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GENERATORS AND RELATIONS FOR SCHUR ALGEBRAS

STEPHEN DOTY AND ANTHONY GIAQUINTO

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ABSTRACT. We obtain a presentation of Schur algebras (and q-Schur algebras) by generators and relations, one which is compatible with the usual presentation of the enveloping algebra (quantized enveloping algebra) corresponding to the Lie algebra $\mathfrak{gl}_n$ of $n \times n$ matrices. We also find several new bases of Schur algebras.

1. Introduction

The classical Schur algebra $S(n,d)$ (over $\mathbb{Q}$) may be defined as the algebra $\mathrm{End}_{\Sigma_d}(V^{\otimes d})$ of linear endomorphisms on the dth tensor power of an n-dimensional $\mathbb{Q}$-vector space V commuting with the action of the symmetric group Σ_d, acting by permutation of the tensor places (see [6]). Schur algebras determine the polynomial representation theory of general linear groups, and they form an important class of quasi-hereditary algebras.

We identify V with $\mathbb{Q}^n$. Then $V_{\mathbb{Z}} = \mathbb{Z}^n$ is a lattice in V. One can define a $\mathbb{Z}$-order $S_{\mathbb{Z}}(n,d)$ (the integral Schur algebra) in $S(n,d)$ by setting $S_{\mathbb{Z}}(n,d) = \mathrm{End}_{\Sigma_d}(V_{\mathbb{Z}}^{\otimes d})$. For any field K, one then obtains the Schur algebra $S_K(n,d)$ over K by setting $S_K(n,d) = S_{\mathbb{Z}}(n,d) \otimes_{\mathbb{Z}} K$. Moreover, $S_{\mathbb{Q}}(n,d) \cong S(n,d)$.

In the quantum case one can replace $\mathbb{Q}$ by $\mathbb{Q}(v)$ (v an indeterminate), V by an n-dimensional $\mathbb{Q}(v)$-vector space, and Σ_d by the corresponding Hecke algebra $\mathbf{H}(\Sigma_d)$. Then the resulting commuting algebra, $\mathbf{S}(n,d)$, is known as the q-Schur algebra, or quantum Schur algebra. It appeared first in work of Dipper and James, and, independently, Jimbo. Beilinson, Lusztig, and MacPherson [1] have given a geometric construction of $\mathbf{S}(n,d)$ in terms of orbits of flags in vector spaces. (See also [5].)

In this situation $\mathbb{Z}$ is replaced by the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and there is a corresponding "integral" form $\mathbf{S}_{\mathcal{A}}(n,d)$ in $\mathbf{S}(n,d)$. The above quantized objects specialize to their classical versions when $v = 1$. We have more detailed information about the Schur algebras in rank 1; see [2] and [3]. Proofs of the main results will appear in [4].

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2. Serre's presentation of U

Let $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$ be the root system of type A_{n-1}, with simple roots $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ denotes the standard basis of $\mathbb{Z}^n$. The corresponding set Φ^+ of positive roots is the set $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$.

The abelian group $\mathbb{Z}^n$ has a bilinear form (,) given by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ (Kronecker delta). We write $\alpha_j := \varepsilon_j - \varepsilon_{j+1}$.

The enveloping algebra $U = U(\mathfrak{gl}_n)$ is the associative algebra (with 1) on generators e_i, f_i $(i = 1, \dots, n-1)$ and H_i $(i = 1, \dots, n)$ with relations

$$H_i H_j = H_j H_i, \tag{R1}$$

$$e_i f_j - f_j e_i = \delta_{ij}(H_j - H_{j+1}), \tag{R2}$$

$$H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j) e_j, \quad H_i f_j - f_j H_i = -(\varepsilon_i, \alpha_j) f_j, \tag{R3}$$

$$\begin{aligned} &e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0 \quad (|i-j| = 1), \\ &e_i e_j - e_j e_i = 0 \quad \text{(otherwise)}, \end{aligned} \tag{R4}$$

$$\begin{aligned} &f_i^2 f_j - 2 f_i f_j f_i + f_j f_i^2 = 0 \quad (|i-j| = 1), \\ &f_i f_j - f_j f_i = 0 \quad \text{(otherwise)}. \end{aligned} \tag{R5}$$

For $\alpha \in \Phi$, let x_α denote the corresponding root vector of $\mathfrak{gl}_n$ viewed as an element of U. We have in particular $e_i = x_{\alpha_i}$ and $f_i = x_{-\alpha_i}$.

