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CHARACTERIZATION OF THE RANGE OF THE RADON TRANSFORM ON HOMOGENEOUS TREES

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Abstract. This article contains results on the range of the Radon transform R on the set $\mathcal{H}$ of horocycles of a homogeneous tree T. Functions of compact support on $\mathcal{H}$ that satisfy two explicit *Radon conditions* constitute the image under R of functions of finite support on T. Replacing functions on $\mathcal{H}$ by distributions, we extend these results to the non-compact case by adding decay criteria.

1. Introduction

We study the Radon transform R on the set $\mathcal{H}$ of horocycles of a homogeneous tree T, and describe its image on various function spaces. We show that the functions of compact support on $\mathcal{H}$ that satisfy two explicit *Radon conditions* constitute the image under R of functions of finite support on T. Larger domains and ranges are described by adding decay criteria to the domain and range, although we show that functions on $\mathcal{H}$ need to be replaced by distributions.

The **Radon transform** (**RT** for short), in its original formulation by Radon [R], associates to each (sufficiently nice) function on $\mathbf{R}^2$ its one-dimensional Lebesgue integrals along all affine straight lines. This transform has been receiving considerable attention for its highly applicable nature and intrinsic interest, leading to a variety of generalizations.

In $\mathbf{H}^2$ lines correspond to two essentially different kinds of one-dimensional submanifolds: geodesics and horocycles, giving rise to two different RTs (cf. [H]).

Homogeneous trees are widely regarded as discrete counterparts of $\mathbf{H}^2$, as well as objects of thorough study in harmonic analysis in their own right. Exactly like $\mathbf{H}^2$, they feature two distinct kinds of RTs, namely the **geodesic RT** (a.k.a. the **X-ray transform**, since it is reminiscent of the CAT-scan procedure (cf. [BC])), and the **horocyclic RT**. Several of the standard RT issues in this setting have been investigated over time by various authors: e.g., [BCCP], [A] for injectivity and inversion, [CCP2] for range characterization, and [CC] for function space setting for the geodesic RT; [BP], [BFP], [CCP1] for injectivity and inversion of the horocyclic

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RT (part of the results therein are rewritten in [CMS] for the Abel transform, which is a multiple of the RT). Another related transform has been studied recently by Cowling and Setti.

In this work, we pursue a description of the range of the horocyclic RT R on a homogeneous tree T of degree $q+1$ with $q \geq 2$. We first state two natural explicit relations (one of which had already been observed in [BFPp] and [BFP] for radial functions) for functions on the space $\mathcal{H}$ of horocycles of T. We then show that among compactly supported functions on $\mathcal{H}$, these conditions completely characterize the range of R on finitely supported functions on (the set of vertices of) T. Similar descriptions are valid for the range of R on larger function spaces, although distributions on $\mathcal{H}$ need then to be taken into account. All the results with complete proofs can be found in [CCC].

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2. Preliminaries

The boundary Ω of T is the set of equivalence classes of infinite paths under the relation $[v_0, v_1, \dots] \simeq [v_1, v_2, \dots]$. For any vertex u, we denote by $[u, \omega)$ the (unique) path starting at u in the class ω. Then Ω can be identified with the set of paths starting at u. Each $\omega \in \Omega$ induces an orientation on the edges of T: $[u, v]$ is positively oriented if $v \in [u, \omega)$.

For $\omega \in \Omega$, and $u, v \in T$, define the **horocycle index** $\kappa_\omega(u, v)$ as the number of positively oriented edges minus the number of negatively oriented edges in the path from u to v. Given $u \in T$ and $\omega \in \Omega$, the **horocycle through u touching** ω is the set $\{v : \kappa_\omega(u, v) = 0\}$. More generally, for any $n \in \mathbb{Z}$, the **horocycle of index n touching ω with respect to u** is $h^u_{\omega,n} = \{w \in T \colon \kappa_\omega(u, w) = n\}$. Then the set of vertices may be decomposed as $\coprod_{n\in\mathbb{Z}} h^u_{\omega,n}$.

