

MATING QUADRATIC MAPS WITH KLEINIAN GROUPS VIA QUASICONFORMAL SURGERY

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ABSTRACT. Let $q : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be any quadratic polynomial and $r : C_2 * C_3 \rightarrow PSL(2, \mathbb{C})$ be any faithful discrete representation of the free product of finite cyclic groups C_2 and C_3 (of orders 2 and 3) having connected regular set. We show how the actions of q and r can be combined, using quasiconformal surgery, to construct a 2 : 2 holomorphic correspondence $z \rightarrow w$, defined by an algebraic relation $p(z, w) = 0$.

1. INTRODUCTION

Given two abstractly isomorphic Fuchsian groups $G_1 \subset PSL(2, \mathbb{R})$ and $G_2 \subset PSL(2, \mathbb{R})$, acting on the upper and lower halves \mathcal{U} and \mathcal{L} of the complex plane respectively, each having limit set $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$, and such that the action of G_1 on $\hat{\mathbb{R}}$ is *topologically* conjugate to that of G_2 , it is well known that one can *mate* the actions of G_1 and G_2 to obtain a Kleinian group $G \subset PSL(2, \mathbb{C})$, isomorphic as an abstract group to both G_1 and G_2 , such that the limit set Λ of G is a simple closed (fractal) curve and the actions of G on the two components of $\Omega = \hat{\mathbb{C}} - \Lambda$ are *conformally* conjugate to those of G_1 on \mathcal{U} and G_2 on \mathcal{L} .

Equally, given two polynomial maps P and Q of the same degree n , having connected filled Julia sets $K(P)$ and $K(Q)$ respectively, it is well known that in certain cases one can *mate* the actions to obtain a *rational* map R such that the complement Ω of the Julia set $J(R)$ is a disjoint union of two open sets, on one of which the action of R is conformally conjugate to that of P on the interior $K(P)^\circ$ of its filled Julia set, and on the other of which the action of R is conformally conjugate to that of Q on $K(Q)^\circ$. A necessary condition for a mating of two quadratic polynomials $P : z \rightarrow z^2 + c$ and $Q : z \rightarrow z^2 + c'$ to exist is that c and c' should not belong to conjugate limbs of the connectivity locus in parameter space: this was first shown also to be a sufficient condition in the case that P and Q are *postcritically finite* [12], and subsequently for more general classes of P and Q .

It is also possible to mate certain Kleinian groups with polynomial maps. To realise such matings we have to move into the larger world of *holomorphic correspondences*. A *holomorphic correspondence*, of bidegree $m : n$, on the Riemann sphere $\hat{\mathbb{C}}$, is a multivalued map $z \rightarrow w$ defined by a relation $p(z, w) = 0$, where p is

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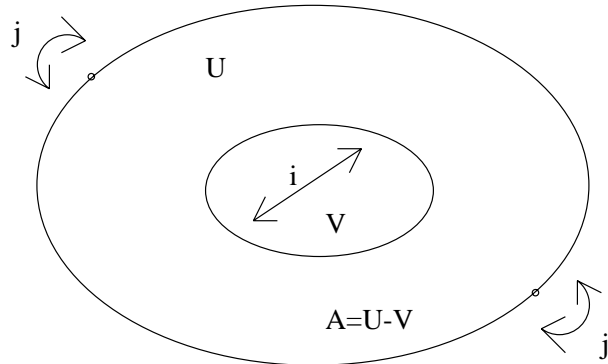


FIGURE 1. An annulus A for the quadratic map $q : z \rightarrow z^2 + c$.

