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PARUSIŃSKI'S "KEY LEMMA" VIA ALGEBRAIC GEOMETRY

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Abstract. The following "Key Lemma" plays an important role in the work by Parusiński on the existence of Lipschitz stratifications in the class of semianalytic sets: For any positive integer n, there is a finite set of homogeneous symmetric polynomials $W_1, \dots, W_N$ in $Z[x_1, \dots, x_n]$ and a constant $M > 0$ such that

$$|dx_i/x_i| \le M \max_{j=1,\dots,N} |dW_j/W_j| ,$$

as densely defined functions on the tangent bundle of $\mathbb{C}^n$. We give a new algebro-geometric proof of this result.

1. Introduction

Parusiński's fundamental work on the existence of Lipschitz stratifications in the class of semianalytic sets relies on the following result.

Theorem 1.1 (Parusiński [P, pp. 202–203]). *For any positive integer n, there is a finite set of homogeneous symmetric polynomials $W_1, \dots, W_N \in \mathbb{Z}[x_1, \dots, x_n]$ and a constant $M > 0$ such that*

$$\left|\frac{dx_i}{x_i}(p,v)\right| \le M \max_{j=1,\dots,N} \left|\frac{dW_j}{W_j}(p,v)\right| \tag{1.1}$$

for all $p \in \mathbb{C}^n$ and $v \in T_p\mathbb{C}^n$ for which both sides are defined. Here for any $P \in \mathbb{C}[x_1, \dots, x_n]$ we view the meromorphic differential form $\frac{dP}{P}$ on $\mathbb{C}^n$ as a densely defined function on the total space of the tangent bundle $T\mathbb{C}^n$.

Parusiński refers to Theorem 1.1 as the "Key Lemma"; the proof of this result in [P, Section 6] is quite difficult, being apparently the hardest part of [P]. The purpose of this paper is to show that Theorem 1.1, in spite of its analytic appearance, has a natural proof in the framework of algebraic geometry. Our argument is an application of the results of [RY] about group actions on algebraic varieties; these results, in turn, rely on canonical resolution of singularities.

We remark that Parusiński proves the inequality (1.1) under the additional assumption that $dV(p,v) = 0$ if $V(p) = 0$ for every V belonging to a finite set $\mathcal{V}$ of polynomials. Since this additional requirement does not affect a dense Zariski open subset of $T\mathbb{C}^m$ (given by $V(p) \neq 0$ for every $V \in \mathcal{V}$), it can be dropped. We also note

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that the statement of the Key Lemma in [P] only asserts the existence of polynomials $W_1, \dots, W_N$ with real coefficients; however, the construction of $W_1, \dots, W_N$ given there, produces polynomials over $\mathbb{Z}$. Thus, while Theorem 1.1 appears to be stronger than the "Key Lemma" in [P], the two are, in fact, equivalent.

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2. PRELIMINARIES

Notational conventions. All algebraic varieties considered in this paper, are assumed to be irreducible and defined over a field k of characteristic 0. The base field k is not assumed to be algebraically closed; the two cases of interest to us are $k = \mathbb{Q}$ and $k = \mathbb{C}$. By a point of a variety we shall always understand a closed point. Given an embedding $k \subset \mathbb{C}$ and a rational function f on X, we shall denote the corresponding rational function on $X_{\mathbb{C}} = X \otimes_k \mathbb{C}$ by f as well.

Throughout this paper G will be a finite group. A G-variety X is a variety with a regular action of G, $G \times X \longrightarrow X$, where $G \times X$ is understood as the disjoint union of $|G|$ copies of X. We will always assume that the G-action is faithful, i.e., every nonidentity element of G acts nontrivially. By a morphism (respectively, rational map, birational isomorphism) of G-varieties we shall mean a G-equivariant morphism (respectively, rational map, birational isomorphism).

Stabilizers. For a point x in a G-variety X, we define its "naive" stabilizer $\mathrm{NStab}(x)$ as the set of all $g \in G$ which preserve x. If k is not algebraically closed, the residue field $k(x)$ may be a nontrivial finite extension of k, and G may act on it nontrivially. We define the "honest" stabilizer $\mathrm{Stab}(x)$ as the set of all $g \in \mathrm{NStab}(x)$ that act on $k(x)$ trivially; cf. [MFK, Definition 0.4]. The subgroups $\mathrm{NStab}(x)$ and $\mathrm{Stab}(x)$ of G are sometimes called the *decomposition group* and the *inertia group* respectively.

