

ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS  
OF THE DIRICHLET PROBLEM IN HALF-SPACE  
FOR LINEAR AND QUASI-LINEAR ELLIPTIC EQUATIONS

VASILY DENISOV AND ANDREY MURAVNIK

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ABSTRACT. We study the Dirichlet problem in half-space for the equation  $\Delta u + g(u)|\nabla u|^2 = 0$ , where  $g$  is continuous or has a power singularity (in the latter case positive solutions are considered). The results presented give necessary and sufficient conditions for the existence of (pointwise or uniform) limit of the solution as  $y \rightarrow \infty$ , where  $y$  denotes the spatial variable, orthogonal to the hyperplane of boundary-value data. These conditions are given in terms of integral means of the boundary-value function.

INTRODUCTION

The phenomenon called *stabilization* is well known for *parabolic* equations both in linear (see e.g. [1] and references therein) and non-linear (see e.g. [2] and references therein) cases; it means the existence of a finite limit of the solution as  $t \rightarrow \infty$ . However, there are well-posed non-isotropic *elliptic* boundary-value problems in unbounded domains (see e.g. [3]) for which we can talk about stabilization in the following sense: the solution has a finite limit as a selected *spatial* variable tends to infinity.

This paper is devoted to the Dirichlet problem in half-space for elliptic equations. We present necessary and sufficient conditions for the stabilization of its solution; here the spatial variable, orthogonal to the hyperplane of boundary-value data, plays the role of time. In Section 1, the linear case is presented; Sections 2 and 3 are devoted to quasi-linear equations with the so-called Burgers-Kardar-Parisi-Zhang non-linearity type (see e.g. [4], [5]). Equations with such non-linearities arise, for example, in modeling of directed polymers and interface growth. They also present an independent theoretical interest because they contain second powers of the first derivatives (see e.g. [6] and references therein).

Note that we deal with the stabilization problem in cylindrical domains with an *unbounded* base (in particular, here the base of the cylinder is the whole  $E^N$ ). As in the parabolic case, this problem is principally different (this refers both to the results and to the methods of research) from the stabilization problem in cylindrical domains with a *bounded* base. The latter problem has been investigated

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for a rather broad class of non-linear elliptic equations by means of methods of dynamical systems (see e.g. [7] and references therein).

### 1. LINEAR EQUATIONS

In the half-space  $\{y \geq 0\}$  we consider the Dirichlet problem

$$(1) \quad p(x) \frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x, y) \frac{\partial u}{\partial x_j} \right) = 0, \quad (x, y) \in \{y > 0\},$$

$$(2) \quad u(x, 0) = \varphi(x), \quad x \in E^N,$$

where the coefficients of equation (1) are measurable and bounded and the uniform ellipticity condition is satisfied, i.e., there exists a positive constant  $\lambda$  such that for any  $x, y \in \{y > 0\}$ ,

$$(3) \quad \lambda^{-1} \leq p(x) \leq \lambda, \quad \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^N a_{ij}(x, y) \xi_i \xi_j \leq \lambda |\xi|^2;$$

the boundary-value function  $\varphi(x)$  is supposed to be continuous and bounded. We say that  $u(x, y)$  is a solution of (1) if  $u$  is bounded on  $\{y > 0\}$ , belongs to  $W_2^1$  in any strictly interior subdomain of  $\{y \geq 0\}$ , and satisfies the integral identity

$$\int_0^{+\infty} \int_{E^N} \left[ \sum_{i,j=1}^N a_{ij}(x, y) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + p(x) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right] dx dy = 0$$

for any  $v(x, y) \in C_0^\infty\{y > 0\}$ .

It is known (see e.g. [3]) that, under the above assumptions, a solution  $u(x, y)$  of (1), (2) exists and is unique; it is bounded and continuous on  $\{y > 0\}$  and coincides at  $y = 0$  with a continuous boundary-value function  $\varphi(x)$  on  $E^N$ . Moreover, it follows from [8] and [9] that  $u(x, y)$  satisfies Hölder's condition with a constant  $\alpha > 0$  depending merely on  $\lambda$  and  $N$ .

We discuss the question of finding necessary and sufficient conditions for the existence of

$$(4) \quad \lim_{y \rightarrow +\infty} u(x, y) = l$$

for any  $x \in E^N$  (or uniformly with respect to  $x \in E^N$ ).

First of all we introduce a few definitions related to the means of multi-variable functions.

