

## THE HOWE DUALITY AND THE PROJECTIVE REPRESENTATIONS OF SYMMETRIC GROUPS

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ABSTRACT. The symmetric group  $\mathfrak{S}_k$  possesses a nontrivial central extension, whose irreducible representations, different from the irreducible representations of  $\mathfrak{S}_k$  itself, coincide with the irreducible representations of the algebra  $\mathfrak{A}_k$  generated by indeterminates  $\tau_{i,j}$  for  $i \neq j$ ,  $1 \leq i, j \leq n$  subject to the relations

$$\begin{aligned} \tau_{i,j} &= -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset; \\ \tau_{i,j}\tau_{j,m}\tau_{i,j} &= \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \text{ for any } i, j, l, m. \end{aligned}$$

Recently M. Nazarov realized irreducible representations of  $\mathfrak{A}_k$  and Young symmetrizers by means of the Howe duality between the Lie superalgebra  $\mathfrak{q}(n)$  and the Hecke algebra  $H_k = \mathfrak{S}_k \circ Cl_k$ , the semidirect product of  $\mathfrak{S}_k$  with the Clifford algebra  $Cl_k$  on  $k$  indeterminates.

Here I construct one more analog of Young symmetrizers in  $H_k$  as well as the analogs of Specht modules for  $\mathfrak{A}_k$  and  $H_k$ .

### 1. SUMMARY

Lately, we have witnessed an increase of interest in the study of representations of symmetric groups. In particular, in their projective representations.

Recall that the symmetric group  $\mathfrak{S}_k$  has a nontrivial central extension whose irreducible representations do not reduce to those of  $\mathfrak{S}_k$  but coincide (may be identified) with the irreducible representations of the algebra  $\mathfrak{A}_k$  determined by generators  $\tau_{i,j}$  for  $i \neq j$ ,  $1 \leq i, j \leq n$  subject to the relations

$$(1.1) \quad \begin{aligned} \tau_{i,j} &= -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{m,l} = -\tau_{m,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{m,l\} = \emptyset; \\ \tau_{i,j}\tau_{j,m}\tau_{i,j} &= \tau_{j,m}\tau_{i,j}\tau_{j,m} = -\tau_{i,m} \text{ for any } i, j, l, m. \end{aligned}$$

In [N1] Nazarov realized irreducible representations of  $\mathfrak{A}_k$  by means of an orthogonal basis constructed in each of the spaces of the representations and indicating the action of the generators  $\tau_{i,i+1}$  on them (an analog of the Young orthogonal form). In [N2], with the help of an “odd” analog of the degenerate affine Hecke algebra Nazarov constructed elements of the algebra  $H_k = \mathfrak{S}_k \circ Cl_k$ , the semidirect product of  $\mathfrak{S}_k$  with the Clifford algebra  $Cl_k$  on  $k$  indeterminates. Certain elements of  $H_k$  serve as analogs of Young symmetrizers.

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Here I construct one more analog of Young symmetrizers in  $H_k$  as well as analogs of Specht modules (cf. [Ja]) for the algebras  $\mathfrak{A}_k$  and  $H_k$ . This construction is based on another form of expression of Young symmetrizers based on the notion of seminormal representation (an analog of Gelfand-Tsetlin basis) for the symmetric group (see [OV], [R], [Mu]) one can express Young symmetrizers in the form distinct from both the classical one (that of Weyl [W]) and that suggested in [Ju2]. Observe that the structure of Nazarov's symmetrizers is rather complicated; Jones simplified them [Jo1], [Jo2] but not completely. The symmetrizers constructed in this paper are simpler in structure. Our method is also applicable to the  $q$ -analogs of the Young symmetrizers of the usual and projective representations of  $\mathfrak{S}_k$ ; cf. [JN].

This paper is an expanded and rewritten (thanks to a shrewd referee) version of my short note math.RT/9810148. The reader might also be interested in the results of [Ya1] and [Ya2], slightly related with this paper; namely, [Ya1] describes  $\mathfrak{A}_k$  as a Howe dual of  $\mathfrak{q}(n)$  and in [Ya2] a Howe dual of  $\mathfrak{q}(m) \oplus \mathfrak{q}(n)$  is established to be the hyperoctahedral group.

1.1. For the general notion of a seminormal representation see [R]. In this section I will formulate it for  $\mathfrak{S}_k$  in convenient terms (cf. [OV]).

Let  $\mathfrak{S}_k$  be the symmetric group on  $k$  symbols,  $s_{ij} \in \mathfrak{S}_k$  the transposition. The Jucys–Murphy elements are defined as follows (cf. [Ju1], [Mu]):

$$x_1 = 0, \quad x_2 = s_{12}, \quad x_3 = s_{13} + s_{23}, \quad \dots \quad x_k = s_{1k} + s_{2k} + \dots + s_{k-1,k}.$$

As in [Ma], we will identify the partition  $\lambda$  of a positive integer  $k$  with a Young diagram of  $k$  cells. A *Young tableau of the form*  $\lambda$  is the Young diagram corresponding to  $\lambda$  filled in with the integers 1 to  $k$ . The tableau is called a *standard* one if the numbers that fill it do not decrease along the rows left to right and along the columns downwards.

For any partition  $\lambda$  denote by  $\mathcal{T}(\lambda)$  the set of standard tableaux of shape  $\lambda$ . For  $\Lambda \in \mathcal{T}(\lambda)$  and  $1 \leq p \leq k$ , we set  $c_\Lambda(p) = j - i$ , where  $(i, j)$  is the slot in  $\Lambda$  occupied by  $p$ .

**1.2. Theorem** ([Mu], [OV], [R]). *Let  $S^\lambda$  be the irreducible representation of  $\mathfrak{S}_k$  corresponding to the partition  $\lambda$ . Then in  $S^\lambda$  there exists a common eigenbasis for the Jucys–Murphy elements  $x_1, \dots, x_k$ ; the basis can be indexed with the elements of  $\mathcal{T}(\lambda)$  as follows:*

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda \quad \text{for any } \Lambda \in \mathcal{T}(\lambda) \text{ and } 1 \leq p \leq k.$$

Let us partially order the partitions with respect to *dominance*, i.e., we set

$$\mu \geq \lambda \iff \sum_{i=1}^l \mu_i \geq \sum_{i=1}^l \lambda_i \text{ for } l = 1, 2, \dots$$

Let  $\Lambda_c$  be a Young tableau of shape  $\lambda$  consecutively filled in along columns (that is where the subscript comes from) from left to right. Set

$$(1.1.1) \quad \kappa_{\Lambda_c} = \prod_{i=1}^k (j - x_i),$$

where  $j$  is the number of the column occupied by  $i$ . Since the elements  $x_1, \dots, x_k$  pairwise commute, the order of factors in (1.1.1) is irrelevant.

**Example 1.** For

$$\Lambda_c = \begin{matrix} & 1 & 4 & 7 \\ 2 & & 5 & 8 \\ 3 & & & 6 \end{matrix}$$

we have

$$\kappa_{\Lambda_c} = 1 \cdot (1 - x_2)(1 - x_3)(2 - x_4)(2 - x_5)(2 - x_6)(3 - x_7)(3 - x_8).$$

**1.3. Theorem.** *Let  $S^\mu$  be the irreducible representation of  $\mathfrak{S}_k$  corresponding to the partition  $\mu$  of  $k$ . Then*

- i)  $\kappa_{\Lambda_c}(S^\mu) = 0$  if  $\mu > \Lambda_c$ ;
- ii)  $\kappa_{\Lambda_c}(v_\Lambda) = 0$  if  $\Lambda_c \neq \Lambda$  for  $\Lambda \in \mathcal{T}(\lambda)$  and the basis vectors  $v_{\Lambda_c}$  as in Theorem 1.2;
- iii)  $\kappa_{\Lambda_c}(v_{\Lambda_c}) = 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n} v_{\Lambda_c}$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  are nonzero integers of the partition  $\lambda$ .

*Proof.* Let  $\mu \geq \lambda$  and  $\Lambda$  be a standard tableau of shape  $\mu$ ; let  $v_\Lambda \in S^\mu$  be the corresponding vector for the seminormal basis (i.e., as in section 1.2). If  $\Lambda \neq \Lambda_c$ , then there exists a  $j$  such that the  $j$ -th columns of the tableaux  $\Lambda$  and  $\Lambda_c$  are distinct whereas all the columns with lesser numbers are identical. Since the orders of the tableaux coincide and  $\mu \geq \lambda$ , it follows that  $\mu'_j \leq \lambda'_j$  (these are the lengths of the  $j$ -th columns of the tableaux  $\Lambda$  and  $\Lambda_c$ , respectively).

Let  $p$  be the least of the numbers from the  $j$ -th column of  $\Lambda$  (such a number exists because  $\Lambda \neq \Lambda_c$  and  $\mu'_j \leq \lambda'_j$ ). Since  $\Lambda$  is standard, the slot that  $p$  occupies is  $(1, j + 1)$ . Hence,

$$x_p v_\Lambda = c_\Lambda(p) v_\Lambda = (j + 1 - 1) v_s = j v_\Lambda, \quad \text{i.e., } (j - x_p) v_\Lambda = 0.$$

Since  $j - x_p$  enters  $\kappa_{\Lambda_c}$  as a factor,  $\kappa_{\Lambda_c} v_\Lambda = 0$ . This proves i) and ii).

