REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 3, Pages 444–456 (November 9, 1999)
S 1088-4165(99)00066-7

THE ADJOINT REPRESENTATION OF A REDUCTIVE GROUP AND HYPERPLANE ARRANGEMENTS

J. MATTHEW DOUGLASS

ABSTRACT. Let G be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$ defined over an algebraically closed field, k, with $\operatorname{char} k = 0$. Fix a parabolic subgroup of G with Levi decomposition $P = LU$ where U is the unipotent radical of P. Let $\mathfrak{u} = \operatorname{Lie}(U)$ and let $\mathfrak{z}$ denote the center of $\operatorname{Lie}(L)$. Let T be a maximal torus in L with Lie algebra $\mathfrak{t}$. Then the root system of $(\mathfrak{g}, \mathfrak{t})$ is a subset of $\mathfrak{t}^*$ and by restriction to $\mathfrak{z}$, the roots of $\mathfrak{t}$ in $\mathfrak{u}$ determine an arrangement of hyperplanes in $\mathfrak{z}$ we denote by $\mathcal{A}^{\mathfrak{z}}$. In this paper we construct an isomorphism of graded $k[\mathfrak{z}]$-modules $\operatorname{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})]) \cong D(\mathcal{A}^{\mathfrak{z}})$, where $D(\mathcal{A}^{\mathfrak{z}})$ is the $k[\mathfrak{z}]$-module of derivations of $\mathcal{A}^{\mathfrak{z}}$. We also show that $\operatorname{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})])$ and $k[\mathfrak{z}] \otimes \operatorname{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic graded $k[\mathfrak{z}]$-modules, so $D(\mathcal{A}^{\mathfrak{z}})$ and $k[\mathfrak{z}] \otimes \operatorname{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic, graded $k[\mathfrak{z}]$-modules. It follows immediately that $\mathcal{A}^{\mathfrak{z}}$ is a free hyperplane arrangement. This result has been proved using case-by-case arguments by Orlik and Terao. By keeping track of the gradings involved, and recalling that $\mathfrak{g}$ affords a self-dual representation of G, we recover a result of Sommers, Trapa, and Broer which states that the degrees in which the adjoint representation of G occurs as a constituent of the graded, rational G-module $k[G \times^P \mathfrak{u}]$ are the exponents of $\mathcal{A}^{\mathfrak{z}}$. This result has also been proved, again using case-by-case arguments, by Sommers and Trapa and independently by Broer.

1. INTRODUCTION

Suppose k is an algebraically closed field with characteristic zero and G is a connected, reductive, algebraic group defined over k with Lie algebra $\mathfrak{g}$. Let $\mathcal{N}$ denote the cone of nilpotent elements in $\mathfrak{g}$. Then G acts on $\mathfrak{g}$ by the adjoint representation and $\mathcal{N}$ is a closed, G-invariant subvariety, so G acts on $k[\mathcal{N}]$, the ring of regular functions on $\mathcal{N}$. Since $\mathcal{N}$ is a cone, $k[\mathcal{N}]$ inherits a grading from $k[\mathfrak{g}]$ and each homogeneous component, $k[\mathcal{N}]_j$, is a G-stable subspace. Kostant [5] proved that $\mathfrak{g}$ occurs as a constituent of $k[\mathcal{N}]_j$ if and only if j is an exponent of the Weyl group of G. Precisely, he proved the equality of polynomials in q:

$$\sum_{j \geq 0} \dim \operatorname{Hom}_G(\mathfrak{g}, k[\mathcal{N}]_j) q^j = q^{n_1} + \cdots + q^{n_l}$$

where $\{ n_1, \ldots, n_l \}$ is the multiset of exponents of W (see [7, §6.2] for a definition of the exponents of W).

Sommers and Trapa [9] and also Broer [4] have generalized this result as follows. Fix a maximal torus, T, and a parabolic subgroup, P, of G with $T \subseteq P$. Let U denote the unipotent radical of P and let L be a Levi subgroup of P containing

Received by the editors March 8, 1999 and, in revised form September 28, 1999.
1991 *Mathematics Subject Classification.* Primary 22E46.

T. We follow the convention that lower case Fraktur letters will denote the Lie algebra of the group denoted by the same upper case Roman letter, so for example $\mathfrak{t} = \mathrm{Lie}(T)$, and $\mathfrak{u} = \mathrm{Lie}(U)$. Let $\mathfrak{z}$ denote the center of $\mathfrak{l}$, so $\mathfrak{z} \subseteq \mathfrak{t}$. Then $\mathfrak{z}$ acts on $\mathfrak{u}$ by restricting the adjoint representation of $\mathfrak{g}$ and the weights of $\mathfrak{z}$ on $\mathfrak{u}$ are the restrictions to $\mathfrak{z}$ of the weights of $\mathfrak{t}$ on $\mathfrak{u}$. The kernels of these weights determine a hyperplane arrangement in $\mathfrak{z}$ we denote by $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$. Orlik and Terao [8] have proven, using case-by-case arguments, that $\mathcal{A}^{\mathfrak{z}}$ is a free hyperplane arrangement and Orlik and Solomon [6] have computed the exponents of $\mathcal{A}^{\mathfrak{z}}$ in all cases.

Now P acts on $\mathfrak{u}$ and so we can form the associated fibre bundle, $G \times^P \mathfrak{u}$, over G/P with fibre $\mathfrak{u}$. Then G acts on $G \times^P \mathfrak{u}$ by left multiplication and so G acts on $k[G \times^P \mathfrak{u}]$, the ring of global regular functions on $G \times^P \mathfrak{u}$. Moreover, $k[G \times^P \mathfrak{u}]$ inherits a grading from the scalar action of k on $\mathfrak{u}$, $k[G \times^P \mathfrak{u}] \cong \bigoplus_{j \geq 0} k[G \times^P \mathfrak{u}]_j$, and each homogeneous component is a G-stable subspace. Sommers and Trapa [9], and independently Broer [4], have proven that $\mathfrak{g}$ occurs as a constituent of $k[G \times^P \mathfrak{u}]_j$ if and only if j is an exponent of $\mathcal{A}^{\mathfrak{z}}$. Precisely, they prove the equality of polynomials in q:

$$\sum_{j \geq 0} \dim \mathrm{Hom}_G(\mathfrak{g}, k[G \times^P \mathfrak{u}]_j) q^j = q^{n_1^{\mathfrak{z}}} + \cdots + q^{n_r^{\mathfrak{z}}} \tag{1.1}$$

where $\{ n_1^{\mathfrak{z}}, \ldots, n_r^{\mathfrak{z}} \}$ is the multiset of exponents of $\mathcal{A}^{\mathfrak{z}}$ as defined in [7, Definition 4.25].

In the special case when P is a Borel subgroup of G, then $\mathfrak{z} = \mathfrak{t}$, the exponents of $\mathcal{A}^{\mathfrak{t}}$ are the exponents of the Weyl group of G, and $k[\mathcal{N}]_j$ and $k[G \times^P \mathfrak{u}]_j$ afford equivalent representations of G for $j \geq 0$, so we recover Kostant's result.

The proofs given by Sommers and Trapa and by Broer of the equality (1.1) are both case-by-case arguments, with separate arguments for each type of root system.

In this paper, our main result is that there is an isomorphism of graded $k[\mathfrak{z}]$-modules $\Gamma \colon \mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})]) \to D(\mathcal{A}^{\mathfrak{z}})$, where $D(\mathcal{A}^{\mathfrak{z}})$ is the $k[\mathfrak{z}]$-module of derivations of $\mathcal{A}^{\mathfrak{z}}$. We also show that there is an isomorphism of graded $k[\mathfrak{z}]$-modules between $\mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})])$ and $k[\mathfrak{z}] \otimes \mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$. Thus $D(\mathcal{A}^{\mathfrak{z}})$ and $k[\mathfrak{z}] \otimes \mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ are isomorphic, graded $k[\mathfrak{z}]$-modules, so by keeping track of the gradings involved, and recalling that $\mathfrak{g}$ affords a self-dual representation of G, we see that the isomorphism Γ explains the computation, (1.1), of Sommers, Trapa, and Broer.

