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KAZHDAN–LUSZTIG CELLS AND DECOMPOSITION NUMBERS

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ABSTRACT. We consider a generic Iwahori–Hecke algebra H associated with a finite Weyl group. Any specialization of H gives rise to a corresponding decomposition matrix, and we show that the problem of computing that matrix can be interpreted in terms of Lusztig's map from H to the asymptotic algebra J. This interpretation allows us to prove that the decomposition matrices always have a lower uni-triangular shape; moreover, we determine these matrices explicitly in the so-called defect 1 case.

1. INTRODUCTION

Let W be a finite Weyl group with generators $S \subset W$. Let H be the corresponding generic Iwahori–Hecke algebra over the ring $A = R[v, v^{-1}]$, where v is an indeterminate and R is the ring of integers in some algebraic number field F. Thus, H has a basis $\{T_w \mid w \in W\}$ with multiplication $(T_s - v^2)(T_s + 1) = 0$ for $s \in S$ and $T_w = T_{s_1} \cdots T_{s_m}$ whenever $w = s_1 \cdots s_m$ (with $s_i \in S$) is a reduced expression in W.

Let K be the field of fractions of A and $\theta\colon A \to k$ be a homomorphism into a field k such that k is the field of fractions of $\theta(A)$. We shall assume throughout that $\theta(v)$ has finite order. (See Gyoja [13] for the remaining case, which is completely solved.)

We regard k as an A-algebra via θ. Since A is integrally closed in K, we have a well-defined decomposition map (see [10, §2]),

$$d_k\colon R_0(H_K) \to R_0(H_k)$$

between the Grothendieck groups of H_K and H_k, respectively. (For any commutative A-algebra B with 1, we denote $H_B := B \otimes_A H$; the class of a module M in the Grothendieck group will be denoted by $[M]$.) In order to avoid any technical problems, we will always want to work with algebras that are split (i.e., the endomorphism ring of any simple module consists just of the scalar multiples of the identity). Note that every finite-dimensional algebra becomes split after a finite field extension. Thus, choosing F sufficiently large and using the canonical isomorphism $H_K \cong K[W]$ of [16], this can always be arranged.

One of the main open problems in the representation theory of Iwahori–Hecke algebras is the determination of decomposition maps d_k as above. This is only solved in special cases, the deepest of which is Ariki's solution [1] of the Lascoux–Leclerc–Thibon conjecture [15]; see [10] for a survey of further results.

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In Theorem 3.3 of this paper, we describe a natural ordering of the rows and columns of the decomposition matrix associated with d_k in terms of the relation $\leq_{LR}$ of Kazhdan and Lusztig [14] and we prove that this yields a lower triangular shape for that matrix, with 1 along the diagonal. In particular, d_k is surjective. Previously, results of this kind could only be proved case by case for certain types (by work of various authors: Dipper, James, Murphy, Pallikaros, Bremke, Geck, Lux, Müller; see, for example, [5, 6, 23] and [10] for further references) or by using the relation of the representation theory of Hecke algebras with that of a corresponding finite Chevalley group (see [10, Proposition 3.8] and [4]). For type D_n with n even, this is a new result.

In Section 4, as an application, we determine explicitly the labelling of the nodes in Brauer trees of blocks of H with defect 1 (as defined in [9]). Again, these trees were known case by case. For the classical types, they can be extracted from the trees determined by Fong and Srinivasan [8]; for the exceptional types, see [9]. We obtain the more precise statement that the ordering of the simple modules on the tree is given in terms of the relation $\leq_{LR}$.

The proofs are based on methods which are all contained in Lusztig's papers on cells in affine Weyl groups [18, 19, 20]. More precisely, we use the following two remarks.

(1) d_k has a natural interpretation in terms of the canonical homomorphism from H to Lusztig's asymptotic algebra J of [19]; see Lemma 2.3 below.
(2) The techniques in the proof of [20, Lemma 1.9] also work for d_k (via the interpretation of (1)), and they yield Theorem 3.3 below.

Note that a variant of (1) is already used by Gyoja [13] (in a different context), but the above interpretation together with (2) seems to be new.

Finally, we remark that these results are non-elementary in the sense that they use the positivity properties established by Springer [24]. This also restricts the applicability of these methods to the case of 1-parameter Iwahori–Hecke algebras while, for example, the results in [6] for type B_n hold without any restriction on the parameters.

2. The asymptotic algebra J

The above definitions and assumptions remain in force. The first results in this section are Lemma 2.3 and Remark 2.5, which show how to reduce the study of d_k to that of a homomorphism from H into Lusztig's asymptotic algebra J. We then state the main result about that homomorphism, due to Lusztig, in Proposition 2.8.

2.1. Let $\{C_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of H; see [14]. For $x, y \in W$, we write $C_xC_y = \sum_z h_{x,y,z}C_z$ with $h_{x,y,z} \in A$. For any $z \in W$, there is a well-defined integer $a(z) \geq 0$ such that

$$\begin{aligned} v^{a(z)}h_{x,y,z} &\in \mathbb{Z}[v] \quad \text{for all } x, y \in W, \\ v^{a(z)-1}h_{x,y,z} &\notin \mathbb{Z}[v] \quad \text{for some } x, y \in W. \end{aligned}$$

Let J be the asymptotic algebra introduced in [19]. It is an algebra over $\mathbb{Z}$ with a basis $\{t_w \mid w \in W\}$, whose structure constants are given in terms of the constant terms of the polynomials $v^{a(z)}h_{x,y,z}$; the identity element is $\sum_d t_d$ where d runs over the set of so-called distinguished involutions $\mathcal{D} \subseteq W$ (see [19, (1.3)]). For any commutative ring B with 1, we write $J_B := B \otimes_{\mathbb{Z}} J$. The algebra J is a "based ring"; see [21, §1 and (3.1j)]. Since F is assumed to be large enough, this implies that J_F is split semisimple (see [21, (1.2a)]).