If $\overline{k}$ is the algebraic closure of k, then x is represented by a set of "conjugate" points of the variety $X_{\overline{k}} = X \otimes_k \overline{k}$ ("geometric points" of X), one for each embedding $k(x) \hookrightarrow \overline{k}$; $\mathrm{Stab}(x)$ fixes each of these points while $\mathrm{NStab}(x)$ permutes them. As an example, consider the action of $G = \mathbb{Z}/2\mathbb{Z}$ on the affine line $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[t]$: the nontrivial element of this group acts by $t \mapsto -t$. Here $\mathrm{Stab}(x) = \{1\}$ for any $x \in \mathbb{A}^1_{\mathbb{Q}} - \{0\}$. On the other hand, $\mathrm{NStab}(x) = G$ iff x corresponds to the ideal in $\mathbb{Q}[t]$ generated by an irreducible polynomial of the form $q(t^2)$ (e.g., $t^2 + 1$).

This phenomenon is entirely arithmetic; we are concerned with it here because the symmetric polynomials $W_1, \dots, W_N$ in Theorem 1.1 are asserted to have integer coefficients. A reader who is only interested in the existence of such polynomials in $\mathbb{C}[x_1, \dots, x_n]$ may skip the rest of this section and assume that "naive" stabilizers always coincide with "honest" ones in the sequel.

Semilinear representations and skew group rings. Let X be a G-variety and let $x \in X$. The "naive stabilizer" $\mathrm{NStab}(x)$ acts upon $T_x(X)^* = \mathfrak{m}_x/\mathfrak{m}_x^2$. However, if $\mathrm{NStab}(x)$ is strictly larger than $\mathrm{Stab}(x)$, then this action is not linear over $k(x)$ but rather "semilinear" in the following sense.

Definition 2.1. Suppose a finite group H acts by automorphisms on a field K. A *semilinear representation* of H over K is a K-vector space V with a K^H-linear action of H on V having the property $g(\lambda v) = g(\lambda)g(v)$ for any $g \in H$, $\lambda \in K$ and $v \in V$.

For the rest of this section we shall assume that K is a field, K^* is the multiplicative group of K, H is a finite group acting on K by automorphisms, and H' is the kernel of this action. In the subsequent applications we will take $K = k(x)$, $H = \mathrm{NStab}(x)$ and $H' = \mathrm{Stab}(x)$.

Recall that the skew group algebra $K * H$ is defined as the set of formal sums $\sum_{h \in H} a_h h$ (where $a_h \in K$), with componentwise addition and with multiplication given, distributively, by $(a_1h_1)(a_2h_2) = a_1h_1(a_2)h_1h_2$. A semilinear representation of H is the same thing as a $(K * H)$-module. (All modules in this paper are understood to be left modules.)

Remark 2.2. Note that $V = K$ has a natural structure of a $(K * H)$-module. This module contains a vector $1 \in K$ which is fixed by H.

Recall that by Wedderburn's Theorem every semisimple ring R is a direct product of simple rings, called *the simple components* of R; see [B, Theorem VIII.5.1].

Lemma 2.3. *$K * H$ is a semisimple K^H-algebra with at most $|H'|$ simple components. (Here H' is the kernel of the H-action on K, as above.)*

Proof. Semisimplicity of $K * H$ is proved by the same averaging argument as the usual Maschke's theorem; for details see, e.g., [Mo, Theorem 0.1 and Corollary 0.2].

Denote the simple components of $K * H$ by $S_1, \dots, S_m$. Then $Z(K * H) = Z(S_1) \times \cdots \times Z(S_m)$, where $Z(A)$ denotes the center of A. It is easy to see directly that $\dim_{K^H}(Z(K * H)) \le |H'|$. Hence, $m \le |H'|$, as claimed. $\square$

The following proposition describes the particular kind of skew group rings we shall encounter in the sequel.

Proposition 2.4. *Suppose that H' is an abelian group of exponent e, K contains a primitive eth root of unity and $\chi_1, \dots, \chi_m$ is a set of generators for the dual group $(H')^* = \mathrm{Hom}(H', K^*)$. Assume further that for each i there is a one-dimensional semilinear representation V_i of H over K such that $h'(v) = \chi_i(h')v$ for every $v \in V_i$ and every $h' \in H'$. Then:*

(a) *For every $\chi \in (H')^*$, there exists a unique semilinear representation V_χ of H such that $\dim_K(V_\chi) = 1$ and $h'(v) = \chi(h')v$ for every $v \in V_\chi$ and every $h' \in H'$.*

(b) *Every simple $(K * H)$-module is isomorphic to V_χ for some $\chi \in (H')^*$.*