a polynomial of degree m in z and n in w . We require that $p(z, w)$ has no square factors, so that a generic point w has m inverse images z and a generic point z has n images w . Equivalently, a holomorphic correspondence on $\hat{\mathbb{C}}$ is defined by a (singular) Riemann surface in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ with the two projections to $\hat{\mathbb{C}}$ branched-coverings of degrees m and n respectively. Examples of holomorphic correspondences are rational maps (defined by $P(z) - w = 0$), their inverses (defined by $P(w) - z = 0$), and finitely generated Kleinian groups (defined by $(w - A_1 z)(w - A_2 z) \cdots (w - A_n z) = 0$, where A_1, \dots, A_n are Möbius transformations generating the group in question). We formulate below (in Section 3) what it means to say that a holomorphic correspondence is a *mating* of a particular Kleinian group and a particular polynomial map. The first examples were described in [3] and more general constructions were presented in [5]. These examples and constructions pick out particular classes of polynomial relations $p(z, w) = 0$ and then in appropriate circumstances identify the resulting correspondences as matings. Below we show how it is possible to create a mating of a quadratic map and a representation of the group $C_2 * C_3$ to order, by fitting the pieces together using quasiconformal surgery [8]. The key observation that enables us to get started (Section 4.1 below) is that a certain ‘pair of pants’ domain associated to a representation of $C_2 * C_3$ double covers an annulus carrying precisely the same combinatorial data as does a ‘fundamental annulus’ for a quadratic-like map.

2. THE INGREDIENTS

2.1. The quadratic map. Given any quadratic map $q : z \rightarrow z^2 + c$, there is a holomorphic conjugacy from $z \rightarrow z^2$ to q on a neighbourhood of ∞ , fixing the point ∞ and tangent to the identity map there [7]. An *equipotential* for q is the image of a circle $\{Re^{2\pi it} : 0 \leq t < 1\}$ under this conjugacy. It is a smooth Jordan curve parametrised by *external angle* t . The region bounded by such an equipotential is a simply-connected domain V , mapped 2 : 1 by q onto a larger domain $U \supset V$ which also has boundary an equipotential parametrised by external angle (the restriction $q : V \rightarrow U$ is an example of a *quadratic-like map* in the sense of Douady and Hubbard [8]). We shall denote the annulus $U - V$ by A and its inner and outer boundaries by $\partial_1 A$ and $\partial_2 A$ respectively (Figure 1). The map q sends $\partial_1 A$ two-to-one onto $\partial_2 A$. For future reference we note the existence of an involution $i : z \rightarrow -z$

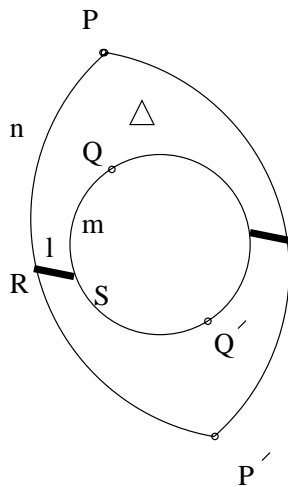


FIGURE 2. A fundamental domain Δ for the group $G = \langle \sigma, \rho, \chi \rangle$.

on V sending each $z \in V$ to the other point which has the same image in U under q , and an involution j on $\partial_2 A$ given by $t \rightarrow 1 - t$ on external angles (in fact in what follows j may be taken to be any *smooth orientation-reversing involution* on $\partial_2 A$).

For simplicity, until the final section of this article we shall assume that the filled Julia set $K(q)$ is connected. The corresponding set \mathcal{M} of values of the parameter c is known as the *connectivity locus* or *Mandelbrot set*.

2.2. The Kleinian group. Up to conjugacy each representation r of $C_2 * C_3$ in $PSL(2, \mathbb{C})$ is determined by a single complex parameter, the cross-ratio between the fixed points on $\hat{\mathbb{C}}$ of the action of the generator σ of C_2 and those of the generator ρ of C_3 . Such a representation comes equipped with a (unique) involution χ which exchanges the two fixed points of σ and also those of ρ , so that $\chi\sigma = \sigma\chi$ and $\chi\rho = \rho^{-1}\chi$ [4]. On the Poincaré 3-disc χ is simply rotation through π around the common perpendicular to the axes of σ and ρ . Write G for the group $\langle \sigma, \rho, \chi \rangle$.