2.2. Lusztig has shown in [19, (2.4)] that the map $\phi\colon H \to J_A$ defined by $C_w \mapsto \sum_{z,d} h_{w,d,z} t_z$ (sum over all $z \in W$ and $d \in \mathcal{D}$ such that $a(z) = a(d)$) is a homomorphism of A-algebras which preserves the identity elements. For any commutative A-algebra B with 1, it gives rise to a homomorphism $\phi_B\colon H_B \to J_B$.

If E is a J_K-module, we denote by E_* the H_K-module with underlying vector space E and where $h \in H_K$ acts via $\phi_K(h)$. The assignment $E \mapsto E_*$ defines a homomorphism $(\phi_K)_*\colon R_0(J_K) \to R_0(H_K)$. Similarly, the algebra homomorphism $\phi_k\colon H_k \to J_k$ also gives rise to a homomorphism $(\phi_k)_*\colon R_0(J_k) \to R_0(H_k)$.

By [19, Theorem 2.8], the map ϕ_K is an isomorphism. Hence, $(\phi_K)_*$ is an isomorphism which preserves the classes of simple modules. Note that this is not necessarily the case for $(\phi_k)_*$.

The following result gives a basic relation between the various maps between the Grothendieck groups of H_K, H_k, J_K, J_k. In order to distinguish the decomposition map between $R_0(H_K)$ and $R_0(H_k)$ from that between $R_0(J_K)$ and $R_0(J_k)$, we denote these maps by d_k^H and d_k^J, respectively.

Lemma 2.3. *The following diagram is commutative.*

$$\begin{tikzcd} R_0(J_K) \arrow[r, "(\phi_K)_*"] \arrow[d, "d_k^J"'] & R_0(H_K) \arrow[d, "d_k^H"] \\ R_0(J_k) \arrow[r, "(\phi_k)_*"] & R_0(H_k) \end{tikzcd}$$

Proof. Let $\rho\colon J_K \to M_n(K)$ be an irreducible representation. Since J_F is already split semisimple and F is large enough, we can assume that $\rho(t_w) \in M_n(R)$ for all $w \in W$. Now $\rho_* := \rho \circ \phi_K\colon H_K \to M_n(K)$ is a representation such that $\rho_*(C_w) \in M_n(A)$ for all $w \in W$. We specialize ρ_* under the map $\theta\colon A \to k$ and obtain a representation $(\rho_*)_k\colon H_k \to M_n(k)$ such that $(\rho_*)_k(C_w) = (\theta(a_{ij}))$ where $\rho_*(C_w) = (a_{ij})$. (Here, we write again C_w for $1 \otimes C_w$.)

On the other hand, we can first specialize ρ under the map $\theta\colon A \to k$ and obtain a representation $\rho_k\colon J_k \to M_n(k)$ such that $\rho_k(t_w) = (\theta(b_{ij}))$ where $\rho(t_w) = (b_{ij})$. (Here, we write again t_w for $1 \otimes t_w$.) Then we compose with ϕ_k and obtain a representation $(\rho_k)_* := \rho_k \circ \phi_k\colon H_k \to M_n(k)$.

We check that the result is the same as before. Indeed, let $\phi_{x,y} \in A$ $(x, y \in W)$ such that $\phi(C_y) = \sum_x \phi_{x,y} t_x$. Then we have

$$(\rho \circ \phi_K)_k(C_y) = \sum_x \theta(\phi_{x,y})\rho_k(t_x) = \rho_k\Big(\sum_x \theta(\phi_{x,y})t_x\Big) = (\rho_k \circ \phi_k)(C_y),$$

for all $y \in W$, as desired. The proof is now complete, since the above discussion shows that $(\phi_k)_* \circ d_k^J \circ (\phi_K)_*^{-1}$ defines a map $R_0(H_K) \to R_0(H_k)$ which makes the diagram (D) in [10, §2] commutative, and this characterizes d_k^H. ☐

Under some mild restrictions on k, the left hand vertical arrow in the above diagram can be seen to be an isomorphism which preserves the classes of simple modules. To prove this, we have to recall some results from [21].

2.4. The algebra J is symmetric, with symmetrizing trace $\tau\colon J \to \mathbb{Z}$ given by $\tau(t_w) = 1$ if w is a distinguished involution and $\tau(t_w) = 0$ otherwise (see [21, (1.1c) and (3.1j)]). For any simple J_K-module E let $0 \neq f_E \in K$ be the scalar in [21, (1.3c)]. Note that since J_K already splits over F, we have $f_E \in R$ by [21, (1.3f)]. Furthermore, we have

$$D_{E_*} = f_E^{-1} v^{2a_E} + \text{higher powers of } v^2,$$

where $a_E \geq 0$ is an integer and $D_{E_*} \in F[v^2]$ denotes the generic degree associated with the simple H_K-module E_*; see [21, (3.4e)].

For later reference, we remark that a_E can also be characterized as follows. For any $w \in W$, let $c_{w,E_*} \in F$ be the "leading coefficient" of the character value of T_w on E_*, as defined in [17, (5.1.21)] or [21]. By [17, Lemma 5.2], there exists some $w \in W$ such that $c_{w,E_*} \neq 0$ and, using the results in [21, (3.4), (3.5)], we have

$$a_E = a(w) \qquad \text{for any } w \in W \text{ such that } c_{w,E_*} \neq 0.$$

It is known that $D_{E_*} \in \mathbb{Q}[v^2]$; see [2]. Hence the constant f_E is in fact an integer. The prime numbers which divide any one of these constants are called bad primes for W. Recall that a prime number > 5 is never bad; moreover, the prime 2 is bad if W has an irreducible component of type $\neq A$, the prime 3 is bad if W has a component of exceptional type, and the prime 5 is bad if W has a component of type E_8. This follows, for example, from the formula in [17, (4.14.2)], which expresses f_E in terms of the finite groups associated with the families of simple modules of W, and the description of these families in [17, Chap. 4].