Proof. Note that if V_1 and V_2 are semilinear representations of H over K, then so is $V_1 \otimes_K V_2$. Indeed,

$$h(\lambda v_1) \otimes h(v_2) = h(\lambda) \cdot (h(v_1) \otimes h(v_2)) = h(v_1) \otimes h(\lambda v_2).$$

To construct V_χ, write $\chi \in (H')^*$ as $\chi = \chi_1^{l_1} \dots \chi_m^{l_m}$ for some nonnegative integers $l_1, \dots, l_m$, and set $V_\chi = V_1^{\otimes l_1} \otimes_K \cdots \otimes_K V_m^{\otimes l_m}$. The subgroup H' acts on V_χ by the character χ, as desired. As $\dim_K(V_\chi) = 1$, V_χ is a simple $(K * H)$-module. Note that the $(K * H)$-modules V_χ are pairwise nonisomorphic because H' acts on them by different characters.

The isomorphism classes of simple $(K * H)$-modules are in 1–1 correspondence with the simple components of $K * H$; see [B, Proposition VIII.5.11]. Thus Lemma 2.3 implies that $K * H$ has $\le |H'|$ nonisomorphic simple modules. On the other hand, we have constructed $|H'|$ nonisomorphic simple modules V_χ. This proves (b) and the uniqueness of V_χ in (a). $\square$

Remark 2.5. One can show that, under the assumptions of Proposition 2.4, H is a semidirect product of H' and H/H', where the the action of H/H' on H' is given by embedding H' into $(K^*)^m$ via $h' \longmapsto (\chi_1(h'), \dots, \chi_m(h'))$.

3. Reduction to an algebro-geometric problem

We begin by restating (1.1) as an inequality involving densely defined functions on the tangent bundle of $\mathbb{P}^{n-1}_{\mathbb{C}}$ rather than the tangent bundle of $\mathbb{C}^n$. Since $\mathbb{P}^{n-1}_{\mathbb{C}}$ is compact in the metric topology, this will allow us to pass from local to global estimates.

Proposition 3.1. *Let X be a projective G-variety over a field $k \subset \mathbb{C}$, and let f be a rational function on X. Then there exist G-invariant rational functions $\beta_1, \dots, \beta_m$ on X and a constant $K > 0$ such that*

$$\left|\frac{df}{f}(p,v)\right| \leq K \max_{j=1,\dots,m}\left|\frac{d\beta_j}{\beta_j}(p,v)\right| \tag{3.1}$$

for any $(p,v) \in T(X_{\mathbb{C}})$ such that p is a smooth point of $X_{\mathbb{C}}$ and does not lie on the divisors of $f, \beta_1, \dots, \beta_m$.

Reduction 3.2. *Proposition 3.1 $\Longrightarrow$ Theorem 1.1.*

Indeed, apply Proposition 3.1 with $k = \mathbb{Q}$, $X = \mathbb{P}^{n-1}_{\mathbb{Q}}$, $G = S_n$ and $f = x_1/s_1$; here S_n acts on $\mathbb{P}^{n-1}_{\mathbb{Q}}$ by permuting the homogeneous coordinates $x_1, \dots, x_n$, and $s_1 = x_1 + \cdots + x_n$. Write each β_j as a quotient of two homogeneous polynomials (of the same degree, with integer coefficients) in $x_1, \dots, x_n$:

$$\beta_1 = W_1/W_2\,,\ \dots\ ,\ \beta_m = W_{2m-1}/W_{2m}.$$

We claim that the polynomials $W_1, \dots, W_{2m}, W_{2m+1} \overset{\text{def}}{=} s_1$ have the property asserted in Theorem 1.1. Indeed, since $df/f = dx_1/x_1 - ds_1/s_1$ and $d\beta_i/\beta_i = dW_{2i-1}/W_{2i-1} - dW_{2i}/W_{2i}$, inequality (3.1) translates into

$$\begin{aligned}\left|\left(\frac{dx_1}{x_1} - \frac{ds_1}{s_1}\right)(p,v)\right| &\leq K \max_{i=1,\dots,m}\left|\left(\frac{dW_{2i-1}}{W_{2i-1}} - \frac{dW_{2i}}{W_{2i}}\right)(p,v)\right| \\ &\leq 2K \max_{j=1,\dots,2m}\left|\frac{dW_j}{W_j}(p,v)\right|.\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\frac{dx_1}{x_1}(p,v)\right| &\leq 2K \max_{j=1,\dots,2m}\left|\frac{dW_j}{W_j}(p,v)\right| + \left|\frac{ds_1}{s_1}(p,v)\right| \\ &\leq (2K+1) \max_{j=1,\dots,2m+1}\left|\frac{dW_j}{W_j}(p,v)\right|.\end{aligned}$$