The faithful discrete actions r with connected regular set $\Omega(G)$ form a single quasiconformal conjugacy class, the class of representations for which one can find simply-connected fundamental domains for σ and ρ with interiors together covering the whole Riemann sphere (i.e. the conditions of the simplest form of the Klein Combination Theorem are satisfied) [9, 10]. Such fundamental domains may be constructed as illustrated in Figure 2. Here P and P' are the fixed points of ρ , Q and Q' are the fixed points of σ , R is a fixed point of (the involution) $\chi\rho$ and S is a fixed point of $\chi\sigma$. The lines l, m and n , joining R to S , Q to S and R to P , are chosen such that they are smooth and remain non-intersecting in the quotient orbifold $\Omega(G)/G$. The region bounded by $n, \rho n, \chi n$ and $\chi\rho n$ is a fundamental domain for ρ , and the region exterior to the loop made up of $m, \sigma m, \chi m$ and $\chi\sigma m$ is a fundamental domain for σ . The intersection of these two regions is a fundamental domain for the (faithful) action of $C_2 * C_3$ on $\Omega(G)$, and the half Δ of this intersection bounded by $n, l, m, \sigma m, \chi l$ and ρn is a fundamental domain for the action of G . The union of all translates of Δ under elements of $C_2 * C_3$ is a

topological disc D which is a fundamental domain for the action of χ on $\Omega(G)$. The complement $\Lambda(G)$ of $\Omega(G) = D \cup \chi(D)$ in $\hat{\mathbb{C}}$ is a Cantor set.

3. THE DEFINITION OF A MATING AND THE STATEMENT OF THE THEOREM

We say that a $2 : 2$ holomorphic correspondence $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a *mating* of the quadratic map $q : z \rightarrow z^2 + c$ (where $c \in \mathcal{M}$) and the faithful discrete representation r of $C_2 * C_3$ (having connected regular set) if $\hat{\mathbb{C}}$ is partitioned into an open set Ω and a closed set Λ , each completely invariant under f and with the following properties:

- (I) Λ is the disjoint union of two sets Λ_+ and Λ_- , on which the $2 : 2$ correspondence $f : \Lambda \rightarrow \Lambda$ decomposes into the following parts:
 - (i) $f : \Lambda_- \rightarrow \Lambda_-$, a $2 : 1$ correspondence (a map of degree two);
 - (ii) $f : \Lambda_+ \rightarrow \Lambda_+$, a $1 : 2$ correspondence (the inverse of a map of degree two);
 - (iii) $f : \Lambda_- \rightarrow \Lambda_+$, a $1 : 1$ correspondence (a bijection).
- (II) There is a homeomorphism from the filled Julia set K of q to Λ_- conjugating $q|_K$ to $f|_{\Lambda_-}$ and a homeomorphism from K to Λ_+ conjugating $q|_K$ to $f^{-1}|_{\Lambda_+}$. Both are conformal on the interior of K .
- (III) The $2 : 2$ correspondence $f : \Omega \rightarrow \Omega$ acts properly discontinuously and there is a conformal homeomorphism h from the orbifold Ω/f to the orbifold $\Omega(G)/G$ compatible with the actions of f and G respectively, in the following sense: there exist a (completely invariant) set of curves \mathcal{C} in Ω and a fundamental domain D for the action of χ on $\Omega(G)$ which is invariant under $C_2 * C_3$, such that h lifts to a conformal homeomorphism $(\Omega - \mathcal{C}) \rightarrow D$ conjugating f to $\{\sigma\rho, \sigma\rho^{-1}\}$.

Theorem 1. *For every quadratic map $q : z \rightarrow z^2 + c$ with $c \in \mathcal{M}$ and every faithful discrete representation r of $C_2 * C_3$ in $PSL(2, \mathbb{C})$ having connected regular set, there exists a polynomial relation $p(z, w) = 0$ defining a $2 : 2$ correspondence $z \rightarrow w$ which is a mating of q with r .*

A modification dealing with the case when c is outside \mathcal{M} will be outlined in the final section of this paper.

4. THE CONSTRUCTION AND THE PROOF OF THE THEOREM

4.1. An annulus associated to the Kleinian group, and a $2 : 2$ correspondence on it. The quotient orbifold $\Omega(G)/G$ is the Riemann surface (with cone point singularities) obtained from Δ by making the boundary identifications corresponding to ρ, σ and χ . Covering this orbifold we have an annulus B consisting of three contiguous copies of Δ in $\Omega(G)$, namely $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$, with the boundary identifications (induced by χ) indicated in Figure 3. Concretely we may regard B as a subset of the quotient Riemann sphere $\hat{\mathbb{C}}/\chi$. We remark that $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ is itself a fundamental domain for the action of the index three subgroup $\langle \chi, \sigma, \rho\sigma\rho \rangle$ of G , and that $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ and its image under χ make up a ‘pair of pants’ fundamental domain for $\langle \sigma, \rho\sigma\rho \rangle$, the annulus B being the quotient of this ‘pair of pants’ by the action of χ .