Our proof that $\mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P (\mathfrak{z} + \mathfrak{u})])$ and $D(\mathcal{A}^{\mathfrak{z}})$ are isomorphic, graded $k[\mathfrak{z}]$-modules does not involve any case-by-case computations. We define Γ explicitly and then show, using that the set of semisimple elements is dense in $\mathfrak{g}$ and some results on the P-orbit structure of $\mathfrak{z} + \mathfrak{u}$, that it's a bijection.

Since $k[\mathfrak{z}] \otimes \mathrm{Hom}_G(\mathfrak{g}^*, k[G \times^P \mathfrak{u}])$ and $D(\mathcal{A}^{\mathfrak{z}})$ are isomorphic $k[\mathfrak{z}]$-modules, it follows immediately that the arrangement $\mathcal{A}^{\mathfrak{z}}$ is free. We thus obtain a proof, free of case-by-case considerations, of the result of Orlik and Terao [8] that the restrictions of the reflection arrangements arising from Weyl groups are free.

After learning of our ideas about the definition of Γ, Broer [3] has given another proof that it's surjective, the difficult part of showing that it's an isomorphism. His proof is elegant, using Saito's criterion, but relies on a previous result of his concerning sums of exponents. Our proof that Γ is surjective is completely different than Broer's and uses only some basic algebraic geometry and properties of root systems and algebraic groups.

Moreover, our technique can be extended to give a new proof of a theorem of Broer's [2, Theorem 1] that characterizes when the restriction mapping from $\mathrm{Mor}_G(\mathfrak{g}, V)$ to $\mathrm{Mor}_W(\mathfrak{t}, V^T)$ is an isomorphism for a rational G-module V. In a subsequent paper we hope to give a parabolic analog of Broer's theorem, where $\mathfrak{g}$ is replaced by $\mathrm{Ad}G(\mathfrak{z}+\mathfrak{u})$ (the closure of a Dixmier sheet in $\mathfrak{g}$), $\mathfrak{t}$ is replaced by $\mathfrak{z}$, and V^T is replaced by V^L.

The rest of this paper is organized as follows. In §2 we collect several results about hyperplane arrangements in the form we need later; in §3 we discuss associated fibre bundles over G/P; in §4 we collect some results about the adjoint action of U on $\mathfrak{z}+\mathfrak{u}$; in §5 we construct Γ and prove it's an isomorphism; and in §6 we explain how the fact that Γ is an isomorphism implies the results discussed above.

Finally, we use the following notation throughout this paper. If X and Y are varieties defined over k, then $k[X]$ denotes the k-algebra of global regular functions on X and $\mathrm{Mor}(X, Y)$ denotes the set of all morphisms from X to Y. If a group, G, acts on X and Y, then $\mathrm{Mor}_G(X, Y)$ denotes the set of all G-equivariant morphisms from X to Y. If V and W are k-vector spaces, then $V \otimes W$ and $\mathrm{Hom}(V, W)$ denote $V \otimes_k W$ and $\mathrm{Hom}_k(V, W)$ respectively. If V and W afford representations of G, then $\mathrm{Hom}_G(V, W)$ denotes the set of all G-equivariant linear transformations from V to W.

2. Hyperplane arrangements

In this paper a "hyperplane arrangement" is a pair, $(V, \mathcal{A})$, where V is a finite dimensional k-vector space and $\mathcal{A}$ is a finite multiset of hyperplanes in V. This slight generalization of the usual meaning has no effect on the invariants of arrangements we consider.

Suppose $(V, \mathcal{A})$ is an arrangement with $\dim V = n$. Let $\{\, \mu_j \mid 1 \leq j \leq m \,\}$ be a multiset of linear functionals so that $\mathcal{A} = \{\, \ker \mu_1, \dots, \ker \mu_m \,\}$. Fix a basis, $\{\, e_1, \dots, e_n \,\}$, of V and let $\{\, x_1, \dots, x_n \,\}$ be the dual basis of V^*. Then for $1 \leq i \leq m$ we may write $\mu_j = \sum_{i=1}^n \xi_{i,j} x_i$ where the $\xi_{i,j}$'s are in k. We may identify $k[V]$ with the polynomial ring $k[x_1, \dots, x_n]$ and if ∂_i denotes partial differentiation with respect to x_i, then $\mathrm{Der}\, k[V]$, the $k[V]$-module of k-linear derivations of $k[V]$, has basis $\{\, \partial_1, \dots, \partial_n \,\}$.

With pointwise operations, $\mathrm{Mor}(V, V)$ is naturally a $k[V]$-module, isomorphic to $k[V] \otimes V$. Suppose $\theta = \sum_{i=1}^n f_i \partial_i$ is in $\mathrm{Der}\, k[V]$. Define $\phi \colon V \to V$ by $\phi(v) = \sum_{i=1}^n f_i(v) e_i$. Then it is easily seen that ϕ is in $\mathrm{Mor}(V, V)$ and that the mapping $\theta \mapsto \phi$ is an isomorphism of $k[V]$-modules.

Recall that the module of $\mathcal{A}$-derivations, $D(\mathcal{A})$, is the $k[V]$-module of all derivations, θ, of $k[V]$ with the property that $\theta(q)$ is in $k[V]q$ where $q = \prod_{j=1}^m \mu_j$ is the defining polynomial of $\mathcal{A}$. It's known that $D(\mathcal{A})$ may be characterized as the set of all θ in $\mathrm{Der}\, k[V]$ with the property that $\theta(\mu_i)$ is in $k[V]\mu_i$ for $1 \leq i \leq m$ [7, Proposition 4.8]. If θ and ϕ are as in the preceding paragraph, then θ is in $D(\mathcal{A})$ if and only if $\sum_{i=1}^n \xi_{i,j} f_i$ is in $k[V]\mu_j$ for $1 \leq j \leq m$. On the other hand, if $\phi^\#$ is the comorphism of ϕ, then $\phi^\#(\mu_j) = \sum_{i=1}^n \xi_{i,j} f_i$. Therefore, the image of $D(\mathcal{A})$ under the isomorphism in the preceding paragraph is the set of all morphisms, $\phi \colon V \to V$, with the property that $\phi^\#(\mu_j)$ is in $k[V]\mu_j$ for $1 \leq j \leq m$.

In the rest of this paper, we'll use only morphisms, and not derivations, so from now on, we'll abuse notation and identify $D(\mathcal{A})$ with its image in $\mathrm{Mor}(V, V)$, so $D(\mathcal{A}) = \{\, \phi \in \mathrm{Mor}(V, V) \mid \phi^\#(\mu_j) \in k[V]\mu_j,\ 1 \leq j \leq m \,\}$.

Next, recall that the lattice of $\mathcal{A}$, $L(\mathcal{A})$, is the set of subspaces of V that arise as intersections of hyperplanes in $\mathcal{A}$. Suppose X is in the lattice of $\mathcal{A}$. Then if $\ker \mu_j$ is in $\mathcal{A}$, either $X \subseteq \ker \mu_j$ or $X \cap \ker \mu_j$ is a hyperplane in X. The "restricted" arrangement, $(X, \mathcal{A}^X)$, is defined to be the arrangement in X consisting of all hyperplanes, $X \cap \ker \mu_j$, where $X \not\subseteq \ker \mu_j$. Let

$$X_{\text{reg}} = \{\, v \in X \mid v \in \ker \mu_j \text{ if and only if } X \subseteq \ker \mu_j,\ 1 \leq j \leq m \,\}.$$

Then V is the disjoint union of the X_{reg}'s as X runs through $L(\mathcal{A})$. Thus, given v in V we can define X_v to be the unique subspace in $L(\mathcal{A})$ with v in $(X_v)_{\text{reg}}$.

Proposition 2.1. *Suppose $\phi\colon V \to V$ is a morphism. Then the following are equivalent:*

1. *ϕ is in $D(\mathcal{A})$,*
2. *$\phi(v) \in X_v$ for all v in V,*
3. *$\phi(X) \subseteq X$ for all X in $L(\mathcal{A})$.*

Proof. Suppose first that ϕ is in $D(\mathcal{A})$. Fix v in V. If $X_v = X$, then the result is obvious. Suppose $X_v \neq X$. Then relabeling if necessary we may assume that v is in $\ker \mu_j$ if and only if $1 \leq j \leq r$, for some r with $1 \leq r \leq m$. Then $X \subseteq \ker \mu_j$ if and only if $1 \leq j \leq r$ and $X_v = \bigcap_{j=1}^r \ker \mu_j$. By assumption, for $1 \leq j \leq r$, $\mu_j(v) = 0$ implies that $\mu_j \phi(v) = 0$, so $\phi(v)$ is in $\bigcap_{j=1}^r \ker \mu_j = X_v$.