To prove iii), observe that

$$(j - x_p) v_{\Lambda_c} = (j - (j - i)) v_{\Lambda_c} = i v_{\Lambda_c},$$

where  $i$  is the number of the row occupied by  $x_p$ . This proves iii). □

**1.4. Lemma.** *Let  $p < k$  and let  $\mathfrak{S}_p$  be considered as the subgroup of  $\mathfrak{S}_k$  that preserves all but the first  $p$  symbols. Let  $U$  be a  $\mathfrak{S}_k$ -module,  $\varepsilon$  the nontrivial one-dimensional representation and  $a = \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \sigma$ . Let  $v \in U$  be a nonzero vector fixed by a transposition  $s_{lk}$  for  $l \leq p$ . Then*

$$a \left( \sum_{i \in \mathfrak{S}_p} s_{ik} \right) v = av.$$

*Proof.* Let  $P = \{1, \dots, p\}$ . Then

$$\begin{aligned} a \left( \sum_{i \in P} s_{ik} \right) v &= a \left( \sum_{i \in P} s_{ik} \right) s_{lk} v \\ &= a \left( 1 + \sum_{i \in P \setminus \{l\}} s_{ik} s_{kl} \right) v = a \left( 1 + \sum_{i \in P \setminus \{l\}} s_{li} s_{ik} \right) v \\ &= a \left( 1 - \sum_{i \in P \setminus \{l\}} s_{ik} \right) v = a \left( 2 - \sum_{i \in P} s_{ik} \right) v; \end{aligned}$$

hence,  $a \left( \sum_{i \in P} s_{ik} \right) v = av$ . □

**1.5. Corollary.** *Let  $R_{\Lambda_c}$  and  $C_{\Lambda_c}$  be the row and column stabilizers of the tableau  $\Lambda_c$ ; let  $U$  be a  $\mathfrak{S}_k$ -module and  $v \in U$  a nonzero  $R_{\Lambda_c}$ -invariant vector. Set  $a_{\Lambda_c} = \sum_{\sigma \in C_{\Lambda_c}} \varepsilon(\sigma)\sigma$ . Then  $\kappa_{\Lambda_c}v = a_{\Lambda_c}v$ .*

*Proof.* Induction on the number of columns in  $\Lambda_c$ . If  $\Lambda_c$  consist of one column, then by induction on  $k$  we easily verify the identity

$$(1 - s_{12})(1 - s_{13} - s_{23}) \dots (1 - s_{1k} - \dots - s_{k-1,k}) = \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\sigma.$$

So for one column the statement is true.

Let tableau  $\Lambda_c$  consist of  $n$  columns  $\Lambda_1, \dots, \Lambda_n$ ; let  $\Lambda_c^*$  be obtained from  $\Lambda_c$  by deleting the last column. Then

$$\begin{aligned} \kappa_{\Lambda_c}v &= \kappa_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i)v = \prod_{i \in \Lambda_n} (n - x_i) \cdot \kappa_{\Lambda_c^*}v \\ &= (\text{by induction}) = \prod_{i \in \Lambda_n} (n - x_i) \cdot a_{\Lambda_c^*}v = a_{\Lambda_c^*} \prod_{i \in \Lambda_n} (n - x_i)v. \end{aligned}$$

Let  $i \in \Lambda_n$ ; then

$$a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(n - \sum_{\alpha \in \Lambda_1} s_{\alpha i} - \sum_{\alpha \in \Lambda_2} s_{\alpha i} - \dots - \sum_{\alpha \in \Lambda_n, \alpha < i} s_{\alpha i})v.$$

By Lemma 1.4  $a_{\Lambda_c^*}(\sum_{\alpha \in \Lambda_j} s_{\alpha i})v = a_{\Lambda_c^*}v$  for  $j = 1, \dots, n-1$ , so  $a_{\Lambda_c^*}(n - x_i)v = a_{\Lambda_c^*}(1 - \sum_{\alpha \in \Lambda_n, \alpha < i} s_{\alpha i})v$ . Hence, if  $\Lambda_n = \{i, i+1, \dots, i+l\}$ , then

$$\begin{aligned} \kappa_{\Lambda_c}v &= a_{\Lambda_c^*}(1 - s_{i,i+1})(1 - s_{i,i+2} - s_{i+1,i+2}) \dots (1 - s_{i,i+l} - \dots - s_{i+l-1,i+l})v \\ &= a_{\Lambda_c^*}(\sum_{\sigma \in \Lambda_n} \varepsilon(\sigma)\sigma)v = a_{\Lambda_c}v. \end{aligned}$$

□

Thus, if  $v = \sum_{\sigma \in R_{\Lambda_c}} \sigma = b_{\Lambda_c}$ , then

$$e_{\Lambda_c} = a_{\Lambda_c}b_{\Lambda_c} = \kappa_{\Lambda_c}b_{\Lambda_c},$$

i.e., we have expressed the Young symmetrizer  $e_{\Lambda_c}$  as  $\kappa_{\Lambda_c}b_{\Lambda_c}$ . In §3 a similar construction will be used to construct projective analogs of Young symmetrizers.

## 2. AUXILIARY DATA

Let  $\mathfrak{S}_k$  be the symmetric group,  $Cl_k$  the Clifford algebra generated by  $k$  indeterminates  $p_1, \dots, p_k$  subject to the relations

$$p_i^2 = -1, p_i p_j + p_j p_i = 0 \text{ for } i \neq j.$$

The symmetric group acts on  $Cl_n$  permuting the generators, so we can form a semidirect product  $H_k = \mathfrak{S}_k \circ Cl_k$ , i.e., the linear span of the elements  $f\sigma$ , where  $f \in Cl_k$  and  $\sigma \in \mathfrak{S}_k$  with the multiplication satisfying

$$\sigma f = f^\sigma \cdot \sigma \text{ (here } f^\sigma \text{ is the result of the action of } \sigma \in \mathfrak{S}_k \text{ on } f \in Cl_k).$$

Set

$$\tau_{i,j} = \frac{1}{\sqrt{2}}(p_i - p_j)s_{i,j}.$$

As is not difficult to verify, the relations (1.1) hold; hence, the algebra generated by the  $\tau_{i,j}$  is isomorphic to  $\mathfrak{A}_n$ . Besides,  $\mathfrak{A}_n$  supercommutes with  $Cl_n$ ; hence,  $H_n = \mathfrak{A}_n \otimes Cl_n$ , as *superalgebras* if we define parity in  $\mathfrak{A}_n$  by setting  $p(\tau_{i,j}) = \bar{1}$ .

Let  $\mathbb{N}$  be the set of positive integers,  $\bar{\mathbb{N}}$ , another, “odd”, copy of  $\mathbb{N}$ ; its elements are barred. Let  $V$  be a superspace of superdimension  $(n, n)$  with the fixed basis  $\{e_i\}_{i=1}^n \amalg \{\bar{e}_i\}_{i=1}^{\bar{n}}$ . Define the odd operator  $Q : V \rightarrow V$  by setting

$$Q(e_i) = e_{\bar{i}}; \quad Q(\bar{e}_i) = -e_i.$$

The (super)centralizer of  $Q$  in  $\text{Mat}(V)$  is denoted by  $Q(V)$ ; cf. [BL]. We denote the Lie superalgebras associated with the associative superalgebras  $\text{Mat}(V)$  and  $Q(V)$  by  $\mathfrak{gl}(V)$  and  $\mathfrak{q}(V)$ , respectively. We set  $\mathfrak{q}(n) = \mathfrak{q}(V)$  for  $n = \dim V$ .

Select a basis in  $\mathfrak{q}(n)$ : let the  $e_i^*$  be the left dual basis to the  $\{e_i : i = 1, \dots, n; \bar{1}, \dots, \bar{n}\}$ ; set

$$E_{i,j} = e_i \otimes e_j^* + \bar{e}_i \otimes \bar{e}_j^*; \quad F_{i,j} = e_i \otimes \bar{e}_j^* + \bar{e}_i \otimes e_j^*.$$

Then  $\mathfrak{h} = \text{Span}(E_{i,i} \text{ and } F_{i,i} : i = 1, \dots, n)$  is a Cartan subalgebra in  $\mathfrak{q}(n)$  and  $\mathfrak{b} = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i \leq j)$  is a Borel subalgebra; cf. [Pe]. Set further  $\mathfrak{n}_- = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i > j)$  and  $\mathfrak{n}_+ = \text{Span}(E_{i,j} \text{ and } F_{i,j} : i < j)$ . Then, clearly,  $\mathfrak{q}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ ; any finite dimensional irreducible  $\mathfrak{q}(n)$ -module  $U = U^\lambda$  is uniquely, up to an isomorphism, determined by its highest weight and the irreducible  $\mathfrak{h}$ -module  $U^+$  of highest weight vectors annihilated by  $\mathfrak{n}_+$ , and for any  $u \in U^+$  we have  $hu = \lambda(h)u$  for any  $h \in \mathfrak{h}_0$ ; cf. [Pe]. Observe that since  $\mathfrak{h}$  is, unlike the case of Lie algebras, not commutative,  $U^+$  is seldom one-dimensional; cf. [S3]. This module  $U$  will be denoted by  $U^\lambda$ .

We will stick to the following terminology; cf [BL], [L]. The representation of a superalgebra  $A$  in the superspace  $V$  is *irreducible* if it does not contain homogeneous (with respect to parity) subrepresentations distinct from 0 and  $V$  itself.

