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THE SMALLEST HYPERBOLIC 6-MANIFOLDS

BRENT EVERITT, JOHN RATCLIFFE, AND STEVEN TSCHANTZ

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Abstract. By gluing together copies of an all right-angled Coxeter polytope a number of open hyperbolic 6-manifolds with Euler characteristic -1 are constructed. They are the first known examples of hyperbolic 6-manifolds having the smallest possible volume.

1. Introduction

The last few decades have seen a surge of activity in the study of finite volume hyperbolic manifolds—that is, complete Riemannian n-manifolds of constant sectional curvature -1. Not surprisingly for geometrical objects, volume has been, and continues to be, the most important invariant for understanding their sociology. The possible volumes in a fixed dimension form a well-ordered subset of $\mathbb{R}$, indeed a discrete subset except in 3 dimensions (where the orientable manifolds at least have ordinal type ω^ω). Thus it is a natural problem with a long history to construct examples of manifolds with minimum volume in a given dimension.

In 2 dimensions the solution is classical, with the minimum volume in the compact orientable case achieved by a genus 2 surface, and in the noncompact orientable case by a once-punctured torus or thrice-punctured sphere (the identities of the manifolds are of course also known in the nonorientable case). In 3 dimensions the compact orientable case remains an open problem with the Matveev-Fomenko-Weeks manifold [16, 30] obtained via $(5, -2)$-Dehn surgery on the sister of the figure-eight knot complement conjecturally the smallest. Amongst the noncompact orientable 3-manifolds the figure-eight knot complement realizes the minimum volume [17], and the Gieseking manifold (obtained by identifying the sides of a regular hyperbolic tetrahedron as in [14, 20]) does so for the nonorientable ones [1]. One could also add "arithmetic" to our list of adjectives and so have eight optimization problems to play with (so that the Matveev-Fomenko-Weeks manifold is known to be the minimum volume orientable, arithmetic compact 3-manifold; see [5]).

When $n \geq 4$ the picture is murkier, although in even dimensions we have recourse to the Gauss-Bonnet Theorem, so that in particular the minimum volume a $2m$-dimensional hyperbolic manifold could possibly have, is when the Euler characteristic χ satisfies $|\chi| = 1$. The first examples of noncompact 4-manifolds with

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$\chi = 1$ were constructed in [23] (see also [13]). The compact case remains a difficult unsolved problem, although if we restrict to arithmetic manifolds, then it is known [2, 8] that a minimum volume arithmetic compact orientable 4-manifold M has $\chi \leq 16$ and M is isometric to the orbit space of a torsion-free subgroup of the hyperbolic Coxeter group $[5, 3, 3, 3]$. The smallest compact hyperbolic 4-manifold currently known to exist has $\chi = 8$ and is constructed in [8]. Manifolds of very small volume have been constructed in 5 dimensions [13, 24], but the smallest volume 6-dimensional example hitherto known has $\chi = -16$ [13].

In this paper we announce the discovery of a number of noncompact nonorientable hyperbolic 6-manifolds with Euler characteristic $\chi = -1$. The method of construction is classical in that the manifolds are obtained by identifying the sides of a 6-dimensional hyperbolic Coxeter polytope.

2. Coxeter polytopes

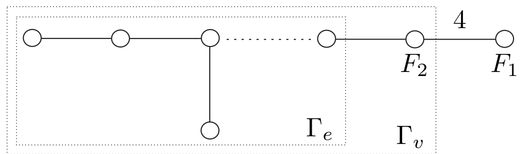
Let C be a convex (not necessarily bounded) polytope of finite volume in a simply connected space X^n of constant curvature. Call C a Coxeter polytope if the dihedral angle subtended by two intersecting $(n-1)$-dimensional sides is π/m for some integer $m \geq 2$. When $X^n = S^n$ or the Euclidean space E^n, such polyhedra have been completely classified [11], but in the hyperbolic space H^n, a complete classification remains a difficult problem (see for example [27] and the references there).

If Γ is the group generated by reflections in the $(n-1)$-dimensional sides of C, then Γ is a discrete cofinite subgroup of the Lie group $\operatorname{Isom} X^n$, and every discrete cofinite reflection group in $\operatorname{Isom} X^n$ arises in this way from some Coxeter polytope, which is uniquely defined up to isometry. The Coxeter symbol for C (or Γ) has nodes indexed by the $(n-1)$-dimensional sides, and an edge labeled m joining the nodes corresponding to sides that intersect with angle π/m (label the edge joining the nodes of nonintersecting sides by ∞). In practice the labels 2 and 3 occur often, so that edges so labeled are respectively removed or left unlabeled.