For u fixed, the map $(n, \omega) \mapsto h^u_{\omega,n}$ is a one-to-one correspondence between $\mathbb{Z} \times \Omega$ and the set $\mathcal{H}$ of horocycles.

Definition 1. The L^1**-horocyclic Radon transform** R on T is given by $Rf(h) = \sum_{v\in h} f(v)$ for $f \in L^1T$, and $h \in \mathcal{H}$.

For $u, v \in T$, set $S(u, v) = \{h \in \mathcal{H} : \exists\omega \in \Omega \text{ s.t. } h = h^u_{\omega,0}, v \in [u, \omega)\}$. The topology generated by the sets $S(u, v)$ makes $\mathcal{H}$ totally disconnected. Then $\mathcal{H}$ is homeomorphic to $\mathbb{Z} \times \Omega$, where Ω is endowed with the compact topology generated by $I^u_v = \{\omega \in \Omega : v \in [u, \omega)\}$. For any $u \in T$, there is a measure μ^u on Ω: $\mu^u(I^u_v) = 1/c_{d(u,v)}$.

The family of horocycles through a fixed ω does not depend on the choice of the reference vertex u, but indices do: $h^v_{\omega,n} = h^u_{\omega,n+\kappa_\omega(u,v)}$.

For simplicity of notation, we fix a root e throughout, and set $h_{\omega,n} = h^e_{\omega,n}$, $\mu = \mu^e$, $d\omega = d\mu^e(\omega)$, $k(v, \omega) = \kappa_\omega(e, v)$, and $I_v = I^e_v$. Notice that $d\mu^v(\omega) = q^{k(v,\omega)}\, d\omega$.

For $\omega \in \Omega$, let $\omega_n \in [e, \omega)$ be the vertex of length n. For $v \in T$, and $0 \leq n \leq |v|$, let $v_n \in [e, v]$ be the vertex of length n. For $v \in T$ and $n \geq |v|$, the set $D_n(v) = \{u : |u| = n \text{ and } u_{|v|} = v\}$ is the set of **descendants** of v of length n.

Definition 2. For a function φ on $\mathcal{H}$, we define the **Radon conditions** as follows:

(R_1) $\sum_n \varphi(h^v_{\omega,n})$ is independent of v and ω.

(R_2) For any $v \in T$, $n \in \mathbb{Z}$,

$$\int_\Omega \varphi(h^v_{\omega,n}) d\mu^v\omega = q^{-n} \int_\Omega \varphi(h^v_{\omega,-n}) d\mu^v\omega.$$

Proposition 1. *If $f \in L^1T$, then Rf is a continuous function satisfying the Radon conditions.*

There are, however, continuous functions satisfying the Radon conditions that are of the form Rf for $f \notin L^1T$.

Proposition 1 is proved by showing first that the Radon conditions are satisfied for the function $\varphi = R\chi_u$, where χ_u is the characteristic function of $\{u\}$, and then extending linearly.

Fix $v \in T$. For $0 \le t \le |v|$, let $I^t_v = \{\omega \in \Omega : k(v,\omega) = 2t - |v|\}$. Then for $t \ne |v|$, $I^t_v = I_{v_t} - I_{v_{t+1}}$, $I^{|v|}_v = I_v$, and $\Omega = \coprod_{t=0}^{|v|} I^t_v$. Using the relations $h^v_{\omega,n} = h_{\omega,n+k(v,\omega)}$ and $d\mu^v\omega = q^{k(v,\omega)}\, d\omega$, condition (R_2) may be rewritten as

$$(R_2') \qquad \sum_{t=0}^{|v|} q^{2t-|v|} \int_{I^t_v} \varphi(h_{\omega,n+2t-|v|}) = q^{-n} \sum_{t=0}^{|v|} q^{2t-|v|} \int_{I^t_v} \varphi(h_{\omega,-n+2t-|v|})\, d\omega.$$

In §3, we characterize the range of the RT on the set of functions on T of finite support, and then in §4, after defining Rf as a distribution on $\mathcal{H}$, we obtain a similar characterization for the case of f of infinite support.

