ and $\dot{\eta}$ or equivalently on $(u, \dot{u})$, but not on $\ddot{\eta}$. Hence the right side of (4.3) involves only spatial derivatives of $u$ and the first time derivative in the gradient term $\nabla_\eta q$. Combining the gradient terms we define $F$ to be $\nabla Q(Du) + \nabla_\eta q$ so (4.3) becomes:

$$\begin{aligned}\ddot{u}^i &= \Delta u^i + c_3(\partial_j u^k + \partial_k u^j)\partial_j \partial_k u^i \\ &+ c_4(\partial_i u^j + \partial_j u^i)\Delta u^j + Q(Du)^{ij}_{k\ell}\partial_k \partial_\ell u^j + F^i.\end{aligned} \tag{4.5}$$

Now we proceed to discuss the initial data for our equation. $\eta(0)$ should be a diffeomorphism of $\mathbf{R}^3$ which is near the identity and has Jacobian identically one. Thus $u(0) = \eta(0) - id$ is small and satisfies equation (4.2). Similarly $\dot{\eta}(0) = \dot{u}(0)$ is small and it satisfies equation (4.4).

We shall show that if $\|u(0)\|_{LS,9}$ and $\|\dot{u}(0)\|_{LS,8}$ [6] are sufficiently small, then equation (4.5) has a solution for all $t$.

[5] $c_n$ will be a sequence of positive constants throughout the paper.

[6] These norms are defined in equation (5.4) below.

## 5. Energy estimates – outline of proof

We have shown (with Saxton) in [EbSa] that our equation (3.14) (or equivalently (4.5)) has a unique solution on some time interval $[0, T)$, where $T \leq \infty$ depends on the initial data. Furthermore, if $T$ is finite and maximal, then the pair $(u(t), \dot{u}(t))$ must become arbitrarily large as $t \to T$. One can prove that if the initial data are small, then $(u(t), \dot{u}(t))$ do not get large and thus $T = \infty$, or the solution exists for all time. We shall now outline this proof.

Given $v(t, x)$ an $\mathbf{R}^3$-valued function on $\mathbf{R} \times \mathbf{R}^3$ we (following Klainerman [Kla3]) define an energy function:

$$E^0(t)(v) = \int_{\mathbf{R}^3} \left\{ \frac{1}{2}(1 + t^2 + |x|^2)(|\nabla v|^2 + |\dot{v}|^2) + 2tx^i \langle \partial_i v, \dot{v} \rangle + 2t\langle \dot{v}, v \rangle - |v^2| \right\} dx.$$

If $\Box v = f$, a direct computation shows that

$$\frac{d}{dt} E^0(t)(v) = \int_{\mathbf{R}^3} \langle Lv, f \rangle dx, \tag{5.1}$$

where $Lv = (1 + t^2 + |x|^2)\dot{v} + 2tx^i \partial_i v + 2tv$.

Furthermore, the energy $E^0(t)$ is equivalent to a certain $L^2$-norm of $v$ and its derivatives, which we now define.

Let $\Gamma_0 = \partial_t$ and let $\Gamma_i = \partial_i$ on $\mathbf{R} \times \mathbf{R}^3, i = 1, 2, 3$.

Let $\Gamma_4, \Gamma_5, \Gamma_6$ be the infinitesimal rotations $\Gamma_{3+k} = x^i \partial_j - x^j \partial_i$, where $\{ijk\}$ are cyclic permutations of $\{123\}$. Also let $\Gamma_7, \Gamma_8, \Gamma_9$ be the infinitesimal Lorentz transformations $\Gamma_{i+6} = t\partial_i + x^i \partial_t$. Finally let $\Gamma_{10} = t\partial_t + x^i \partial_i$. Then we define the first Lorentz-Sobolev norm of $v$ by

$$\|v\|^2_{LS,1} = \int_{\mathbf{R}^3} \left( |v|^2 + \sum_{\ell=0}^{\ell=10} |\Gamma_\ell v|^2 \right) dx. \tag{5.2}$$

