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RESIDUES AND EFFECTIVE NULLSTELLENSATZ

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ABSTRACT. Let $\mathbf{K}$ be a commutative field; an algorithmic approach to residue symbols defined on a Noetherian $\mathbf{K}$-algebra $\mathbf{R}$ has been developed. It is used to prove an effective Nullstellensatz for polynomials defined over infinite factorial rings $\mathbf{A}$ equipped with a size. This result extends (and slightly improves) the previous work of the authors in the case $\mathbf{A} = \mathbf{Z}$.

Let $p_1, \ldots, p_M$ be polynomials in n variables with coefficients in an integral domain $\mathbf{A}$, and respective degrees $D_1 \geq D_2 \geq \cdots \geq D_M$, with no common zeros in an integral closure of the quotient field $\mathbf{K}$ of $\mathbf{A}$. It follows from the versions of the Hilbert Nullstellensatz in [B], [CGH], [Ko] that one can find an element $r_0 \in \mathbf{A} \setminus \{0\}$ and polynomials $q_j \in \mathbf{A}[x]$ such that

$$r_0 = \sum_{j=1}^{M} q_j p_j \tag{1}$$

with a priori estimates on the degrees

$$\max_j \deg q_j \leq (3/2)^\iota D_1 \cdots D_\mu,$$

where $\mu = \min\{n, M\}$ and $\iota = \max\{j : 1 \leq j < \mu - 1,\ D_j = 2\}$. When $\mathbf{A} = \mathbf{Z}$, using analytic methods, and mainly integral representation formulas and multidimensional residues in $\mathbf{C}^n$, one can show [BGVY, Section 5] that system (1) can be solved with the estimates

$$\begin{cases} \max_j \deg q_j \leq n(2n+1)(3/2)^\iota \prod_{j=1}^{\mu} D_j, \\ \max h(q_j) \leq \kappa(n) D_1^4 \Bigl(\prod_{j=1}^{\mu} D_j \Bigr)^8 (h + \log M + D_1 \log D_1), \end{cases} \tag{2}$$

for the Mahler size $h(q_j)$ in terms of the maximal Mahler size h of the original polynomials p_j. Recall that the Mahler size of $p \in \mathbf{Z}[x_1, \ldots, x_n]$ is given by

$$h(p) = \int_{[0,1]^n} \log |p(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta \, .$$

For $\mathbf{A} = \mathbf{Z}$, one can also recover the estimate for $\log |r_0|$, implicit in (2), from the Arithmetic Bézout Theorem (see [Ph2],[BGS, Theorem 5.4.4]), which shows that

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the Faltings height H of the intersection of the arithmetic cycles X_j in $\mathbf{P}^n(\mathbf{Z})$ corresponding to the polynomials ${}^h p_j$ (homogeneous versions of the original polynomials) has the bound

$$H \le c_n \mathcal{H}\Big(\prod_{j=1}^{\nu} D_j\Big)\Big(1/\mathcal{H} + \sum_{j=1}^{\nu} 1/D_j\Big),$$

for some constant c_n, where $\mathcal{H} := \max_j H(X_j)$ and $\nu := \min\{n+1, M\}$.

The origin of the results we announce here was the desire to obtain estimates similar to the Effective Nullstellensatz (2) for factorial rings $\mathbf{A}$ equipped with a size (in the sense of [Ph1]) and simultaneously eliminate Analysis from the proof of such results. Apart from the case where $\mathbf{A} = \mathbf{Z}$ and the size is the Mahler size defined above, the natural example of such a ring is $\mathbf{A} = \mathbf{F}_p[\tau_1, \dots, \tau_q]$, where p is a prime number and the size is the total degree in the parameters τ. The main idea is to keep the structure of the proof in [BY1], while eliminating all the *analytic artifacts*. In this process we develop the computational aspects for a Residue Calculus, where the residues are defined purely algebraically, following [H] and, especially, [L]. We are sure that this Residue Calculus will have many applications beyond the scope of this paper, in particular, the algorithms obtained permit the computation of residue symbols in the polynomial case, without appealing to elimination theory or using Gröbner bases. In particular, one should expect interesting geometric consequences of a generalized form of the Jacobi vanishing theorem obtained here, as one knows in the classical case [G],[Ku]. The reader will find complete proofs of these results in [BY2].