The second statement clearly implies the third since if v is in X, then X_v is a subset of X.

Finally, suppose $\phi(X) \subseteq X$ for all X in $L(\mathcal{A})$. Then $\phi(\ker \mu_j) \subseteq \ker \mu_j$ for $1 \leq j \leq m$. It follows that $\ker \mu_j$ is contained in the zero set of $\phi^{\#}(\mu_j)$ and so by the Nullstellensatz, some power of $\phi^{\#}(\mu_j)$ is a multiple of μ_j. Since μ_j is irreducible, it must be the case that $\phi^{\#}(\mu_j)$ is a multiple of μ_j. Therefore ϕ is in $D(\mathcal{A})$. □

3. Fibre bundles

In this section, G is an affine algebraic group and P is a closed subgroup of G.

Suppose W is a finite dimensional k-vector space that affords a rational representation of P. Then P acts on $G \times W$ by $p(g, w) = (gp^{-1}, pw)$ for p in P, g in G, and w in W. We can form the associated fibre bundle, $G \times^P W$, over G/P, which as a set is the set of P-orbits on $G \times W$. For g in G and w in W, $g * w$ will denote the image of (g, w) in $G \times^P W$.

For a non-negative integer, j, let $k[W]_j$ denote the vector space of all homogeneous, degree j polynomial functions on W. Since P acts linearly on W, $k[W]_j$ is a P-invariant subspace of $k[W]$. The grading, $k[W] \cong \bigoplus_{j \geq 0} k[W]_j$, induces a grading, $(k[G] \otimes k[W])^P \cong \bigoplus_{j \geq 0} (k[G] \otimes k[W]_j)^P$, of $(k[G] \otimes k[W])^P$. The natural map from $(k[G] \otimes k[W])^P$ to $k[G \times^P W]$ is an isomorphism of k-algebras and it's straightforward to check that the image of $(k[G] \otimes k[W]_j)^P$ in $k[G \times^P W]$ is

$$\{\, f \in k[G \times^P W] \mid f(g * \xi w) = \xi^j f(g * w) \text{ for all } g \in G, w \in W, \xi \in k \,\}.$$

Denoting this last set by $k[G \times^P W]_j$ we see that $k[G \times^P W]$ is graded by $k[G \times^P W] = \bigoplus_{j \geq 0} k[G \times^P W]_j$.

Left multiplication defines a regular action of G on $G \times^P W$ and so the coordinate ring, $k[G \times^P W]$, affords a representation of G. Clearly each $k[G \times^P W]_j$ is a G-invariant subspace. Moreover, if $f = \sum_{j \geq 0} f_j$ is in $k[G \times^P W]^G$ with f_j

in $k[G\times^P W]_j$, then for fixed g and w, and any ξ in k, $f(g*\xi w)$ is a polynomial in ξ with coefficients $f_j(g*w)$. Since $f(g*\xi w)=f(1*\xi w)$, it follows that the coefficients don't depend on g and so each f_j is in $k[G\times^P W]^G$. Therefore, $k[G\times^P W]^G \cong \bigoplus_{j\geq 0} k[G\times^P W]^G_j$ is a graded k-algebra.

Now suppose that V is a finite dimensional k-vector space that affords a rational representation of G. It's easy to see that with pointwise operations $\mathrm{Mor}_G(G\times^P W, V)$ is a $k[G\times^P W]^G$-module. Also, $\mathrm{Mor}_G(G\times^P W, V)$ is isomorphic to the vector space of all G-equivariant linear transformations from V^* to $k[G\times^P W]$ since V is an affine space. Thus, $\mathrm{Mor}_G(G\times^P W, V)\cong \mathrm{Hom}_G(V^*, k[G\times^P W])$. It's straightforward to check that

$$\mathrm{Hom}_G(V^*, k[G\times^P W]) \cong \bigoplus_{j\geq 0} \mathrm{Hom}_G(V^*, k[G\times^P W]_j)$$

and that the image of $\mathrm{Hom}_G(V^*, k[G\times^P W]_j)$ in $\mathrm{Mor}_G(G\times^P W, V)$ is

$$\{\, \phi\in \mathrm{Mor}_G(G\times^P W, V) \mid \phi(g*\xi w)=\xi^j\phi(g*w) \text{ for all } g\in G, w\in W, \xi\in k\,\}.$$

Thus, if we define $\mathrm{Mor}_G(G\times^P W, V)_j$ to be this last vector space, it follows that $\mathrm{Mor}_G(G\times^P W, V)\cong \bigoplus_{j\geq 0}\mathrm{Mor}_G(G\times^P W, V)_j$. Moreover, it's easy to see that in fact $\mathrm{Mor}_G(G\times^P W, V)$ is a graded $k[G\times^P W]^G$-module.

Similarly, again since P acts linearly on W, $k[W]^P\cong\bigoplus_{j\geq 0} k[W]^P_j$ and if

$$\mathrm{Mor}_P(W,V)_j=\{\,\psi\in \mathrm{Mor}_P(W,V)\mid \psi(\xi w)=\xi^j w \text{ for all } w\in W,\ \xi\in k\,\},$$

then $\mathrm{Mor}_P(W,V)\cong\bigoplus_{j\geq 0}\mathrm{Mor}_P(W,V)_j$ and therefore $\mathrm{Mor}_P(W,V)$ is a graded $k[W]^P$-module.

Let $\tau\colon W\to G\times^P W$ be the "inclusion" of the fibre over P defined by $\tau(w)=1*w$ for w in W, so τ is a morphism. Suppose ϕ is in $\mathrm{Mor}_G(G\times^P W, V)$. Then, for p in P and w in W,

$$p\phi\tau(w)=\phi(p*w)=\phi(1*pw)=\phi\tau(pw),$$

so $\phi\tau$ is in $\mathrm{Mor}_P(W,V)$, the $k[W]^P$-module of all P-equivariant morphisms from W to V. We need the following version of Frobenius reciprocity.

Proposition 3.1. *The mapping $\phi\mapsto\phi\tau$ defines an isomorphism of graded vector spaces between* $\mathrm{Mor}_G(G\times^P W, V)$ *and* $\mathrm{Mor}_P(W,V)$.

Proof. First, suppose ϕ is in $\mathrm{Mor}_G(G\times^P W, V)_j$. Then for w in W and ξ in k,

$$\phi\tau(\xi w)=\phi(1*\xi x)=\xi^j\phi(1*w)=\xi^j\phi\tau(w),$$

so $\phi\tau$ is in $\mathrm{Mor}_P(W,V)_j$.

It's straightforward to check that the mapping is a homomorphism of graded vector spaces. To show that it's injective, suppose ϕ_1 and ϕ_2 are in $\mathrm{Mor}_G(G\times^P W, V)$ and $\phi_1\tau=\phi_2\tau$. Then for g in G and w in W,

$$\begin{aligned}\phi_1(g*w)&=\phi_1(1*w)\\&=\phi_1\tau(w)\\&=\phi_2\tau(w)\\&=\phi_2(g*w),\end{aligned}$$

so $\phi_1=\phi_2$.

To show the mapping is surjective, suppose $\widetilde{\phi}$ is in $\mathrm{Mor}_P(W,V)$. We define $\phi\colon G\times^P W \to V$ by $\phi(g*w) = g\widetilde{\phi}(w)$. Then ϕ is easily seen to be a well-defined morphism and clearly $\phi\tau = \widetilde{\phi}$. If g_1 is in G, then

$$\begin{aligned}\phi(g_1(g*w)) &= \phi(g_1 g * w)\\ &= g_1 g\widetilde{\phi}(w)\\ &= g_1\phi(g*w),\end{aligned}$$

so ϕ is G-equivariant. This completes the proof of the proposition. □

Corollary 3.2. *The mapping $f \mapsto f\tau$ defines an isomorphism of graded k-algebras between $k[G\times^P W]^G$ and $k[W]^P$.*

Proof. This is just the preceding proposition when $V = k$ with the trivial action of G. □

Notice that if ϕ is in $\mathrm{Mor}_G(G\times^P W, V)$ and f is in $k[G\times^P W]^G$, then $(f\phi)\circ\tau = (f\circ\tau)(\phi\circ\tau)$, so the isomorphism above intertwines the $k[G\times^P W]^G$-module structure on $\mathrm{Mor}_G(G\times^P W, V)$ and the $k[W]^P$-module structure on $\mathrm{Mor}_P(W,V)$.