Remark 2.5. In the set-up of Lemma 2.3, assume that the characteristic of k is 0 or a good prime for W. Then all constants f_E are non-zero in k and, hence, J_k is split semisimple. (This follows from the fact that J_F is split semisimple (see 2.1) and a general semisimplicty criterion for symmetric algebras; see [11, Proposition 4.3] or [12].) Tits' Deformation Theorem (see [3, §68A]) now implies that the decomposition map $d_k^J\colon R_0(J_K) \to R_0(J_k)$ is an isomorphism which preserves the classes of simple modules.

Since ϕ_K also is an isomorphism, we see that the problem of determining the map $d_k^H\colon R_0(H_K) \to R_0(H_k)$ is equivalent to that of determining the map $(\phi_k)_*$.

Corollary 2.6. *Assume that the characteristic of k is 0 or a prime which does not divide the order of W. Assume also that v is mapped to $1 \in k$ so that $H_k = k[W]$. Let k_0 be the prime field of k. Then J_{k_0} is split semisimple and the algebra homomorphism $\phi_{k_0}\colon k_0[W] \to J_{k_0}$ is an isomorphism.*

Proof. First note that all of the algebras J_K, J_k, H_K, H_k are semisimple. (We already know this for J_K and H_K; as far as $k[W]$ is concerned, this is just Maschke's Theorem. Finally note that if the characteristic of k is a prime not dividing the order of W, then it is also a good prime for W; and see Remark 2.5.) Consider the commutative diagram in Lemma 2.3. Tits' Deformation Theorem [3, §68A] implies that the two vertical arrows d_k^J and d_k^H are isomorphisms which preserve the classes of simple modules. Since $(\phi_K)_*$ also has this property, we conclude that $(\phi_k)_*$ does also. It is easily seen that, consequently, ϕ_k itself must be an isomorphism.

Since ϕ_k is defined over k_0 (by its construction), we conclude that we already have an isomorphism $\phi_{k_0}\colon k_0[W] \to J_{k_0}$. Finally, this also implies the assertion about splitting fields since $k[W]$ already splits over k_0 (in fact, W is split over any field; see [2]). □

The above result is also obtained by Fleischmann [7, Lemma 3.6], using a slightly different argument, and by Gyoja [13, Lemma 2.1]. If J_k is not semisimple, then J_{k_0} is still split, by [13, Theorems B' and C'].

2.7. For any $i \geq 0$, we introduce the following subspaces, following [20]:

$$\begin{aligned} J_k^i &= \text{subspace of } J_k \text{ generated by all } t_w \text{ with } a(w) = i, \\ H_k^{\geq i} &= \text{subspace of } H_k \text{ generated by all } C_w \text{ with } a(w) \geq i. \end{aligned}$$

(Again we write C_w and t_w for $1 \otimes C_w$ and $1 \otimes t_w$, respectively.) Then J_k^i and $H_k^{\geq i}$ are two-sided ideals. Let $H_k^i := H_k^{\geq i}/H_k^{\geq i+1}$; this is an (H_k, H_k)-bimodule in a natural way. It is also a (J_k, J_k)-bimodule in a natural way; see [20, (1.4)]. Denoting the multiplication by $\circ$, we have for all $f \in H_k^i$, $h \in H_k$ and $j \in J_k$:

$$\text{(1)} \qquad hf = \phi_k(h) \circ f, \quad j \circ (fh) = (j \circ f)h, \quad (hf) \circ j = h(f \circ j).$$

Thus, H_k^a is a (J_k, H_k)-bimodule. Furthermore, we have $t_w \circ f = f \circ t_w = 0$ unless $a(w) = i$.

To any simple J_k-module E we can attach an integer a_E by the requirement that $t_w E \neq 0$ for some $w \in W$ with $a(w) = a_E$. This is well-defined since $J_k = \bigoplus_i J_k^i$; see [20, (1.3)(d)]. We can also attach an integer a_M to any simple H_k-module M by the requirement that

$$\begin{aligned} C_w M = 0 &\quad \text{for all } w \in W \text{ with } a(w) > a_M, \\ C_w M \neq 0 &\quad \text{for some } w \in W \text{ with } a(w) = a_M. \end{aligned}$$

The following construction is taken from the proof of [20, Lemma 1.9]. Let M be a simple H_k-module and set $a = a_M$. Let $\tilde{M} := H_k^a \otimes_{H_k} M$, where we regard H_k^a as a right H_k-module and M as a left H_k-module. Then $\tilde{M}$ is naturally a left H_k-module. Let $\tilde{M}_J$ be the J_k-module whose underlying vector space is $\tilde{M}$ and J_k acts via $j\colon (f \otimes m) \mapsto (j \circ f) \otimes m$. By the argument in condition (b) of the proof of [20, Corollary 3.6], we have:

$$\text{(2)} \qquad \text{If } E \text{ is a composition factor of } \tilde{M}_J, \text{ then } a_E = a_M.$$

(Indeed, if $z \in W$ is such that $t_z E \neq 0$, then we also have $t_z \tilde{M}_J \neq 0$ and so $t_z \circ H_k^a \neq 0$, where $a = a_M$. Since $t_z \circ f = 0$ unless $a(z) = a$, we must have $a(z) = a_M$.)

Proposition 2.8 (Lusztig [20, Lemma 1.9])**.** *We have a surjective homomorphism of H_k-modules $p\colon \tilde{M} \to M$ given by $p(f \otimes m) = \dot{f}m$, where $\dot{f} \in H_k^{\geq a}$ is a representative of $f \in H_k^a$. In $R_0(H_k)$, we have $(\phi_k)_*([\tilde{M}_J]) = [\tilde{M}] = [M]+$ sum of terms $[M']$ where M' are simple H_k-modules with $a_{M'} < a_M$.*

Proof. In [*loc. cit.*], it is generally assumed that (W, S) is an affine Weyl group and that $k = \mathbb{C}$, but the proof works, word by word, in general. ∎

Remark 2.9. The above result shows, in particular, that the map $(\phi_k)_*\colon R_0(J_k) \to R_0(H_k)$ is surjective. This in turn implies that the kernel of $\phi_k\colon H_k \to J_k$ acts as 0 on every simple H_k-module. Hence that kernel is contained in the radical of H_k. Consequently, ϕ_k is an isomorphism if H_k is semisimple. This gives another proof of [19, Theorem 2.8] (at least for the case where W is finite; see also [7]).