The rotation ρ maps $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ to itself in the obvious way, with a single fixed point at P , but since ρ *anticommutes* with χ , this action does not descend to a rotation on the quotient annulus B . Rather, the action of the pair $\{\rho, \rho^{-1}\}$ on $\Delta \cup \rho\Delta \cup \rho^{-1}\Delta$ descends to the action of a $2 : 2$ correspondence g on B . Under this

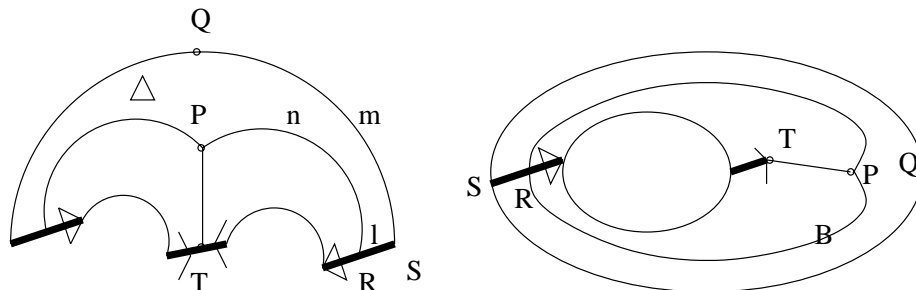


FIGURE 3. The set $(\Delta \cup \rho\Delta \cup \rho^{-1}\Delta)$ and its quotient, the annulus B .

2 : 2 correspondence each $z \in B$ is mapped to the points ρz and $\rho^{-1}z$ (or rather to their equivalence classes under the action of χ). The points P and T are singular for g , having *unique* images P and R , respectively, but all other points of B have two distinct images under g , and also 2 distinct inverse images, since $g = g^{-1}$. Note that in a neighbourhood of P the correspondence g behaves like a pair of rotations through $2\pi/3$ and $-2\pi/3$, but in a neighbourhood of T it behaves like a square root map. Generic orbits of g have cardinality three, the correspondence g sending each point of an orbit to the other two, and the image of Δ in B is a ‘fundamental domain’ for the action. The boundary of B is divided into three segments (two inner and one outer, Figure 3), each of which is mapped to the other two by g . Thus when its domain is restricted to the inner boundary $\partial_1 B$, and its range is restricted to the outer boundary $\partial_2 B$, the correspondence g defines a two to one map. When restricted to a correspondence from the inner boundary to itself, g defines a (fixed point free) bijection. Moreover, since the involution σ commutes with χ , it descends to an involution (which we shall also denote σ) on the outer boundary $\partial_2 B$ of B , having fixed points Q and S . Observe that since the composition $\sigma \circ g : \partial_1 B \rightarrow \partial_2 B$ is an orientation preserving two to one map, and the bijection $g : \partial_1 B \rightarrow \partial_1 B$ is the covering involution of this map, the annulus B carries the same data as that furnished on the annulus A by the quadratic map q .

4.2. A bijection between the annuli A and B . In general the annuli A and B will not be conformally equivalent: the conformal equivalence class of an annulus is determined by its *modulus*, a positive real number [2]. However, any two annuli are *quasiconformally* equivalent.

Lemma 1. *There exists a quasiconformal homeomorphism h from A to B which restricts to a smooth homeomorphism from ∂A to ∂B conjugating the boundary maps ($q : \partial_1 A \rightarrow \partial_2 A$, $j : \partial_2 A \rightarrow \partial_2 A$) to the boundary maps ($\sigma \circ g : \partial_1 B \rightarrow \partial_2 B$, $\sigma : \partial_2 B \rightarrow \partial_2 B$).*