Now suppose that W_1 is a subspace of W with the property that $pw - w$ is in W_1 for all p in P and w in W. Notice that W_1 is a P-stable subspace of W and that P acts trivially on W/W_1. Let W_0 be a complement to W_1 in W. Then we may identify $k[W]$ and $k[W_0]\otimes k[W_1]$. If $k[W_0]$ is given the trivial P-action and P acts diagonally on $k[W_0]\otimes k[W_1]$, then this identification is P-equivariant.

Let $\theta\colon G\times^P W \to W_0$ by $\theta(g*(w_0+w_1)) = w_0$. Notice that θ is well-defined since P acts trivially on W/W_1. The comorphism, $\theta^\#\colon k[W_0]\to k[G\times^P W]$ defines the structure of a graded $k[W_0]$-module on $k[G\times^P W]$.

Since P acts trivially on $k[W_0]$ in the decomposition $k[W]\cong k[W_0]\otimes k[W_1]$ we have G-equivariant isomorphisms of graded $k[W_0]$-modules,

$$\begin{aligned}k[G\times^P W] &\cong \left(k[G]\otimes k[W_0]\otimes k[W_1]\right)^P\\ &\cong k[W_0]\otimes\left(k[G]\otimes k[W_1]\right)^P\\ &\cong k[W_0]\otimes k[G\times^P W_1],\end{aligned}$$

where G acts trivially on $k[W_0]$ and diagonally on $k[W_0]\otimes k[G\times^P W_1]$.

More generally, using the isomorphism in the last paragraph, we have isomorphisms of graded $k[W_0]$-modules,

$$\begin{aligned}\mathrm{Mor}_G(G\times^P W, V) &\cong \mathrm{Hom}_G(V^*, k[G\times^P W])\\ &\cong \left(V\otimes k[W_0]\otimes k[G\times^P W_1]\right)^G\\ &\cong k[W_0]\otimes\left(V\otimes k[G\times^P W_1]\right)^P\\ &\cong k[W_0]\otimes \mathrm{Hom}_G(V^*, k[G\times^P W_1]).\end{aligned}$$

This proves the next lemma.

Lemma 3.3. *There is an isomorphism graded $k[W_0]$-modules,*

$$\mathrm{Mor}_G(G\times^P W, V)\cong k[W_0]\otimes\mathrm{Hom}_G(V^*, k[G\times^P W_1]).$$

Proposition 3.4. *The comorphism, $\theta^\#\colon k[W_0]\to k[G\times^P W]$, is an injective homomorphism of graded k-algebras. Moreover, if PW_0 is a dense subvariety of W,* *then* $\theta^{\#}(k[W_0]) = k[G \times^P W]^G$. *Thus, if* PW_0 *is dense in* W*, then* $k[W_0]$ *and* $k[G \times^P W]^G$ *are isomorphic, graded* k*-algebras.*

Proof. It's easy to see that $\theta^{\#}$ is an injective homomorphism of graded k-algebras and that $\theta^{\#}(k[W_0]) \subseteq k[G \times^P W]^G$. Now suppose that PW_0 is dense in W. For f in $k[W_0]^G$, define $f_1 \colon W_0 \to k$ by $f_1(w_0) = f(1 * w_0)$, so f_1 is in $k[W_0]$. Suppose g is in G, p is in P, and w_0 is in W_0. Then,

$$\begin{aligned} f_1\theta(g * pw_0) &= f_1\theta(gp * w_0) \\ &= f_1(w_0) \\ &= f(1 * w_0) \\ &= f(gp * w_0) \\ &= f(g * pw_0). \end{aligned}$$

Therefore $f_1\theta$ and f agree on

$$G * PW_0 = \{\, g * pw_0 \mid g \in G, p \in P, w_0 \in W_0 \,\}.$$

Since PW_0 is dense in W, $G * PW_0$ is dense in $G \times^P W$, and so $f_1\theta = f$. Therefore $\theta^{\#}(f_1) = f$ and hence $k[G \times^P W]^G \subseteq \theta^{\#}(k[W_0])$. This completes the proof of the proposition. □

4. The action of U on $\mathfrak{z} + \mathfrak{u}$

For the rest of this paper, we return to the notation of the introduction: G is a connected, reductive, algebraic group; T is a maximal torus in G; P is a parabolic subgroup of G with Levi decomposition $P = LU$; and $\mathfrak{z}$ is the center of $\mathfrak{l}$. Also, let Φ denote the root system of $(\mathfrak{g}, \mathfrak{t})$ and let $\Phi_{\mathfrak{u}}$ be the subset of Φ consisting of the weights of $\mathfrak{t}$ on $\mathfrak{u}$.

Lemma 4.1. *Suppose* l *is a semisimple element in* $\mathfrak{l}$*. Define* $a_l \colon U \times Z_{\mathfrak{u}}(l) \to l + \mathfrak{u}$ *by* $a_l(u, z) = \operatorname{Ad}u(l + z)$*. Then* a_l *is surjective with fibres isomorphic to* $Z_U(l)$*.*

Proof. To show that a_l is surjective, suppose n is in $\mathfrak{u}$. Let $l+n = (l+n)_s + (l+n)_n$ be the Jordan decomposition on $l + n$. Then Borho [1, §2] has shown that there is a u in U with $\operatorname{Ad}u(l) = (l + n)_s$. Set $z = \operatorname{Ad}u^{-1}((l + n)_n)$, so $\operatorname{Ad}u(z) = (l + n)_n$. Then clearly $\operatorname{Ad}u(l+z) = l+n$. Moreover, $[\operatorname{Ad}u(l), \operatorname{Ad}u(z)] = [(l+n)_s, (l+n)_n] = 0$. Thus, z is in $Z_{\mathfrak{u}}(l)$ and $a_l(u, z) = l + n$.

Suppose n is in $\mathfrak{u}$ and consider $F = {a_l}^{-1}(l+n)$. Fix (u, z) in F. Since $\operatorname{Ad}u(l+z)$ has Jordan decomposition $\operatorname{Ad}u(l+z) = \operatorname{Ad}u(l) + \operatorname{Ad}u(z)$, if (u_1, z_1) is in $U \times Z_{\mathfrak{u}}(l)$, then (u_1, z_1) is in F if and only if $\operatorname{Ad}u_1(l) = \operatorname{Ad}u(l)$ and $\operatorname{Ad}u_1(z_1) = \operatorname{Ad}u(z)$. Thus $\rho \colon F \to uZ_U(l)$ by $\rho(u_1, z_1) = u_1$ is a well-defined morphism. If $\rho(u_1, z_1) = \rho(u_2, z_2)$, then $u_1 = u_2$ and so $\operatorname{Ad}u_1(z_1) = \operatorname{Ad}u_1(z_2)$, so $z_1 = z_2$ and hence ρ is injective. If v is in $Z_U(l)$, set $u_1 = uv$ and $z_1 = \operatorname{Ad}v^{-1}(l+z) - l$. It's straightforward to check that (u_1, z_1) is in F and so ρ is surjective. Finally, ρ is an isomorphism by Zariski's Main Theorem. □

Corollary 4.2. *The* P*-saturation of* $\mathfrak{z}_{reg}$*,* $\operatorname{Ad}P(\mathfrak{z}_{reg})$*, is* $\mathfrak{z}_{reg} + \mathfrak{u}$*, so* $\operatorname{Ad}P(\mathfrak{z}_{reg})$ *is a dense open subvariety of* $\mathfrak{z} + \mathfrak{u}$*.*

Proof. Recall that $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$ is an arrangement in $\mathfrak{z}$, so $\mathfrak{z}_{\mathrm{reg}}$ is defined as in §2. It's easy to see that $\mathfrak{z}_{\mathrm{reg}} = \{\, t \in \mathfrak{t} \mid \alpha(t) \neq 0 \text{ for all } \alpha \in \Phi_{\mathfrak{u}} \,\}$ and hence that t is in $\mathfrak{z}_{\mathrm{reg}}$ if and only if $Z_{\mathfrak{u}}(t) = 0$. If t is in $\mathfrak{z}_{\mathrm{reg}}$, then it follows from Lemma 4.1 that $\operatorname{Ad}P(t) = t + \mathfrak{u}$, so $\operatorname{Ad}P(\mathfrak{z}_{\mathrm{reg}}) = \mathfrak{z}_{\mathrm{reg}} + \mathfrak{u}$. □

For α in Φ let $\mathfrak{g}_\alpha$ denote the α-weight space in $\mathfrak{g}$ and fix a non-zero root vector, e_α in $\mathfrak{g}_\alpha$. Also, let U_α be the corresponding root subgroup in G and suppose $x_\alpha \colon k \to U_\alpha$ is a fixed isomorphism from k to U_α satisfying $\operatorname{Ad} x_\alpha(\xi) = \exp(\operatorname{ad}(\xi e_\alpha))$ for ξ in k, where ad is the adjoint representation of $\mathfrak{g}$.