Thus, there are two types of irreducible representations: those that do not contain any nontrivial subrepresentations (called of *general type* or of type  $G$ ) and those that contain *inhomogeneous* invariant subspaces (call them of *type Q*). If  $V$  is of finite dimension, then in the first case its centralizer, as of  $A$ -module, is isomorphic to  $\mathfrak{gl}(1)$ , whereas in the second case to  $\mathfrak{q}(1)$ .

Let  $V_1$  and  $V_2$  be finite dimensional irreducible modules over  $A_1$  and  $A_2$ , respectively. Then  $V_1 \otimes V_2$  is an irreducible  $A_1 \otimes A_2$ -module except for the case where both  $V_1$  and  $V_2$  are of type  $Q$ . In the latter case, the centralizer of the  $A_1 \otimes A_2$ -module  $V_1 \otimes V_2$  is isomorphic to  $Cl_2$ , the Clifford superalgebra with 2 generators. (For further interesting details see [Sch].)

If  $e \in Cl_2$  is a minimal idempotent, then  $e(V_1 \otimes V_2)$  is an irreducible  $A_1 \otimes A_2$ -module of type  $G$  that we will denote by  $2^{-1}(V_1 \otimes V_2)$ .

Let  $\mathbb{M} = \mathbb{N} \amalg \bar{\mathbb{N}}$  be ordered as follows:

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 \dots$$

We will call the elements from  $\mathbb{N}$  “even” and those from  $\bar{\mathbb{N}}$  “odd” ones; so we can encounter an “even” odd element, etc.

The partition  $\lambda$  of  $k$  is called a *strict* one if its parts are pairwise distinct. To a strict partition  $\lambda$  we assign a *shifted diagram* (see [Ma]) obtained from the ordinary diagram by shifting its  $i$ -th row by  $i - 1$  cell to the right for  $i = 1, 2, \dots$ . A shifted tableau of shape  $\lambda$  is a filling of a shifted diagram of shape  $\lambda$  with numbers from  $\mathbb{M}$ . The shifted tableau is called a *standard* one if:

i) the numbers that fill it do not decrease from left to right along rows and downwards along columns;

ii) the symbols from  $\mathbb{N}$  strictly increase along columns (downwards);

iii) the symbols from  $\overline{\mathbb{N}}$  strictly increase along rows (from left to right).

For the strict partition  $\lambda$  denote by  $ST(\lambda)$  the set of strict standard tableaux of shape  $\lambda$  filled with the numbers  $1, \dots, k$ , where  $k = \lambda_1 + \lambda_2 + \dots$ . If  $\Lambda \in ST(\lambda)$  and  $1 \leq p \leq k$ , then set  $c_\Lambda(p) = \sqrt{(j-i)(j-i+1)}$ , where  $(i, j)$  is the slot in  $\Lambda$  occupied by  $p$ .

Denote by  $l(\lambda)$  the number of nonzero parts of the partition  $\lambda$  and set

$$\delta(\lambda) = \begin{cases} 0 & \text{if the number of nonzero parts of } \lambda \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$$

As is shown in [S1], the algebras  $\mathfrak{q}(n)$  and  $H_k$  constitute a Howe-dual pair in the superspace  $W_k = V^{\otimes k}$ . The  $\mathfrak{q}(n)$ -action is uniquely determined by the identity action in  $V$  extended as a superdifferentiation onto the tensor algebra  $T(V) = \sum_{k \geq 0} W_k$ . The symmetric group  $\mathfrak{S}_k$  acts on  $W_k$  by the formula

$$\begin{aligned} s_{i,i+1}(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_k) \\ = (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k, \end{aligned}$$

where  $p(v_i)$  and  $p(v_{i+1})$  are parities of respective vectors. Recall that  $\mathfrak{q}(n)$  is defined as the centralizer of an odd operator  $Q$  whose square is  $-\text{id}$ . The element  $p_i \in H_k$  acts in  $W_k$  as the operator

$$1 \otimes \cdots \otimes 1 \otimes Q \otimes 1 \otimes \cdots \otimes 1 \quad (Q \text{ occupies the } i\text{-th slot}).$$

**2.1. Lemma.** *Let  $w \in W_{k-1}$ , set  $\pi_k = \sum_{\alpha < k} \tau_{\alpha k}$ , and also*

$$E_{im}^{(2)} = \sum_{j=1}^n (E_{ij}E_{jm} - F_{ij}F_{jm}), \quad F_{im}^{(2)} = \sum_{j=1}^n (F_{ij}E_{jm} - E_{ij}F_{jm}).$$

Then

$$\begin{aligned} \text{i) } \pi_k(w \otimes e_i) &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j); \\ \text{ii) } \pi_k^2(w \otimes e_i) &= \frac{1}{2} \sum_{m=1}^n \left( (E_{im}^{(2)} w) \otimes e_m + (F_{im}^{(2)} w) \otimes e_{\overline{m}} \right) + \sum_{m=1}^n (E_{mm} w) \otimes e_i. \end{aligned}$$

*Proof.* i) It suffices to assume that  $w = e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}$ , where all the indices  $i_1, \dots, i_{k-1}$  are ‘‘even’’, i.e., belong to  $\mathbb{N}$ , not  $\overline{\mathbb{N}}$ . Then

$$\begin{aligned} \pi_k(w \otimes e_i) &= \sum_{\alpha < k} \tau_{\alpha k}(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) s_{\alpha k}(w \otimes e_j) \\ &= \frac{1}{\sqrt{2}} \sum_{\alpha < k} (p_\alpha - p_k) w_\alpha^i \otimes e_{i_\alpha}, \end{aligned}$$

where  $w_\alpha^i$  is obtained from  $w = e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}$  by replacing  $e_{i_\alpha}$  with  $e_i$ . Hence,

$$\pi_k(w \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{j=1}^n \sum_{i_\alpha=j} (p_\alpha - p_k) w_\alpha^i \otimes e_j.$$

But, on the other hand,  $\sum_{i_\alpha=j} w_\alpha^i = E_{ij}w$  and  $\sum_{i_\alpha=j} p_\alpha w_\alpha^i = F_{ij}w$  implying that

$$\begin{aligned} \pi_k(w \otimes e_i) &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij}w \otimes e_j - E_{ij}w \otimes e_{\bar{j}}) \\ &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij}(w \otimes e_j) - w \otimes e_{\bar{i}} - p_k E_{ij}(w \otimes e_{\bar{i}}) + w \otimes e_{\bar{i}}) \\ &= \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j) \end{aligned}$$

and heading i) is proven.

ii)

$$\begin{aligned} \pi_k^2(w \otimes e_i) &= \pi_k \left( \frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij})(w \otimes e_j) \right) \\ &= -\frac{1}{\sqrt{2}} \sum_{j=1}^n (F_{ij} - p_k E_{ij}) \pi_k(w \otimes e_j) \\ &= -\frac{1}{2} \sum_{j=1}^n (F_{ij} - p_k E_{ij}) \sum_{m=1}^n (F_{jm} - p_k E_{jm})(w \otimes e_m) \\ &= -\frac{1}{2} \sum_{j,m=1}^n (F_{ij} - p_k E_{ij})(F_{jm} - p_k E_{jm})(w \otimes e_m) \\ &= -\frac{1}{2} \sum_{j,m=1}^n ((F_{ij}F_{jm} - E_{ij}E_{jm}) - p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m) \\ &= \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm} - F_{ij}F_{jm}) + p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m). \end{aligned}$$

Let us calculate the sum by parts. We have

$$\begin{aligned} &(E_{ij}E_{jm} - F_{ij}F_{jm})(w \otimes e_m) \\ &= E_{ij}(E_{jm}w \otimes e_m + w \otimes e_j) - F_{ij}(F_{jm}w \otimes e_m + w \otimes e_{\bar{j}}) \\ &= E_{ij}E_{jm}w \otimes e_m + E_{jm}w \otimes E_{ij}e_m + E_{ij}w \otimes e_j + w \otimes e_i \\ &\quad - F_{ij}F_{jm}w \otimes e_m + F_{jm}w \otimes F_{ij}e_m - F_{ij}w \otimes e_{\bar{j}} - w \otimes e_i \\ &= E_{ij}E_{jm}w \otimes e_m - F_{ij}F_{jm}w \otimes e_m + E_{mm}w \otimes e_i \\ &\quad + F_{mm}w \otimes e_{\bar{i}} + E_{ij}w \otimes e_j - F_{ij}w \otimes e_{\bar{j}}. \end{aligned}$$

and

$$\begin{aligned} &p_k(E_{ij}F_{jm} - F_{ij}E_{jm})(w \otimes e_m) \\ &= p_k E_{ij}(F_{jm}w \otimes e_m + w \otimes e_{\bar{j}}) - p_k F_{ij}(E_{jm}w \otimes e_m + w \otimes e_j) \\ &= -E_{ij}E_{jm}w \otimes e_{\bar{m}} - F_{mm}w \otimes e_{\bar{i}} - E_{ij}w \otimes e_j - w \otimes e_i \\ &\quad + F_{ij}E_{jm}w \otimes e_{\bar{m}} + E_{mm}w \otimes e_i + F_{ij}w \otimes e_{\bar{j}} + w \otimes e_i \\ &= (F_{ij}E_{jm} - E_{ij}F_{jm})w \otimes e_{\bar{m}} + E_{mm}w \otimes e_i \\ &\quad + F_{ij}w \otimes e_{\bar{j}} - E_{ij}w \otimes e_j - F_{mm}w \otimes e_{\bar{i}}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm} - F_{ij}F_{jm}) + p_k(E_{ij}F_{jm} - F_{ij}E_{jm}))(w \otimes e_m) \\ &= \frac{1}{2} \sum_{j,m=1}^n ((E_{ij}E_{jm}w \otimes e_m - F_{ij}F_{jm}w \otimes e_m) \\ & \quad + (F_{ij}E_{jm} - E_{ij}F_{jm})w \otimes e_{\bar{m}} + 2E_{mm}w \otimes e_i) \\ &= \frac{1}{2} \sum_{m=1}^n (E_{im}^{(2)}w \otimes e_m + F_{im}^{(2)}w \otimes e_{\bar{m}}) + \sum_{m=1}^n E_{mm}w \otimes e_i. \end{aligned}$$