1. Residue calculus

Let $\mathbf{R}$ be a Noetherian $\mathbf{K}$-algebra, where $\mathbf{K}$ is a commutative field. A sequence $h_1, \dots, h_n$ generating an ideal I in $\mathbf{R}$ is said to be quasiregular [M] if whenever one has a relation of the form

$$\sum_{k \in \mathbf{N}^n : |k| = p} r_k h_1^{k_1} \cdots h_n^{k_n} \in I^{p+1}\,, r_k \in \mathbf{R}\,,$$

for some $p \in \mathbf{N}$, then all the $r_k \in I$. Given $x_1, \dots, x_n, h_1, \dots, h_n$ in $\mathbf{R}$ such that $(h_1, \dots, h_n)$ is a quasiregular sequence generating the ideal $(h) = I$ and $\mathbf{P} := \mathbf{R}/I$ is a finite-dimensional $\mathbf{K}$-vector space, we follow Lipman [L, Chapter 3] to define the residue symbols as traces of certain $\mathbf{K}$-linear operators from $\mathbf{P}$ into itself. Namely, let $\mathbf{E} = \mathrm{Hom}_{\mathbf{K}}(\mathbf{P}, \mathbf{P})$ and $\sigma : \mathbf{P} \to \mathbf{R}$ be a $\mathbf{K}$-linear section of the quotient map; then one can associate to any $Q \in \mathbf{R}$ the element $Q^\sharp := \sum_{k \in \mathbf{N}^n} q_k^\sharp h^k \in \mathbf{E}[[h]]$, where the operators $q_k^\sharp \in \mathbf{E}$ are defined by the relations

$$Q\sigma(t) = \sum_{k \in \mathbf{N}^n} \sigma(q_k^\sharp(t)) h^k\,, t \in \mathbf{P}\,,$$

in the I-adic completion of $\mathbf{R}$. Consider now the formal expansion in $\mathbf{E}[[h]]$,

$$Q^\sharp \circ \det\Big(\partial x_i^\sharp / \partial h_j\Big) = \sum_{k \in \mathbf{N}^n} \delta_k h^k\,,$$

where the determinant is understood to take into account the noncommutativity of the operator product. Although the operators δ_k depend on the choice of the section σ, their traces do not. Now we can define the residue symbols as

$$\operatorname{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ h_1^{k_1+1}, \ldots, h_n^{k_n+1} \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q dx \\ h^{k+\underline{1}} \end{bmatrix} := \operatorname{Tr}(\delta_k), \quad Q \in \mathbf{R},\ k \in \mathbf{N}^n .$$

This definition extends the one given by Grothendieck [H]. In the polynomial case, computations of residues are usually performed by taking as a section σ the remainder with respect to a Gröbner basis of I. One can also show that in the analytic case (that is, $\mathbf{R} = \mathbf{C}[x_1, \ldots, x_n]$ or $\mathbf{R} = \mathcal{O}$, the ring of germs of holomorphic functions about $0 \in \mathbf{C}^n$) this definition of residue coincides with the usual one, defined, for instance, using Dolbeault cohomology [GH]. One can also see this using the following Transformation Law [L, Corollary 2.8].

Transformation Law. *Let $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ be two quasiregular sequences in $\mathbf{R}$, such that $g = Af$, where A is an $n \times n$ matrix with coefficients in $\mathbf{R}$, and such that the quotients $\mathbf{R}/(f)$ and $\mathbf{R}/(g)$ are finite-dimensional $\mathbf{K}$-vector spaces. Let Δ be the determinant of the matrix A; then, for any $x_1, \ldots, x_n, Q \in \mathbf{R}$,*

$$\operatorname{Res}\begin{bmatrix} Q dx_1 \wedge \cdots \wedge dx_n \\ f_1, \ldots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q \Delta dx_1 \wedge \cdots \wedge dx_n \\ g_1, \ldots, g_n \end{bmatrix}.$$

Extending the base ring $\mathbf{R}$ with transcendental parameters, we can derive an extension of the Transformation Law.