Let  $g(x)$  be bounded on  $E^N$ , let  $K$  be a compact subset of  $E^N$ , and let  $R$  be a large positive parameter. We consider the following family of functions:

$$\{g(Rx)\} = \{g(Rx_1, Rx_2, \dots, Rx_N)\}, \quad R > 0.$$

Suppose  $g(x)$  has a weak limit at infinity in  $L_{loc}^2$ , i.e.,

$$(5) \quad g(Rx) \rightharpoonup \bar{g}(x),$$

where the symbol  $\rightharpoonup$  denotes weak convergence in  $L^2(K)$ . It follows from the boundedness of  $g(x)$  that (5) is equivalent to the following two assertions:

$$(6) \quad \lim_{R \rightarrow \infty} \int_{E^N} g(Rx) \psi(x) dx = \int_{E^N} \bar{g}(x) \psi(x) dx \quad \text{for any } \psi \in C_0^\infty(E^N),$$

$$(7) \quad \lim_{R \rightarrow \infty} \int_K g(Rx) dx = \int_K \bar{g}(x) dx.$$

We say that  $g(x)$  has a mean value, if it has a weak limit (5) at infinity in  $L^2_{\text{loc}}$  and that limit is a constant. Relation (7) yields that the above definition of the mean value is equivalent to the following one:

$$(8) \quad \bar{g} = \lim_{R \rightarrow \infty} \frac{1}{|K_R|} \int_{K_R} g(y) dy,$$

where  $K_R \stackrel{\text{def}}{=} \{y \in E^N \mid y = Rx, x \in K\}$ ,  $|K_R|$  denotes the Lebesgue measure of  $K_R$ .

We say that  $g(x)$  has a uniform mean value  $\bar{g}$  on  $E^N$  if for any  $\psi \in C_0^\infty(E^N)$  there exists

$$(9) \quad \lim_{R \rightarrow \infty} \int_{E^N} g(\beta + Rx) \psi(x) dx = \bar{g} \int_{E^N} \psi(x) dx,$$

and this limit is uniform with respect to  $\beta \in E^N$ . The latter definition is equivalent to the following one:

$$(10) \quad \lim_{R \rightarrow \infty} \frac{1}{|K_R|} \int_{K_R} g(\beta + y) dy = \bar{g}$$

uniformly with respect to  $\beta \in E^N$ .

Let  $f(x, y) \stackrel{\text{def}}{=} f(x_1, \dots, x_N; y)$  be bounded on the half-space  $\{y \geq 0\}$ . We say that it satisfies condition  $A$  if there exists a constant  $\bar{f}$  such that

$$(11) \quad \lim_{R \rightarrow \infty} \int_{|x| \leq 1} \int_0^1 [f(Rx, Ry) - \bar{f}]^2 dy dx = 0.$$

Note that an  $A$ -type condition was introduced in [10] and [11]. It was developed further in [12].

Now we are ready to formulate our main results concerning linear equations.

**Theorem 1.** *Suppose  $p(x)$  has a mean value and the elliptic operator in (1) is the Laplacian. Then limit (4) exists at any point  $x \in E^N$  if and only if*

$$(12) \quad \lim_{R \rightarrow \infty} \frac{\int_{|x| \leq R} p(x) \varphi(x) dx}{\int_{|x| \leq R} p(x) dx} = l.$$

**Theorem 2.** *Assume  $p(x)$  has a mean value and the coefficients  $a_{ij}(x, y)$  satisfy condition  $A$  with a positive definite matrix  $\{\bar{a}_{ij}\}_{i,j=1}^n$ . Then limit (4) exists at any point  $x \in E^N$  if and only if*

$$(13) \quad \lim_{R \rightarrow \infty} \frac{\int_{(Bx, x) \leq R^2} \varphi(x) p(x) dx}{\int_{(Bx, x) \leq R^2} p(x) dx} = l,$$

where  $B$  denotes the inverse matrix of  $\{\bar{a}_{ij}\}_{i,j=1}^n$ .

**Theorem 3.** *Limit (4) exists uniformly with respect to  $x \in E^N$  if and only if*

$$(14) \quad \lim_{R \rightarrow \infty} \frac{\int_{|y| \leq R} p(x+y) \varphi(x+y) dy}{\int_{|y| \leq R} p(x+y) dy} = l$$

uniformly with respect to  $x \in E^N$ .

*Remark 1.* In the case of Cauchy problem for *parabolic* equations, assertions similar to Theorems 1–3 were proved in [10]–[12].

## 2. QUASI-LINEAR EQUATIONS WITH REGULAR COEFFICIENTS AT NON-LINEARITIES

Hereinafter the point  $x = (x_1, \dots, x_N, x_{N+1})$  is denoted by  $(x', x_{N+1})$ , and the half-space  $E^N \times (0, +\infty)$  is denoted by  $E_+^{N+1}$ .

In this section we consider the following problem:

$$(15) \quad \Delta u + g(u)|\nabla u|^2 = 0, \quad x \in E_+^{N+1};$$

$$(16) \quad u(x', 0) = \varphi(x'), \quad x' \in E^N;$$

where  $g$  is continuous in  $(-\infty, +\infty)$ , and  $\varphi$  is continuous and bounded in  $E^N$ .