3. The decomposition map d_k^H

Recall that we assume throughout that the image of v in k has finite order, and that H_k is split. From now on, we will also assume that the characteristic of k is 0 or a good prime for W. Then the discussion in Remark 2.5 applies and so the decomposition map $d_k^J \colon R_0(J_K) \to R_0(J_k)$ is an isomorphism which preserves the classes of simple modules.

Recall that we have attached a-values to the simple modules for J_K (see 2.4) and H_k, J_k (see 2.7). We have the following relations between these a-values (compare with the similar properties established in the proof of [20, Corollary 3.6]).

Lemma 3.1. *Let E be a simple J_k-module and M be a simple H_k-module.*

(1) *There exists some $z \in W$ such that* $\operatorname{trace}(t_z, E) \neq 0$. *For any such z, we have $a_E = a(z) = a_{E'}$, where E' is a simple J_K-module such that $d_k^J([E']) = [E]$.*
(2) *Assume that $[M]$ occurs in $(\phi_k)_*([E])$ and that $w \in W$ is such that $C_w M \neq 0$. Then, for any z as in* (1)*, we have $z \leq_{LR} w$, with $\leq_{LR}$ as defined in* [14]*. In particular, this implies that $a_M \leq a_E$.*

Note that we have $\operatorname{trace}(t_z, E) = \theta(\operatorname{trace}(t_z, E'))$ and, by [21, (3.5b)], we also have $\operatorname{trace}(t_z, E') = \pm c_{z,E'_*}$, with E'_* as in 2.4.

Proof. As in the proof of Lemma 2.3, there exists a representation $\rho \colon J_K \to M_n(K)$ affording E' such that $\rho(t_z) \in M_n(R)$ for all $z \in W$. Let $\chi_{E'}$ be its character. Consider the representation $\rho_k \colon J_k \to M_n(k)$ induced by $\theta \colon A \to k$. Its character is given by $t_z \mapsto \theta(\chi_{E'}(t_z))$. Now Tits' Deformation Theorem states more precisely that the latter map is the character afforded by E. Hence ρ_k is a representation affording E.

This implies that if $z \in W$ is any element such that $t_z E \neq 0$, then we also have $t_z E' \neq 0$. (If $\rho_k(t_z)$ is not the zero matrix, then neither is $\rho(t_z)$.)

(1) Let $z \in W$ be such that $\chi_E(t_z) \neq 0$. Then we certainly have $t_z E \neq 0$ and so $a_E = a(z)$, by definition. The above remarks show that $t_z E' \neq 0$. By [21, Proposition 3.3 and (3.4e)], this implies that $a_E = a(z) = a_{E'}$, as required. Since J_k is split semisimple, the characters of the various simple J_k-modules are linearly independent. Hence there exists some $z \in W$ such that $\chi_E(t_z) \neq 0$.

(2) The following arguments are just a slight modification of those in the proof of [20, Corollary 3.6]. Let $w \in W$ be such that $C_w M \neq 0$. Then we also have $\phi_k(C_w)E \neq 0$, since M is a constituent of E, regarded as an H_k-module via ϕ_k. Using the defining equation of ϕ, this yields $\sum_{x,d} \theta(h_{w,d,x}) t_x E \neq 0$, where the sum is over all $x \in W$ and $d \in \mathcal{D}$ such that $a(x) = a(d)$. It follows that for some x_0, d_0 we have $t_{x_0} E \neq 0$ and $h_{w,d_0,x_0} \neq 0$. The latter condition means that C_{x_0} occurs in the product $C_w C_{d_0}$ (expressed as a linear combination of C_y's). Thus, we have $x_0 \leq_{LR} w$, by the definition in [14].

We have remarked above that the condition $t_{x_0} E \neq 0$ implies that $t_{x_0} E' \neq 0$. Now consider any element $z \in W$ such that $\chi_E(t_z) \neq 0$. Then $t_z E \neq 0$ and, hence, $t_z E' \neq 0$. By [21, (3.11)], this implies that $z \sim_{LR} x_0 \leq_{LR} w$, as desired. Finally, by [18, Theorem 5.4], we then have $a(w) \leq a(z)$. Choosing w such that $a(w) = a_M$ also shows that $a_M \leq a_E$. ∎

We can now describe our main application to the decomposition matrix associated with d_k^H.

3.2. Let $M_1, \dots, M_m$ be a set of representatives for the isomorphism classes of simple H_k-modules. Consider the J_k-modules $(\tilde{M}_1)_J, \dots, (\tilde{M}_m)_J$, as defined in 2.7.

By Proposition 2.8, there exist simple J_k-modules $E_1, \dots, E_m$, uniquely determined up to isomorphism by the conditions that E_j is a constituent of $(\tilde{M}_j)_J$ and that $[M_j]$ occurs as a summand in the decomposition of $(\phi_k)_*([E_j])$. Note that, in $R_0(H_k)$, we have $(\phi_k)_*([E_j]) = [M_j]+$ sum of terms $[M']$ where M' are simple H_k-modules with $a_{M'} < a_{M_j}$. This also implies, by a simple induction on the a-values, that $E_1, \dots, E_m$ are pairwise non-isomorphic.

Let $E_{m+1}, \dots, E_n$ $(n \geq m)$ be simple J_k-modules such that $E_1, \dots, E_n$ is a set of representatives for the isomorphism classes of all simple J_k-modules. For $1 \leq i \leq n$, we have

$$(1) \qquad (\phi_k)_*([E_i]) = \sum_{j=1}^{m} d_{ij}[M_j] \qquad \text{where } d_{ij} \in \mathbb{N}_0.$$

Let $V_1, \dots, V_n$ be simple H_K-modules such that $(\phi_K)_* \circ (d_k^J)^{-1}([E_i]) = [V_i]$ for $1 \leq i \leq n$. Note that $V_1, \dots, V_n$ are representatives for the isomorphism classes of simple H_K-modules, since d_k^J and $(\Phi_K)_*$ are both isomorphisms which preserve the classes of simple modules. Let $w_1, \dots, w_n \in W$ be such that $\theta(c_{w_i,V_i}) \neq 0$ for all i. For each i let a_{V_i} be the exponent of the largest power of v^2 which divides the generic degree of V_i. Then, by 2.4, 2.7(2) and Lemma 3.1(1), we have

$$(2) \qquad a_{E_i} = a_{V_i} = a(w_i) \; (1 \leq i \leq n) \quad \text{and} \quad a_{M_j} = a_{E_j} \; (1 \leq j \leq m).$$

We can now state the following.