3. The quantized enveloping algebra

The Drinfeld-Jimbo quantized enveloping algebra $\mathbf{U} = \mathbf{U}_v(\mathfrak{gl}_n)$, by definition, is the $\mathbb{Q}(v)$-algebra with generators E_i, F_i $(1 \le i \le n-1)$, $K_i^{\pm 1}$ $(1 \le i \le n)$ and relations

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{Q1}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \tag{Q2}$$

$$K_i E_j = v^{(\varepsilon_i, \alpha_j)} E_j K_i, \qquad K_i F_j = v^{-(\varepsilon_i, \alpha_j)} F_j K_i, \tag{Q3}$$

$$\begin{aligned} &E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad (|i-j| = 1), \\ &E_i E_j - E_i E_j = 0 \quad \text{(otherwise)}, \end{aligned} \tag{Q4}$$

$$\begin{aligned} &F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad (|i-j| = 1), \\ &F_i F_j - F_j F_i = 0 \quad \text{(otherwise)}. \end{aligned} \tag{Q5}$$

For $\alpha \in \Phi^+$, let E_α and F_α be the positive and negative quantum root vectors of $\mathbf{U}$ as defined by Jimbo [8]. We have in particular $E_i = E_{\alpha_i}$ and $F_i = F_{\alpha_i}$.

4. Main results: classical case

We now give a precise statement of our main results in the classical case. The first result describes a presentation by generators and relations of the Schur algebra over the rational field $\mathbb{Q}$. This presentation is compatible with the usual presentation (see section 2) of $U = U(\mathfrak{gl}_n)$.

Theorem 1. *Over $\mathbb{Q}$, the Schur algebra $S(n,d)$ is isomorphic to the associative algebra (with 1) on the same generators as for U subject to the same relations* (R1)–(R5) *as for U, together with the additional relations:*

(R6) $$H_1 + H_2 + \cdots + H_n = d,$$

(R7) $$H_k(H_k - 1)\cdots(H_k - d) = 0 \qquad (k = 1, \ldots, n).$$

The next result gives a basis for the Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) basis of U.

Theorem 2. *The algebra $S(n,d)$ has a "truncated PBW" basis (over $\mathbb{Q}$) which can be described as follows. Fix an integer k_0 with $1 \le k_0 \le n$ and set*

$$G = \{x_\alpha \mid \alpha \in \Phi\} \cup \{H_k \mid k \in \{1, \ldots, n\} - \{k_0\}\}$$

and fix some ordering for this set. Then the set of all monomials in G (with specified order) of total degree at most d is a basis for $S(n,d)$.

Our next result constructs the integral Schur algebra $S_{\mathbb{Z}}(n,d)$ in terms of the generators given above. We need some more notation. For B in $\mathbb{N}^n$, we write

$$H_B = \prod_{k=1}^{n} \binom{H_k}{b_k}.$$

Let $\Lambda(n,d)$ be the subset of $\mathbb{N}^n$ consisting of those $\lambda \in \mathbb{N}^n$ satisfying $|\lambda| = d$; this is the set of n-part compositions of d. Given $\lambda \in \Lambda(n,d)$ we set $1_\lambda := H_\lambda$. One can show that the collection $\{1_\lambda\}$ as λ varies over $\Lambda(n,d)$ forms a set of pairwise orthogonal idempotents in $S_{\mathbb{Z}}(n,d)$ which sum to the identity element.