Following e.g. [13], we introduce the function

$$(17) \quad f(s) \stackrel{\text{def}}{=} \int_0^s e^{\int_0^\tau g(\sigma) d\sigma} d\tau,$$

and prove the following assertions.

**Theorem 4.** *There exists a unique classical bounded solution of problem (15), (16).*

**Theorem 5.** *Let  $x' \in E^N$ ,  $l \in (-\infty, +\infty)$ , and let  $u(x)$  be the classical bounded solution of problem (15), (16). Then*

$$\lim_{x_{N+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} f[\varphi(y)] dy = f(l).$$

**Theorem 6.** *Let  $l \in (-\infty, +\infty)$ , and let  $u(x)$  be the classical bounded solution of problem (15), (16). Then*

$$u(x) \xrightarrow{x_{N+1} \rightarrow +\infty} l \text{ uniformly with respect to } x' \in E^N$$

*if and only if*

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} f[\varphi(x' + y)] dy \xrightarrow{R \rightarrow +\infty} f(l) \text{ uniformly with respect to } x' \in E^N.$$

*Remark 2.* The results of this section are valid also for the equation

$$\frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( a_{ij}(x, y) \frac{\partial u}{\partial x_j} \right) + g(u) \left[ \sum_{i,j=1}^N a_{ij}(x, y) \left( \frac{\partial u}{\partial x_j} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] = 0,$$

where, besides (3) and condition  $A$ , we have  $a_{ij} \in C^{1,q}(\overline{E_+^N})$  with a positive  $q$  ( $i, j = 1, \dots, N$ ).

## 3. THE CASE OF SINGULAR COEFFICIENTS AT NON-LINEARITIES

In this section, we study the equation

$$(18) \quad \Delta u + \alpha u^\beta |\nabla u|^2 = 0,$$

where  $\beta \in [-1, 0)$  and  $\alpha > -1$  for  $\beta = -1$ .

We will also assume that  $\varphi(x') \geq 0$  (apart from its continuity and boundedness), and consider positive solutions of (18), (16).

For  $\beta \in (-1, 0)$  we use (17) again, but for  $\beta = -1$  the function  $f(s)$  is defined as  $s^{\alpha+1}$ . Then the following assertions are valid.

**Theorem 7.** *If  $\varphi$  is different from the identical zero, then there exists a unique classical positive bounded solution of problem (18), (16).*

**Theorem 8.** *Let  $\beta = -1$  and let  $u(x)$  be the classical positive bounded solution of problem (18), (16),  $x' \in E^N$ ,  $l \geq 0$ . Then*

$$\lim_{x_{N+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} \varphi^{\alpha+1}(y) dy = l^{\alpha+1}.$$

**Theorem 9.** *Let  $\beta \in (-1, 0)$  and let  $u(x)$  be the classical positive bounded solution of problem (18), (16),  $x' \in E^N$ ,  $l \geq 0$ . Then*

$$\lim_{x_{N+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} \tilde{f}[\varphi(y)] dy = f(l),$$

where

$$(19) \quad \tilde{f}(s) = \int_0^s e^{\frac{\alpha}{1-\beta} \tau^{1-\beta}} d\tau.$$

**Theorem 10.** *Let  $\beta = -1$  and let  $u(x)$  be the classical positive bounded solution of problem (18), (16),  $l \geq 0$ . Then*

$$u(x) \xrightarrow{x_{N+1} \rightarrow +\infty} l \text{ uniformly with respect to } x' \in E^N$$

if and only if

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} \varphi^{\alpha+1}(x' + y) dy \xrightarrow{R \rightarrow +\infty} l^{\alpha+1} \text{ uniformly with respect to } x' \in E^N.$$

**Theorem 11.** *Let  $\beta \in (-1, 0)$  and let  $u(x)$  be the classical positive bounded solution of problem (18), (16),  $l \geq 0$ . Then*

$$u(x) \xrightarrow{x_{N+1} \rightarrow +\infty} l \text{ uniformly with respect to } x' \in E^N$$

if and only if

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}} R^N} \int_{|y| \leq R} \tilde{f}[\varphi(x' + y)] dy \xrightarrow{R \rightarrow +\infty} \tilde{f}(l) \text{ uniformly with respect to } x' \in E^N.$$

*Remark 3.* In the case of Cauchy problem for *parabolic* equations, assertions similar to Theorems 5–11 were proved in [2].

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MOSCOW STATE UNIVERSITY, FACULTY OF COMPUTATIONAL MATHEMATICS AND CYBERNETICS,  
 LENINSKIE GORY, MOSCOW 119899, RUSSIA  
*E-mail address*: V.Denisov@g23.relcom.ru

DEPARTMENT OF DIFFERENTIAL EQUATIONS, MOSCOW STATE AVIATION INSTITUTE, VOLOKO-  
 LAMSKOE SHOSSE 4, MOSCOW, A-80, GSP-3, 125993, RUSSIA  
*E-mail address*: abm@mailru.com