3. Functions of compact support

Theorem 1. *The image of R on the space of functions on T of finite (i.e. compact) support is the space of functions on $\mathcal{H}$ of compact support satisfying the Radon conditions.*

The proof is based on the use of a generalization of radiality:

Definition 3. Let N be a non-negative integer.

(1) A function f on T is N**-radial** if for all $v \in T$ with $|v| \ge N$, $f(v)$ depends only on v_N and $|v|$.

(2) f has **radius** N if $\{v : |v| \le N\}$ is the smallest disk centered at e containing the support of f (so $f(v) = 0$ for $|v| > N$).

(3) A function φ on $\mathcal{H}$ is N**-radial** if $\varphi(h_{\omega,n})$ depends only on ω_N and n.

(4) φ has **radius** N if $[-N,\dots,N] \times \Omega$ is the smallest such set containing the support of φ (so $\varphi(h_{\omega,n}) = 0$ for $|n| > N$).

In particular, a 0-radial function on T is what is generally called *radial*.

We actually prove a more precise version of Theorem 1, specifically that the image under R of the set of functions on T of radius less than or equal to N is the set of continuous functions on $\mathcal{H}$ of radius less than or equal to N satisfying the Radon conditions. This result is established by means of Propositions 2 and 3, whose proofs are outlined below.

For $N \ge 0$, let E^N be the set of N-radial functions on $\mathcal{H}$ of radius less than or equal to N satisfying (R_1) and (R_2).

Proposition 2. $E^N = E^{N-1} \oplus \bigoplus_{|v|=N} \mathbb{C}\, R\chi_v$.

It follows by induction that E^N is the image under R of the set of functions of radius less than or equal to N.

Proposition 3. *If φ is a function on $\mathcal{H}$ of compact support satisfying the Radon conditions, then there exists N such that $\varphi \in E^N$.*

Let $\{v^1, \ldots, v^{c_N}\}$ be an enumeration of the vertices of length N. If $v \in T$, $|v| \leq N$, let $A_v^t = \{j : I_{v^j} \subset I_v^t\}$. Thus $I_v^t = \coprod_{j \in A_v^t} I_{v^j}$. If j_0 is the index such that $v = v^{j_0}$, then $A_v^N = \{j_0\}$. Observe that $\{1, 2, \ldots, c_N\} = \coprod_{t=0}^{|v|} A_v^t$ and recall that $\Omega = \coprod_{t=0}^{|v|} I_v^t$. Let $\varphi \in E^N$, and set $a_{n,j} = \varphi(h_{\omega,n})$ for $\omega_N = v^j$. Then (R_2') becomes

$$(R_2'') \quad \sum_{t=0}^{M} q^{2t} \sum_{j \in A_v^t} a_{n+2t-M,j} = q^{-n} \sum_{t=0}^{M} q^{2t} \sum_{j \in A_v^t} a_{-n+2t-M,j},$$

for $|v| = M \leq N$.

The proof of Propositions 2 and 3 is based on repeated applications of (R_2'') for various values of n and M. For instance, if we set $M = N$ and $n = 2N$, the left-hand side of (R_2'') reduces to $\sum_{j \in A_v^0} a_{N,j}$, since $n + 2t - M > N$ except for $t = 0$. On the right-hand side, $a_{-n+2t-M} = 0$, except for $t = M = N$, leaving just $\sum_{j \in A_v^N} a_{-N,j}$, which is a_{-N,j_0}, where $v = v^{j_0}$. Thus $\sum_{j \in A_v^0} a_{N,j} = a_{-N,j_0} q^N$. In particular, if $a_{N,j} = 0$ for all j, then $a_{-N,j} = 0$ for all j.