Let Λ be an $(n+1)$-dimensional Lorentzian lattice, that is, an $(n+1)$-dimensional free $\mathbb{Z}$-module equipped with a $\mathbb{Z}$-valued bilinear form of signature $(n, 1)$. For each n, there is a unique such Λ, denoted $I_{n,1}$, that is odd and self-dual (see [25, Theorem V.6], or [18, 19]). By [4], the group $\mathrm{O}_{n,1}\mathbb{Z}$ of automorphisms of $I_{n,1}$ acts discretely, cofinitely by isometries on the hyperbolic space H^n obtained by projectivising the negative norm vectors in the Minkowski space-time $I_{n,1} \otimes \mathbb{R}$ (to get a faithful action one normally passes to the centerless version $\mathrm{PO}_{n,1}\mathbb{Z}$).

Vinberg and Kaplinskaya showed [28, 29] that the subgroup Reflec_n of $\mathrm{PO}_{n,1}\mathbb{Z}$ generated by reflections in positive norm vectors has finite index if and only if $n \leq 19$, thus yielding a family of cofinite reflection groups and corresponding finite volume Coxeter polytopes in the hyperbolic spaces H^n for $2 \leq n \leq 19$. Indeed, Conway and Sloane showed ([9, Chapter 28] or [10]) that for $n \leq 19$ the quotient of $\mathrm{PO}_{n,1}\mathbb{Z}$ by Reflec_n is a subgroup of the automorphism group of the Leech lattice. Borcherds [3] showed that the (non-self-dual) even sublattice of $I_{21,1}$ also acts cofinitely, yielding the highest-dimensional example known of a Coxeter group acting cofinitely on hyperbolic space.

When $4 \leq n \leq 9$ the group $\Gamma = \text{Reflec}_n$ has Coxeter symbol,



with $n+1$ nodes and C a noncompact, finite volume n-simplex Δ^n (when $n > 9$, the polytope C has a more complicated structure).

Let v be the vertex of Δ^n opposite the side F_1 marked on the symbol, and let Γ_v be the stabilizer in Γ of this vertex. This stabilizer is also a reflection group with symbol as shown, and is finite for $4 \leq n \leq 8$ (being the Weyl group of type A_4, D_5, E_6, E_7 and E_8 respectively) and infinite for $n = 9$ (when it is the affine Weyl group of type $\widetilde{E}_8$). Let

$$P_n = \bigcup_{\gamma \in \Gamma_v} \gamma(\Delta^n),$$

a convex polytope obtained by gluing $|\Gamma_v|$ copies of the simplex Δ^n together. Thus, P_n has finite volume precisely when $4 \leq n \leq 8$, although it is noncompact, with a mixture of finite vertices in H^n and cusped ones on ∂H^n. In any case, P_n is an all right-angled Coxeter polytope: its sides meet with dihedral angle $\pi/2$ or are disjoint. This follows immediately from the observation that the sides of P_n arise from the Γ_v-images of the side of Δ^n opposite v, and this side intersects the other sides of Δ^n in dihedral angles $\pi/2$ or $\pi/4$. Vinberg has conjectured that $n = 8$ is the highest dimension in which finite volume all right-angled polytopes exist in hyperbolic space.

The volume of the polytope P_n is given by

$$\text{vol}(P_n) = |\Gamma_v|\text{vol}(\Delta^n) = |\Gamma_v|[\text{PO}_{n,1}\mathbb{Z} : \Gamma]\text{covol}(\text{PO}_{n,1}\mathbb{Z}),$$

where $\text{covol}(\text{PO}_{n,1}\mathbb{Z})$ is the volume of a fundamental region for the action of $\text{PO}_{n,1}\mathbb{Z}$ on H^n (and for $4 \leq n \leq 9$ the index $[\text{PO}_{n,1}\mathbb{Z} : \Gamma] = 1$). When n is even, we have by [26] and [22],

$$\text{covol}(\text{PO}_{n,1}\mathbb{Z}) = \frac{(2^{\frac{n}{2}} \pm 1)\pi^{\frac{n}{2}}}{n!} \prod_{k=1}^{\frac{n}{2}} |B_{2k}|,$$

with B_{2k} the $2k$-th Bernoulli number and with the plus sign if $n \equiv 0, 2 \mod 8$ and the minus sign otherwise.