For $m \in \mathbb{N}$ and $\alpha \in \Phi$, set $x_\alpha^{(m)} := x_\alpha^m/(m!)$. Any product in U of elements of the form

$$x_\alpha^{(m)}, \quad \binom{H_k}{m} \qquad (m \in \mathbb{N}, \alpha \in \Phi, k \in \{1, \ldots, n\}),$$

taken in any order, will be called a *Kostant monomial*. Note that the set of Kostant monomials is multiplicatively closed, and spans $U_{\mathbb{Z}}$. We define a function χ (content function) on Kostant monomials by setting

$$\chi(x_\alpha^{(m)}) := m\,\varepsilon_{\max(i,j)}, \quad \chi(\binom{H_k}{m}) := 0,$$

where $\alpha = \varepsilon_i - \varepsilon_j$ $(i \ne j)$, and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever X, Y are Kostant monomials.

We write any $A \in \mathbb{N}^{\Phi^+}$ in "multi-index" form $A = (a_\alpha)_{\alpha \in \Phi^+}$ and set $|A| := \sum_{\alpha \in \Phi^+} a_\alpha$. For $A, C \in \mathbb{N}^{\Phi^+}$ we write

$$e_A = \prod_{\alpha \in \Phi^+} x_\alpha^{(a_\alpha)}, \quad f_C = \prod_{\alpha \in \Phi^-} x_\alpha^{(c_\alpha)}$$

where the products in e_A and f_C are taken relative to any fixed orders on Φ^+ and Φ^-.

Theorem 3. *The integral Schur algebra $S_{\mathbb{Z}}(n,d)$ is the subring of $S(n,d)$ generated by all divided powers $e_j^{(m)}$, $f_j^{(m)}$ $(j \in \{1, \ldots, n-1\}, m \in \mathbb{N})$. Moreover, this algebra has a $\mathbb{Z}$-basis consisting of all*

(B1) $$e_A 1_\lambda f_C \qquad (\chi(e_A f_C) \preceq \lambda)$$

and another such basis consisting of all

$$f_A 1_\lambda e_C \qquad (\chi(f_A e_C) \preceq \lambda) \tag{B2}$$

where $A, C \in \mathbb{N}^{\Phi^+}$, $\lambda \in \Lambda(n,d)$, *and where* $\preceq$ *denotes the* componentwise *partial ordering on* $\mathbb{N}^n$.

Finally, we have another presentation of the Schur algebra by generators and relations. This presentation has the advantage that it possesses a quantization of the same form, in which we can set $v = 1$ to recover the classical version.

Theorem 4. *For each* $i \in \{1, \dots, n-1\}$ *write* $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. *The algebra* $S(n,d)$ *is the associative algebra (with* 1*) given by generators* 1_λ *(*$\lambda \in \Lambda(n,d)$*),* e_i, f_i *(*$i \in \{1, \dots, n-1\}$*) subject to the relations*

$$\begin{gathered} 1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda \quad (\lambda, \mu \in \Lambda(n,d)), \\ \sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1, \end{gathered} \tag{S1}$$

$$\begin{aligned} e_i 1_\lambda &= \begin{cases} 1_{\lambda+\alpha_i} e_i & \textit{if } \lambda + \alpha_i \in \Lambda(n,d), \\ 0 & \textit{otherwise}, \end{cases} \\ f_i 1_\lambda &= \begin{cases} 1_{\lambda-\alpha_i} f_i & \textit{if } \lambda - \alpha_i \in \Lambda(n,d), \\ 0 & \textit{otherwise}, \end{cases} \\ 1_\lambda e_i &= \begin{cases} e_i 1_{\lambda-\alpha_i} & \textit{if } \lambda - \alpha_i \in \Lambda(n,d), \\ 0 & \textit{otherwise}, \end{cases} \\ 1_\lambda f_i &= \begin{cases} f_i 1_{\lambda+\alpha_i} & \textit{if } \lambda + \alpha_i \in \Lambda(n,d), \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{S2}$$

$$e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} (\lambda_j - \lambda_{j+1}) 1_\lambda, \tag{S3}$$

along with the Serre relations (R4), (R5), *for* $i, j \in \{1, \dots, n-1\}$.