Proposition 1. *Let $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ be two quasiregular sequences in $\mathbf{R}$, such that*

$$g_j = \sum_{l=1}^{n} a_{jl} f_l, \quad j = 1, \ldots, n,$$

where the coefficients a_{jl} are in $\mathbf{R}$ and we let Δ be the determinant of the matrix $A = [a_{jl}]$. Then, for any $x_1, \ldots, x_n, Q \in \mathbf{R}$ and any $k \in \mathbf{N}^n$,

$$\operatorname{Res}\begin{bmatrix} Q dx \\ f^{k+\underline{1}} \end{bmatrix} = \sum_{\substack{|q_{;j}| = k_j \\ 1 \le j \le n}} \prod_{i=1}^{n} \binom{\mu_i}{q_{i;}} \operatorname{Res}\begin{bmatrix} Q \Delta \prod_{1 \le i,j \le n} (a_{ij})^{q_{i,j}} dx \\ g_1^{\mu_1+1}, \ldots, g_n^{\mu_n+1} \end{bmatrix},$$

where we have introduced the following notation for the matrix of indices $q_{i,j} \in \mathbf{N}$:

$$q_{;j} = (q_{1,j}, \ldots, q_{n,j}),\ q_{i;} = (q_{i,1}, \ldots, q_{i,n}), \quad \mu_i = |q_{i;}| ,$$

and

$$\binom{\mu_i}{q_{i;}} = \mu_i! / q_{i,1}! \cdots q_{i,n}!.$$

This proposition was stated in the analytic case in [Ky]. The proof given there is not complete. One can also prove a very useful variant of the Transformation Law, which we have not been able to find in the literature.

Proposition 2. *Let $f_0, f_1, \ldots, f_n$ be a regular sequence in $\mathbf{R}$. Let $g_1, \ldots, g_n$ in $\mathbf{R}$ be such that the sequence $(f_0, g_1, \ldots, g_n)$ is quasiregular. Assume that there are nonnegative integers $s_1, \ldots, s_n$ and an $n \times n$ matrix A of elements in $\mathbf{R}$ such that*

$$f_0^{s_j} g_j = \sum_{l=1}^{n} a_{jl} f_l, \quad j = 1, \ldots, n.$$

Let $x_0, \ldots, x_n, Q$ be elements of $\mathbf{R}$; then, for any $k_0 \in \mathbf{N}$ one has

$$\operatorname{Res}\begin{bmatrix} Q dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1}, f_1, \ldots, f_n \end{bmatrix} = \operatorname{Res}\begin{bmatrix} Q\Delta dx_0 \wedge \cdots \wedge dx_n \\ f_0^{k_0+1+|s|}, g_1, \ldots, g_n \end{bmatrix},$$

where $|s| = s_1 + \cdots + s_n$ and Δ is the determinant of the matrix A.

In the case where $\mathbf{R} = \mathbf{K}[x_1, \ldots, x_n]$, which is our main interest here, one can extend the residue symbol to act on rational functions, as follows. Given a quasiregular sequence $P_1, \ldots, P_n$ in $\mathbf{R}$, let Q_1/Q_2 be a rational function such that the ideal $(P_1, \ldots, P_n, Q_2)$ is $\mathbf{K}[x_1, \ldots, x_n]$. Namely, we define

$$\operatorname{Res}\begin{bmatrix} (Q_1/Q_2) dx_1 \wedge \cdots \wedge dx_n \\ P_1, \ldots, P_n \end{bmatrix} := \operatorname{Res}\begin{bmatrix} Q_1 V dx_1 \wedge \cdots \wedge dx_n \\ P_1, \ldots, P_n \end{bmatrix},$$

where V is any polynomial such that for some $U_1, \ldots, U_n$ in $\mathbf{R}$ one has $1 = U_1 P_1 + \cdots + U_n P_n + V Q_2$. (This definition does not depend on the choice of V.)

In the previous transformation laws, one can replace Q by a rational function, provided the two residue symbols involved can be defined.

From now on, we deal only with the case $\mathbf{R} = \mathbf{K}[x_1, \ldots, x_n]$, where $\mathbf{K}$ is a field of arbitrary characteristic. Recall that a polynomial map $P = (P_1, \ldots, P_n)$ from $\mathbf{K}^n$ into itself is dominant if and only if $\mathbf{K}(x)$ is a finite-dimensional $\mathbf{K}(P)$-vector space. It is proper if and only if $\mathbf{K}[x]$ is a finite type $\mathbf{K}[P]$-module.