In [Kla3] it is shown that there are positive constants $c_\ell$ and $c_u$ such that

$$c_\ell \|v\|^2_{LS,1} \leq E^0(t)(v) \leq c_u \|v\|^2_{LS,1}. \tag{5.3}$$

Higher Lorentz-Sobolev norms can be defined by

$$\|v\|^2_{LS,k} = \sum_{|\alpha| \leq k} \int_{\mathbf{R}^3} |\Gamma^\alpha v|^2, \tag{5.4}$$

where $\alpha$ is a multi-index $(\alpha_1, \ldots, \alpha_{10})$ and $\Gamma^\alpha = \prod_{i=0}^{10} \Gamma_i^{\alpha_i}$. Similarly we define higher energy functions:

$$E^0_s(t)(v) = \sum_{|\alpha| \leq s} E^0(t)(\Gamma^\alpha v). \tag{5.5}$$

The equation (4.5) for $u$ is a wave equation with additional non-linear terms, so we modify $E^0(t)$ accordingly. We define

$$E(t)(v) = E^0(t)(v) + \int_{\mathbf{R}^3} \frac{1}{2}(1 + t^2 + |x|^2) \Big( c_3(\partial_i u^j + \partial_j u^i)\partial_i v^k \partial_j v^k + c_4(\partial_\ell u^k + \partial_k u^\ell)\partial_i v^k \partial_i v^\ell + Q(Du)^{jk}_{i\ell} \partial_j v^i \partial_k v^\ell \Big) \, dx$$

so that if $v$ satisfies

$$\ddot{v}^i = \Delta v^i + c_3(\partial_j u^k + \partial_k u^j)\partial_j\partial_k v^i + c_4(\partial_i u^j + \partial_j u^i)\Delta v^j + Q(Du)^{ij}_{kl}\partial_k\partial_\ell v^j + F,$$

then

$$\frac{d}{dt}E(t)(v) = \int_{\mathbf{R}^3}\langle Lv, F\rangle + I + II + III, \tag{5.6}$$

where:

$$I = \int_{\mathbf{R}^3}(1+t^2+|x|^2)c_3(1/2(\partial_i\dot{u}^j + \partial_j\dot{u}^i)\partial_j v^i\partial_k v^i - (\partial_j^2 u^k + \partial_j\partial_k u^j)\partial_k v^i\dot{v}^i)dx,$$

$$II = \int_{\mathbf{R}^3}(1+t^2+|x|^2)c_4(1/2(\partial_i\dot{u}^j + \partial_j\dot{u}^i)\partial_k v^i\partial_k v^j - (\partial_k\partial_i u^j + \partial_k\partial_j u^i)\partial_k v^i\dot{v}^j)dx,$$

and

$$III = \int_{\mathbf{R}^3}(1+t^2+|x|^2)(1/2\langle \dot{Q(Du)}Dv, Dv\rangle - \partial_k Q(Du)^{jk}_{i\ell}\partial_j v^i\dot{v}^\ell)dx.$$

Clearly $I$ and $II$ are bounded by a constant times the sup-norms of $D^2u$ and $D\dot{u}$ times $E(t)(v)$. Also $III$ is bounded by a constant times the square of the sup-norms times $E(t)(v)$. To estimate higher derivatives we introduce

$$E_s(t)(v) = \sum_{|\alpha|\le s} E(t)(\Gamma^\alpha v).$$

In order to estimate the sup-norms above, we use Klainerman's inequality:

$$\sup_{x\varepsilon\mathbf{R}^3}\{w(t,x)\} \le \frac{c_5}{1+t}\|w\|_{LS,2}. \tag{5.7}$$

Also we note that one can get an improved estimate for derivatives of a function in light-like directions. Let $\partial_r = \frac{1}{|x|}x^i\partial_i$ denote differentiation in the (purely spatial) radial direction and let $\partial_{r^\perp}$ denote a spatial derivative in a perpendicular direction $r^\perp$.

Then $\partial_r + \partial_t$ and $\{\partial_{r^\perp}\}$ span the derivatives in the light-like directions. We fix a point $(t,x)$ in $\mathbf{R}^1\times\mathbf{R}^3$ and consider light-like derivatives computed at this point. Clearly we can pick spatial coordinates so that $x = (0,0,|x|)$. Then $\partial_r = \partial_3$ so

$$\partial_t + \partial_r = \partial_t + \partial_3 = \frac{1}{t+|x|}(\Gamma_{10} + \Gamma_7). \tag{5.8}$$

Without loss of generality, we can assume that $\partial_r^\perp = \partial_1$. Then

$$\partial_{r^\perp} = \frac{1}{t+|x|}(\Gamma_5 + \Gamma_7) \tag{5.9}$$

From (5.8) and (5.9) we find that Klainerman's inequality can be strengthened for light-like derivatives. If $\partial_\ell$ is such a derivative, we get

$$\sup_{x\epsilon\mathbf{R}^3}\|\partial_\ell w(t,x)\| \le \frac{c_6}{(1+t)^2}\|w\|_{LS,3}. \tag{5.10}$$

There is a similar inequality for the sup-norm. Letting

$$|||w|||_{LS,k} = \sup_{x\epsilon\mathbf{R}^3}\{\sum_{\alpha\le k}|\Gamma^\alpha w(t,x)|\}$$

we find that

$$\sup_{x\epsilon\mathbf{R}^3} \|\partial_\ell w(t,x)\| \le \frac{c_7}{1+t}|||w|||_{LS,1}. \tag{5.11}$$

The final ingredient needed for our result is an estimate for the inhomogeneous wave equation: $\Box w = F$, with zero initial data.