Alternatively (when n is even), we have recourse to the Gauss-Bonnet Theorem, so that $\text{vol}(P_n) = \kappa_n|\Gamma_v|\chi(\Gamma)$, where $\chi(\Gamma)$ is the Euler characteristic of the Coxeter group Γ and $\kappa_n = 2^n(n!)^{-1}(-\pi)^{n/2}(n/2)!$. The Euler characteristic of Coxeter groups can be easily computed from their symbol (see [6, 7] or [13, Theorem 9]). Indeed, when $n = 6$, $\chi(\Gamma) = -1/\mathcal{L}$ where $\mathcal{L} = 2^{10}\, 3^4\, 5$ and so $\text{vol}(P_6) = 8\pi^3|E_6|/15\mathcal{L} = \pi^3/15$.

The Coxeter symbol for P_n has a nice description in terms of finite reflection groups. If v' is the vertex of Δ^n opposite the side F_2, let Γ_e be the pointwise stabilizer of $\{v, v'\}$: the elements thus stabilize the edge e of Δ^n joining v and v'.

Now consider the Cayley graph $\mathcal{C}_v$ for Γ_v with respect to the generating reflections in the sides of the symbol for Γ_v. Thus, $\mathcal{C}_v$ has vertices in one-to-one correspondence with the elements of Γ_v and for each generating reflection s_α, an undirected edge labeled s_α connecting vertices γ_1 and γ_2 if and only if $\gamma_2 = \gamma_1 s_\alpha$ in Γ_v. In particular, $\mathcal{C}_v$ has s_2 labeled edges corresponding to the reflection in F_2. Removing these s_2-edges decomposes $\mathcal{C}_v$ into components, each of which is a copy of the Cayley graph $\mathcal{C}_e$ for Γ_e, with respect to the generating reflections.

Take as the nodes of the symbol for P_n these connected components. If two components have an s_2-labeled edge running between any two of their vertices in $\mathcal{C}_v$, then leave the corresponding nodes unconnected; otherwise, connect them by an edge labeled ∞. The resulting symbol (respectively the polytope P_n) thus has $|\Gamma_v|/|\Gamma_e|$ nodes (resp. sides). The number of sides of P_n for $n = 4, 5, 6, 7, 8$ is $10, 16, 27, 56$ and 240 respectively.

3. Constructing the manifolds

We now restrict our attention to the case $n = 6$. We work in the hyperboloid model of hyperbolic 6-space

$$H^6 = \{x \in \mathbb{R}^7 : x_1^2 + x_2^2 + \cdots + x_6^2 - x_7^2 = -1 \text{ and } x_7 > 0\}$$

and represent the isometries of H^6 by Lorentzian 7×7 matrices that preserve H^6. The right-angled polytope P_6 has 27 sides each congruent to P_5. We position P_6 in H^6 so that 6 of its sides are bounded by the 6 coordinate hyperplanes $x_i = 0$ for $i = 1, \ldots, 6$ and these 6 sides intersect at the center e_7 of H^6. Let K_6 be the group of 64 diagonal Lorentzian 7×7 matrices $\mathrm{diag}(\pm 1, \ldots, \pm 1, 1)$. The set $Q_6 = K_6 P_6$, which is the union of 64 copies of P_6, is a right-angled convex polytope with 252 sides. We construct hyperbolic 6-manifolds, with $\chi = -8$, by gluing together the sides of Q_6 by a proper side-pairing with side-pairing maps of the form rk with k in K_6 and r a reflection in a side S of Q_6. The side-pairing map rk pairs the side $S' = kS$ to S (see §11.1 and §11.2 of [21] for a discussion of proper side-pairings). We call such a side-pairing of Q_6 simple. We searched for simple side-pairings of Q_6 that yield a hyperbolic 6-manifold M with a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 64 copies of P_6 making up M in such a way that the resulting quotient manifold is obtained by gluing together 8 copies of P_6. Such a quotient manifold has $\chi = -8/8 = -1$. This is easier said than done, since the search space of all possible side-pairings of Q_6 is very large. We succeeded in finding desired side-pairings of Q_6 by employing a strategy that greatly reduces the search space. The strategy is to extend a side-pairing in dimension 5 with the desired properties to a side-pairing in dimension 6 with the desired properties.