□

2.2. For the convenience of the reader we formulate the main result of [S1] in the following suitable form. Observe that if  $\lambda$  is a strict partition of  $k$  and  $\lambda_{n+1} = 0$ , then  $\lambda$  can be interpreted as a linear functional on  $\mathfrak{h}_{\bar{0}}$ : set

$$(2.2.1) \quad \lambda(E_{ii}) = \lambda_i.$$

**Theorem.** *The superalgebras  $H_k$  and  $\mathfrak{q}(n)$  are mutual centralizers in the superspace  $W_k$  and the following decomposition takes place:*

$$(2.2.2) \quad W_k = \bigoplus_{\lambda} 2^{-\delta(\lambda)} T^{\lambda} \otimes V^{\lambda},$$

where  $\lambda$  runs over strict partitions of  $k$  such that  $\lambda_{n+1} = 0$ ; here  $T^{\lambda}$  is an irreducible (in the super sense)  $H_k$ -module corresponding to  $\lambda$ , and  $V^{\lambda}$  an irreducible  $\mathfrak{q}(n)$ -module.

If  $\delta(\lambda) = 0$ , then the corresponding representation  $T^{\lambda}$  is of type  $G$ , it is of type  $Q$  if  $\delta(\lambda) = 1$ .

The dimension of the space (superstructure ignored) of the highest weight vectors  $(V^{\lambda})^+$  in  $V^{\lambda}$  is equal to  $2^{(l(\lambda)+\delta(\lambda))/2}$  copies of  $T^{\lambda}$ .

**2.3. Corollary.** *The  $H_k$ -module of highest weight vectors in  $W_k$  is a free  $Cl_k$ -module.*

*Proof.* By Theorem  $W_k^+ = \bigoplus_{\lambda} 2^{-\delta(\lambda)} T^{\lambda} \otimes (V^{\lambda})^+$ , so the  $H_k$ -module of highest weight vectors in  $W_k$  is equal to  $2^{-\delta(\lambda)} T^{\lambda} \otimes (V^{\lambda})^+$ . By considering the dimension of  $(V^{\lambda})^+$  and taking into account the type ( $G$  or  $Q$ ) of its irreducibility we get the statement desired. □

2.4. Let us denote the  $H_k$ -module of highest weight vectors of weight  $\lambda$  in  $W_k$  by  $R^{\lambda}$ .

**Lemma.** *Let  $\mathfrak{g} = \mathfrak{q}(n)$ ; let  $\mathfrak{b}$  and  $\mathfrak{h}$  be defined as above and  $V$  a  $\mathfrak{g}$ -module on which  $\mathfrak{h}_{\bar{0}}$  acts diagonally. Let  $(V^{\lambda})^+$  be the set of  $\mathfrak{b}$ -highest vectors of weight  $\lambda$ .*

*If  $u \in U(\mathfrak{g})$  and  $u((V^{\lambda})^+) \subset (V^{\lambda})^+$ , then there exists  $w \in U(\mathfrak{h})$  such that  $u|_{(V^{\lambda})^+} = w|_{(V^{\lambda})^+}$ .*

*Proof.* Let  $u = \sum u_{\alpha}$  be the weight decomposition of  $u \in U(\mathfrak{g})$  with respect to  $\mathfrak{h}_{\bar{0}}$ . Observe that  $(V^{\lambda})^+$  is the  $\lambda$  weight space of  $V$ . Therefore, if  $v \in V_{\lambda}$ , then  $u(v) = \sum_{\alpha} u_{\alpha}v$  and, if  $u_{\alpha}v \neq 0$ , then the weight of  $u_{\alpha}v$  is equal to  $\lambda + \alpha$ . Thus, thanks to the condition  $u(v) \in (V^{\lambda})^+$ , we may assume that  $u = u_0 \in U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$ , where  $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$  is the centralizer of  $\mathfrak{h}_{\bar{0}}$ .

Further, thanks to [S2], we know that  $U(\mathfrak{g})^{\mathfrak{h}_0} \cong U(\mathfrak{h}) \oplus L$ , where  $L = U(\mathfrak{g})^{\mathfrak{h}_0} \cap U(\mathfrak{g})\mathfrak{b}^+$  and where  $\mathfrak{b}^+$  is the linear span of the positive roots in  $\mathfrak{b}$ , is a two-sided ideal in  $U(\mathfrak{g})^{\mathfrak{h}_0}$ .

Hence,  $u = w + u_1$ , where  $w \in U(\mathfrak{h})$  and  $u_1 \in L$ ; this implies that  $uv = vw$  for  $v \in (V^\lambda)^+$ . □

Let  $T$  be a finite dimensional  $H_k$ -module. The  $\mathfrak{q}(n)$ -module  $U$  corresponding to it via Howe duality is determined by the formula

$$U = \text{Hom}_{H_k}(T, W_k) \text{ for } W_k = V^{\otimes k}.$$

It is not difficult to deduce from Theorem 2.2 that likewise  $T$  is uniquely recoverable from  $U$ , namely,

$$T = \text{Hom}_{\mathfrak{q}(n)}(U, W_k).$$

Further on, since  $U$  is a  $\mathfrak{q}(n)$ -submodule of  $W_k$ , it follows that all its weights with respect to  $\mathfrak{h}_0$  are of the form  $m_1\varepsilon_1 + \dots + m_n\varepsilon_n$ , where the  $\varepsilon_i$  are the weights of the standard  $\mathfrak{q}(n)$ -module  $V$  and the  $m_i$  are nonnegative integers. This implies that the formal character of the  $\mathfrak{q}(n)$ -module  $U$  equal to  $\sum m_\chi e^\chi$  (here  $\chi$  runs over the set of weights of  $U$  with respect to  $\mathfrak{h}_0$ ) is a polynomial in  $x_1 = e^{\varepsilon_1}, \dots, x_n = e^{\varepsilon_n}$ . This polynomial is called the *characteristic* of  $T$ .

**Example 2.** Consider the Clifford superalgebra  $Cl_k$  as a module over  $H_k$  ( $\mathfrak{S}_k$  acts by permuting the generators of  $Cl_k$  while  $Cl_k$  acts on itself via the left regular representation). The corresponding characteristic  $q_k$  is the coefficient of  $t^k$  in the decomposition

$$\frac{\prod_{i=1}^n (1 + x_i t)}{\prod_{i=1}^n (1 - x_i t)} = \sum_{k \geq 0} q_k(x) t^k.$$

In [S1] I showed that the  $x$ -character of  $T^\lambda$  for the strict partition  $\lambda$  is equal to  $2^{(l(\lambda) - \delta(\lambda))/2} Q_\lambda(x)$ , where  $Q_\lambda(x)$  is the projective Schur function (for its definition see [Ma]). Therefore, the characteristic of  $R^\lambda$  is equal precisely to  $Q_\lambda(x)$ .

2.5. Set  $P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x)$ . Denote by  $SST(\lambda)$  the set of standard tableaux of arbitrary shape with elements from  $\mathbb{M}$  and such that

- a) scanning the tableau from left to right and downwards the last of the symbols encountered should be an “even” one;
- b) for every  $i = 1, 2, \dots$  the total number of occurrences of any of the symbols  $i$  or  $\bar{i}$  is equal to  $\lambda_i$ .

**Lemma.** *Let  $\lambda$  be a strict partition of  $k$  and  $\lambda_1 > \dots > \lambda_n > 0$ . Then*

$$P_{\lambda_1}(x) \dots P_{\lambda_n}(x) = \sum_{\mu} m_{\lambda\mu} P_{\mu}(x),$$

where the sum runs over strict partitions  $\mu$  such that  $\mu \geq \lambda$  (with respect to dominance partial order) and  $m_{\lambda\mu}$  is equal to the number of tableaux from  $SST(\lambda)$  of shape  $\mu$ .

Proof of this statement is an easy corollary of the multiplication table for projective Schur functions; cf. [P], §4.

2.6. **Even and odd Jucys-Murphy elements** (cf. [N2]). Set

$$x_1 = 0, x_2 = s_{12}(1 + p_1p_2), \dots, x_k = \sum_{i < k} s_{ik}(1 + p_ip_k).$$

The *odd analogs* of the Jucys-Murphy elements  $\pi_i$  are defined as

$$\pi_1 = 0, \pi_2 = \tau_{12}, \dots, \pi_k = \sum_{i < k} \tau_{ik}.$$

Observe that  $\pi_i = -\frac{1}{\sqrt{2}}p_ix_i$  and the distinction of  $\pi_i$  from  $x_i$  is that the former supercommute with the elements of the Clifford superalgebra  $Cl_k$ .