From Hörmander [Hö], Section 6, we have:

$$|w(t,x)| \le \frac{c_8}{1+t+|x|}\int_0^t \int_{\mathbf{R}^3} \sum_{|\alpha|\le 2} (\Gamma^\alpha F(s,y))/(1+s+|y|)\,dy\,ds. \tag{5.12}$$

Now that we have assembled the items needed for our theorem, we proceed to outline the proof. Our scheme follows that of Hörmander [Hö], Section 6.

First we note that if $u_1$ is the solution of the wave equation with the same initial data as $u$, then $u_1$ satisfies

$$|||u_1(t)|||_{LS,4} \le \frac{c_9\varepsilon}{1+t}. \tag{5.13}$$

This is because the initial data are of size $\varepsilon$. Now we write $u$, the solution of (4.5), as $u_1+u_2$, where $u_1$ is as above and $u_2$ has zero initial data.

Thus

$$\begin{aligned}\Box u_2^i = {} & c_3(\partial_j u^k + \partial_k u^j)\partial_j\partial_k u^i + c_4(\partial_i u^j + \partial_j u^i)\Delta u^j \\ & + Q(Du)^{ij}_{k\ell}\partial_k\partial_\ell u^j + F.\end{aligned} \tag{5.14}$$

We write the terms on the right side of (5.14) as $F_1+F_2+F_3$, where $F_1$ consists of terms that are quadratic in $u$ and its derivatives, $F_2$ consists of terms that are at least cubic, and $F_3 = F$.

The terms of $F_1$ have the form $(\partial_j u^k + \partial_k u^j)\partial_j\partial_k u^i$ or $(\partial_i u^j + \partial_j u^i)\Delta u^j$. Each of these terms contains at least one light-like derivative (actually a $\partial_{r^\perp}$ derivative) except the terms of the form $\partial_r u^j \partial_r\partial_r u^j$. However, if we subtract from $F_1$ the expression $\frac{1}{2}\nabla(\nabla u^j \nabla u^j)$, we eliminate terms of this form. We alter $F_1$ in this way and add the same term to $F_3$, so as to keep equation (5.14) unchanged. Thus all the terms of $F_1$ are of the form $\frac{1}{t+|x|}Du\Gamma Du$ or $\frac{1}{t+|x|}\Gamma u D^2 u$.

The terms of $F_2$ have the form $(Du)^m D^2 u$, where $m$ is at least 2. The terms of $F_3$, being gradients, can be bounded by their divergences, which have the same form as the terms of $F_1$ or $F_2$.

Thus applying $\Gamma^\alpha$ to (5.14) (with $|\alpha| \le 4$) and using the inequality (5.12), we find that

$$|||u_2|||_{LS,4} \le c_{10}E_8/(1+t)^2. \tag{5.15}$$

Thus we get

$$|||u|||_{LS,4} \le \frac{c_{11}}{1+t}\left(\varepsilon + \frac{E_8(t)}{1+t}\right). \tag{5.16}$$

On the other hand, by taking a time derivative of $E_8(t)$ $(= E_8(t)(u))$ and doing the usual energy norm type calculations, we get

$$\frac{d}{dt}E_8(t) \le c_{12}|||u(t)|||_{SL,4}E_8(t) \le c_{13}\left(\frac{\varepsilon}{1+t} + \frac{E_8(t)}{(1+t)^2}\right)E_8(t).$$

Integrating this inequality we find that for small $\varepsilon$

$$E_8(t) \le (1+t)^{c_{14}\varepsilon}E_8(0). \tag{5.17}$$

Thus $E_8(t)$ remains bounded on any finite time interval. Furthermore, using (5.7) and noting that $E(0)$ is of size $\varepsilon^2$ we see that

$$|||u(t)|||_{LS,4} \leq c_{15}\varepsilon(1+t)^{c_{14}\varepsilon-1}, \tag{5.18}$$

so that if $\varepsilon$ is small, then $|||u(t)|||_{LS,4}$ actually decays for large time. Thus (4.5) has a solution for all time, which remains small.

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