For α and β in $\Phi_\mathfrak{u}$, define $\alpha \sim \beta$ if $\ker \alpha|_\mathfrak{z} = \ker \beta|_\mathfrak{z}$. Clearly this is an equivalence relation on $\Phi_\mathfrak{u}$. It's also clear that for α and β in $\Phi_\mathfrak{u}$, $\alpha \sim \beta$ if and only if $\alpha|_\mathfrak{z}$ is a scalar multiple of $\beta|_\mathfrak{z}$. Let $\Psi_1, \dots, \Psi_s$ be the equivalence classes for this relation on $\Phi_\mathfrak{u}$.

Notice that each Ψ_i is a closed set of roots (that is, if α and β are in Ψ_i and $\alpha+\beta$ is a root, then $\alpha+\beta$ is in Ψ_i). It follows that if we define $\mathfrak{u}_i$ to be the span of the root vectors e_α with α in Ψ_i and U_i to be the subgroup of U generated by the root subgroups, U_α, where α is in Ψ_i, then $\mathfrak{u}_i$ is a subalgebra of $\mathfrak{u}$, U_i is a subgroup of U isomorphic to the product $\prod_{\alpha\in\Psi_i} U_\alpha$ (the product can be taken in any order), and $\operatorname{Lie}(U_i) = \mathfrak{u}_i$. Moreover, if $U^i = \prod_{\beta\in\Phi_\mathfrak{u}\setminus\Psi_i} U_\beta$ (the product taken in some fixed order), then every element in U has a unique factorization $u = u^i u_i$ with u^i in U^i and u_i in U_i.

For $1 \le i \le s$, let H_i denote $\ker \alpha|_\mathfrak{z}$ for α in Ψ_i, so $H_1, \dots, H_s$ are the distinct hyperplanes in $\mathcal{A}^\mathfrak{z}$. Set $\mathfrak{z}^i = \{\, t \in \mathfrak{z} \mid \beta(t) \neq 0 \text{ for all } \beta \in \Phi_\mathfrak{u} \setminus \Psi_i \,\} = \mathfrak{z} \setminus \bigcup_{i\neq j} H_j$. Then $\mathfrak{z}^i$ is an open subvariety of $\mathfrak{z}$ containing $\mathfrak{z}_{\mathrm{reg}}$. Finally, define $a \colon U \times (\mathfrak{z}+\mathfrak{u}) \to \mathfrak{z}+\mathfrak{u}$ by $a(u, t+n) = \operatorname{Ad} u(t+n)$ and let a_i denote the restriction of a to $U^i \times (\mathfrak{z}^i + \mathfrak{u}_i)$.

Proposition 4.3. *For $1 \le i \le s$, the morphism $a_i \colon U^i \times (\mathfrak{z}^i + \mathfrak{u}_i) \to \mathfrak{z}+\mathfrak{u}$ determines an isomorphism between $U^i \times (\mathfrak{z}^i+\mathfrak{u}_i)$ and $\mathfrak{z}^i + \mathfrak{u}$.*

Proof. We show that a_i is injective with image equal to $\mathfrak{z}^i+\mathfrak{u}$. Then by restricting the range we obtain a bijective, birational morphism of normal varieties between $U^i \times (\mathfrak{z}^i+\mathfrak{u}_i)$ and $\mathfrak{z}^i+\mathfrak{u}$ which must be an isomorphism by Zariski's Main Theorem.

To show that a_i is injective, suppose $\operatorname{Ad} u(t+n) = \operatorname{Ad} u_1(t_1+n_1)$ for u, u_1 in U^i, t, t_1 in $\mathfrak{z}^i$, and n, n_1 in $\mathfrak{u}_i$. Then clearly $t = t_1$ and $\operatorname{Ad} u_1{}^{-1}u(t+n) = t+n_1$. Just suppose $u_1{}^{-1}u$ is not in U_i and choose β in $\Phi_\mathfrak{u}$ with β not in Ψ_i, so that the height of β is minimal and such that $u_1{}^{-1}u = x_\beta(c)u'$ for some u' in U, with $c \neq 0$. We show that $\operatorname{Ad} x_\beta(c)u'(t+n) = t - c\beta(t) + \sum_{\alpha\neq\beta} r_\alpha e_\alpha$. First, if e_α occurs with a non-zero coefficient in $\operatorname{Ad} u'(t)$, then $\alpha = \gamma_1 + \dots + \gamma_m$ with $\gamma_i \neq \beta$ for $1 \le i \le m$. If some γ_j are not in Ψ_i, then either $m = 1$ and $\alpha \neq \beta$, or $m > 1$ in which case the height of α is strictly greater than the height of β, so again $\alpha \neq \beta$. If every γ_j is in Ψ_i, then so is their sum, so again $\alpha \neq \beta$. It follows that $\operatorname{Ad} u'(t) = t + \sum_{\alpha\neq\beta} s_\alpha e_\alpha$ for some s_α's in k. Thus $\operatorname{Ad} x_\beta(c)u'(t) = t - c\beta(t) + \sum_{\alpha\neq\beta} s'_\alpha e_\alpha$ where the s'_α's are in k. Next, if e_α occurs with a non-zero coefficient in $\operatorname{Ad} x_\beta(c)u'(n)$, then again, either α is in Ψ_i in which case $\alpha \neq \beta$, or α is not in Ψ_i in which case the height of α is strictly greater than the height of β, so $\alpha \neq \beta$ in this case either. Thus, e_β does not appear in $\operatorname{Ad} x_\beta(c)u'(n)$. It follows that $\operatorname{Ad} x_\beta(c)u'(t+n) = t - c\beta(t) + \sum_{\alpha\neq\beta} r_\alpha e_\alpha$ as desired. Since $\beta \neq \alpha$, it must be that $c\beta(t) = 0$ and since t is in $\mathfrak{z}^i$, $\beta(t) \neq 0$, so $c = 0$, a contradiction. It follows that $u_1{}^{-1}u$ is in U_i and so it follows from the factorization $U = U^i U_i$ that $u = u_1$. Therefore $n = n_1$ also and so a_i is injective.

To show that the image of a_i is $\mathfrak{z}^i+\mathfrak{u}$, suppose that t is in $\mathfrak{z}^i$. If t is in H_i, then $Z_\mathfrak{u}(t) = \mathfrak{u}_i$ since t is not in H_j for $j \neq i$, so by Lemma 4.1 given n in $\mathfrak{u}$, there is a u in U and an n_1 in $\mathfrak{u}_i$ so that $\operatorname{Ad} u(t+n_1) = t+n$. Write $u = u^i u_i$ with u^i in U^i and u_i in U_i. Then $\operatorname{Ad} u_i(t+n_1)$ is in $t+\mathfrak{u}_i$ (recall that Ψ_i is a closed set of roots), so say $\operatorname{Ad} u_i(t+n_1) = t+n'_1$ where n'_1 is in $\mathfrak{u}_i$. It follows that $t+n = \operatorname{Ad} u^i(t+n'_1)$

is in the image of a_i. If t is not in H_i, then t is in $\mathfrak{z}_{\mathrm{reg}}$ and so again by Lemma 4.1, given n in $\mathfrak{u}$ there is a u in U with $\mathrm{Ad}u(t) = t + n$. Factor $u = u^i u_i$ as above. Then $\mathrm{Ad}u_i(t)$ is in $t + \mathfrak{u}_i$, say $\mathrm{Ad}u_i(t) = t + n_1$. Then $t + n = \mathrm{Ad}u^i(t + n_1)$, and so $t + n$ is in the image of a_i. □

For $1 \leq i \leq s$, fix a root, say α_i, in Ψ_i. This set of representatives will remain fixed for the rest of this paper.