When P is dominant, the residue symbol

$$\operatorname{Res}\begin{bmatrix} Q(x) dx \\ P_1 - u_1, \ldots, P_n - u_n \end{bmatrix}$$

(where $u_1, \ldots, u_n$ are n independent transcendental parameters) is an element of $\mathbf{K}(u)$. When P is proper, this symbol is a polynomial in u. In order to obtain more information about this polynomial, we shall assume that $\mathbf{K}$ is infinite, algebraically closed, and equipped with a nontrivial absolute value $|\cdot|$. We consider the norm defined on $\mathbf{K}^n$ by $|x| = \max_{1 \le i \le n} |x_i|$. Recall that one can check properness by means of inequalities, namely, the map P is proper if and only if there exist three constants $K, \gamma, \delta > 0$ such that

$$|x| \ge K \implies |P(x)| \ge \gamma |x|^{\delta}. \tag{3}$$

Any $\delta > 0$ for which (3) holds is called a *Lojasiewicz exponent* for P.

Let $P \in \mathbf{K}[x_1, \ldots, x_n]$. We will denote as ${}^hP$ the homogeneous polynomial in $n+1$ variables corresponding to the polynomial P in $\mathbf{K}[X_0, \ldots, X_n]$, namely

$${}^hP(X_0, \ldots, X_n) := X_0^{\deg P} P(X_1/X_0, \ldots, X_n/X_0).$$

As a consequence of the Lipman-Teissier theorem [LT] one has the following result.

Proposition 3. *Let $P = (P_1, \ldots, P_n)$ be a proper polynomial map in $\mathbf{K}^n$ such that condition (3) holds for constants K, γ, δ, with $\delta \in \mathbf{N}^*$. Assume that $D = \deg P_j$ for every j. Then, for any $1 \le j \le n$, one can find a homogeneous polynomial $\mathcal{R}_j$ in two variables, with coefficients in $\mathbf{K}$, distinguished in X_j, such that*

$$\left(\mathcal{R}_j(X_0, X_j) X_0^{D-\delta}\right)^{n+1} \in I({}^hP_1, \ldots, {}^hP_n),$$

the homogeneous ideal generated by the ${}^hP_j$.

This proposition, combined with the Transformation Law and residue calculus in one variable, leads to the following vanishing theorem for residue symbols.

Proposition 4. *Let $P_1, \dots, P_n$ be polynomials in $\mathbf{K}[x_1, \dots, x_n]$ with $\deg P_j = D$, for $1 \le j \le n$, which satisfy (3) with an integral exponent δ. Assume that*

$$(1 - \epsilon_n)D < \delta, \quad \textit{for } \epsilon_n := 1/n(n+1).$$

Then, for any $(k_1, \dots, k_n) \in \mathbf{N}^n$, one has

$$\deg Q \le n(n+1)(|k| + n)(\delta - (1 - \epsilon_n)D) - n - 1 \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P^{k+\underline{1}} \end{bmatrix} = 0.$$

Moreover, under the stronger hypothesis that

$$(1 - \epsilon_n/(n+1))D < \delta,$$

one has

$$\left.\begin{array}{l} \deg Q \le n(D-1) \\ k \ne 0 \end{array}\right\} \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P^{k+\underline{1}} \end{bmatrix} = 0. \tag{4}$$

The case where $D = \delta$ corresponds to the situation where the P_j have no common zeros at ∞. In this case, (4) is a particular case (here all the polynomials have the same degree) of the Jacobi vanishing theorem. For the convenience of the reader, let us recall this theorem here [KK].