The following statement proven by M. Nazarov in [N2] (Th. 7.2) describes the action of  $x_i$  in  $R^\lambda$ .

**Theorem.** *The  $Cl_k$ -module  $R^\lambda$  is free with a basis  $\{v_\Lambda : \Lambda \in ST(\lambda)\}$  and*

$$x_p v_\Lambda = c_\Lambda(p)v_\Lambda \text{ for } 1 \leq p \leq k.$$

### 3. SPECHT MODULES OVER $H_k$

Let  $\lambda$  be a strict partition of  $k$  and  $\lambda_{n+1} = 0$ . Define the functional  $\lambda$  on the Cartan subalgebra  $\mathfrak{h}$  by setting

$$\lambda(E_{ii}) = \lambda_i; \quad \lambda(F_{ii}) = 0.$$

In  $W_k = V^{\otimes k}$ , where  $V$  is the standard  $\mathfrak{q}(n)$ -module of dimension  $(n, n)$ , consider the  $H_k$ -submodule  $M^\lambda$  consisting of the vectors of weight  $\lambda$ , i.e.,  $w \in W_k$  such that  $E_{ii}w = \lambda_i w$ . Let  $\Lambda_r$  be a shifted tableau of shape  $\lambda$  filled in consecutively along rows from left to right with the numbers 1 to  $k$ .

Let  $\Lambda$  be an arbitrary shifted tableau of shape  $\lambda$  and let  $R_\Lambda$  be the row stabilizer of  $\Lambda$ . Let  $I$  be the sequence obtained by reading the tableau  $\Lambda$  from left to right and downwards. For  $i \in I$  define

$$\pi_i = \sum_{\alpha \in I, \alpha \text{ precedes } i} \tau_{\alpha,i}.$$

Set

$$\kappa_\Lambda = \prod_{i \in I} \left( \frac{1}{2}j(j+1) - \pi_i^2 \right),$$

where  $j$  is the number of the column occupied by  $i$ .

**Example 3.** For

$$\Lambda = \begin{array}{cccc} 1 & 2 & 4 & 7 \\ & 3 & 5 & 8 \\ & & & 6 \end{array}$$

we have

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(3 - \pi_3^2)(6 - \pi_4^2)(6 - \pi_5^2)(6 - \pi_6^2)(10 - \pi_7^2)(10 - \pi_8^2).$$

For every shifted tableau  $\Lambda$  of shape  $\lambda$  denote by  $w_\Lambda$  the vector from  $W_k$  equal to  $v_1 \otimes \dots \otimes v_k$ , where  $v_p = e_i$  if  $p$  occupies the  $i$ -th row. Clearly,  $w_\Lambda \in M^\lambda$ .

**3.1. Theorem.**  $\kappa_\Lambda(M^\lambda) = Cl_k \kappa_\Lambda(w_\Lambda)$  and  $\kappa_\Lambda(w_\Lambda) \in R^\lambda$  is nonzero.

*Proof.* Since  $M^\lambda$  is the module induced from the trivial representation of the row stabilizer of  $\Lambda_r$ , it follows that

$$\text{Hom}_{H_k}(T, W_k) = S^{\lambda_1}(V) \otimes \dots \otimes S^{\lambda_n}(V),$$

where  $S^p$  is the operator of the  $p$ -th symmetric power and  $V$  is the standard  $\mathfrak{q}(n)$ -module. Hence, the characteristic of  $M^\lambda$  is equal to  $q_{\lambda_1} \dots q_{\lambda_n}$  (see (2.4)). Lemma 2.5 implies that

$$q_{\lambda_1} \dots q_{\lambda_n} = \sum_{\mu \geq \lambda} 2^{l(\lambda)-l(\mu)} m_{\lambda\mu} Q_\mu.$$

Hence,  $M^\lambda$  is equal to the direct sum of modules isomorphic to either  $T^\mu$  or  $\Pi(T^\mu)$ . Further, let  $\varphi_i : M^\lambda \rightarrow M^\lambda$  be a homomorphism such that  $\varphi_i(v_{\Lambda_r}) = p_{\Lambda_r}^i$  for  $i = 1, \dots, l(\lambda)$ , where  $p_{\Lambda_r}^i = \sum_{j \text{ lies in the } i\text{-th row}} p_j$ .

It is not difficult to verify that  $\varphi_1, \dots, \varphi_{l(\lambda)}$  generate the Clifford algebra  $Cl_{l(\lambda)}$ . Therefore,  $M^\lambda$  is equal to the direct sum of modules isomorphic to  $R^\mu$  (or  $\Pi(R^\mu)$ ) for  $\mu \geq \lambda$  and the multiplicity of  $R^\lambda$  in  $M^\lambda$  is equal to 1.

It suffices to consider the case when  $\Lambda = \Lambda_c$  is a shifted tableau of shape  $\lambda$  filled from left to right and downwards, as usual. In each module  $R^\mu$ , select a basis  $\{v_\Lambda\}$ , where  $\Lambda$  runs over standard shifted tableaux, as in Theorem 2.6. Let us demonstrate that  $\kappa_{\Lambda_c}(R^\mu) = 0$  if  $\mu > \lambda$  and  $\kappa_{\Lambda_c}(R^\lambda) = Cl_k v_{\Lambda_c}$ .

Consider a vector  $v_\Lambda \in R^\mu$  for  $\mu \geq \lambda$  and  $\Lambda \neq \Lambda_c$ . There exists a number  $j$  such that the  $j$ -th columns of the tableaux  $\Lambda$  and  $\Lambda_c$  are distinct and all the respective columns with lesser numbers are identical. Since the tableaux are of the same order and  $\mu \geq \lambda$ , it follows that the length of the  $j$ -th column of the *shifted* tableau  $\Lambda$  is not less than the length of the  $j$ -th column of the shifted tableaux  $\Lambda$ . Let  $p$  be the least of the numbers occupying the  $j$ -th column of  $\Lambda_c$  which is not contained in the  $j$ -th column of  $\Lambda$ . Since  $\Lambda$  is standard, we see that  $p$  occupies the  $(j+1)$ -st column and the first row of  $\Lambda$ .

By Theorem 2.6

$$\pi_p^2 v_\Lambda = \frac{1}{2} x_p^2 v_\Lambda = \frac{1}{2} (j+1-1)(j+1-1+1) v_\Lambda = \frac{1}{2} j(j+1) v_\Lambda.$$

Hence,

$$\left(\frac{1}{2} j(j+1) - \pi_p^2\right) v_\Lambda = 0.$$

Since  $\frac{1}{2} j(j+1) - \pi_p^2$  enters  $\kappa_{\Lambda_c}$  as a factor,  $\kappa_{\Lambda_c}(v_\Lambda) = 0$ . Moreover, since the  $\pi_p$  for  $p = 1, \dots, k$  supercommute with  $Cl_k$ , it follows that  $\kappa_{\Lambda_c}(R^\mu) = 0$  if  $\mu > \lambda$  and  $\kappa_{\Lambda_c}(v_\Lambda) = 0$  if  $v_\Lambda \in R^\lambda$  and  $\Lambda \neq \Lambda_c$ . Moreover,

$$\left(\frac{1}{2} j(j+1) - \pi_p^2\right) v_{\Lambda_c} = \left(\frac{1}{2} j(j+1) - \frac{1}{2} (j-i)(j-i+1)\right) v_{\Lambda_c} = \frac{1}{2} i(2j+i-1) v_{\Lambda_c},$$

where  $(i, j)$  is the position of  $p$ . Hence,  $\kappa_{\Lambda_c}(v_{\Lambda_c}) = \alpha v_{\Lambda_c}$  for a nonzero  $\alpha$ .

Thus, it is shown that  $\kappa_{\Lambda_c}$  projects  $M^\lambda$  onto the subspace  $Cl_k v_{\Lambda_c}$ . To prove the theorem, it suffices to establish that  $\kappa_{\Lambda_c}(w_{\Lambda_c}) \neq 0$ . We induct on  $k$ . For  $k = 1$  the statement is obvious. Let  $k > 1$ , let  $\Lambda_c^*$  be obtained from  $\Lambda_c$  by deleting the last cell in the last column and let  $(i, j)$  be the position of  $k$ . Then

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \left(\frac{1}{2} j(j+1) - \pi_k^2\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*} \otimes e_i) = \left(\frac{1}{2} j(j+1) - \pi_k^2\right) (\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i).$$

By induction,  $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0$  while by Lemma 2.1 and in notations of Lemma 2.1 we have

$$\left(\frac{1}{2}j(j+1) - \pi_k^2\right)(\kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i) = \sum_{m=1}^n (w_m \otimes e_m + \bar{w}_m \otimes e_{\bar{m}}),$$

where

$$w_m = \left(\frac{1}{2}j(j+1) - \frac{1}{2}E_{im}^{(2)} - \delta_{il} \sum_{l=1}^n E_{ll}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$$

and  $\bar{w}_m = -\frac{1}{2}F_{im}^{(2)} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$ .