This means that (1.1) holds for $i = 1$, with $M = 2K + 1$ and $N = 2m + 1$. By symmetry, (1.1) holds for all i. $\square$

Definition 3.3. Let X be a projective G-variety and f be a rational function on X. We shall say that the pair (X, f) has property (*) if there is a Zariski open covering $X = \bigcup_i U_i$ and, for each i, rational functions $\beta_{i1}, \dots, \beta_{i,q_i} \in k(X)^G$ and regular functions $\gamma_{i1}, \dots, \gamma_{i,q_i} \in \mathcal{O}_X(U_i)$ such that

$$\frac{df}{f} = \gamma_{i1}\frac{d\beta_{i1}}{\beta_{i1}} + \cdots + \gamma_{i,q_i}\frac{d\beta_{i,q_i}}{\beta_{i,q_i}}. \tag{3.2}$$

In other words, the pair (X, f) has property (*) if df/f is a global section of the sheaf of differentials on X generated over $\mathcal{O}_X$ by $d\beta/\beta$, as β ranges over some finite subset of $k(X)^G$ (or, equivalently, as β ranges over all of $k(X)^G$).

Reduction 3.4. *Proposition 3.1 holds, assuming the pair (X, f) that appears there, has property (*).*

Indeed, the Zariski open cover $\bigcup_i U_i$ of X, as in Definition 3.3, gives rise to a Zariski open cover $\bigcup_i U_{i,\mathbb{C}}$ of $X_{\mathbb{C}}$. The functions γ_{ij} are continuous on $U_{i,\mathbb{C}}$ with respect to the metric topology. Thus any point $x \in X$ has an open neighborhood U_x (in the metric topology) such that $U_x \subset U_{i_x,\mathbb{C}}$ for some i_x, and

$$\left|\frac{df}{f}(p,v)\right| \leq K_x \max_{j=1,\dots,q_{i_x}} \left|\frac{d\beta_{i_x,j}}{\beta_{i_x,j}}(p,v)\right|$$

whenever $v \in T_p(X)$ and p is a smooth point of U_x which does not lie on the divisors of $f, \beta_{i_x,1}, \dots, \beta_{i_x,q_{i_x}}$. The open sets U_x form a cover of X; since X is compact in the metric topology, we can choose a finite subcover $U_{x_1}, \dots, U_{x_r}$. Now if $K > K_{x_1}, \dots, K_{x_r}$, then

$$\left|\frac{df}{f}(p,v)\right| \leq K \max_{i,j} \left|\frac{d\beta_{ij}}{\beta_{ij}}(p,v)\right|.$$

This shows that Proposition 3.1 holds. □

Reduction 3.5. *Suppose X and X' are birationally isomorphic G-varieties over $k \subset \mathbb{C}$. If Proposition 3.1 holds for X and $f \in k(X)$, then it holds for X' and the same $f \in k(X') = k(X)$.*

Indeed, X and X' have isomorphic Zariski-open subsets U and U'. After passing to smaller subsets if necessary, we may assume that U and U' are smooth and do not intersect the divisors of $f, \beta_1, \dots, \beta_n$ on X and X' respectively. Thus if inequality (3.1) holds for every (p, v) such that $p \in U_{\mathbb{C}}$ and $v \in T_p(X_{\mathbb{C}})$, then it holds for every (p, v) such that $p \in U'_{\mathbb{C}}$ and $v \in T_{p'}(X'_{\mathbb{C}})$. The subset $U'_{\mathbb{C}}$ is dense in $X'_{\mathbb{C}}$ with respect to the metric topology; hence, by continuity the same inequality (with the same β_j and the same K) holds for every $(p, v) \in T(X'_{\mathbb{C}})$ such that p is a smooth point of $X'_{\mathbb{C}}$ and does not lie in the union of divisors of $f, \beta_1, \dots, \beta_m$. This means that Proposition 3.1 holds for X' as claimed. □

We have thus shown that Theorem 1.1 is a consequence of the following, purely algebraic statement (see Reductions 3.2, 3.4 and 3.5).