Jacobi vanishing theorem. *Let $P_1, \dots, P_n$ be elements of $\mathbf{K}[x_1, \dots, x_n]$ such that the homogeneous parts of highest degree define a power of the maximal ideal $I(x_1, \dots, x_n)$. Then*

$$\deg Q \le \sum_{j=1}^{n} \deg P_j - n - 1 \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P_1, \dots, P_n \end{bmatrix} = 0.$$

Using analytic methods, we were able to obtain in [BY1] a result similar to Proposition 4, without any restriction on the quotient δ/D. That result can be stated as follows. If P is a proper polynomial map from $\mathbf{C}^n$ to $\mathbf{C}^n$ such that the degrees D_j of the polynomials P_j are in decreasing order and δ is a Łojasiewicz exponent, then, for any polynomial $Q \in \mathbf{C}[x_1, \dots, x_n]$ and multiindex k, one has

$$(|k| + 2n - 1)\delta > \deg Q + D_1 + \cdots + D_{n-1} + n \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P^{k+\underline{1}} \end{bmatrix} = 0. \tag{5}$$

This statement was crucial in the proof of the effective Nullstellensatz over $\mathbf{C}$ given in [BY1]. On the other hand, one can see that this vanishing theorem is not the best one could expect. For example, (5) implies that

$$\deg Q < (2n - 1)\delta - (D_1 + \cdots + D_{n-1}) - n \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P_1, \dots, P_n \end{bmatrix} = 0.$$

But a more careful analysis of the Bochner-Martinelli representation of the residue currents yields the statement

$$\deg Q \le n\delta - n - 1 \implies \operatorname{Res}\begin{bmatrix} Qdx \\ P_1, \dots, P_n \end{bmatrix} = 0.$$

The point here is that this result depends on the Łojasiewicz exponent δ, but not on the degrees of the P_j. We do not know how to prove such result when $\mathbf{K}$ has positive characteristic, though it is possibly true. Nevertheless, Proposition 4, which holds for any characteristic, is enough to prove the effective Nullstellensatz theorem below. We use it now to obtain a global version of the Kronecker interpolation formula.

Proposition 5. *Let $P_1, \dots, P_n$ be n polynomials in $\mathbf{K}[x_1, \dots, x_n]$ of degree D, with the property that there exist strictly positive constants K, γ such that (3) holds for some integer $\delta > 0$ satisfying*

$$1 - 1/n(n+1)^2 < \delta/D \leq 1.$$

Suppose that g_{jl}, $1 \leq j, l \leq n$, are elements in $\mathbf{K}[x_1, \dots, x_n, y_1, \dots, y_n]$, with degree less or equal than $D - 1$, and such that

$$P_j(x) - P_j(y) = \sum_{l=1}^{n} (x_l - y_l) g_{jl}(x, y), \quad 1 \leq j \leq n, \ x, y \in \mathbf{K}^n.$$

Then, if $\Delta(x, y) := \det[g_{jl}(x, y)]_{\substack{1 \leq j \leq n \\ 1 \leq l \leq n}}$ (such Δ is called a Bézoutian *for the map P), the following polynomial identity holds:*

$$1 = \operatorname{Res}\begin{bmatrix} \Delta(x, y) dx \\ P_1(x), \dots, P_n(x) \end{bmatrix}, \quad y \in \mathbf{K}^n.$$

The transformation laws are also used to compute the residue symbols

$$\operatorname{Res}\begin{bmatrix} Q dx \\ P^{k+\underline{1}} \end{bmatrix}$$

from the relations of integral dependence of the coordinates over $\mathbf{K}(P)$ when P is a dominant map. For instance, one can use the following lemma.

Lemma 1. *Let $P_1, \dots, P_n$ be a quasiregular sequence in $\mathbf{K}[x_1, \dots, x_n]$; then for any $q \in \mathbf{N}$ and for any $\alpha \in \mathbf{K}^n$ the sequence $(t^{q+1}, P_1(x) - \alpha_1 t, \dots, P_n(x) - \alpha_n t)$ is a quasiregular sequence in $\mathbf{K}[x, t]$. Moreover, we have the formula*

$$\operatorname{Res}\begin{bmatrix} Q(x) dt \wedge dx \\ t^{q+1}, P_1(x) - \alpha_1 t, \dots, P_n(x) - \alpha_n t \end{bmatrix} = \sum_{|k|=q} \operatorname{Res}\begin{bmatrix} Q dx \\ P^{k+\underline{1}} \end{bmatrix} \alpha^k. \tag{6}$$

The left-hand side of (6) can be computed using relations of the form

$$A_{j0}(u) x_j^{N_j} - \sum_{l=1}^{N_j} A_{jl}(u) x_j^{N_j - l} = \sum_{l=1}^{n} A_j^l(x_j, P, u)(P_l - u_l), \quad j = 1, \dots, n,$$

so that it can be rewritten as

$$t^{s_j}(R_j(x_j, \alpha) - t S_j(x_j, \alpha, t)) = \sum_{l=1}^{n} A_j^l(x_j, P, \alpha t)(P_l - \alpha_l t), \quad R_j \not\equiv 0.$$

