

COBOUNDING ODD CYCLE COLORINGS

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ABSTRACT. We prove that the $(n - 2)$ nd power of the Stiefel-Whitney class of the space of all n -colorings of an odd cycle is 0 by presenting a cochain whose coboundary is the desired power of the class. This gives a very short self-contained combinatorial proof of a conjecture by Babson and the author.

1. PRELIMINARIES

The study of the following family of complexes has recently been undertaken in connection with equivariant obstructions to graph colorings.

Definition 1.1. For any graphs T and G , $\mathbf{Hom}(T, G) \subseteq \prod_{x \in V(T)} \Delta^{V(G)}$ consists of all cells $\sigma = \prod_{x \in V(T)} \sigma_x$ such that for any $x, y \in V(T)$, if $(x, y) \in E(T)$, then (σ_x, σ_y) is a complete bipartite subgraph of G .

In particular, the cells of $\mathbf{Hom}(T, G)$ are indexed by functions $\sigma : V(T) \rightarrow 2^{V(G)}$ satisfying that additional property, and $\dim \sigma = \sum_{v \in V(T)} (|\sigma(v)| - 1)$. We refer the reader to the survey [4] for an introduction to the subject of \mathbf{Hom} complexes.

The study of the complexes $X_{r,n} := \mathbf{Hom}(C_{2r+1}, K_n)$, $n \geq 3$, has been of special interest. Here for $r \in \mathbb{N}$, we let C_{2r+1} denote both the cyclic graph with $2r + 1$ vertices and the additive cyclic group with $2r + 1$ elements. The adjacent vertices of $v \in C_{2r+1}$ get labels $v + 1$ and $v - 1$. Taking the negative in the cyclic group gives an involution γ of the graph with a fixed vertex 0 and a flipped edge $(r, r + 1)$. Then $(X_{r,n}, \gamma)$ is a \mathbb{Z}_2 -space; hence the Stiefel-Whitney characteristic class $w_1(X_{r,n}) \in H^1(X_{r,n}/\mathbb{Z}_2; \mathbb{Z}_2)$ of the associated line bundle can be considered.

Theorem 1.2. *We have $w_1^{n-2}(X_{r,n}) = 0$.*

The case $r = 1$ was settled by Babson and the author in [2]. For $r \geq 2$ and odd n , it was proved by the same authors in [3]; see also [4], where the remaining case: $r \geq 2$, $n \geq 4$, n is even, was conjectured. The latter was then proved by Schultz in [5, 6]. In the next section we give a short self-contained combinatorial proof of Theorem 1.2 covering all cases: we simply take a cochain representative of $w_1^{n-2}(X_{r,n})$ and certify that it is a coboundary.

First we fix notation. For $t \in \mathbb{N}$, we set $[t] := \{1, \dots, t\}$. For a cell complex X , we let X^d denote the set of d -dimensional cells of X . Since we are working over \mathbb{Z}_2 ,

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we may identify d -cochains with their support subsets of X^d . Then, the cochain addition is replaced by the symmetric difference of sets, denoted \oplus . For $S \subseteq X_{r,n}^d$ the coboundary operator translates to $dS = \bigoplus_{\sigma \in S} \{\tau \in X_{r,n}^{d+1} \mid \tau \supset \sigma\}$.

If r is even, we set $t := r/2$, and $v_i := r - 2i + 1$ for $i \in [t]$; otherwise, we set $t := (r + 1)/2$, and $v_i := r + 2i - 1$ for $i \in [t]$. For any $v \in C_{2r+1}$, we put

$$\begin{aligned} A_v &:= \{\sigma \in X_{r,n}^{n-2} \mid \sigma(v) = [n-1]\}, \\ B_v &:= \{\sigma \in X_{r,n}^{n-3} \mid \sigma(v-1) \cup \sigma(v+1) = [n-1]\}. \end{aligned}$$

For $S \subseteq X_{r,n}^d$, let $q(S) := \bigoplus_{\sigma \in S} \{\{\sigma, \gamma\sigma\}\} \in C^d(X_{r,n}/\mathbb{Z}_2)$. We see that $q(A_0) = \emptyset$, and $q(S \oplus T) = q(S) \oplus q(T)$ for any $S, T \subseteq X_{r,n}^d$. Furthermore, since $\tau \cap \gamma\tau = \emptyset$ for any $\tau \in X_{r,n}^{d+1}$, we have $q(dS) = dq(S)$ for any $S \subseteq X_{r,n}^d$.