Proposition 3.6. *Let G be a finite group, X a projective G-variety, and $f \in k(X)$. Then there exists a birational morphism $\pi\colon X' \longrightarrow X$ of G-varieties such that the pair $(X', \pi^*(f))$ has property (*) (see Definition 3.3).*

A proof of Proposition 3.6 (and thus of Theorem 1.1) will be given in the next section. The idea is to construct $\pi\colon X' \longrightarrow X$ by resolving the G-action on X to "standard form" with respect to a divisor containing the divisor of f; see below. The simplest (affine) example of such X' is $X' = \mathbb{A}^1 = \operatorname{Spec} k[t]$, where k contains a primitive mth root of unity, $G = \mathbb{Z}/m\mathbb{Z}$ acts on $\mathbb{A}^1$ linearly by a faithful character, and $f = t$. In this case we can take $\beta = t^m \in k(X)^G$; the equality $df/f = \frac{1}{m} d\beta/\beta$ shows that (X', f) has property (*).

4. Conclusion of the proof

G-varieties in standard form.

Definition 4.1 ([RY, Definition 3.1])**.** We say that a generically free G-variety X is *in standard form with respect to a divisor Y* if

(a) X is smooth and Y is a normal crossing divisor on X,
(b) the G-action on $X - Y$ is free, and
(c) for every $g \in G$ and for every irreducible component Y_0 of Y either $g(Y_0) = Y_0$ or $g(Y_0) \cap Y_0 = \emptyset$.

Theorem 4.2 ([RY, Corollary 3.6])**.** *Let X be a G-variety and $Y \subset X$ be a Zariski closed G-invariant subvariety such that the action of G on $X - Y$ is free. Then there is a sequence of blowups*

$$\pi\colon X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X \tag{4.1}$$

with smooth G-invariant centers $C_i \subset X_i$ such that X_n is in standard form with respect to a divisor $\tilde{Y}$ containing $\pi^{-1}(Y)$.

Theorem 4.3. *Let X be a G-variety in standard form with respect to a divisor Y, let x be a point of X, let $Y_1, \dots, Y_m$ be the irreducible components of Y passing through x, and let $W = Y_1 \cap \dots \cap Y_m$. Then*

(a) [RY, Theorem 4.1] $\mathrm{Stab}(x)$ *is commutative.*
(b) [RY, Remark 4.4] *The action of* $\mathrm{Stab}(x)$ *on the normal space to W at x is faithful and decomposes into the sum of one-dimensional representations as follows:*

$$T_x(X)/T_x(W) = \bigoplus_{i=1}^{m} \frac{T_x(Y_1) \cap \dots \cap \widehat{T_x(Y_i)} \cap \dots \cap T_x(Y_m)}{T_x(W)}. \tag{4.2}$$

(c) [RY, Remark 4.5] *Let e be the exponent of* $\mathrm{Stab}(x)$*. Then the residue field $k(x)$ of x contains a primitive eth root of unity.*

Remark 4.4. Under the assumptions of Theorem 4.3, set $H = \mathrm{NStab}(x)$ and $H' = \mathrm{Stab}(x)$. Recall that H acts on $T_x(X)$ semilinearly; see Definition 2.1. Property (c) of Definition 4.1 implies that Y_i is preserved by the action of H, and hence, the subspace $T_x(Y_i)$ is H-invariant for each i. It follows that all spaces appearing in (4.2), are $(k(x) * H)$-modules.

The conormal space $(T_x(X)/T_x(Y_i))^*$ is dual to the ith summand in (4.2); it is a $(k(x) * H)$-module of dimension 1 over $k(x)$, and H' acts on it by a character. Denote this character by ξ_i. Theorem 4.3(b) implies that the characters $\xi_1, \dots, \xi_m$ generate the dual group $(H')^*$. Combining this observation with Theorem 4.3(c), we conclude that Proposition 2.4 applies in this setting.

A local coordinate system. Suppose that X is an algebraic variety and x is a point of X. Recall that $u_1, \dots, u_n \in \mathfrak{m}_x$ are said to form a local coordinate system on X at x if their classes modulo $\mathfrak{m}_x^2$ form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $k(x)$-vector space.

Proposition 4.5. *Let X be a quasiprojective G-variety in standard form with respect to a divisor Y. Suppose $Y_1, \dots, Y_m$ are the irreducible components of Y passing though a point x of X. Then there exists a local coordinate system $u_1, \dots, u_m$, $v_1, \dots, v_l$ at x with the following properties:*