Theorem 3.3. *Recall that we assume that the characteristic of k is 0 or a good prime for W. We have*

$$d_k^H([V_i]) = \sum_{j=1}^{m} d_{ij}[M_j] \qquad \text{for } 1 \leq i \leq n,$$

i.e., the matrix $D = (d_{ij})$ is the decomposition matrix associated with d_k^H. We have $d_{jj} = 1$ for $1 \leq j \leq m$ and $d_{ij} = 0$ unless $a_{V_j} \leq a_{V_i}$. Furthermore,

$$(*) \qquad d_{ij} \neq 0 \; (1 \leq i,j \leq m, \; i \neq j) \quad \Rightarrow \quad w_i \leq_{LR} w_j \text{ and } a_{V_j} < a_{V_i}.$$

In particular, assuming that $a_{V_1} \leq \dots \leq a_{V_m}$, this means that D has a lower uni-triangular shape, and that d_k^H is surjective.

Proof. The first assertion is just the commutativity of the diagram in Lemma 2.3; recall also Remark 2.5. By the definition of E_j (for $1 \leq j \leq m$) we have $d_{jj} = 1$. Now assume that $d_{ij} \neq 0$ for some i, j. This means that $[M_j]$ occurs as a summand in $(\phi_k)_*([E_i])$. Hence Lemma 3.1(2) implies that $a_{M_j} \leq a_{E_i}$, and so $a_{V_j} \leq a_{V_i}$ by 3.2(2).

Now consider $(*)$. Let $1 \leq i, j \leq m$ be such that $d_{ij} \neq 0$ and $i \neq j$. Then $[M_j]$ occurs as a summand in $(\phi_k)_*([(\tilde{M}_i)_J])$. By Proposition 2.8, we have $a_{M_j} \leq a_{M_i}$ with equality only for $i = j$. Thus, we must have $a_{M_j} < a_{M_i}$. Again, 3.2(2) shows that $a_{V_i} < a_{V_j}$.

It remains to check the assertion about w_i, w_j. Our assumption means that $[M_j]$ occurs as a summand in both $(\phi_k)_*([E_j])$ and $(\phi_k)_*([E_i])$. Let $w \in W$ be such that $a(w) = a_{M_j}$ and $C_w M_j \neq 0$. Then $w_i \leq_{LR} w$ and $w_j \leq_{LR} w$ by Lemma 3.1(2). Now note that $a(w_j) = a_{M_j}$ by 3.2(2). Hence we have $a(w_j) = a(w)$. By [18, Corollary 6.3(b)], this and the condition $w_j \leq_{LR} w$ imply that $w_j \sim_{LR} w$, and so $w_i \leq_{LR} w_j$, as desired.

Finally, assume that $a_{V_1} \leq \ldots \leq a_{V_m}$. Let $1 \leq i, j \leq m$. Then $d_{ij} \neq 0$ implies that $i = j$ or $a_{V_j} < a_{V_i}$, and hence $j \leq i$. Thus, D has a lower uni-triangular shape. □

Remark 3.4. Recall from 3.2 that $a_{E_i} = a_{V_i}$ is the exponent of the lowest power of v^2 in the generic degree associated with V_i; hence these a-values are explicitly known by the tables in [17, Chap. 4]. Now consider a simple H_k-module M_j. A priori, we do not have a more direct way of computing the constant a_{M_j} other than just using the definition in 2.7. However, once the decomposition matrix (d_{ij}) is known, a_{M_j} can be computed as

$$a_{M_j} = \min\{a_{V_i} \mid d_{ij} \neq 0\}.$$

This follows from the conditions on d_{ij} in Theorem 3.3.

By the construction in 3.2, the simple J_k-modules $E_1, \ldots, E_m$ (and hence the simple H_K-modules $V_1, \ldots, V_m$) are canonically associated with the simple H_k-modules. It would be desirable to characterize these H_K-modules directly, in terms of the labellings in [17, Chap. 4] for example. In type A, such a characterization is known:

Example 3.5. Assume that H is the Iwahori–Hecke algebra associated with the symmetric group S_r ($r \geq 1$). Then a statement similar to that in Theorem 3.3 has been proved by Dipper and James [5]. In this case, the simple H_K-modules $V_1, \ldots, V_n$ have a natural labelling by the partitions of r. We write μ_i for the partition which labels V_i. (For example, the trivial module corresponds to (r) and the module giving the sign representation corresponds to (1^r).) Dipper and James show that the simple H_k-modules $M_1, \ldots, M_m$ have a natural labelling by the e-regular partitions of r, where $e \geq 1$ is minimal such that $1+\theta(v)^2+\cdots+\theta(v)^{2(e-1)} = 0$. We write λ_j for the e-regular partition which labels M_j. By [5, Theorem 7.6], we have

$$\begin{aligned} d_{ij} &= 1 \qquad \text{if } \mu_i = \lambda_j, \\ d_{ij} &= 0 \qquad \text{unless } \mu_i \trianglelefteq \lambda_j, \end{aligned}$$

where $\trianglelefteq$ denotes the usual dominance order on partitions. Using the formula for the a-values in [17, (4.4.2)], one checks immediately that if $\mu_j \trianglelefteq \mu_i$, then $a_{V_i} \leq a_{V_j}$, with equality only for $j = i$. So we must have

$$\mu_j = \lambda_j \qquad \text{for } 1 \leq j \leq m.$$

(Indeed, fix j and let i be such that $\mu_i = \lambda_j$; then $d_{jj} = 1$ and the above relations imply $\mu_j \trianglelefteq \lambda_j = \mu_i$, and so $a_{V_i} \leq a_{V_j}$. On the other hand, $d_{ij} = 1$ implies $a_{V_j} \leq a_{V_i}$ by Theorem 3.3; hence $a_{V_i} = a_{V_j}$ and $i = j$.) Similar results also hold for type B_r. The analogue of [5, Theorem 7.6] is established by Dipper, James and Murphy in [6, Theorem 6.5]; for the parametrization of the simple modules over k, see [6, Conjecture 8.13] and Mathas [22].