5. Main results: quantized case

Our main results in the quantized case are similar in form to those in the classical case. The first result describes a presentation by generators and relations of the quantized Schur algebra over the rational function field $\mathbb{Q}(v)$. This presentation is compatible with the usual presentation (see section 3) of $\mathbf{U} = \mathbf{U}_v(\mathfrak{gl}_n)$.

Theorem 1′. *Over* $\mathbb{Q}(v)$, *the* q*-Schur algebra* $\mathbf{S}(n,d)$ *is isomorphic with the associative algebra (with* 1*) on the same generators as for* $\mathbf{U}$ *subject to the same relations* (Q1)–(Q5) *as for* $\mathbf{U}$, *together with the additional relations:*

$$K_1 K_2 \cdots K_n = v^d, \tag{Q6}$$

$$(K_j - 1)(K_j - v)(K_j - v^2) \cdots (K_j - v^d) = 0 \qquad (j = 1, \dots, n). \tag{Q7}$$

The next result gives a basis for the q-Schur algebra which is the analogue of the Poincare-Birkhoff-Witt (PBW) type basis of $\mathbf{U}$, given in Lusztig [9, Prop. 1.13].

Theorem 2′. *The algebra* $\mathbf{S}(n,d)$ *has a "truncated PBW type" basis which can be described as follows. Fix an integer* k_0 *with* $1 \le k_0 \le n$ *and set*

$$G' = \{E_\alpha, F_\alpha \mid \alpha \in \Phi^+\} \cup \{K_k \mid k \in \{1,\dots,n\} - \{k_0\}\}$$

and fix some ordering for this set. Then the set of all monomials in G' *(with specified order) of total degree at most* d *is a basis for* $\mathbf{S}(n,d)$.

Note that setting $v = 1$ in the basis of Theorem 2′ does not yield the basis of Theorem 2 since K_i acts as the identity when $v = 1$.

Our next result constructs the integral q-Schur algebra $\mathbf{S}_\mathcal{A}(n,d)$ in terms of the generators given above. For B in $\mathbb{N}^n$, we write

$$K_B = \prod_{j=1}^{n} \begin{bmatrix} K_j \\ b_j \end{bmatrix},$$

where for indeterminates X, X^{-1} satisfying $XX^{-1} = X^{-1}X = 1$ and any $t \in \mathbb{N}$ we formally set

$$\begin{bmatrix} X \\ t \end{bmatrix} := \prod_{s=1}^{t} \frac{Xv^{-s+1} - X^{-1}v^{s-1}}{v^s - v^{-s}},$$

an expression that makes sense if X is replaced by any invertible element of a $\mathbb{Q}(v)$-algebra.

Given $\lambda \in \Lambda(n,d)$ we set (when we are in the quantum case) $1_\lambda := K_\lambda$. It turns out that, just as in the classical case, the collection $\{1_\lambda\}$ as λ varies over $\Lambda(n,d)$ forms a set of pairwise orthogonal idempotents in $\mathbf{S}_\mathcal{A}(n,d)$ which sum to the identity element.

Let $[m]$ denote the quantum integer $[m] := (v^m - v^{-m})/(v - v^{-1})$ and set $[m]! := [m][m-1]\cdots[1]$. Then the q-analogues of the divided powers of root vectors are defined to be $E_\alpha^{(m)} := E_\alpha/[m]!$ and $F_\alpha^{(m)} := F_\alpha/[m]!$. Any product in $\mathbf{U}$ of elements of the form

$$E_\alpha^{(m)}, \quad F_\alpha^{(m)}, \quad K_j^{\pm 1}, \quad \begin{bmatrix} K_j \\ m \end{bmatrix} \qquad (m \in \mathbb{N},\ \alpha \in \Phi,\ j \in \{1,\dots,n\}),$$

taken in any order, will be called a *Kostant monomial*. As before, the set of Kostant monomials is multiplicatively closed, and spans $\mathbf{U}_\mathcal{A}$. The definition of content χ of a Kostant monomial is obtained similarly, by setting