(a) *Let $\overline{u}_i = u_i \bmod \mathfrak{m}_x^2$ and $\overline{v}_j = v_j \bmod \mathfrak{m}_x^2$. Then each of $\overline{u}_1, \dots, \overline{u}_m, \overline{v}_1, \dots, \overline{v}_l$ generates a one-dimensional H-invariant $k(x)$-subspace of $\mathfrak{m}_x/\mathfrak{m}_x^2$.*
(b) *For every $i = 1, \dots, m$ and every $g \in G$, u_i is a local equation of $g(Y_i)$ at gx.*
(c) *For every $j = 1, \dots, l$ there are integers $e_{j1}, \dots, e_{jm} \geq 0$ such that $h_j = u_1^{e_{j1}} \cdots u_m^{e_{jm}} v_j$ is a G-invariant rational function on X.*

Proof. We begin by constructing $u_1, \dots, u_m$. Consider the divisor $D_i = \sum g(Y_i)$, where each summand of the form $g(Y_i)$ (for some $g \in G$) appears in this sum exactly once. Since X is quasiprojective, the divisor D_i can be "moved off" the finite set Gx; see [Sh, Theorem III.1.1]. In other words, for every $i = 1, \dots, m$ there is a rational function u_i on X such that the support of the divisor $D_i - (u_i)$ does not intersect Gx. It is now easy to see that $u_1, \dots, u_m$ satisfy (a) and (b).

Next we turn to the construction of $v_1, \dots, v_l$. Each $\overline{u}_i$ generates a one-dimensional $k(x)$-subspace $\langle \overline{u}_i \rangle = (T_x(X)/T_x(Y_i))^* \subset \mathfrak{m}_x/\mathfrak{m}_x^2$. We have seen in Remark 4.4 that $\langle \overline{u}_i \rangle$ is H-invariant; H' acts on it by the character ξ_i. In view of Lemma 2.3 and Proposition 2.4, we can write

$$\mathfrak{m}_x/\mathfrak{m}_x^2 = \langle \overline{u}_1 \rangle \oplus \cdots \oplus \langle \overline{u}_m \rangle \oplus V_1 \oplus \cdots \oplus V_l,$$

where each V_i is a simple $(k(x) * H)$-module and $\dim_{k(x)}(V_i) = 1$. Choose $\overline{v}_1, \dots, \overline{v}_l \in \mathfrak{m}_x/\mathfrak{m}_x^2$ so that $\overline{v}_i$ generates V_i as a $k(x)$-vector space. Denote the character of H' associated to $\overline{v}_j$ by η_j. By Remark 4.4, $\xi_1, \dots, \xi_m$ generate the dual group $(H')^*$; consequently, each η_j can be written in the form

$$\eta_j = \xi_1^{-e_{j1}} \cdots \xi_m^{-e_{jm}}$$

for some integers $e_{j1}, \dots, e_{jm} \geq 0$. Note that $\overline{h}_j = \overline{u}_1^{e_{j1}} \cdots \overline{u}_m^{e_{jm}} \overline{v}_j$ is an H'-invariant element of $\mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$, where $d_j = e_{j1} + \cdots + e_{jm} + 1$. Clearly, $\overline{h}_j$ generates an H-invariant one-dimensional $k(x)$-subspace $\langle \overline{h}_j \rangle \subset \mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$ on which H' acts trivially. By the uniqueness statement in Proposition 2.4(a), $\langle \overline{h}_j \rangle \cong k(x)$ as $(k(x) * H)$-modules, and by Remark 2.2, after replacing $\overline{v}_j$ by $\lambda \overline{v}_j$ for some $\lambda \in k(x)$, we may assume

$$\overline{h}_j = \overline{u}_1^{e_{j1}} \cdots \overline{u}_m^{e_{jm}} \overline{v}_j \text{ is an } H\text{-invariant element of } \mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}. \tag{4.3}$$

We claim that we can choose $v_1, \dots, v_l \in \mathfrak{m}_x$ so that $\overline{v}_j = v_j \bmod \mathfrak{m}_x^2$ and each $h_j = u_1^{e_{j1}} \cdots u_m^{e_{jm}} v_j$ is G-invariant. If we can do this, then $u_1, \dots, u_m, v_1, \dots, v_l$ will clearly satisfy the requirements of the proposition.

To prove the claim, let $R = \bigcap_{y \in Gx} \mathcal{O}_{y,X}$ be the ring of rational functions on X that are well defined on Gx, and let $\mathcal{I}_{Gx} = \bigcap_{y \in Gx} \mathfrak{m}_y$ be a G-invariant ideal in R consisting of all elements that vanish on Gx. By our choice of $u_1, \dots, u_m$,

$$g^*(u_i)/u_i \text{ is defined and invertible at every point of } Gx \tag{4.4}$$

for every $g \in G$, and every $i = 1, \dots, m$. Consequently, $u_1^{e_{j1}} \cdots u_m^{e_{jm}} \mathcal{I}_{Gx}$ is a G-invariant ideal of R.