Let $Q_5 = \{x \in Q_6 : x_1 = 0\}$. Then Q_5 is a right-angled convex 5-dimensional polytope with 72 sides. Note that Q_5 is the union $K_5 P_5$ of 32 copies of P_5 where $P_5 = \{x \in P_6 : x_1 = 0\}$ and K_5 is the group of 32 diagonal Lorentzian 7×7 matrices $\mathrm{diag}(1, \pm 1, \ldots, \pm 1, 1)$. A simple side-pairing of Q_6 that yields a hyperbolic 6-manifold M restricts to a simple side-pairing of Q_5 that yields a hyperbolic 5-manifold which is a totally geodesic hypersurface of M. All the orientable hyperbolic 5-manifolds that are obtained by gluing together the sides of Q_5 by a simple side-pairing are classified in [24].

We started with the hyperbolic 5-manifold N, numbered 27 in [24], obtained by gluing together the sides of Q_5 by the simple side-pairing with side-pairing code `2B7JB47JG81`. The manifold N has a freely acting $\mathbb{Z}/8$ symmetry group that permutes the 32 copies of P_5 making up N in such a way that the resulting quotient

TABLE 1. Side-pairing codes and homology groups of the seven examples.

N	SP	H_1	H_2	H_3	H_4	H_5
		0248	0248	0248	0248	0248
1	`GW8dNEEdN4ZJO1k2l1PIY`	0401	1910	4821	1500	0000
2	`HX9dNFEcM5aKU6f3f6UKa`	0401	1810	8710	5500	0000
3	`HX9dNFEcM5YIO1l3l1OIY`	0401	2900	7810	4400	1000
4	`HX9dNFEcM5YIO6l3l6OIY`	0401	2800	7910	4400	1000
5	`HX9dNFEcM5YIOxl3lyOIY`	0211	2800	4821	1400	1000
6	`HX9dNFEcM5YIOyl3lxOIY`	0211	2800	4930	1400	1000
7	`HX9dNFEcM5aKUxf3fyUKa`	0301	1900	5630	2500	0000

manifold is obtained by gluing together 4 copies of P_5. A generator of the $\mathbb{Z}/8$ symmetry group of N is represented by the Lorentzian 6×6 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

The strategy is to search for simple side-pairings of Q_6 that yield a hyperbolic 6-manifold with a freely acting $\mathbb{Z}/8$ symmetry group with generator represented by the following Lorentzian 7×7 matrix that extends the above Lorentzian 6×6 matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

For such a side-pairing the resulting quotient manifold can be obtained by gluing together 8 copies of P_6 by a proper side-pairing. By a computer search we found 14 proper side-pairings of 8 copies of P_6 in this way, and hence we found 14 hyperbolic 6-manifolds with $\chi = -1$. Each of these 14 manifolds is noncompact with volume $8\mathrm{vol}(P_6) = 8\pi^3/15$ and five cusps. These 14 hyperbolic 6-manifolds represent at least 7 different isometry types, since they represent 7 different homology types. Table 1 lists side-pairing codes for 7 simple side-pairings of Q_6 whose $\mathbb{Z}/8$ quotient manifold has homology groups isomorphic to $\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/4)^c \oplus (\mathbb{Z}/8)^d$ for nonnegative integers a, b, c, d encoded by $abcd$ in the table. In particular, all 7 manifolds in Table 1 have a finite first homology group.

All of our examples, with $\chi = -1$, can be realized as the orbit space H^6/Γ of a torsion-free subgroup Γ of $\mathrm{PO}_{6,1}\mathbb{Z}$ of minimum index. These manifolds are the first examples of hyperbolic 6-manifolds having the smallest possible volume. All these manifolds are nonorientable. In the near future, we hope to construct orientable examples of noncompact hyperbolic 6-manifolds having $\chi = -1$.

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Department of Mathematics, University of York, York YO10 5DD, England
E-mail address: bje1@york.ac.uk

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
E-mail address: ratclifj@math.vanderbilt.edu

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
E-mail address: tschantz@math.vanderbilt.edu