If $\varphi \in E^N$, then the function $\tilde{\varphi} = \varphi - \sum_{j=1}^{c_N} a_{M,j} R(\chi_{v^j})$ has the property that $\tilde{\varphi}(h_{\omega,n}) = 0$ for $n = N$ as well as for $|n| > N$. Hence $\tilde{a}_{N,j} = 0$ for all j, and so, by what we just proved, $\tilde{a}_{-N,j} = 0$ for all j. Thus $\tilde{\varphi} \in E^{N-1}$, proving Proposition 2.

Now let φ be a function with compact support satisfying the Radon conditions. Since topologically $\mathcal{H} \simeq \mathbb{Z} \times \Omega$ with Ω compact, there is some positive integer N such that the support of φ is contained in $[-N, N] \times \Omega$, i.e. $\varphi(h_{\omega,n}) = 0$ for $|n| > N$. Then φ has radius less than or equal to N. Again using (R_2''), it is possible to show that φ is N-radial. Thus $\varphi \in E^N$, proving Proposition 3, and hence Theorem 1.

4. Non-compact support

In this section we develop a parallel theory for distributions on $\mathcal{H}$ and define certain **decay conditions** for functions on T and distributions on $\mathcal{H}$.

For $r > 0$, define $\mathcal{A}_r$ as the class of all functions $f : T \to \mathbb{C}$ satisfying the **decay condition**:

$$\sum_{n=|v|}^{\infty} t^n \left| \sum_{u \in D_n(v)} f(u) \right| < \infty \quad \forall t \in [0, r), \ \forall v \in T.$$

Observe that $L^1 T \subset \mathcal{A}_1$, since for $f \in L^1 T$ and $0 \leq t < 1$,

$$\sum_{n=|v|}^{\infty} t^n \left| \sum_{u \in D_n(v)} f(u) \right| \leq \sum_{n=|v|}^{\infty} \sum_{u \in D_n(v)} |f(u)| \leq \sum_{u \in T} |f(u)| = \|f\|_1.$$

The elementary measurable sets in $\mathcal{H}$ can be generated by all sets of the form $\{h_{\omega,n} \in \mathcal{H} : \omega \in I_v\}$, which may be identified with $\{n\} \times I_v$. A **distribution** on $\mathcal{H}$ is an element of the dual of the vector space generated by the characteristic functions of the elementary measurable sets of $\mathcal{H}$. Thus, since $I_v = \coprod_{u^- = v} I_u$, we may think of a distribution on $\mathcal{H}$ as a function φ on the sets $\{n\} \times I_v$ satisfying the property

$$\varphi(\{n\} \times I_v) = \sum_{u^- = v} \varphi(\{n\} \times I_u).$$

If $f \in L^1T$, then Rf is defined on each horocycle and is bounded. By abuse of notation, we define Rf as the distribution given by

$$Rf(\{n\} \times I_u) = \int_{I_u} Rf(h_{\omega,n}) d\omega.$$

Now for a larger class of functions on T, this leads to the following definition of the Radon transform as a distribution:

Definition 4. For a function f on T, let

$$Rf(\{n\} \times I_u) = \sum_{m=0}^{\infty} \sum_{|v|=m} f(v) R\chi_v(\{n\} \times I_u),$$

if this is defined for all $u \in T$, and all $n \in \mathbb{Z}$.

This definition is consistent with the previous formula, since

$$f = \sum_{m=0}^{\infty} \sum_{|v|=m} f(v)\chi_v.$$

We extend the Radon conditions to the case of distributions as follows:

(R_1) $\sum_{n\in\mathbb{Z}} \varphi(\{n\} \times I_v)/\mu(I_v)$ is independent of v.