The above results have the following application to splitting fields. First recall that $\mathbb{Q}$ is a splitting field for W (see, for example, [2]). Using Lusztig's canonical isomorphism [16], this implies that H_K is already split over $\mathbb{Q}(v)$.

Corollary 3.6. *With the assumptions of Theorem 3.3, let $k_0 \subseteq k$ be the field of fractions of the image of $\mathbb{Z}[v, v^{-1}]$ under the map $\theta\colon A \to k$. Then H_{k_0} is split.*

Proof. We must show that every simple H_k-module M_j can be realized over k_0. The facts that H_K is already split over $\mathbb{Q}(v)$ and that $\mathbb{Z}[v, v^{-1}]$ is integrally closed in $\mathbb{Q}(v)$ imply that $\mathrm{trace}(C_w, V_i) \in \mathbb{Z}[v, v^{-1}]$ for all w, i (see [11, Lemma 2.10]). Since the matrix (d_{ij}) has all elementary divisors 1, we conclude that $\mathrm{trace}(C_w, M_j) \in k_0$ for all w. It is well-known that this implies that if k is a finite field, then M_j can already be realized over k_0 (see the argument in [3, (74.9)]). Note that the assumption that k is finite is satisfied if the characteristic of k is a prime number (since $\theta(v)$ has finite order in k).

If k has characteristic 0, we argue as follows. Let E_j be the simple J_k-module as in 3.2. If we regard E_j as an H_k-module via ϕ_k, then M_j occurs in that module as a composition factor with multiplicity 1. But, by Corollary 2.6, we know that J_k is already split over $\mathbb{Q}$. So E_j can be realized over $\mathbb{Q}$ and, hence, also over $\mathbb{Z}$. It follows that M_j occurs as a composition factor with multiplicity 1 in a module which can be realized over k_0. The familiar properties of Schur indices (see [3, §74]) imply that M_j can also be realized over k_0. □

3.7. We close this section with some remarks. Assume that we are in the set-up of 3.2 and Theorem 3.3.

(1) By the results for types A_{r-1}, B_r in Example 3.5 and the known decomposition matrices for types F_4, E_6 and E_7, one is lead to expect that, for $i > m$, we always have $(\phi_k)_*([E_i]) =$ sum of terms $[M']$ where M' are simple H_k-modules such that $a_{M'} < a_{E_i}$.

(2) If H_k is semisimple, it can be easily seen that each J_k-module $(\tilde{M}_j)_J$ (for $1 \leq j \leq m$) is simple (see the argument in the proof of [20, Corollary 3.6]). There is no a priori reason why this should be so in general. But this would follow once (1) is known.

R. Rouquier has found general proofs of these statements, which will be discussed elsewhere.

4. Brauer trees

The main result of this section is Theorem 4.4 which determines the Brauer trees of blocks of "defect 1". Before we come to this, we need some preliminary results which are also useful in their own right.

4.1. Consider the A-algebra automorphism $\gamma\colon H \to H$ defined by $T_s \mapsto -v^2 T_s^{-1}$ for $s \in S$. It gives rise to algebra automorphisms $\gamma_K\colon H_K \to H_K$ and $\gamma_k\colon H_k \to H_k$. Let $(\gamma_K)_*$ and $(\gamma_k)_*$ be the induced maps on the Grothendieck groups of H_K and H_k, respectively. Then we have the commutation rule:

$$d_k^H \circ (\gamma_K)_* = (\gamma_k)_* \circ d_k^H.$$

(This follows by an argument similar to that in the proof of Lemma 2.3.) For any simple H_K-module V let V' be the simple H_K-module with underlying vector space V but where $h \in H_K$ acts via $\gamma_K(h)$. The a-values of V, V' are related as follows. Recall that these values can be characterized in terms of the "leading coefficients" as in 2.4. Let $w_0 \in W$ be the longest element. Then we have

$$a_{V'} = a(w_0 w) \qquad \text{where } w \in W \text{ is such that } c_{w,V} \neq 0.$$

This follows from the identity $c_{w_0w,V'} = (-1)^{l(w)}c_{w,V}$ (for all $w \in W$), which in turn is proved using the defining equation and the formulas in [17, (3.3.2), (5.12.5)].

Lemma 4.2. *Let V_1, V_2 be simple H_K-modules and M be a simple H_k-module such that $[M]$ occurs as a summand in both $d_k^H([V_1])$ and $d_k^H([V_2])$. Assume that $a_M = a_{V_1}$.*

1. *We have $z_2 \leq_{LR} z_1$, for any $z_1, z_2 \in W$ with $\theta(c_{z_1,V_1}) \neq 0$, $\theta(c_{z_2,V_2}) \neq 0$.*
2. *We have $a_{V_1} \leq a_{V_2}$ and $a_{V_2'} \leq a_{V_1'}$; moreover, $a_{V_1} < a_{V_2} \Leftrightarrow a_{V_2'} < a_{V_1'}$.*

Proof. We have $a_{V_1} = a(z_1)$ and $a_{V_2} = a(z_2)$. Now let $w \in W$ be such that $a(w) = a_M$ and $C_w M \neq 0$. By Lemma 3.1(2), we have $z_1 \leq_{LR} w$ and $z_2 \leq_{LR} w$. Since $a(w) = a_M = a_{V_1} = a(z_1)$, this implies that $z_2 \leq_{LR} z_1$ (see the argument in the proof of Theorem 3.3).