It remains to show that  $w_i \neq 0$ . Observe that since  $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$  is a highest weight vector with respect to  $\mathfrak{q}(n)$ , i.e.,

$$E_{ij} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = F_{ij} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0 \text{ for } i < j,$$

we have

$$\begin{aligned} w_i &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r=1}^n (E_{ir} E_{ri} - F_{ir} F_{ri}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i}^n (E_{ir} E_{ri} - F_{ir} F_{ri}) - \frac{1}{2}(E_{ii}^2 - F_{ii}^2) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i} ([E_{ir}, E_{ri}] - [F_{ir}, F_{ri}]) \right. \\ &\quad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i} (E_{ii} - E_{rr} - E_{ii} - E_{rr}) \right. \\ &\quad \left. - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r>i} E_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii}) - \sum_{r=1}^n E_{rr}\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}j(j+1) - \frac{1}{2} \sum_{r=1} i E_{rr} - \frac{1}{2}(E_{ii}^2 - E_{ii})\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \left(\frac{1}{2}\lambda_1(\lambda_1 + 1) - (\lambda_1 + \dots + \lambda_i - 1) - \frac{1}{2}(\lambda_i - 1)(\lambda_i - 2)\right) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2}(\lambda_1(\lambda_1 + 1) - 2(\lambda_1 + \dots + \lambda_i) + 2 - (\lambda_i - 1)(\lambda_i - 2)) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}). \end{aligned}$$

If  $i$  is the number of the row occupied by  $k$ , then

$$\lambda_1 = \lambda_i + i - 1, \lambda_2 = \lambda_i + i - 2, \dots, \lambda_{i-1} = \lambda_i + 1,$$

and, therefore,

$$\begin{aligned} w_i &= \frac{1}{2} ((\lambda_i + i - 1)(\lambda_i + i) - 2(i\lambda_i + 1 + 2 + \cdots + i - 1) \\ &\quad + 2 - (\lambda_i - 1)(\lambda_i - 2)) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \neq 0. \end{aligned}$$

□

3.2. *Remark.* Observe that

$$\begin{aligned} \bar{w}_i &= -\frac{1}{2} F_{ii}^{(2)} \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = -\frac{1}{2} \sum_{r=1}^n (F_{ir} E_{ri} - E_{ir} F_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2} \sum_{i=1}^n (E_{ir} F_{ri} - F_{ir} E_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = \frac{1}{2} \sum_{r>i} (E_{ir} F_{ri} - F_{ir} E_{ri}) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \\ &= \frac{1}{2} \sum_{r>i} ([E_{ir}, F_{ri}] - [F_{ir}, E_{ri}]) \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) = 0. \end{aligned}$$

Hence,

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_i \kappa_{\Lambda_c^*}(w_{\Lambda_c^*}) \otimes e_i + \sum_{r \neq i} (w_r + \bar{w}_r p_k) \otimes e_r$$

and by the inductive hypothesis applied to  $\kappa_{\Lambda_c^*}(w_{\Lambda_c^*})$  we deduce that

$$\kappa_{\Lambda_c}(w_{\Lambda_c}) = \lambda_1! \cdots \lambda_n! w_{\Lambda_c} + \sum f_{\Lambda} w_{\Lambda}, \quad \text{where } f_{\Lambda} \in Cl_k \text{ and } w_{\Lambda} \neq w_{\Lambda_c}$$

and where the summation runs over the tableaux  $\Lambda$  of order  $k$  which contain  $\lambda_1$  symbols 1 or  $\bar{1}$ ,  $\lambda_2$  symbols 2 or  $\bar{2}$ , etc.,  $\lambda_n$  symbols  $n$  or  $\bar{n}$ .

3.3. **Specht modules.** Recall that for a strict partition  $\lambda$  of  $k$  the *Specht module* is the submodule in  $M^{\lambda}$  generated by the vectors  $\kappa_{\Lambda} w_{\Lambda}$  for all shifted tableaux  $s$  of shape  $\lambda$ .

**3.3.1. Theorem.** *Specht module is equal to  $R^{\lambda}$ . It is isotypical and its  $H_k$ -centralizer is isomorphic to the Clifford algebra with  $l(\lambda)$  generators.*

*Proof.* Recall that  $R^{\lambda}$  is the set of highest weight vectors in  $W_k = V^{\otimes k}$ . By the Howe duality between  $U(\mathfrak{q}(n))$  and  $H_k$  the algebra of  $H_k$ -homomorphisms is generated by  $U(\mathfrak{q}(n))$ . But thanks to Lemma 2.4 we may assume that the algebra of  $H_k$ -homomorphisms is generated by  $U(\mathfrak{h})$  for the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{q}(n)$ . If  $\lambda_i = 0$ , then, in our notations for the basis of  $\mathfrak{h}$ , we have

$$F_{ii}^2 = E_{ii} = \lambda_i = 0 \text{ in } R^{\lambda}.$$

But  $\ker F_{ii}$  is an  $\mathfrak{h}$ -submodule in the set  $(V^{\lambda})^+$  of highest weight vectors. But  $(V^{\lambda})^+$  is irreducible as an  $\mathfrak{h}$ -module by Theorem 2.2. Hence,  $F_{ii}|_{(V^{\lambda})^+} = 0$  which implies  $F_{ii}|_{R^{\lambda}} = 0$ . Thus, the algebra of  $H_k$ -homomorphisms is generated by the  $F_{ii}$  for  $i$  such that  $\lambda_i \neq 0$ .

By Theorem 3.1 the Specht module is contained in  $R^{\lambda}$ . Moreover, if  $\Lambda$  is a shifted tableau of shape  $\lambda$ , then

$$F_{ii} \kappa_{\Lambda} w_{\Lambda} = \kappa_{\Lambda} F_{ii} w_{\Lambda} = \left( \sum_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_{\alpha} \right) \kappa_{\Lambda} w_{\Lambda}.$$

Therefore, any homomorphism of  $M^{\lambda}$  sends  $R^{\lambda}$  into itself. Hence, the Specht module coincides with  $R^{\lambda}$ . □

**3.3.2. Corollary.** *Let  $\Lambda$  be a shifted tableau. Set  $p_\Lambda^i = \sum_{\alpha \text{ lies in the } i\text{-th row of } \Lambda} p_\alpha$ . Then for any  $H_k$ -module endomorphism  $\varphi$  of  $M^\lambda$  we have*

$$\varphi(\kappa_\Lambda w_\Lambda) = f \cdot \kappa_\Lambda w_\Lambda,$$

where  $f$  belongs to the subalgebra of  $Cl_k$  generated by  $p_\Lambda^1, \dots, p_\Lambda^{l(\lambda)}$ .

*Proof.* Recall that  $M^\lambda$  is the subset of vectors of weight  $\lambda$  in  $W_k$ . Therefore, any endomorphism of  $M^\lambda$  may be identified with an element of  $U(\mathfrak{q}(n))$ ; the restriction of this endomorphism onto  $R^\lambda$  may be identified with an element of  $U(\mathfrak{h})$ .

But  $U(\mathfrak{h})$  is generated by the  $F_{ii}$  and to prove the corollary, it suffices to verify it for these elements. We have

$$F_{ii}(\kappa_\Lambda w_\Lambda) = \kappa_\Lambda(F_{ii}w_\Lambda) = \kappa_\Lambda(p_\Lambda^i w_\Lambda) = p_\Lambda^i \kappa_\Lambda(w_\Lambda).$$

□

**3.3.3. Corollary.** *Let  $b_\Lambda$  be the sum of the elements of the row stabilizer of a shifted tableau  $\Lambda$  and  $\varphi : M^\lambda \rightarrow M^\lambda$  be an  $H_k$ -module endomorphism given by the formula  $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda)$ . Then  $\varphi|_{R^\lambda} = c \in \mathbb{C}$ .*

*Proof.* Let us show that  $\varphi$  commutes with the endomorphisms  $F_{ii}$ . Indeed,

$$\begin{aligned} \varphi \cdot F_{ii}(w_\Lambda) &= \varphi(p_\Lambda^i w_\Lambda) = p_\Lambda^i \varphi(w_\Lambda) = p_\Lambda^i b_\Lambda \kappa_\Lambda(w_\Lambda); \\ F_{ii} \cdot \varphi(w_\Lambda) &= F_{ii} b_\Lambda \kappa_\Lambda(w_\Lambda) = b_\Lambda \kappa_w(F_{ii} w_\Lambda) \\ &= b_\Lambda \kappa_\Lambda(p_\Lambda^i w_\Lambda) = b_\Lambda p_\Lambda^i \kappa_\Lambda(w_\Lambda) = p_\Lambda^i b_\Lambda \kappa_\Lambda(w_\Lambda). \end{aligned}$$

The latter identity holds thanks to the fact that  $p_\Lambda^i$  commutes with  $b_\Lambda$ .