$$\chi(E_\alpha^{(m)}) = \chi(F_\alpha^{(m)}) := m\,\varepsilon_{\max(i,j)}, \quad \chi(K_l^{\pm 1}) = \chi(\begin{bmatrix} K_l \\ m \end{bmatrix}) := 0$$

where $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$, and by declaring that $\chi(XY) = \chi(X) + \chi(Y)$ whenever X, Y are Kostant monomials. For $A, C \in \mathbb{N}^{\Phi^+}$ we write

$$E_A = \prod_{\alpha \in \Phi^+} E_\alpha^{(a_\alpha)}, \quad F_C = \prod_{\alpha \in \Phi^+} F_\alpha^{(c_\alpha)}$$

where the products in E_A and F_C are taken relative to any two specified orderings on Φ^+.

Theorem 3′. *The integral q-Schur algebra* $\mathbf{S}_\mathcal{A}(n,d)$ *is the subring of* $\mathbf{S}(n,d)$ *generated by all quantum divided powers* $E_j^{(m)}$, $F_j^{(m)}$ $(j \in \{1, \dots, n-1\}, m \in \mathbb{N})$*, along* *with the elements* $\begin{bmatrix} K_j \\ m \end{bmatrix}$ *(*$j \in \{1, \dots, n\}$*,* $m \in \mathbb{N}$*). Moreover, this algebra has a basis over* $\mathcal{A}$ *consisting of all*

$$E_A 1_\lambda F_C \qquad (\chi(e_A f_C) \preceq \lambda) \tag{B1$'$}$$

and another such basis consisting of all

$$F_A 1_\lambda E_C \qquad (\chi(f_A e_C) \preceq \lambda) \tag{B2$'$}$$

where $A, C \in \mathbb{N}^{\Phi^+}$*,* $\lambda \in \Lambda(n, d)$*, and where* $\preceq$ *denotes the* componentwise *partial ordering on* $\mathbb{N}^n$*.*

Unlike the truncated PBW basis, the bases of Theorem 3′ do specialize when $v = 1$ to their classical analogues given in Theorem 3.

Finally, we have another presentation of the q-Schur algebra by generators and relations. This presentation has the advantage that by setting $v = 1$, we recover the classical version given in Theorem 4. The relations of the presentation are similar to relations that hold for Lusztig's modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$. (See [9, Chap. 23].)

Theorem 4′. *For each* $i \in \{1, \dots, n-1\}$ *write* $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$*. The algebra* $\mathbf{S}(n, d)$ *is the associative algebra (with 1) given by generators* 1_λ *(*$\lambda \in \Lambda(n, d)$*),* E_i*,* F_i *(*$i \in \{1, \dots, n-1\}$*) subject to the relations*

$$\begin{gathered} 1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda \quad (\lambda, \mu \in \Lambda(n, d)), \\ \sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1, \end{gathered} \tag{S1$'$}$$

$$\begin{aligned} E_i 1_\lambda &= \begin{cases} 1_{\lambda+\alpha_i} E_i & \textit{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \textit{otherwise}, \end{cases} \\ F_i 1_\lambda &= \begin{cases} 1_{\lambda-\alpha_i} F_i & \textit{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \textit{otherwise}, \end{cases} \\ 1_\lambda E_i &= \begin{cases} E_i 1_{\lambda-\alpha_i} & \textit{if } \lambda - \alpha_i \in \Lambda(n, d), \\ 0 & \textit{otherwise}, \end{cases} \\ 1_\lambda F_i &= \begin{cases} F_i 1_{\lambda+\alpha_i} & \textit{if } \lambda + \alpha_i \in \Lambda(n, d), \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{S2$'$}$$

$$E_i F_j - F_j E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} [\lambda_j - \lambda_{j+1}] 1_\lambda, \tag{S3$'$}$$

along with the q*-Serre relations* (Q4)*,* (Q5)*, for* $i, j \in \{1, \dots, n-1\}$*.*