It is easy to describe a cochain representing $w_1^{n-2}(X_{r,n})$. Let $\iota : K_2 \hookrightarrow C_{2r+1}$ be given by $\iota(1) = r$, $\iota(2) = r + 1$, where $V(K_2) = [2]$. This induces an algebra homomorphism $\varphi : H^*(\text{Hom}(K_2, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \rightarrow H^*(X_{r,n}/\mathbb{Z}_2; \mathbb{Z}_2)$. It is well known that $\text{Hom}(K_2, K_n)/\mathbb{Z}_2 \cong \mathbb{RP}^{n-2}$. Let $\tau \in \text{Hom}(K_2, K_n)^{n-2}$ be given by $\tau(1) = [n-1]$, $\tau(2) = \{n\}$. Since the dual of any cell generates $H^{n-2}(\mathbb{RP}^{n-2}; \mathbb{Z}_2)$, we have $w_1^{n-2}(\text{Hom}(K_2, K_n)) = [\{\{\tau, \gamma\tau\}\}]$. By functoriality of w_1 we get $w_1^{n-2}(X_{r,n}) = [\varphi(\{\{\tau, \gamma\tau\}\})]$. Comparing this with our notation we derive $w_1^{n-2}(X_{r,n}) = [q(A_r)]$.

2. PROOF OF THEOREM 1.2

Lemma 2.1. *We have $dB_v = A_{v-1} \oplus A_{v+1}$ for any $v \in C_{2r+1}$.*

Proof. The cells in dB_v are obtained by taking a cell $\sigma \in B_v$ and adding x to $\sigma(w)$, for some $x \in [n]$, $w \in C_{2r+1}$. When $w \neq v \pm 1$, we get a cell τ , which appears in dB_v twice: in $d\sigma_1$ and in $d\sigma_2$, where σ_1, σ_2 are obtained from τ by deleting one of the elements from $\tau(w)$. When $w = v \pm 1$, we also get a cell τ , which appears in dB_v twice: in $d\sigma_1$ and in $d\sigma_2$, where σ_1, σ_2 are obtained from τ by deleting $\{x\} = \tau(v-1) \cap \tau(v+1)$ either from $\tau(v-1)$ or from $\tau(v+1)$, unless $|\tau(v-1)| = 1$ or $|\tau(v+1)| = 1$. The latter cells appear once and yield $A_{v-1} \oplus A_{v+1}$. \square

Proof of Theorem 1.2. Set $K := \bigoplus_{i=1}^t q(B_{v_i})$; then $dK = \bigoplus_{i=1}^t dq(B_{v_i}) = \bigoplus_{i=1}^t q(dB_{v_i}) = \bigoplus_{i=1}^t (q(A_{v_i-1}) \oplus q(A_{v_i+1})) = q(A_r) \oplus q(A_0) = q(A_r)$, hence $w_1^{n-2}(X_{r,n}) = [q(A_r)] = [dK] = 0$. \square

We remark that Theorem 1.2 implies the Lovász Conjecture: *for any graph G and any positive integer r , if $\text{Hom}(C_{2r+1}, G)$ is k -connected, then $\chi(G) \geq k + 4$.* This conjecture was originally settled by Babson and the author in [1, 3].

Going beyond the mere proof of the Lovász Conjecture, the new ideas and techniques in [1, 2, 3] laid the foundation for all further studies of Hom complexes.

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