Using 4.1, we see that $a_{V_1'} = a(w_0z_1)$ and $a_{V_2'} = a(w_0z_2)$.

By [14, Remark 3.3(a)], the map $z \mapsto w_0z$ reverses the preorder $\leq_{LR}$. Thus, the condition $z_2 \leq_{LR} z_1$ implies that $w_0z_1 \leq_{LR} w_0z_2$. By [18, Theorem 5.4], we have $x \leq_{LR} y \Rightarrow a(y) \leq a(x)$ for all $x, y \in W$. Thus, we conclude that $a(z_1) \leq a(z_2)$ and $a(w_0z_2) \leq a(w_0z_1)$.

If we have $a(z_1) = a(z_2)$, then the condition $z_2 \leq_{LR} z_1$ already implies that $z_1 \sim_{LR} z_2$; see [18, Corollary 6.3(b)]. But then we also have $w_0z_1 \sim_{LR} w_0z_2$ and so $a(w_0z_2) = a(w_0z_1)$. Reversing the argument shows that we have $a(z_1) = a(z_2)$ if and only if $a(w_0z_2) = a(w_0z_1)$. □

4.3. From now on, we assume that $k = F$ has characteristic 0, and that $\theta(v)$ is a primitive $2d$-th root of unity in F. With every simple H_K-module V we can attach an integer, its "d-defect", as follows. Let $P = \sum_{w \in W} v^{2l(w)}$ be the Poincaré polynomial and $D_V \in \mathbb{Q}[v^2]$ be the generic degree associated with V; it is known that D_V divides P_W (see [2]). Then the d-defect of V is the exponent of the largest power of $\Phi_{2d}(v)$ which divides P_W/D_V.

The Brauer graph associated with d_k^H is defined as follows. It has vertices labelled by the simple H_K-modules (up to isomorphism) and edges labelled by the simple H_k-modules (up to isomorphism). Two vertices, labelled by the simple modules V_1, V_2 say, are joined by an edge, labelled by the simple H_k-module M say, if $d_k([V_1])$ and $d_k([V_2])$ have $[M]$ as a common summand in $R_0(H_k)$. The connected components of this graph define partitions of the sets of isomorphism classes of simple modules for H_K and H_k, which are called "blocks".

By [9, Proposition 7.4], all simple H_K-modules belonging to a fixed block have the same d-defect. (Note that the proof of this result still seems to require the use of results from the representation theory of finite Chevalley groups.) This common d-defect is called the d-defect of the block.

It is shown in [9, Theorem 9.6] that in a block of d-defect 1 all decomposition numbers are 0 and 1, and the Brauer graph is an open polygon, i.e., a so-called "Brauer tree". However, this result does not yield the labelling of the vertices of that tree by the simple H_K-modules. For any simple H_K-module V, let a_V be the exponent of the largest power of v^2 which divides the generic degree of V. Now we can state:

Theorem 4.4. *Consider a block of d-defect 1 containing $e + 1 \geq 2$ simple H_K-modules (up to isomorphism). Label these modules by $V_1, \ldots, V_{e+1}$ such that the* *corresponding Brauer tree is given by*

Assume that $a_{V_1} \leq a_{V_{e+1}}$. Then the following conditions must be satisfied.

$$\dim V_1 < \dim V_2 \quad \textit{and} \quad \dim V_e > \dim V_{e+1} \quad (\textit{if } e \geq 2),$$
$$a_{V_1} \leq a_{V_2} < a_{V_3} < \ldots < a_{V_{e-1}} < a_{V_e} \leq a_{V_{e+1}}.$$

(Conversely, it is clear that these conditions determine the tree.) Moreover, if $w_i \in W$ is such that $c_{w_i,V_i} \neq 0$, then we must have $w_{e+1} \leq_{LR} \ldots \leq_{LR} w_1$.

Proof. First note, since the above graph has $e+1$ edges, our block contains precisely e simple H_k-modules (up to isomorphism), which we denote by $M_1, \ldots, M_e$ (from left to right). We now forget the chosen ordering of the modules V_i and M_j. The proof will proceed by reconstructing that original ordering from properties of the a-values and similar things.

To simplify the notation, we say that M_j is a modular constituent of V_i, if $[M_j]$ occurs as a summand in $d_k^H([V_i])$. Furthermore, we write $a_i = a_{V_i}$ for all i.

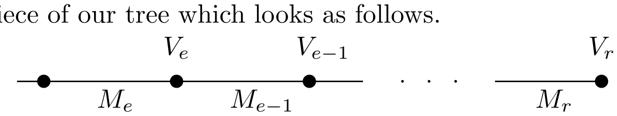
For the moment, the only assumption about the labelling of the modules V_i and M_j is the following. For $1 \leq j \leq e$, we assume that V_j has M_j as a modular constituent, that $a_{M_j} = a_j$, and that any other modular constituent of V_j has a strictly smaller a-value. Thus, the labelling is such that condition $(*)$ in Theorem 3.3 holds for $1 \leq i, j \leq e$. We proceed in several steps.

Step 1. We claim that there exists some $1 \leq r \leq e$ such that if we go from left to right on the Brauer tree, the simple H_K-modules are ordered as

$$V_1,\ V_2,\ \ldots,\ V_{r-1},\ V_{e+1},\ V_e,\ V_{e-1},\ \ldots,\ V_r,$$

where $a_{e+1} \geq a_e > \ldots > a_r$ and $a_{e+1} \geq a_{r-1} > \ldots > a_1$. In particular, this implies that if $e \geq 3$, then at least one of the two end points must be joined to a vertex with a strictly bigger a-value. We shall call this "property (E)".