6. Other results

6.1. **Triangular decomposition.** One can define the plus, minus, and zero parts of Schur algebras in terms of the generators, as follows. (These subalgebras have been studied before.) The plus part $S^+(n, d)$ (resp., the minus part $S^-(n, d)$) is the subalgebra of $S(n, d)$ generated by all x_α, $\alpha \in \Phi^+$ (resp., $\alpha \in \Phi^-$). The zero part $S^0(n, d)$ is the subalgebra generated by all H_j, $j = 1, \dots, n$. We also have the Borel Schur algebras $S^{\geqslant 0}(n, d)$ (resp., $S^{\leqslant 0}(n, d)$), the subalgebra generated by $S^+(n, d)$ (resp., $S^-(n, d)$) together with $S^0(n, d)$.

The appellations $S^+_{\mathbb{Z}}(n,d)$, $S^-_{\mathbb{Z}}(n,d)$, $S^0_{\mathbb{Z}}(n,d)$, $S^{\geqslant 0}_{\mathbb{Z}}(n,d)$, $S^{\leqslant 0}_{\mathbb{Z}}(n,d)$ will denote the intersection of the appropriate algebra from above with the integral form $S_{\mathbb{Z}}(n,d)$.

The algebra $S = S(n,d)$ has a triangular decomposition $S = S^+S^0S^-$. We show that in this decomposition one can permute the three factors in any order. Moreover, the same result holds over $\mathbb{Z}$.

The zero part $S^0_{\mathbb{Z}}(n,d)$ is the algebra generated by all $\binom{H_j}{m}$ for $j = 1,\dots,n$ and $m \in \mathbb{N}$. We give in [4] a presentation of $S^0(n,d)$ by generators and relations. In particular, we prove that $H_B = 0$ whenever $|B| > d$ ($B \in \mathbb{N}^n$) and that the set of (pairwise orthogonal) idempotents 1_λ, $\lambda \in \Lambda(n,d)$, is a $\mathbb{Z}$-basis of $S^0_{\mathbb{Z}}(n,d)$.

$S^+_{\mathbb{Z}}(n,d)$ (resp., $S^-_{\mathbb{Z}}(n,d)$) is the algebra generated by all divided powers $x_\alpha^{(m)}$ for $\alpha \in \Phi^+$ (resp., $\alpha \in \Phi^-$) and $m \in \mathbb{N}$. It is an easy consequence of the commutation formulas (S2) that each generator x_α is nilpotent of index $d+1$; see [4] for details. Moreover, from our main results we see easily that the set of all e_A (resp., f_A) such that $|A| \le d$ is a $\mathbb{Z}$-basis for the algebra $S^+_{\mathbb{Z}}(n,d)$ (resp., $S^-_{\mathbb{Z}}(n,d)$).

It also follows immediately from our results that the set of all $e_A 1_\lambda$ (resp., $1_\lambda f_A$) satisfying $\chi(e_A) \preceq \lambda$ is an integral basis for the Borel Schur algebra $S^{\geqslant 0}_{\mathbb{Z}}(n,d)$ (resp., $S^{\leqslant 0}_{\mathbb{Z}}(n,d)$).

Similar statements to the above hold in the quantum case. In particular, as a corollary of the commutation formulas (S2′) one can give a simple proof of [7, Prop. 2.3].

6.2. **Explicit reduction formulas.** Fix a positive root α and write $\alpha = \varepsilon_i - \varepsilon_j$ for $i < j$.

Then from [2] we have the following reduction formulas in $S_{\mathbb{Z}}(n,d)$, for any $a,b,c \in \mathbb{N}$:

$$f_\alpha^{(a)} \binom{H_j}{b} e_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f_\alpha^{(a-k)} \binom{H_j}{b+k} e_\alpha^{(c-k)}, \tag{1}$$

$$e_\alpha^{(a)} \binom{H_i}{b} f_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} e_\alpha^{(a-k)} \binom{H_i}{b+k} f_\alpha^{(c-k)}, \tag{2}$$

where $s = a+b+c-d$ and $s \ge 1$.

We do not have a q-analogue of these formulas.