Thus,  $\varphi \cdot F_{ii}(w_\Lambda) = F_{ii} \cdot \varphi(w_\Lambda)$  and, since the elements  $w_\Lambda$  generate the  $H_k$ -module  $M^\lambda$ , we have

$$\varphi \cdot F_{ii} = F_{ii} \cdot \varphi.$$

Further,  $\varphi(R^\lambda) \subset R^\lambda$  for any endomorphism  $\varphi$  of  $M^\lambda$ , so  $\varphi|_{R^\lambda}$  is an element from the centralizer of  $R^\lambda$ . But by Theorem 3.3.1 the centralizer is the Clifford algebra  $Cl_{l(\lambda)}$ . But  $\varphi$  is an even central element of  $Cl_{l(\lambda)}$ , hence,  $\varphi$  is a constant. □

**3.3.4. Corollary.** *Set  $e_\Lambda = \kappa_\Lambda b_\Lambda$ . Then*

$$e_\Lambda^2 = c \cdot e_\Lambda \text{ for } c \in \mathbb{C}, c \neq 0;$$

the algebra  $e_\Lambda H_k e_\Lambda$  is isomorphic to  $Cl_{l(\lambda)}$  and is generated by the  $p_\Lambda^i$  for  $1 \leq i \leq l(\lambda)$ .

*Proof.* Thanks to Corollary 3.3.3

$$\varphi(\kappa_\Lambda(w_\Lambda)) = \kappa_\Lambda b_\Lambda \kappa_\Lambda(w_\Lambda) = c \kappa_\Lambda(w_\Lambda)$$

or, equivalently,

$$e_\Lambda^2 = c \cdot e_\Lambda.$$

If  $c = 0$ , then  $\varphi(\kappa_\Lambda(w_\Lambda)) = 0$ . But the  $H_k$ -submodule generated by  $\kappa_\Lambda(w_\Lambda)$  coincides with the Specht module, which, thanks to Theorem 3.3.1 is equal to  $R^\lambda$ . So in this case  $\varphi(R^\lambda) = 0$ . On the other hand, Theorem 3.1 implies that  $\kappa_\Lambda(w_\Lambda) = \alpha w_\Lambda + \sum \beta_T w_T$ , where  $\alpha \in \mathbb{C}$  and  $\beta_T \in Cl_k$  and where  $w_\Lambda \neq w_T$ .

Therefore,

$$b_\Lambda \kappa_\Lambda(w_\Lambda) = \alpha \lambda_1! \dots \lambda_n! w_\Lambda + \sum \gamma_T w_T,$$

where  $\gamma_T \in Cl_k$ , the sum runs over the tableaux  $T$  of order  $k$  which contain  $\lambda_1$  symbols 1 or  $\bar{1}$ ,  $\lambda_2$  symbols 2 or  $\bar{2}$ , etc.,  $\lambda_n$  symbols  $n$  or  $\bar{n}$ , and where  $w_\Lambda \neq w_T$  and  $\varphi(w_\Lambda) = b_\Lambda \kappa_\Lambda(w_\Lambda) \neq 0$ . This contradicts the requirement  $\varphi(R^\lambda) = 0$ . So  $c \neq 0$ .

The algebra  $e_\Lambda H_k e_\Lambda$  is anti-isomorphic to the algebra  $H_k$  of endomorphisms of  $R^\lambda$  and, thanks to Theorem 3.3.1, it is the Clifford superalgebra  $Cl_{l(\lambda)}$ .  $\square$

3.3.5. Let  $\lambda = (k)$  be the partition consisting of one part,  $\Lambda = 12 \dots k$  the corresponding one-row tableau. Then  $b_\Lambda = \sum_{\sigma \in \mathfrak{S}_k} \sigma$  and

$$\kappa_\Lambda = 1 \cdot (3 - \pi_2^2)(6 - \pi_3^2)(10 - \pi_4^2) \dots \left(\frac{1}{2}k(k+1) - \pi_k^2\right).$$

Further, for any  $i, j$  and  $l$  we have  $(\tau_{ij} + \tau_{jl} + \tau_{li})b_\Lambda = 0$ , so

$$\begin{aligned} \pi_p^2 b_\Lambda &= \left(\sum_{i < p} \tau_{ip}\right)^2 b_\Lambda = \left(p - 1 - \sum_{i < j < p} (\tau_{ip}\tau_{jp} + \tau_{jp}\tau_{ip})\right) b_\Lambda \\ &= \left(p - 1 - \sum (\tau_{ij}(\tau_{ij} + \tau_{jp} + \tau_{pi}) - 1)\right) b_\Lambda = \frac{1}{2}p(p-1)b_\Lambda. \end{aligned}$$

Therefore,  $\kappa_\Lambda b_\Lambda = k! b_\Lambda = e_\Lambda$  and, up to a factor,  $e_\Lambda$  coincides with the respective Young symmetrizer.

Observe that the symmetrizer Nazarov constructed for the one-row tableau is equal to

$$\frac{1}{k!} \prod_{j=2}^k \left( \prod_{i=1}^{j-1} \left( 1 + \frac{s_{ij}}{u_j - u_i} - \frac{p_i p_j s_{ij}}{u_j + u_i} \right) \right),$$

where  $u_p = \sqrt{p(p-1)}$  and  $s_{ij}$  is the transposition; so it does not coincide with  $b_\Lambda$ .

3.3.6. In the case  $\lambda = (p+1, p)$  and

$$\Lambda = \begin{matrix} 1 & 2 & 4 & 6 & \dots & p \\ & 3 & 5 & 7 & \dots & 2p+1 \end{matrix}$$

we can simplify the expression for  $\kappa_\Lambda b_\Lambda$ . Namely, set

$$\begin{aligned} a_1 &= 1 + \tau_{12}\tau_{23} + \tau_{23}\tau_{12}, \\ a_2 &= 2 + (\tau_{14} + \tau_{24} + \tau_{34})\tau_{45} + \tau_{45}(\tau_{14} + \tau_{24} + \tau_{34}), \\ a_3 &= 3 + \pi_6\tau_{67} + \tau_{67}\pi_6, \\ &\dots\dots\dots \\ a_p &= p + \pi_{2p}\tau_{2p,2p+1} + \tau_{2p,2p+1}\pi_{2p}. \end{aligned}$$

Then it is possible to verify that

$$\kappa_\Lambda b_\Lambda = a_1 a_2 \dots a_p b_\Lambda.$$

The author does not know how to simplify the expression for  $\kappa_\Lambda b_\Lambda$  in the general case similarly to the case of the symmetric group; cf. section 1.5.

4. THE SPECHT MODULES OVER  $\mathfrak{A}_k$

In this section, we construct for  $\mathfrak{A}_k$  certain analogs of the modules  $M^\lambda$  and the Specht modules  $R^\lambda$ . First of all, we need the following statement.

**4.1. Lemma.** *Let  $\pi_p = \sum_{\alpha < p} \tau_{\alpha p}$  for  $p = 2, \dots, k$  be odd analogs of Jucys–Murphy’s elements. Then for  $k \geq 2$*

$$e_k = \prod_{p \geq 2} \frac{2}{p(p-1)} \pi_p^2$$

is an idempotent and  $e_k \mathfrak{A}_k e_k$  is isomorphic to the Clifford algebra  $Cl_{k-1}$ .

*Proof.* Set  $e_1 = e_2 = 1$ . It is easy to verify by induction that for  $k \geq 2$  we have

$$e_k = \frac{1}{k!} \sum_{p \geq 1} (-1)^p 2^{k-p-1} \Sigma_{2p+1},$$

where  $\Sigma_{2p+1}$  is the sum of all elements from  $\mathfrak{A}_k$  of the form  $\tau_{i_1 i_2} \tau_{i_2 i_3} \dots \tau_{i_{2\alpha} i_{2\alpha+1}}$ . This implies that  $e_k$  is a central element that does not vary under automorphisms of  $\mathfrak{A}_k$  that send  $\tau_{ij}$  into  $\tau_{\sigma(i)\sigma(j)}$  for any  $\sigma \in \mathfrak{S}_k$ . It is not difficult to verify that

$$(\tau_{12} + \tau_{23} + \tau_{31}) \pi_2^2 = 0.$$

Hence,  $(\tau_{12} + \tau_{23} + \tau_{31}) e_k = 0$ . Having applied the above described automorphisms to this relation we deduce that

$$(\tau_{ij} + \tau_{il} + \tau_{li}) e_k = 0.$$

Further, arguments as in section 3.3.5 yield

$$\pi_p^2 e_k = \frac{1}{2} p(p-1) e_k,$$

hence,

$$e_k^2 = \left( \prod_{p \geq 2} \frac{2}{p(p-1)} \pi_p^2 \right) e_k = e_k.$$

Furthermore, since  $e_k$  is a central element, then  $e_k \mathfrak{A}_k e_k = \mathfrak{A}_k e_k$ . Let  $I$  be the two-sided ideal in  $\mathfrak{A}_k$  generated by the elements  $\tau_{ij} + \tau_{jl} + \tau_{li}$ . Then  $e_k I = 0$ . Set  $\bar{\mathfrak{A}} = \mathfrak{A}_k / I$ . In  $\bar{\mathfrak{A}}$ , then, the following relations hold:

$$\tau_{12} = \pi_2, \tau_{23} = \frac{1}{2}(\pi_3 - \pi_2), \tau_{34} = \frac{1}{3}(\pi_4 - \pi_3), \dots, \tau_{k-1,k} = \frac{1}{k-1}(\pi_k - \pi_{k-1}).$$

Hence,  $\bar{\mathfrak{A}}$  is generated by the  $\pi_p$  for  $2 \leq p \leq k$ . We showed above that condition  $\tau_{ij} + \tau_{il} + \tau_{li} = 0$  in  $\bar{\mathfrak{A}}$  implies that  $\pi_i^2 = \frac{1}{2}(i-1)(i-2)$  in  $\bar{\mathfrak{A}}$ . Moreover,  $\pi_i \pi_j + \pi_j \pi_i = 0$  for  $i \neq j$ . So  $\bar{\mathfrak{A}}$  is the Clifford algebra generated by the images of the  $\pi_i$  for  $2 \leq i \leq k$ .