Then, we use Proposition 2 and residue computations in one variable, to compute the left-hand side of (6). Note that such expression is, in principle, an element of $\mathbf{K}(\alpha_1, \dots, \alpha_n)$, say $r_1(\alpha)/r_2(\alpha)$, while we know from (6) that it is a polynomial in α. If the P_j have coefficients in some integral domain $\mathbf{A}$ (with quotient field $\mathbf{K}$) equipped with a size, then $r_1(\alpha)$ and $r_2(\alpha)$ have coefficients in $\mathbf{A}$ and a common denominator for all residue symbols on the right-hand side of (6) divides $r_2(\alpha)$ in $\mathbf{A}[\alpha]$, that is, it divides all the coefficients of $r_2(\alpha)$. This remark is crucial with respect to size estimates for numerators or denominators of residue symbols. It is clear also that a good control on the size of the polynomials A_{jl}, for $j = 1, \dots, n$, $0 \leq l \leq N_j$, and A_j^l, for $1 \leq j, l \leq n$ (which can be chosen with coefficients in $\mathbf{A}$ in this case) will provide good estimates for the sizes of $r_1(\alpha)$ and $r_2(\alpha)$.

2. Łojasiewicz inequalities

Another important tool in our proof of the Nullstellensatz is the *Łojasiewicz inequality* [JKS]. It will be used in the following form. Given n integers $D_1 \geq D_2 \geq \cdots \geq D_n \geq 1$ we define, as in [JKS],

$$B := B(D_1, \dots, D_n) = (3/2)^{\iota} D_1 \cdots D_n,$$

where $\iota = \#\{j < n-1 \,:\, D_j = 2\}$. Let $P_1, \dots, P_n$ be polynomials with respective degrees $\deg P_j = D_j$. Then there is a constant $\gamma > 0$ such that for any $x \in \mathbf{K}^n$

$$|P(x)| \geq \gamma \left(\frac{\min\{1, \operatorname{dist}(x, V(P))\}}{1 + |x|} \right)^B,$$

where $V(P)$ denotes the set of common zeros of the polynomials P_j and the distance corresponds to the absolute value $|\cdot|$.

Such an inequality, which is directly related to Kollár's work [Ko] on the Nullstellensatz, has the following consequence, based on the arguments given in [BY1] and [BGVY, Propositions 5.7 and 5.8] when $\mathbf{K} = \mathbf{C}$. Small modifications are required by the fact that we are now working with fields of arbitrary characteristic.

Proposition 6. *Let $P_1, \dots, P_n$ be a quasiregular sequence in $\mathbf{K}[x_1, \dots, x_n]$; then one can find n linear combinations (with coefficients in $\mathbf{K}$) of the P_j, namely, $\tilde{P}_j$, for $1 \leq j \leq n$, and n linearly independent $\mathbf{K}$-linear forms $L_1, \dots, L_n$, as well as a positive constant K such that for any $N \in \mathbf{N}^*$ and any $x \in \mathbf{K}^n$ with $|x| > K$ one has*

$$\max_{1 \leq j \leq n} |L_j(x)|^{NB} \left| \tilde{P}_j(x) \right| \geq \gamma_N \, |x|^{(N-1)B}$$

for some constant $\gamma_N > 0$.

In fact, the linear combinations $\tilde{P}_j$, as well as the linear forms L_j, can be chosen with generic coefficients. Moreover, if we combine this result with Proposition 5 and the Arithmetic Bézout Theorem [Ph1, Theorem 4], we get the following technical but important result.