The results of [3] give another type of reduction formula for $\mathbf{S}_{\mathcal{A}}(n,d)$. If $b_1, b_2 \in \mathbb{N}$ satisfy $b_1 + b_2 = d$, set $\lambda := b_1\varepsilon_i + b_2\varepsilon_j \in \Lambda(n,d)$. Then for all $s \ge 1$ we have:

$$E_\alpha^{(a)} 1_\lambda F_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} b_1+k \\ k \end{bmatrix} E_\alpha^{(a-k)} 1_{\lambda+k\alpha} F_\alpha^{(c-k)}, \tag{3}$$

$$F_\alpha^{(a)} 1_\lambda E_\alpha^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \begin{bmatrix} k-1 \\ s-1 \end{bmatrix} \begin{bmatrix} b_2+k \\ k \end{bmatrix} F_\alpha^{(a-k)} 1_{\lambda-k\alpha} E_\alpha^{(c-k)}, \tag{4}$$

where $s = a+b_1+c-d$ in (3) and $s = a+b_2+c-d$ in (4).

The classical analogues of formulas (3) and (4) hold in $S(n,d)$.

6.3. **Another presentation (for $n = 2$).** We have the following result from [2], which presents $S(2,d)$ as a quotient of $U(\mathfrak{sl}_2)$.

Theorem 5. *Over $\mathbb{Q}$, the Schur algebra $S(2,d)$ is isomorphic with the associative algebra (with 1) generated by e, f, h subject to the relations:*

$$he - eh = 2e; \qquad ef - fe = h; \qquad hf - fh = -2f;$$
$$(h+d)(h+d-2)\cdots(h-d+2)(h-d) = 0.$$

Moreover, this algebra has a "truncated PBW" basis over $\mathbb{Q}$ consisting of all $f^a h^b e^c$ such that $a+b+c \le d$.

Note that if we eliminate the last relation, we have the usual presentation of $U(\mathfrak{sl}_2)$ over $\mathbb{Q}$. The last relation is the minimal polynomial of h in the representation on tensor space. The problem of presenting $S(n,d)$ as a quotient of $U(\mathfrak{sl}_n)$ seems to be more difficult for $n > 2$.

We also have from [3] the following q-version of the above, which presents the q-Schur algebra $\mathbf{S}(2,d)$ as a quotient of the quantized enveloping algebra $\mathbf{U}(\mathfrak{sl}_2)$.

Theorem 5′. *Over $\mathbb{Q}(v)$, the quantum Schur algebra $\mathbf{S}(2,d)$ is isomorphic to the algebra generated by E, F, $K^{\pm 1}$ subject to the relations:*

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = v^2 E, \qquad KFK^{-1} = v^{-2}F,$$
$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$
$$(K - v^d)(K - v^{d-2})\cdots(K - v^{-d+2})(K - v^{-d}) = 0.$$

6.4. **Hecke algebras.** Suppose that $n \ge d$. Let $\omega = (1^d)$. Then the subalgebra $1_\omega \mathbf{S}(n,d) 1_\omega$ is isomorphic with the Hecke algebra $\mathbf{H}(\Sigma_d)$. The nonzero elements of the basis (B1′) of the form $1_\omega E_A F_C 1_\omega$ form a basis of the Hecke algebra; similarly for elements of the basis (B2′) of the form $1_\omega F_A E_C 1_\omega$.

Moreover, taking $d = n$, we can see that $\mathbf{H} = \mathbf{H}(\Sigma_n)$ is generated by the elements $1_\omega E_i F_i 1_\omega$ $(1 \le i \le n-1)$. Alternatively, $\mathbf{H}$ is generated by the $1_\omega F_i E_i 1_\omega$ $(1 \le i \le n-1)$. One can easily describe the relations on these generators, thus obtaining a presentation of $\mathbf{H}$ which is closely related to one in [10].

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Department of Mathematics, Loyola University, Chicago, IL 60626
E-mail address: doty@math.luc.edu

Department of Mathematics, Loyola University, Chicago, IL 60626
E-mail address: tonyg@math.luc.edu