Further on,  $(1 - e_k)I = I$ ; so  $\mathfrak{A}_k(1 - e_k) \supset I$ , and since  $\bar{\mathfrak{A}}$  is simple as a superalgebra, we deduce that  $\mathfrak{A}_k(1 - e_k) = I$ . Therefore,  $\mathfrak{A}_k e_k \cong \mathfrak{A}_k / \mathfrak{A}_k(1 - e_k) \cong \mathfrak{A}_k / I \cong \bar{\mathfrak{A}}$ .  $\square$

Let  $\lambda$  be a strict partition of  $k$  and  $\Lambda$  a shifted  $\lambda$ -tableau. Let  $\Lambda^{(i)} = \{\alpha_1, \dots, \alpha_{\lambda_i}\}$  be the  $i$ -th row. Recall that

$$p_\Lambda^i = \sum_{\alpha \in \Lambda^{(i)}} p_\alpha.$$

Let further  $\pi_p^{(i)} = \sum_{j < p} \tau_{\alpha_j \alpha_p}$  and

$$e_\Lambda^i = \prod_{p=2}^{\lambda_i} \frac{2}{p(p-1)} \left(\pi_p^{(i)}\right)^2, \quad e_\Lambda = \prod_{i=1}^{l(\lambda)} e_\Lambda^i.$$

Let  $V$  be an  $(n, n)$ -dimensional superspace with  $n \geq l(\lambda)$  and  $W_k = V^{\otimes k}$ , where  $k = \sum \lambda_i$ . Let  $M^\Lambda$  be the subspace of  $W_k$  spanned by the vectors  $w \in W_k$  such that

$$E_{ii}w = \lambda_i w; \quad F_{ii}w = p_\Lambda^i w \text{ for } 1 \leq i \leq l(\lambda).$$

Let  $R^\Lambda = \{w \in M^\Lambda : E_{ij}w = F_{ij}w = 0\}$  be the set of highest weight vectors from  $M^\Lambda$ . Recall that the irreducible representations of superalgebra  $\mathfrak{A}_k$  are labelled by strict partitions  $\lambda$  of  $k$ ; moreover, the representation is of type  $G$  if  $k$  and  $l(\lambda)$  are of the same parity, and of type  $Q$ , otherwise; see [N2].

**4.2. Specht modules.** Let  $\lambda$  be a strict partition of  $k$  and  $\Lambda$  a shifted tableau of shape  $\lambda$ . The Specht module for tableau  $\Lambda$  is the submodule in  $M^\Lambda$  generated by the vectors  $\kappa_T w_\Lambda$ , where the tableau  $T$  is of shape  $\lambda$  and its rows have the same numbers as the respective rows of  $\Lambda$ .

**Theorem.** i) As  $\mathfrak{A}_k$ -module,  $M^\Lambda$  is isomorphic to  $\mathfrak{A}_k e_\Lambda$ .

ii) The Specht submodule of  $M^\Lambda$  is equal to  $R^\Lambda$  and its centralizer is isomorphic to the Clifford superalgebra  $Cl_{k-l(\lambda)}$ .

*Proof.* i) Let  $\mathfrak{A}_i$  be the subalgebra of  $\mathfrak{A}_k$  generated by the  $\tau_{\alpha\beta}$ , where  $\alpha$  and  $\beta$  lie in the  $i$ -th row of  $\Lambda$ . Define the homomorphism  $\varphi_i : \mathfrak{A}_i \rightarrow Cl_{\lambda_i}$  into the Clifford superalgebra on  $\lambda_i$  generators  $p_{\alpha_1}, \dots, p_{\alpha_{\lambda_i}}$ , where  $\alpha_i$  runs numbers in the  $i$ -th row of  $\Lambda$  by setting

$$\varphi_i(\tau_{\alpha\beta}) = \frac{1}{\sqrt{2}}(p_\alpha - p_\beta).$$

From Lemma 4.1 we deduce that this homomorphism induces an isomorphism of  $\mathfrak{A}_i e_\Lambda^i$  with the subalgebra of  $Cl_{\lambda_i}$  generated by  $p_\alpha - p_\beta$  for  $\alpha, \beta \in \Lambda^{(i)}$  (recall that  $\Lambda^{(i)}$  is the  $i$ -th row of  $\Lambda$ ).

Therefore, we have an homomorphism

$$\varphi_1 \otimes \dots \otimes \varphi_{l(\lambda)} : \mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} \rightarrow Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}}$$

which induces an isomorphism of  $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)} e_\Lambda$  with a subalgebra  $Cl_\Lambda$  in  $Cl_{\lambda_1} \otimes \dots \otimes Cl_{\lambda_{l(\lambda)}} \cong Cl_k$ . Moreover, we can see that this subalgebra is generated by the  $p_\alpha - p_\beta$ , where  $\alpha$  and  $\beta$  lie in the same row of  $\Lambda$ . The dimension of this subalgebra is equal to  $2^{\lambda_1-1} 2^{\lambda_2-1} \dots 2^{\lambda_{l(\lambda)}-1} = 2^{k-l(\lambda)}$ . This implies that the ideal  $\mathfrak{A}_k e_\Lambda$  is isomorphic, as an  $\mathfrak{A}_k$ -module, to the module induced from the module  $Cl_\Lambda$  over  $\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_{l(\lambda)}$ , is of dimension

$$\frac{k!}{\lambda_1! \lambda_2! \dots \lambda_{l(\lambda)}!} 2^{k-l(\lambda)}.$$

It is subject to direct verification that  $\dim M^\Lambda$  is the same and the homomorphism  $\varphi : \mathfrak{A}_k e_\Lambda \rightarrow M^\Lambda$  such that  $\varphi(e_\Lambda) = w_\Lambda$  is surjective. Hence,  $\varphi$  is an isomorphism.

ii) By Theorem 3.1  $\kappa_T w_\Lambda = \kappa_T w_T \in R^\Lambda$ , so the Specht module is contained in  $R^\Lambda$ .

Now, let us show that the centralizer of  $R^t$  is isomorphic to the Clifford algebra  $Cl_{k-l(\lambda)}$ .

Obviously,  $\mathfrak{A}_k$  and  $Cl_k \otimes U(\mathfrak{q}(n))$  form a Howe-dual pair in  $W_k$ . By Theorem 3.3.1 the centralizer of the  $H_k$ -module  $R^\lambda$  is the Clifford algebra generated by the  $F_{ii}$  for  $1 \leq i \leq l(\lambda)$ . Therefore, the centralizer of the  $\mathfrak{A}_k$ -module  $R^\lambda$  is the Clifford algebra  $Cl_{k+l(\lambda)}$  generated by the  $F_{ii}$  for  $1 \leq i \leq l(\lambda)$  and  $p_1, \dots, p_k$ .

The condition  $F_{ii}w = p_\Lambda^i w$  is equivalent to the condition  $ew = w$  for

$$e = \prod_{1 \leq i \leq l(\lambda)} \frac{1}{2} \left( 1 - \frac{1}{\lambda_i} p_\Lambda^i F_{ii} \right) \text{ and } e^2 = e.$$

Therefore, the centralizer of  $R^\Lambda$  is isomorphic to  $eCl_{k+l(\lambda)}e$ .

Let  $Cl_\Lambda$  be the subalgebra of  $Cl_k$  generated by the  $p_i - p_j$  for  $i, j$  that belong to the same row of  $\Lambda$ . Then it is not difficult to see that  $eCl_{k+l(\lambda)}e \cong Cl_\Lambda$  and this proves that the centralizer of  $R^\Lambda$  is isomorphic to  $Cl_{k-l(\lambda)}$ .

Let us prove now that the Specht module is actually equal to  $R^\Lambda$ . Indeed,  $Cl_\Lambda$  naturally acts on  $R^\Lambda$  by left multiplications and this action commutes with that of  $\mathfrak{A}_k$ . Hence, by the above, any endomorphism of the  $\mathfrak{A}_k$ -module  $R^\Lambda$  is of the form  $w \mapsto fw$ , where  $w \in R^\Lambda$  and  $f \in Cl_\Lambda$ , and the algebra of such endomorphisms is generated by multiplications by  $p_\alpha - p_\beta$ , where  $\alpha$  and  $\beta$  lie in the same row of  $\Lambda$ .

To prove that the Specht module is equal to  $R^\Lambda$ , it suffices to verify that every endomorphism of  $R^\Lambda$  fixes the Specht module. By the above, it suffices to consider the homomorphism  $\varphi$  given by multiplication by  $p_\alpha - p_\beta$ , where  $\alpha$  and  $\beta$  lie in the same row of  $\Lambda$ . We have

$$\varphi(p_\alpha - p_\beta)\kappa_T w_\Lambda = (p_\alpha - p_\beta)\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta}s_{\alpha\beta}\kappa_T w_\Lambda = \sqrt{2}\tau_{\alpha\beta}\kappa_{s_{\alpha\beta}T} w_\Lambda.$$

But the rows of  $s_{\alpha\beta}T$  consist of the same elements that constitute the respective rows of  $\Lambda$ .  $\square$

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