Proposition 4.4. *Suppose $1 \leq i \leq s$ and u is in U_i. Then there is an element, n_1, in $\mathfrak{u}_i$, so that for t in $\mathfrak{z}$, $\mathrm{Ad}u(t) = t + \alpha_i(t) n_1$.*

Proof. We can write $u = x_{\beta_n}(c_n) \cdots x_{\beta_1}(c_1)$ where the c_j's are in k and $\beta_1, \ldots, \beta_n$ are in Ψ_i. We prove the result using induction on n. If $n = 1$, then $\mathrm{Ad}x_{\beta_1}(c_1)(t) = t - \beta_1(t) c_1 e_{\beta_1}$. Since β_1 is in Ψ_i, there is a rational number, r, so that $\beta|_{\mathfrak{z}} = r\alpha|_{\mathfrak{z}}$. Now we can take $n_1 = -rc_1 e_{\beta_1}$.

Suppose that $n > 1$, $u = x_{\beta_n}(c_n) \cdots x_{\beta_1}(c_1)$, and

$$\mathrm{Ad}\left(x_{\beta_{n-1}}(c_{n-1}) \cdots x_{\beta_1}(c_1)\right)(t) = t + \alpha_i(t) n_1'$$

for some n_1' in $\mathfrak{u}_i$ and all t in $\mathfrak{z}$. Then

$$\begin{aligned}\mathrm{Ad}u(t) &= \mathrm{Ad}x_{\beta_n}(c_n)\left(t + \alpha_i(t) n_1'\right) \\ &= t - \beta_n(t) c_n e_{\beta_n} + \alpha_i(t) \mathrm{Ad}x_{\beta_n}(c_n) n_1'.\end{aligned}$$

Now β_n is in Ψ_i, so $\beta_n|_{\mathfrak{z}} = r\alpha|_{\mathfrak{z}}$ for some rational number, r, and $\mathrm{Ad}x_{\beta_n}(c_n) n_1' = n_1''$ for some n_1'' in $\mathfrak{u}_i$. The result follows by taking $n_1 = -c_n r e_{\beta_n} + n_1''$. □

5. The isomorphism

Let $\Phi_{\mathfrak{l}}$ denote the root system of $(\mathfrak{l}, \mathfrak{t})$, so Φ is the disjoint union of $\Phi_{\mathfrak{l}}$, $\Phi_{\mathfrak{u}}$, and $-\Phi_{\mathfrak{u}}$. Recall that $\mathfrak{z} \subseteq \mathfrak{t}$ and that $\mathfrak{z} = \bigcap_{\alpha \in \Phi_{\mathfrak{l}}} \ker \alpha|_{\mathfrak{z}}$. Also, for α in Φ, $\mathfrak{z} \subseteq \ker \alpha$ if and only if $\alpha \in \Phi_{\mathfrak{l}}$. Thus the hyperplanes, $\ker \alpha|_{\mathfrak{z}}$, for α in $\Phi_{\mathfrak{u}}$ determine a hyperplane arrangement in $\mathfrak{z}$, $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$. As in §2 we consider elements in $D(\mathcal{A}^{\mathfrak{z}})$ as morphisms from $\mathfrak{z}$ to itself satisfying the equivalent conditions of Proposition 2.1.

For t in $\mathfrak{z}$, set $\Psi_t = \{\, \alpha \in \Phi_{\mathfrak{u}} \mid \alpha(t) = 0 \,\}$. Then, with the notation of §2, $X_t = \bigcap_{\alpha \in \Psi_t} \ker \alpha|_{\mathfrak{z}}$. Recall that for α in Φ, the root subgroup of G corresponding to α is U_α. Then $Z_U(t) = \prod_{\alpha \in \Psi_t} U_\alpha$ and $Z_P(t) = L Z_U(t)$.

Let $\sigma\colon \mathfrak{z} \to G \times^P (\mathfrak{z} + \mathfrak{u})$ by $\sigma(t) = 1 * t$, so σ is a morphism. Suppose that $\phi\colon G \times^P (\mathfrak{z} + \mathfrak{u}) \to \mathfrak{g}$ is a G-equivariant morphism. Then $\phi\sigma\colon \mathfrak{z} \to \mathfrak{g}$ is a morphism and since ϕ is G-equivariant and the stabilizer in G of $\sigma(t)$ is $Z_P(t)$, it follows that $\phi\sigma(t)$ is in $\mathfrak{g}^{Z_P(t)}$ for t in $\mathfrak{z}$. Now $\mathfrak{g}^{Z_P(t)} \subseteq \mathfrak{g}^L = \mathfrak{z}$ and so

$$\mathfrak{g}^{Z_P(t)} = \{\, t_1 \in \mathfrak{z} \mid \alpha(t_1) = 0 \text{ for all } \alpha \in \Psi_t \,\} = X_t.$$

Therefore, $\phi\sigma(\mathfrak{z}) \subseteq \mathfrak{z}$ and for t in $\mathfrak{z}$, $\phi\sigma(t) \in X_t$, so by restricting the range of $\phi\sigma$ to $\mathfrak{z}$ we obtain a morphism, $\Lambda(\phi)$, in $D(\mathcal{A}^{\mathfrak{z}})$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.1. *The mapping $\Lambda\colon \mathrm{Mor}_G(G \times^P (\mathfrak{z} + \mathfrak{u}), \mathfrak{g}) \to D(\mathcal{A}^{\mathfrak{z}})$ is an isomorphism of graded $k[\mathfrak{z}]$-modules.*

It's well-known that P acts trivially on $(\mathfrak{z} + \mathfrak{u})/\mathfrak{u}$. Therefore, we can apply the results in §3 with $W = \mathfrak{z} + \mathfrak{u}$, $W_0 = \mathfrak{z}$, and $W_1 = \mathfrak{u}$, and conclude that $\mathrm{Mor}_G(G \times^P (\mathfrak{z} + \mathfrak{u}), \mathfrak{g})$ has the structure of a graded $k[\mathfrak{z}]$-module via the comorphism of $\theta\colon G\times^P(\mathfrak{z}+\mathfrak{u})\to\mathfrak{z}$. Moreover, by Corollary 4.2, $\mathrm{Ad}P(\mathfrak{z})$ is dense in $\mathfrak{z}+\mathfrak{u}$, and so $k[\mathfrak{z}]$ and $k[G\times^P(\mathfrak{z}+\mathfrak{u})]^G$ are isomorphic graded k-algebras by Proposition 3.4.

It's straightforward to check that Λ is a homomorphism of $k[\mathfrak{z}]$-modules. Moreover, if ϕ is in $\mathrm{Mor}_G(G\times^P(\mathfrak{z}+\mathfrak{u}),\mathfrak{g})_j$ for some non-negative integer, j, then

$$\Lambda(\phi(\xi t))=\phi(1*\xi t)=\xi^j\Lambda(\phi)(t)$$

for t in $\mathfrak{z}$ and ξ in k, so $\Lambda(\phi)$ is in $D(\mathcal{A}^{\mathfrak{z}})_j$. Thus Λ is a homomorphism of graded $k[\mathfrak{z}]$-modules.

To show that Λ is injective, suppose ϕ_1 and ϕ_2 are in $\mathrm{Mor}_G(G\times^P(\mathfrak{z}+\mathfrak{u}),\mathfrak{g})$ with $\Lambda(\phi_1)=\Lambda(\phi_2)$. Define $G*(\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u})=\{\,g*(t+n)\mid g\in G,\ t\in\mathfrak{z}_{\mathrm{reg}},\ n\in\mathfrak{u}\,\}$. Then if $g*(t+n)$ is in $G*(\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u})$, by Lemma 4.1 there is a u in U with $\mathrm{Ad}u(t)=t+n$ and so

$$\begin{aligned}\phi_1(g*(t+n))&=\phi_1(gu*t)\\&=\phi_1(1*t)\\&=\phi_1\sigma(t)\\&=\phi_2\sigma(t)\\&=\phi_2(gu*t)\\&=\phi_2(g*(t+n)).\end{aligned}$$

Therefore, ϕ_1 and ϕ_2 agree on $G*(\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u})$. Since $\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u}$ is a dense subvariety of $\mathfrak{z}+\mathfrak{u}$, $G*(\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u})$ is a dense subvariety of $G\times^P(\mathfrak{z}+\mathfrak{u})$ and so $\phi_1=\phi_2$.