Proof. We first use the fixed points of j to divide the outer boundary $\partial_2 A$ of A into two intervals and choose any smooth homeomorphism h from one of these intervals to the corresponding half of $\partial_2 B$ (which has end points Q and S). Now extend h to a smooth homeomorphism from the whole of $\partial_2 A$ to the whole of $\partial_2 B$, using the involutions j and σ , and then to a smooth homeomorphism from $\partial_1 A$ to $\partial_1 B$ by pulling back via q and $\sigma \circ g$. This gives a smooth $h : \partial A \rightarrow \partial B$ equivariant with respect to the boundary data. But any smooth homeomorphism of boundaries of annuli extends to a quasiconformal homeomorphism h of the interiors: this follows

at once from the corresponding result for discs [2], since an annulus can be converted into a disc by cutting along any smooth path joining the inner boundary to the outer. \square

Let μ denote the *complex dilatation* $(\partial h/\partial \bar{z})/(\partial h/\partial z)$ of h . By standard theory of quasiconformal maps [2] μ is of class L^∞ (bounded almost everywhere) and $\|\mu\|_\infty < 1$.

We shall abuse notation to the extent of denoting by g not only the 2 : 2 correspondence on B defined in Section 4.1, but also the correspondence $g_A = h \circ g_B \circ h^{-1}$ on A obtained by transporting $g (= g_B)$ from B . Thus

$$\{q : \partial_1 A \rightarrow \partial_2 A\} = \{j \circ g : \partial_1 A \rightarrow \partial_2 A\}$$

and this degree two map has covering involution

$$\{i : \partial_1 A \rightarrow \partial_1 A\} = \{g : \partial_1 A \rightarrow \partial_1 A\}.$$

Moreover the Beltrami differential μ on A is preserved by $g = g_A : A \rightarrow A$, in the sense that $g^* \mu = \mu$. Since g_B^* is the identity, g_B being holomorphic, this follows at once from the fact that

$$g_A^* \mu = (h^{-1})^* \circ (g_B)^* \circ h^*(\mu).$$

4.3. Constructing the correspondence at the combinatorial/topological level.

We first glue together U and a second copy U' of U , via the boundary involution j , to obtain a sphere $U \cup U'$, equipped with an involution, which we also denote j , exchanging U with U' and restricting to the original j on the common boundary. Inside U' we have a simply-connected subdomain V' , corresponding to $V \subset U$. Let $q' = j \circ q \circ j : V' \rightarrow U'$ denote the quadratic map corresponding to $q : V \rightarrow U$ and A' denote the annulus $U' - V'$. To define a 2 : 2 correspondence f on $U \cup U'$ we fit together:

- $q : V \rightarrow U$ (a 2 : 1 correspondence);
- $(q')^{-1} = j \circ q^{-1} \circ j : U' \rightarrow V'$ (a 1 : 2 correspondence);
- $j \circ i : V \rightarrow V'$ (a 1 : 1 correspondence), and
- $j \circ g : A \rightarrow A'$ (a 2 : 2 correspondence),

where $g : A \rightarrow A$ is the 2 : 2 correspondence constructed in Section 4.2 above. We remark that conjugation by the involution j sends f to f^{-1} . Thus j is a *time-reversing symmetry* of f .

Using the boundary data identities of Section 4.2 it is a straightforward exercise to check that the restrictions of f defined above fit together to define a continuous 2 : 2 correspondence f on the whole Riemann sphere. The next step is to identify the space of grand orbits of mixed iteration of f and f^{-1} on the complement Ω of $K(q) \cup K(q')$. Let A/\sim denote the quotient space obtained from the (closed) annulus A by applying the equivalence relation g on A and the equivalence relation $\langle g, j \rangle$ on ∂A , and let Δ/\approx denote the quotient space obtained from Δ (Figure 2) by identifying l with χl , m with σm and n with ρn .

Lemma 2. *The grand orbit space of the correspondence f , acting on Ω by arbitrary combinations of forward and backward iteration, is homeomorphic to A/\sim and hence to $\Delta/\approx = \Omega(G)/G$.*

Proof. We first observe that if $z \in A$, then $g(z) (\subset A)$ and $j(z) (\in A')$ lie on the grand orbit of z under f . This is because $g(z) \subset f^{-1} \circ f(z)$ and $j(z) \subset f^{-1} \circ f \circ f^{-1}(z)$. Now we must show that the grand orbit of any point in Ω meets A in a single g -orbit.