Proposition 7. *Let $P_1, \dots, P_n$ be a quasiregular sequence in $\mathbf{K}[x_1, \dots, x_n]$, with $D_1 \geq D_2 \geq \cdots \geq D_n$, with $D_j := \deg P_j$. Then one can find a polynomial Φ in $n(n+1) + n^2$ variables u_{jl}, v_{jk}, for $1 \leq j, k \leq n$, and $0 \leq l \leq n$, with coefficients in $\mathbf{K}$, with $\deg \Phi \leq 2^{n+1}(n+1)^4 D_1^n$, such that, for any $(U, V) \in \mathbf{K}^{n \times (n+1)} \oplus \mathcal{M}_n(\mathbf{K})$, with $\Phi(U, V) \neq 0$, the polynomials*

$$\Pi^j_{U,V}(x) := U^j(x)\langle V^j, P(x)\rangle := \left(u_{j0} + \sum_{l=1}^n u_{jl} x_l \right) \left(\sum_{l=1}^n v_{jl} P_l \right)$$

have degree exactly $D_1 + 1$, define a quasiregular sequence in $\mathbf{K}[x_1, \dots, x_n]$, and moreover, if $N \in \mathbf{N}^*$ is such that

$$(B + D_1)/(NB + D_1) < 1/n(n+1)^2,$$

then the following polynomial identity holds in $\mathbf{K}[x_1, \dots, x_n]$:

$$1 = \operatorname{Res} \left[\begin{matrix} \Delta_{N,U,V}(x, y) dx_1 \wedge \cdots \wedge dx_n \\ U^1(x)^{NB} \langle V^1, P \rangle, \dots, U^n(x)^{NB} \langle V^n, P \rangle \end{matrix} \right].$$

This formula, which is a polynomial identity in y, holds whenever $\Delta_{N,U,V}(x,y)$ is the determinant of an arbitrary $n \times n$ matrix whose coefficients $\delta_{jl} \in \mathbf{K}[U,V,x,y]$ have degree in x,y at most $NB+D_1-1$ and for all $j=1,\dots,n$ satisfy the relations

$$U^j(x)^{NB}\langle V^j, P(x)\rangle - U^j(y)^{NB}\langle V^j, P(y)\rangle = \sum_{l=1}^{n}(x_l-y_l)\delta_{jl}(x,y).$$

3. Effective Nullstellensatz

In this section, $\mathbf{A}$ will be a unitary factorial regular integral domain with a size $\mathbf{t}$; its quotient field will be denoted by $\mathbf{K}$ and assumed to be infinite. The size $\mathbf{t}$ can be extended to $\mathrm{Pol}(\mathbf{A})$ with values in $\{-\infty\}\cup[0,+\infty[$; we refer to [Ph1] for this purpose. In the following theorem we need an auxiliary function defined for $s \in \mathbf{N}\cup\{+\infty\}$ by

$$\vartheta(s) = \inf\{\xi \in [0,\infty[:\ \vartheta_0(\xi) \geq s\},$$

where

$$\vartheta_0(\xi) = \#\{a \in \mathbf{A} : \mathbf{t}(a) \leq \xi\}, \quad \xi \in [0,\infty[.$$

Let us state now our final result, that is, our solution of the Bézout identity with good degree and size estimates.

Effective Nullstellensatz. *Let $p_1,\dots,p_M \in \mathbf{A}[x_1,\dots,x_n]$ and let $\mathbf{A}$ be an integral domain with infinite quotient field $\mathbf{K}$. The ring $\mathbf{A}$ is assumed to be a factorial regular ring with Krull dimension κ and equipped with a size $\mathbf{t}$ (with corresponding c, c', ϑ). The degrees $D_j = \deg p_j$ are assumed to be in decreasing order and $h := \max\{\mathbf{t}(p_j), c'\log(n+2)\}$. If $p_1,\dots,p_M$ have no common zero in some algebraic closure $\overline{\mathbf{K}}$ of $\mathbf{K}$, there exists $r_0 \in \mathbf{A}$, and $q_1,\dots,q_N \in \mathbf{A}[x]$, such that*

$$r_0 = \sum_{j=1}^{M} q_j p_j,$$

with the estimates

$$\begin{cases} \deg(p_jq_j) \leq n(n+1)^3 B(D_1,\dots,D_n) + n(D_1-1), \\ \mathbf{t}(p_jq_j) \leq C_0\varpi^4 2^n n^{17} c^{16} B^4 D_1^2\big(h + \vartheta[(\gamma_0 D_1)^n] + c'\log M + c' D_1\log(2n+2)\big), \\ \mathbf{t}(r_0) \leq C_0\varpi^4 2^n n^{17} c^{16} B^4 D_1^2\big(h + \vartheta[(\gamma_0 D_1)^n] + c'\log M + c' D_1\log(2n+2)\big), \end{cases}$$

where γ_0, C_0 are absolute integral constants, $B = B(D_1,\dots,D_n)$ is defined in Section 2, and the constants c, c' depend only on the size $\mathbf{t}$, while ϖ depends both on n and $\mathbf{t}$.