For each $y \in Gx$, the functions $u_1, \dots, u_m$ vanish at y, and hence,

$$u_1^{e_{j1}} \cdots u_m^{e_{jm}} \mathcal{I}_{Gx} \subset \mathfrak{m}_y^{d_j}.$$

Consider the G-equivariant projection map

$$\psi\colon u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathcal{I}_{Gx}\longrightarrow \textstyle\bigoplus_{y\in Gx}\mathfrak{m}_y^{d_j}/\mathfrak{m}_y^{d_j+1}.$$

If $v\in\mathcal{I}_{Gx}$, then $\psi(u_1^{e_{j1}}\cdots u_m^{e_{jm}}v)$ depends only on the image of v in $\bigoplus_{y\in Gx}\mathfrak{m}_y/\mathfrak{m}_y^2$. Moreover, since X is quasiprojective, the finite set Gx lies in an affine open subset of X and hence, by the Chinese Remainder Theorem, the projection map $\mathcal{I}_{Gx}\longrightarrow\bigoplus_{y\in Gx}\mathfrak{m}_y/\mathfrak{m}_y^2$ is surjective. Thus

$$\operatorname{Im}(\psi)=\textstyle\bigoplus_{y\in Gx}\big(u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathfrak{m}_y\big)\bmod \mathfrak{m}_y^{d_j+1}\subset\bigoplus_{y\in Gx}\mathfrak{m}_y^{d_j}/\mathfrak{m}_y^{d_j+1}.$$

We shall denote elements of $\operatorname{Im}(\psi)$ by $a=\big(a_y\mid y\in Gx\big)$, where

$$a_y\in\big(u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathfrak{m}_y\big)\bmod\mathfrak{m}_y^{d_j+1}.$$

Recall that by (4.3), $\overline{h}_j$ is a nonzero H-invariant element of $\big(u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathfrak{m}_x\big)$ mod $\mathfrak{m}_x^{d_j+1}$. Let a_j be the element of $\operatorname{Im}(\psi)$ such that $(a_j)_y=(g^{-1})^*(\overline{h}_j)$, where $y=gx$; in view of (4.4),

$$(a_j)_y\in(g^{-1})^*[\big(u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathfrak{m}_x\big)\bmod\mathfrak{m}_x^{d_j+1}]=\big(u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathfrak{m}_y\big)\bmod\mathfrak{m}_y^{d_j+1}.$$

Note that since $\overline{h}_j$ is H-invariant, $(a_j)_y$ is independent of the choice of g. By our construction a_j is G-invariant and $(a_j)_x=\overline{h}_j\in\mathfrak{m}_x^{d_j}/\mathfrak{m}_x^{d_j+1}$.

The homomorphism ψ has a G-equivariant k-linear splitting and consequently, there exists a G-invariant element $h_j=u_1^{e_{j1}}\cdots u_m^{e_{jm}}v_j\in u_1^{e_{j1}}\cdots u_m^{e_{jm}}\mathcal{I}_{Gx}$ such that $\psi(h_j)=a_j$. In particular, $\overline{h}_j=\overline{u}_1^{e_{j1}}\cdots\overline{u}_m^{e_{jm}}\overline{v}_j=h_j\bmod\mathfrak{m}_x^{d_j+1}$ and hence $\overline{v}_j=v_j\bmod\mathfrak{m}_x^2$. This proves the claim and thus shows that $u_1,\dots,u_m,v_1,\dots,v_l$ have the required properties. □

Property (*). We are now ready to revisit property (*) of Definition 3.3.

Lemma 4.6. *Suppose X is a quasiprojective G-variety in standard form with respect to a divisor Y, $x\in X$, $u_1,\dots,u_m$, $v_1,\dots,v_l$, and $h_1,\dots,h_l$, are as in Proposition 4.5, and $w_i=\prod_{g\in G}g^*(u_i)$. Let f be a rational function on X whose divisor is supported on Y. Then*

$$df/f\in\mathcal{M},$$

where $\mathcal{M}$ is the $\mathcal{O}_{x,X}$-module generated by dw_i/w_i and dh_j/h_j with $i=1,\dots,m$ and $j=1,\dots,l$.

Note that w_i and h_j are G-invariant rational functions on X for every $i=1,\dots,m$ and $j=1,\dots,l$.