Clearly for each $z \in V \cap \Omega$ there is a unique positive integer n such that $q^n(z) \in A$ and for each $z' \in V' \cap \Omega$ there is a unique positive integer n such that $(q')^n(z') \in A'$, and hence $j \circ (q')^n(z') \in A$ (the claim of uniqueness needs qualification if $q^n(z) \in \partial A$ or $(q')^n(z') \in \partial A'$ but it is not hard to make the appropriate changes). However, in order to show that $q^n(z)$ and $j \circ (q')^n(z')$ are the *only* points of the grand orbits of z and z' to lie in A , we need to check that no other points of A can be reached by mixing iterations of q and q' with the other branch, $j \circ i$, of f (which, we recall, carries V bijectively to V'). It will suffice to verify that for any $z \in V$,

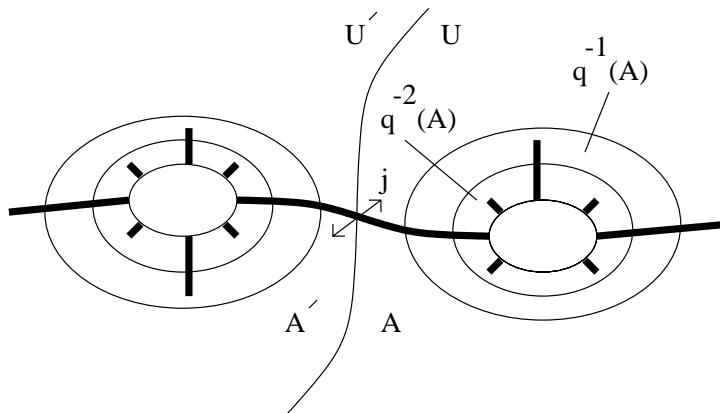
$$j \circ (q' \circ (j \circ i))(z) = q(z).$$

However, $j \circ q' \circ j = q$ and $q \circ i = q$, so we are done. \square

4.4. Making the correspondence holomorphic. Since ∂U is smooth and the boundary involution $j : \partial U \rightarrow \partial U$ is smooth, the complex structure on U extends to a complex structure on the sphere $U \cup U'$. (It first descends to a complex structure on the quotient U/j and then lifts to the double cover $U \cup U'$.)

Consider the Beltrami differential μ on A provided by the complex dilatation of the quasiconformal homeomorphism $h : A \rightarrow B$ (Lemma 1). We may extend μ to $q^{-1}(A)$ by setting its value there to be that of the pull-back $q^*\mu$, and we may extend it to A' by defining its value there to be that of $j^*\mu$. Indeed by repeatedly pulling back using q^* and $(q')^*$ we may extend μ to $U - K = \bigcup q^{-n}(A)$ and $U' - K' = \bigcup (q')^{-n}(A')$, where K and K' are the filled Julia sets of q and q' respectively. Finally by defining it to be zero on $K \cup K'$ we may extend μ to an L^∞ Beltrami differential on the whole of the Riemann sphere. Since $\|\mu\| < 1$, we may now apply the Measurable Riemann Mapping Theorem [1, 2] and deduce that there exists a quasiconformal homeomorphism $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with complex dilatation μ . But $f^*\mu = \mu$, since $g^*\mu = \mu$, $j^*\mu = \mu$ and $q^*\mu = \mu$ on the appropriate regions. Thus, by the chain rule, $\phi \circ f \circ \phi^{-1}$ is holomorphic, except possibly at branch points. But the latter are removable and hence the $2 : 2$ correspondence $\phi \circ f \circ \phi^{-1}$ is holomorphic everywhere. Since μ vanishes on $K \cup K'$, and is the complex dilatation of h on A , the correspondence is a mating (in the sense of the definition in Section 3) of the quadratic map q and the representation r used in its construction, with the union of the images of Δ under $C_2 * C_3$ as the fundamental domain D for the action of χ on $\Omega(G)$, and with the grand orbit under f of the image (under h^{-1}) in A of the curve $l \subset \partial\Delta$ as the set of curves \mathcal{C} such that $\Omega - \mathcal{C}$ is conformally homeomorphic to D . In Figure 4, where the coordinates have been chosen so that j is the map $z \rightarrow -z$, we illustrate the annuli A and A' , their first few pre-images under q^{-1} and $(q')^{-1}$ respectively, and the intersection of \mathcal{C} with these annuli and pre-images. Note that each $q^{-n}(A)$ is an annulus, regularly 2^n -fold covering A itself, and that the union of all the annuli $q^{-n}(A)$ and $(q')^{-n}(A')$, when cut along the set of curves \mathcal{C} , opens out to form a disc (containing the point ∞).