To see this, we begin by constructing a piece of the tree and fixing further labellings of modules V_i and M_j as we go along. Assume that M_e is such that a_{M_e} is maximal among all a_{M_j}. Then M_e is a modular constituent of V_e and we have $a_{M_e} = a_e$. If V_e has no other modular constituent, then we stop and our piece is complete. Otherwise, we obtain a new modular constituent, and we assume that this is M_{e-1}. We have $a_{M_{e-1}} < a_e$, by condition $(*)$ in Theorem 3.3. Now consider V_{e-1}. We can argue in the same way as we just did for V_e. If V_{e-1} has no other modular constituent than M_{e-1}, then we stop; otherwise, we assume that the new constituent is M_{e-2}, and we have $a_{M_{e-2}} < a_{M_{e-1}} = a_{e-1}$. We go on until we have found a piece of our tree which looks as follows.



$(r \geq 1)$

where $a_e > a_{e-1} > \ldots > a_r$, $a_{M_j} = a_j$ for all j, and V_r labels an end point.

If $r = 1$, then the tree is complete. If $r > 1$, we proceed as follows. Let V_i be the second module which has M_e as a modular constituent. If we had $i \leq e$, then V_i would also have M_i as a modular constituent and this would yield $a_i = a_{M_i} > a_{M_e}$ (by condition $(*)$ in Theorem 3.3), contrary to the choice of M_e. So, we must have $i = e + 1$. Then V_{e+1} has another modular constituent, and we assume that this is M_{r-1}.

Now we can construct a second piece of our tree. The module M_{r-1} is a modular constituent of V_{r-1} and we have $a_{M_{r-1}} = a_{r-1}$. If V_{r-1} has no other modular constituents, then we stop and our piece is complete. Otherwise, we assume that the new constituent is M_{r-2}. We have $a_{M_{r-2}} < a_{r-1}$, by condition $(*)$ in Theorem 3.3. We can now go on as above, and obtain a piece of the tree which joins V_{e+1} with modules $V_{r-1}, V_{r-2}, \ldots$ until we arrive at an end point. Since our tree has just two end points, it is now complete, and the other end point must be V_1. Thus, the tree is labelled as desired. The fact that $a_{e+1} \geq a_e = a_{M_e}$ and $a_{e+1} \geq a_{r-1} = a_{M_{r-1}}$ follows from Lemma 3.1(2).

Step 2. Assume that $r = 1$. Then the simple H_K-modules are ordered on the tree as

$$V_{e+1},\ V_e,\ V_{e-1},\ \ldots,\ V_1,$$

where $a_{e+1} \geq a_e > \ldots > a_1$. It is clear that $\dim V_{e+1} < \dim V_e$ and $\dim V_1 < \dim V_2$ (if $e \geq 2$). Hence we have the desired properties. Moreover, since $a_{M_j} = a_j$ for $1 \leq j \leq e$, we have $w_{e+1} \leq_{LR} \ldots \leq_{LR} w_1$ by Lemma 4.2(1).

Step 3. Assume that $2 \leq r \leq e$. We then show that we must have $e = r = 2$. For the proof, we use the automorphism γ. The commutation rule in 4.1 implies that γ induces a symmetry of the Brauer graph of H.

For any i, let V_i' be the simple H_K-module with underlying vector space V_i but where $h \in H_K$ acts as $\gamma_K(h)$. Since γ induces a symmetry of the Brauer graph of H, the modules $\{V_1', \ldots, V_{e+1}'\}$ form a block of H, which also has d-defect 1. (The latter assertion follows from the fact that the generic degrees of V_i and V_i' are equal, up to a power of v^2; see [3, Theorem 71.17].) Thus, we have a corresponding Brauer tree, and if we go from left to right on that tree, the simple H_K-modules are ordered as

$$V_1',\ V_2',\ \ldots,\ V_{r-1}',\ V_{e+1}',\ V_e',\ V_{e-1}',\ \ldots,\ V_r'.$$

For any i, let $a_i' := a_{V_i'}$. We claim that now we have $a_{e+1}' \leq a_e' < \ldots < a_r'$ and $a_{e+1}' \leq a_{r-1}' < \ldots < a_1'$. Indeed, let $r \leq j \leq e$. Then, by the construction in Step 1, M_j is a modular constituent of V_{j+1} and V_j, and we have $a_{M_j} = a_j \leq a_{j+1}$. Hence Lemma 4.2(2) implies that $a_j' \geq a_{j+1}'$. Moreover, these inequalities are strict for $j < e$. The same argument also works for $1 \leq j \leq r-1$. Thus, the claim is established.

We have just proved that none of the two end points of the new Brauer tree is joined to a vertex with a strictly bigger a-value. But the discussion in Step 1 also applies to that new tree. Hence, if $e \geq 3$, we have a contradiction to "property (E)". So we must have $e = r = 2$, as desired.

Step 4. Finally, assume that $e = r = 2$. Thus, our tree has three vertices. In this case, the tree is clearly determined by the conditions on the dimensions of V_1, V_2, V_3. It remains to prove the assertion about the a-values and the w_i.

Since $r = 2$, the three vertices of the tree are labelled by V_1, V_3, V_2 (from left to right) and we must have $a_1 \leq a_3$ and $a_2 \leq a_3$. If both of these inequalities were strict, then again we apply γ and, by a similar argument as in Step 3, we find a new tree, which also has to satisfy the conditions that we obtained so far. But Lemma 4.2(2) shows that $a_1' > a_3' < a_2'$, a contradiction. So we have $a_1 = a_3$ or $a_2 = a_3$. Assume that $a_1 = a_3$. We have $a(w_i) = a_i$ for $i = 1, 2, 3$. Since $a_{M_1} = a_1$ we have $w_3 \leq_{LR} w_1$ by Lemma 4.2(1). Then the condition that $a_{M_1} = a_3$ implies that $w_1 \leq_{LR} w_3$, again by Lemma 4.2(1). Now we also have $w_3 \leq_{LR} w_2$, and so $w_1 \leq_{LR} w_3 \leq_{LR} w_2$, as desired. The argument for the case that $a_2 = a_3$ is completely analogous.

The above steps cover all possibilities. So the proof is complete. □

Remark 4.5. Suppose that 3.7(1) holds in general. Then it would follow that each of the two end points of the tree has an a-value which is different from that for the vertex adjacent to it. This can be seen to hold in all trees, as computed in [8, 9].

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