To show that Λ is surjective we proceed as follows. Suppose η is in $D(\mathcal{A}^{\mathfrak{z}})$. We first construct a P-equivariant morphism, $\widetilde{\phi}\colon\mathfrak{z}+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$ with $\widetilde{\phi}|_{\mathfrak{z}}=\eta$. Extending the range of $\widetilde{\phi}$ to all of $\mathfrak{g}$ we can apply Proposition 3.1 and obtain a G-equivariant morphism, $\phi\colon G\times^P(\mathfrak{z}+\mathfrak{u})\to\mathfrak{g}$, with $\phi(1*(t+n))=\widetilde{\phi}(t+n)$ for all $t+n$ in $\mathfrak{z}+\mathfrak{u}$. Then clearly $\Lambda(\phi)=\eta$ and so Λ is surjective.

Now, to complete the proof of Theorem 5.1 it remains to construct a morphism, $\widetilde{\phi}$, in $\mathrm{Mor}_P(\mathfrak{z}+\mathfrak{u},\mathfrak{z}+\mathfrak{u})$, given a morphism, η, in $D(\mathcal{A}^{\mathfrak{z}})$. So for the rest of this section, fix η in $D(\mathcal{A}^{\mathfrak{z}})$.

Let a_{reg} denote the restriction of $a\colon U\times(\mathfrak{z}+\mathfrak{u})\to\mathfrak{z}+\mathfrak{u}$ to $U\times\mathfrak{z}_{\mathrm{reg}}$. Then by Corollary 4.2, a_{reg} determines an isomorphism of varieties between $U\times\mathfrak{z}_{\mathrm{reg}}$ and $\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u}$. Thus, if $1_U\colon U\to U$ is the identity morphism, then $a\circ(1_U\times\eta|_{\mathfrak{z}_{\mathrm{reg}}})\circ {a_{\mathrm{reg}}}^{-1}\colon\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$ is a morphism. Set $\widetilde{\phi}_{\mathrm{reg}}=a\circ(1_U\times\eta|_{\mathfrak{z}_{\mathrm{reg}}})\circ a^{-1}$. Then $\widetilde{\phi}_{\mathrm{reg}}(\mathrm{Ad}u(t))=\mathrm{Ad}u(\eta(t))$ for u in U and t in $\mathfrak{z}_{\mathrm{reg}}$ so $\widetilde{\phi}_{\mathrm{reg}}|_{\mathfrak{z}_{\mathrm{reg}}}=\eta|_{\mathfrak{z}_{\mathrm{reg}}}$.

We next show that $\widetilde{\phi}_{\mathrm{reg}}$ is P-equivariant. Then, since $\mathfrak{z}_{\mathrm{reg}}+\mathfrak{u}$ is dense in $\mathfrak{z}+\mathfrak{u}$, it will follow that if $\widetilde{\phi}_{\mathrm{reg}}$ extends to a morphism, $\widetilde{\phi}\colon\mathfrak{z}+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$, then $\widetilde{\phi}$ is also P-equivariant. So suppose p is in P, t is in $\mathfrak{z}_{\mathrm{reg}}$, and n is in $\mathfrak{u}$. Then there is a u in U so that $t+n=\mathrm{Ad}u(t)$. Write $pu=u_1l$ where u_1 is in U and l is in L. Then

$$\begin{aligned}\widetilde{\phi}_{\mathrm{reg}}(\mathrm{Ad}p(t+n))&=\widetilde{\phi}_{\mathrm{reg}}(\mathrm{Ad}pu(t))\\&=\widetilde{\phi}_{\mathrm{reg}}(\mathrm{Ad}u_1(t))\\&=\mathrm{Ad}u_1(\eta(t))\\&=\mathrm{Ad}u_1l(\eta(t))\\&=\mathrm{Ad}pu(\eta(t))\\&=\mathrm{Ad}p(\widetilde{\phi}_{\mathrm{reg}}(t+n)).\end{aligned}$$

Therefore, $\widetilde{\phi}_{\mathrm{reg}}$ is P-equivariant.

Our strategy for showing that $\widetilde{\phi}_{\text{reg}}$ extends to all of $\mathfrak{z}+\mathfrak{u}$ is to show that $\widetilde{\phi}_{\text{reg}}$ extends to a morphism $\widetilde{\phi}_i\colon \mathfrak{z}^i+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$ for all α in $\Phi_{\mathfrak{u}}$ where as in §4, $\mathfrak{z}^i = \{\, t\in\mathfrak{z} \mid \beta(t)\neq 0 \text{ for all } \beta\in\Phi_{\mathfrak{u}}\setminus\Psi_i \,\}$.

Assuming this has been done and that $\widetilde{\phi}_{\text{reg}}$ denotes the extension to $\bigcup_{i=1}^s \left(\mathfrak{z}^i+\mathfrak{u}\right)$, we can complete the proof as follows. If f is in $k[\mathfrak{z}+\mathfrak{u}]$, then $f\widetilde{\phi}_{\text{reg}}$ is a rational function on $\mathfrak{z}+\mathfrak{u}$ whose domain contains $\bigcup_{i=1}^s \left(\mathfrak{z}^i+\mathfrak{u}\right)$. If $f\widetilde{\phi}_{\text{reg}}$ is not a regular function on $\mathfrak{z}+\mathfrak{u}$, then the set of points at which it's not defined is a closed subset of $\mathfrak{z}+\mathfrak{u}$ with codimension 1. Now $\bigcup_{i=1}^s \left(\mathfrak{z}^i+\mathfrak{u}\right)$ is contained in the domain of $f\widetilde{\phi}_{\text{reg}}$ and so the set of points in $\mathfrak{z}+\mathfrak{u}$ at which $f\widetilde{\phi}_{\text{reg}}$ is not defined is contained in $(\mathfrak{z}+\mathfrak{u})\setminus\bigcup_{i=1}^s \left(\mathfrak{z}^i+\mathfrak{u}\right)$. However

$$(\mathfrak{z}+\mathfrak{u})\setminus\bigcup_{i=1}^s \left(\mathfrak{z}^i+\mathfrak{u}\right) = \left(\bigcup_{i\neq j}(H_i\cap H_j)\right)+\mathfrak{u}.$$

Thus, the set of points at which $f\widetilde{\phi}_{\text{reg}}$ is not defined has codimension at least 2. Therefore $f\widetilde{\phi}_{\text{reg}}$ must be defined on all of $\mathfrak{z}+\mathfrak{u}$ and so $f\widetilde{\phi}_{\text{reg}}$ is in $k[\mathfrak{z}+\mathfrak{u}]$. It follows that $\widetilde{\phi}^{\#}_{\text{reg}}\left(k[\mathfrak{z}+\mathfrak{u}]\right)\subseteq k[\mathfrak{z}+\mathfrak{u}]$, and so $\widetilde{\phi}_{\text{reg}}$ extends to a morphism $\widetilde{\phi}\colon\mathfrak{z}+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$.