With the complex structure defined above, the correspondence f has graph an analytic subvariety \mathcal{S} of $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Such a subvariety is algebraic, by Chow's Theorem [6, 11], and therefore defined by a polynomial relation $p(z, w) = 0$, quadratic in each of z and w since f is a $2 : 2$ correspondence. This completes the proof of Theorem 1. Moreover since the projection $(z, w) \rightarrow z$ of \mathcal{S} to $\hat{\mathbb{C}}$ is a double cover with one double point, over the fixed point P of ρ , and two branch points, over T and the critical value of q , it follows by a calculation of Euler characteristic that \mathcal{S} is of genus zero and hence, from the analysis in [3], that following a change in

FIGURE 4. Pre-images of the annuli A and A' , and cut lines C .

variable the relation $p(z, w) = 0$ can be put in the form

$$(1) \quad \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k$$

for some value of the (complex) parameters a and k . When the correspondence is taken in this form, the (time-reversing) involution j mapping the complementary subsets U and U' of the complex plane bijectively to one another is $z \rightarrow -z$ (as in Figure 4).

In Figure 5 we display a computer plot of orbits of a correspondence f in the family (1), with the values of the parameters a and k chosen such that the correspondence is one of the matings described in the theorem: indeed in this example the quadratic map is $z \rightarrow z^2$. The figure illustrates the grand orbits of the curves $n, l, m, \sigma m, \chi l, \rho n$ which make up the boundary of Δ in Figure 2, plotted to a certain depth, and a single grand orbit on $\partial K \cup \partial K'$, plotted to a greater depth.

In [3] it was observed that *all* quadratic maps with connected Julia sets could be realised in the family of correspondences (1). The advantage of the present analysis is that the surgery approach shows that matings of *all* quadratic polynomials having connected Julia sets with *all* faithful discrete representations of $C_2 * C_3$ having connected regular sets are realised in this family. We remark that computer experiment suggests we can go further: densely in the boundary of the space of representations of $C_2 * C_3$ with connected regular set Ω lie the circle-packing representations, each still discrete and faithful but now having Ω a disjoint union of (round) discs. Each such representation is obtained by contracting an appropriate closed geodesic on the orbifold Ω/G to a point, and is characterised by a (rational) rotation number ν specifying the geodesic. Computer experiment strongly suggests that within the family (1) we can find a mating of each of these circle-packing representations with any quadratic map $z \rightarrow z^2 + c$ such that c does not lie in the $(1 - \nu)$ -limb of the Mandelbrot set, the latter being impossible for elementary combinatorial reasons. This topic will be explored elsewhere.

Analogous constructions can be made mating representations of $C_p * C_q$ with polynomial maps of degree $(p-1)(q-1)$ for arbitrary p and q . See [5] for a related method which applies a generalisation of Klein's Combination Theorem.