(R_2) For all $v \in T,\ n \in \mathbb{Z}$,

$$\sum_{t=0}^{|v|} q^{2t-|v|} \varphi(\{n + 2t - |v|\} \times I_v^t) = q^{-n} \sum_{t=0}^{|v|} q^{2t-|v|} \varphi(\{-n + 2t - |v|\} \times I_v^t).$$

For $r > 0$, define $\mathcal{B}_r$ as the class of all distributions φ on $\mathcal{H}$ satisfying the **decay condition**:

$$\sum_{n=|v|}^{\infty} t^n q^n \left|\varphi(\{n\} \times I_v)\right| < \infty \qquad \text{for all } t \in [0, r), v \in T.$$

Theorem 2. *For $r > 1/\sqrt{q}$, $R(\mathcal{A}_r)$ is the set of all $\varphi \in \mathcal{B}_r$ satisfying the Radon conditions.*

A distribution φ on $\mathcal{H}$ is N**-radial** if $\varphi(\{n\} \times I_v)$ depends only on n and v_N.

The proof of Theorem 2 is based on the use of N-radial functions and N-radial distributions. Given a positive number r, and a non-negative integer N, let $\mathcal{A}_r^N$ be the space of N-radial functions in $\mathcal{A}_r$, and let $\mathcal{B}_r^N$ be the space of N-radial distributions in $\mathcal{B}_r$. The key result in proving Theorem 2 is the following

Proposition 4. *For $r > 1/\sqrt{q}$, the image of the Radon transform on $\mathcal{A}_r^N$ is the set of all $\varphi \in \mathcal{B}_r^N$ satisfying the Radon conditions.*

The following example shows that the use of distributions is necessary:

Example. Let $l_1, \ldots, l_q$ be complex numbers of absolute value one, such that $\sum_{j=1}^q l_j = 2/3$, and set $l_{q+1} = l_1$. Label the vertices as follows: let $x_1, \ldots, x_{q+1}$ be the vertices of length 1. If $v \neq e$ has already been labeled, write the immediate descendants of v as $vx_1, \ldots, vx_q$. Thus a typical vertex v of length N is labeled as

$x_{i_1}\dots x_{i_N}$, where the i_j are between 1 and q, except for i_1 which can also be $q+1$. Then define $f(v)$ as $l_{i_1}\dots l_{i_N}\left(\frac{4}{3}\right)^N$, $f(e)=1$. Thus

$$\left|\sum_{u\in D_n(v)} f(u)\right| = |f(v)|(8/9)^{n-N} = \left(\frac{8}{9}\right)^n\left(\frac{3}{2}\right)^N .$$

If $0<t<9/8$, then $\sum_n t^n\left(\frac{8}{9}\right)^n\left(\frac{3}{2}\right)^N$ converges, and so $f\in\mathcal{A}_{9/8}$. By Theorem 2, $Rf\in\mathcal{B}_{9/8}$.

On the other hand, we now show that Rf cannot be evaluated at any horocycle. A horocycle $h_{\omega,n}$ is the disjoint union of the sets $D_{n+2k}(\omega_{n+k})-D_{n+2k}(\omega_{n+k+1})$ over the set of all non-negative integers k, for $n\geq 0$. Now

$$\left|\sum_{v\in D_{n+2k}(\omega_{n+k})} f(v)\right| = \left(\frac{4}{3}\right)^n\left(\frac{32}{27}\right)^k$$

and

$$\left|\sum_{v\in D_{n+2k}(\omega_{n+k+1})} f(v)\right| = \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)^n\left(\frac{32}{27}\right)^k .$$

Since the second sum has a larger absolute value, the absolute value of the difference is at least $\frac{1}{2}\left(\frac{4}{3}\right)^n\left(\frac{32}{27}\right)^k$. Thus the series for defining $Rf(h_{\omega,n})$ does not converge, for $n\geq 0$. For $n<0$,

$$h_{\omega,n} = \coprod_{k=0}^{\infty}\left(D_{n+2k}(\omega_k)-D_{n+2k}(\omega_{k+1})\right),$$

and the same conclusion holds. Since point evaluation cannot be defined, Rf cannot be induced by a function.

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