In the two examples mentioned earlier, one can make the constants c, c', ϖ and the function ϑ explicit. Namely, for $\mathbf{A} = \mathbf{Z}$, we have $c = 3$, $c' = 1$, and $\varpi = 9(n+1)2^{n+2}(1+4\log(n+1))^{n+2}$. For $\mathbf{A} = \mathbf{F}_p[\tau_1,\dots,\tau_q]$, we can take $c = 1$, $c' = 0$, and $\varpi = 2n+q+1$. Similarly, in the first example, $\vartheta(s) \simeq \log s$, and in the second, $\vartheta(s) \simeq (\log s/\log p)^{1/q}$.

Sketch of the proof. The key idea of the proof is to consider n affine forms (with coefficents in $\mathbf{A}$), $U_1,\dots,U_n$, and n linear combinations $\langle V^j, P\rangle$ (with coefficients in $\mathbf{A}$) of the entries p_j, such that the hypotheses of Proposition 7 are fulfilled. Then, we start with the Kronecker identity

$$1 = \operatorname{Res}\begin{bmatrix} \Delta_{N,U,V}(x,y)dx_1 \wedge \cdots \wedge dx_n \\ U^1(x)^{NB}\langle V^1, P\rangle, \dots, U^n(x)^{NB}\langle V^n, P\rangle \end{bmatrix}, \tag{7}$$

which is valid for $N = (n+1)^3$. From now on, we assume that the variables U, V have been fixed and so we will drop them, as well as N, from the notation, when convenient. In particular, we will denote by $\Theta_j(x) = U^j(x)^{NB}\langle V^j, P(x)\rangle$, for $j = 1, \dots, n$. Correspondingly, we let $\Delta(x,y) = \Delta_{N,U,V}(x,y)$, and δ_{ij} be the entries of the corresponding matrix. We introduce a linear combination q of the p_j such that the ideal $I(\Theta_1, \dots, \Theta_n, q) = \mathbf{K}[x_1, \dots, x_n]$, and polynomials $g_{n+1,l}$, for $1 \le l \le n$, in $2n$ variables such that $q(x) - q(y) = \sum_{l=1}^n g_{n+1,l}(x,y)(x_l - y_l)$. Thus, we can rewrite the determinant Δ as

$$(1/q(x))\begin{vmatrix} \delta_{11} & \dots & \delta_{1n} & g_{n+1,1}(x,y) \\ \vdots & \ddots & \vdots & \vdots \\ \delta_{n1} & \dots & \delta_{nn} & g_{n+1,n}(x,y) \\ \Theta_1(y) - \Theta_1(x) & \dots & \Theta_n(y) - \Theta_n(x) & q(y) \end{vmatrix}$$

and develop this new $(n+1) \times (n+1)$ determinant along the last row to obtain

$$\Delta(x,y) = (1/q(x))\Biggl(\Biggl(\sum_{j=1}^n (\Theta_j(y) - \Theta_j(x))\Delta_j(x,y)\Biggr) + q(y)\Delta(x,y)\Biggr).$$

Since the residue symbol is annihilated by the ideal, we can rewrite (7) as a Bézout identity

$$1 = \sum_{j=1}^n \operatorname{Res}\begin{bmatrix} \Delta_j(x,y)dx/q(x) \\ \Theta_1(x), \dots, \Theta_n(x) \end{bmatrix}\Theta_j(y) + \operatorname{Res}\begin{bmatrix} \Delta(x,y)dx/q(x) \\ \Theta_1(x), \dots, \Theta_n(x) \end{bmatrix} q(y).$$

It is clear that this is an identity of the form $1 = \sum_{j=1}^M p_j(y)q_j(y)$, where the q_j are in $\mathbf{K}[x]$. This is the formula that solves the Nullstellensatz with good estimates. Note that, up to this point, we already have the estimates for the degrees. The size estimates are obtained from the results in Section 1. □

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