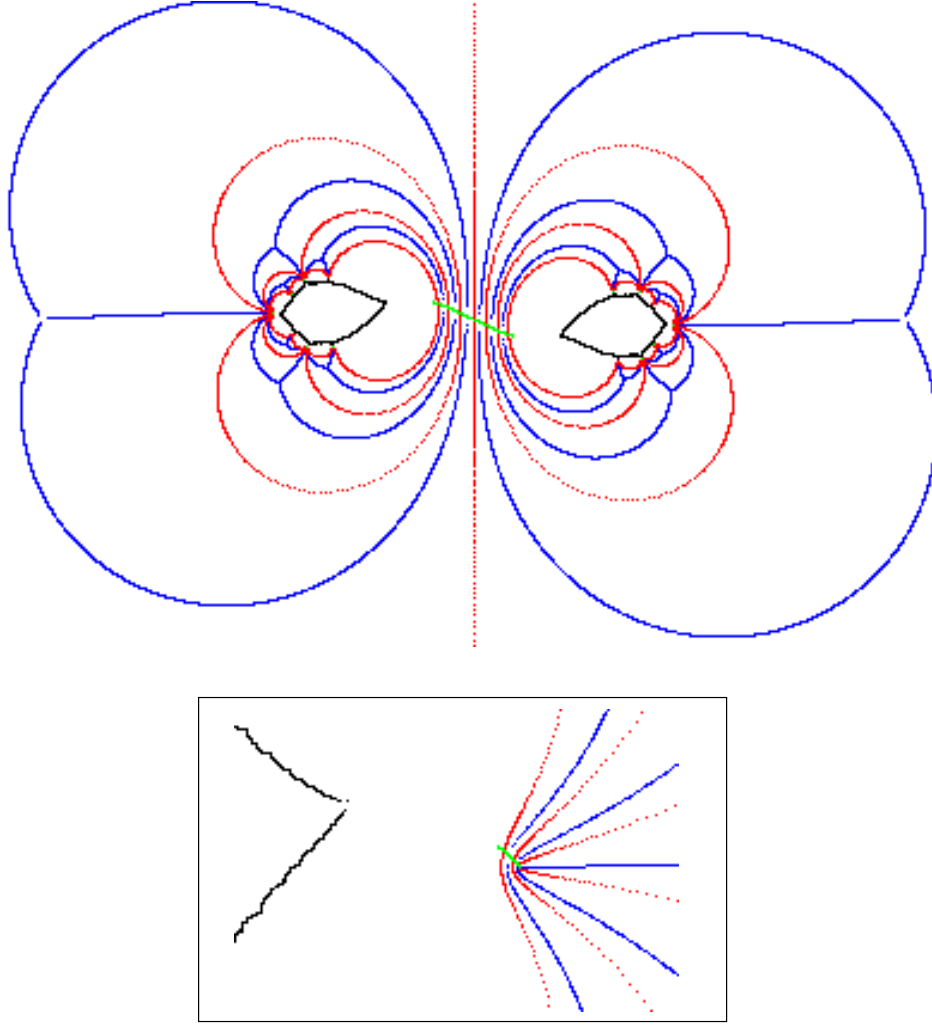


FIGURE 5. Orbits of correspondence (1) (and zoom around cusp on right), when $a = 4.38 + 0.09i$ and $k = 0.91 + 0.04i$.

5. THE CASE WHEN THE QUADRATIC MAP HAS DISCONNECTED JULIA SET

In the case considered so far, where q has a *connected* Julia set, the construction of the mating is independent of the choice of equipotential made in order to define the domain U (Section 1.1). When the Julia set is not connected, the critical value c of q lies in the basin of attraction of ∞ and the choice of equipotential used to define U becomes significant, since the number n such that $q^n(c) \in A = U - V$ is a topological invariant of the correspondence constructed. Thus when c lies outside the Mandelbrot set, our initial data need to include not just the quadratic map $z \rightarrow z^2 + c$ but also a choice of equipotential, which should lie *outside* the point c so that both U and V are simply-connected and $A = U - V$ is an annulus. We can now construct both a $2 : 2$ correspondence f and a complex structure respected by it, just as we did in the case of connected Julia sets. This correspondence is no

longer a mating of the quadratic map q with the representation r in the strict sense of the definition we gave earlier, since the presence in Ω of the critical value c and its pre-images prevent us from obtaining a conjugacy to an action of $C_2 * C_3$ in the way we did before. Nevertheless there is still a conformal homeomorphism between the orbit space Ω/f of the correspondence and that of the group $G = \langle \sigma, \rho, \chi \rangle$ on its regular set $\Omega(G)$, so it is clear how to recover the representation r of G from the correspondence. We also remark that when the representation r is deformed to one lying on the boundary of moduli space, by contracting an appropriate geodesic on the orbit space to a point, certain restrictions come into play as to what positions are allowed for the critical value c . It seems likely that the effect is to exclude matings of the circle-packing representation of $C_2 * C_3$ having rotation number ν with quadratic maps $z \rightarrow z^2 + c$ having c lying in the $(1 - \nu)$ -wake of \mathcal{M} . This question, like that towards the end of the previous section, will be further explored elsewhere.

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