Proof. Let $(\Omega_X^1)_x$ be the $\mathcal{O}_{x,X}$-module of germs at x of regular differential forms on X. Since X is smooth, $(\Omega_X^1)_x$ is a free $\mathcal{O}_{x,X}$-module generated by $du_1,\dots,du_m$, $dv_1,\dots,dv_l$. Let $\mathcal{M}'$ be the $\mathcal{O}_{x,X}$-module generated by du_i/u_i, and dv_j/v_j, where $i=1,\dots,m$ and $j=1,\dots,l$. Clearly, $(\Omega_X^1)_x\subset\mathfrak{m}_x\mathcal{M}'$.

We claim that $\mathcal{M}=\mathcal{M}'$. It is clear that $\mathcal{M}\subset\mathcal{M}'$; to prove the opposite inclusion, we shall show that $dw_1/w_1,\dots,dw_m/w_m,dh_1/h_1,\dots,dh_l/h_l$ generate $\mathcal{M}'$ as an $\mathcal{O}_{x,X}$-module.

Note that by our choice of $u_1,\dots,u_m$, we can write $w_i=a_iu_i^{|G|}$ for some $a_i\in\mathcal{O}_{X,x}-\mathfrak{m}_x$; see (4.4). Thus

$$\frac{dw_i}{w_i}=\frac{da_i}{a_i}+|G|\frac{du_i}{u_i}\equiv|G|\,\frac{du_i}{u_i}\pmod{(\Omega_X^1)_x}.$$

In particular, since $(\Omega^1_X)_x \subset \mathfrak{m}_x\mathcal{M}'$, we conclude that

$$\frac{dw_i}{w_i} \equiv |G|\frac{du_i}{u_i} \pmod{\mathfrak{m}_x\mathcal{M}'}. \tag{4.5}$$

On the other hand, since $h_j = u_1^{e_{j1}} \cdots u_m^{e_{jm}} v_j$, we have

$$\frac{dh_j}{h_j} = \frac{dv_j}{v_j} + \sum_i e_{ji}\frac{du_i}{u_i}. \tag{4.6}$$

Examining (4.5) and (4.6), we see that dw_i/w_i $(i = 1, \dots, m)$, and dh_j/h_j $(j = 1, \dots, l)$ generate $\mathcal{M}'/\mathfrak{m}_x\mathcal{M}'$ as a $k(x)$-vector space. Consequently, by Nakayama's lemma these elements generate $\mathcal{M}'$ as an $\mathcal{O}_{x,X}$-module. Thus $\mathcal{M}' = \mathcal{M}$, as claimed.

Since the divisor of f is supported on Y, locally near x it is a union of smooth hypersurfaces of the form $\{u_i = 0\}$. This means that $f = au_1^{e_1} \cdots u_m^{e_m}$ for some $a \in \mathcal{O}_{x,X} - \mathfrak{m}_x$ and $e_1, \dots, e_m \geq 0$; hence,

$$\frac{df}{f} = \frac{da}{a} + e_1\frac{du_1}{u_1} + \cdots + e_m\frac{du_m}{u_m} \in \mathcal{M}' = \mathcal{M}.$$

□

Proposition 4.7. *Let X be a projective variety in standard form with respect to a divisor Y and let f be a rational function on X whose divisor is supported on Y. Then the pair (X, f) has property (*); see Definition 3.3.*

Proof. By Zariski compactness, it is enough to show that for any $x \in X$, there exist $\beta_1, \dots, \beta_q \in k(X)^G$ and $\gamma_1, \dots, \gamma_q \in \mathcal{O}_{x,X}$ such that

$$\frac{df}{f} = \gamma_1\frac{d\beta_1}{\beta_1} + \cdots + \gamma_q\frac{d\beta_q}{\beta_q}.$$

The last assertion is immediate from Lemma 4.6. □

Proof of Theorem 1.1. As we showed in Section 3, it is enough to prove Proposition 3.6.

Assume X is a projective G-variety and $f \in k(X)$. Let X_0 be the subvariety of all points in X with nontrivial stabilizers. Let Y be the union of X_0 and (the supports of) the divisors of $g^*(f)$ for every $g \in G$; it is a G-invariant Zariski closed subvariety of X. By Theorem 4.2 there exists a birational morphism $\pi\colon X' \longrightarrow X$ and a divisor $Y' \subset Y$, such that X' is in standard form with respect to Y' and $\pi^{-1}(Y) \subset Y'$. Note that the divisor of $\pi^*(f)$ is contained in $\pi^{-1}(Y)$ and hence in Y'. Proposition 3.6 (and thus Theorem 1.1) now follows from Proposition 4.7, which asserts that the pair $(X', \pi^*(f))$ has property (*). □

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