It remains to show that $\widetilde{\phi}_{\text{reg}}$ can be extended to a morphism from $\mathfrak{z}^i+\mathfrak{u}$ to $\mathfrak{z}+\mathfrak{u}$ for $1\leq i\leq s$. In order to show that $\widetilde{\phi}_{\text{reg}}$ extends to $\mathfrak{z}^i+\mathfrak{u}$, we'll finally use the hypothesis that η is in $D(\mathcal{A}^{\mathfrak{z}})$. Since η is in $D(\mathcal{A}^{\mathfrak{z}})$, for α in $\Phi_{\mathfrak{u}}$, $\eta^{\#}(\alpha|_{\mathfrak{z}})$ is a $k[\mathfrak{z}]$-multiple of $\alpha|_{\mathfrak{z}}$. Say $\eta^{\#}(\alpha_i|_{\mathfrak{z}})=f_i\alpha_i|_{\mathfrak{z}}$ for $1\leq i\leq s$. Define $\eta_i\colon\mathfrak{z}^i+\mathfrak{u}_i\to\mathfrak{z}+\mathfrak{u}$ by $\eta_i(t+n)=\eta(t)+f_i(t)n$, so η_i is a morphism. Recall that $a_i\colon U^i\times(\mathfrak{z}^i+\mathfrak{u}_i)\to\mathfrak{z}+\mathfrak{u}$ determines an isomorphism between $U^i\times(\mathfrak{z}^i+\mathfrak{u}_i)$ and $\mathfrak{z}^i+\mathfrak{u}$ by Proposition 4.3. Define $\widetilde{\phi}_i = a\circ(1_U\times\eta_i)\circ {a_i}^{-1}\colon\mathfrak{z}^i+\mathfrak{u}\to\mathfrak{z}+\mathfrak{u}$.

Lemma 5.2. *The restriction of $\widetilde{\phi}_i$ to $\mathfrak{z}_{reg}+\mathfrak{u}$ is equal to $\widetilde{\phi}_{reg}$.*

Proof. Suppose $t+n$ is in $\mathfrak{z}_{\text{reg}}+\mathfrak{u}$. Then $t+n=\operatorname{Ad}u(t)$ for some u in U. Write $u=u^iu_i$ where u^i is in U^i and u_i is in U_i. It follows from Proposition 4.4 that there is an n_1 in $\mathfrak{u}_i$ so that $\operatorname{Ad}u_i(t')=t'+\alpha_i(t')n_1$ for every t' in $\mathfrak{z}$. Then $t+n=\operatorname{Ad}u^i(t+\alpha_i(t)n_1)$ and so

$$\begin{aligned}\widetilde{\phi}_i(t+n)&=\operatorname{Ad}u^i\left(\eta(t)+f_i(t)\alpha_i(t)n_1\right)\\&=\operatorname{Ad}u^i\left(\eta(t)+\alpha_i(\eta(t))n_1\right)\\&=\operatorname{Ad}u^iu_i(\eta(t))\\&=\operatorname{Ad}u(\eta(t))\\&=\widetilde{\phi}_{\text{reg}}(t+n).\end{aligned}$$

This completes the proof of the lemma. □

It follows from the lemma that $\widetilde{\phi}_{\text{reg}}$ extends to $\mathfrak{z}^i+\mathfrak{u}$ for α in $\Phi_{\mathfrak{u}}$, and so the proof of Theorem 5.1 is complete.

Corollary 5.3. *The graded $k[\mathfrak{z}]$-modules,* $\operatorname{Hom}_G(\mathfrak{g}^*, k[G\times^P(\mathfrak{z}+\mathfrak{u})])$ *and* $D(\mathcal{A}^{\mathfrak{z}})$ *are isomorphic.*

Proof. It is easily seen that the natural map between $\operatorname{Hom}_G(\mathfrak{g}^*, k[G\times^P(\mathfrak{z}+\mathfrak{u})])$ and $\operatorname{Mor}_G(G\times^P(\mathfrak{z}+\mathfrak{u}),\mathfrak{g})$ is an isomorphism of graded $k[\mathfrak{z}]$-modules, and so if Γ is the composition of Λ with this isomorphism, then

$$\Gamma\colon \operatorname{Hom}_G(\mathfrak{g}^*, k[G\times^P(\mathfrak{z}+\mathfrak{u})]) \to D(\mathcal{A}^{\mathfrak{z}})$$

is an isomorphism of graded $k[\mathfrak{z}]$-modules. □

6. Conclusions

As in §5, since P acts trivially on $(\mathfrak{z}+\mathfrak{u})/\mathfrak{u}$, we can take $W=\mathfrak{z}+\mathfrak{u}$, $W_0=\mathfrak{z}$, and $W_1=\mathfrak{u}$ in Lemma 3.3 and conclude that there is an isomorphism of graded $k[\mathfrak{z}]$-modules, $k[\mathfrak{z}]\otimes \operatorname{Hom}_G(\mathfrak{g}^*, k[G\times^P\mathfrak{u}]) \cong \operatorname{Mor}_G(G\times^P(\mathfrak{z}+\mathfrak{u}), \mathfrak{g})$. Composing this isomorphism with the isomorphism of Theorem 5.1 and using that $\mathfrak{g}$ affords a self-dual representation of G, we obtain an isomorphism of graded $k[\mathfrak{z}]$-modules, $k[\mathfrak{z}]\otimes \operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}]) \cong D(\mathcal{A}^{\mathfrak{z}})$. It follows that $(\mathfrak{z}, \mathcal{A}^{\mathfrak{z}})$ is a free hyperplane arrangement. We can now prove the following result of Orlik and Terao [8].

Corollary 6.1. *If $(V,\mathcal{A})$ is a reflection arrangement arising from a finite Weyl group, then for X in the lattice of $\mathcal{A}$, the restricted arrangement, $(X,\mathcal{A}^X)$ is free.*

Proof. Suppose Φ is a root system in a k-vector space V. Let $\mathcal{A}$ be the arrangement of hyperplanes in V consisting of the hyperplanes orthogonal to the vectors in Φ. It's straightforward to check that if G is the adjoint group with root system $\Phi=\Phi(G,T)$, then every restriction of $(V,\mathcal{A})$ arises as $(\mathfrak{z},\mathcal{A}^{\mathfrak{z}})$ for a suitable choice of P and L.

Alternately, if W is the Weyl group of $(V,\mathcal{A})$, then W acts on the lattice of $\mathcal{A}$ and it follows from standard results about root systems that if Π is a base of Φ, then every W-orbit in the lattice of $\mathcal{A}$ contains a representative that's an intersection of the hyperplanes orthogonal to the vectors in a subset of Π (see [6]). Clearly, if X is in the lattice of $\mathcal{A}$ and w is in W, then $\mathcal{A}^X$ is free if and only if $\mathcal{A}^{wX}$ is free. Now if G is the adjoint group with root system $\Phi=\Phi(G,T)$, B is a Borel subgroup of G containing T, P is a parabolic subgroup of G containing B, and L is the standard Levi factor in P, then $(\mathfrak{z},\mathcal{A}^{\mathfrak{z}})$ is free. Since every orbit of W on the lattice of $\mathcal{A}$ contains a representative of this form, the result follows. □

Finally, we show that the isomorphism of graded $k[\mathfrak{z}]$-modules between $k[\mathfrak{z}]\otimes \operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}])$ and $D(\mathcal{A}^{\mathfrak{z}})$ can be used to prove the result of Sommers, Trapa, and Broer mentioned in the introduction.

Corollary 6.2. *The multiplicity of the adjoint representation of G in the graded representation $k[G\times^P\mathfrak{u}]$ is given by*

$$\sum_{j\geq 0} \dim \operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}]_j)q^j = q^{n_1^{\mathfrak{z}}}+\cdots+q^{n_r^{\mathfrak{z}}}$$

where q is an indeterminate and $\{\, n_1^{\mathfrak{z}},\dots,n_r^{\mathfrak{z}} \,\}$ is the multiset of exponents of $\mathcal{A}^{\mathfrak{z}}$.

Proof. As in §2, $\operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}])$ inherits a grading from the scalar action of k on $\mathfrak{u}$. Suppose $\{\,\phi_1,\dots,\phi_s\,\}$ is a homogeneous basis of $\operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}])$ with ϕ_i in $\operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}]_{d_i})$ for $1\leq i\leq s$. Then clearly

$$\sum_{j\geq 0} \dim \operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}]_j)q^j = q^{d_1}+\cdots+q^{d_s}.$$

On the other hand, the set $\{\,1\otimes\phi_1,\dots,1\otimes\phi_s\,\}$ is a homogeneous basis of $k[\mathfrak{z}]\otimes \operatorname{Hom}_G(\mathfrak{g}, k[G\times^P\mathfrak{u}])$ and so corresponds to a homogeneous basis of $D(\mathcal{A}^{\mathfrak{z}})$ under the isomorphism $k[\mathfrak{z}] \otimes \mathrm{Hom}_G(\mathfrak{g}, k[G\times^P \mathfrak{u}]) \cong D(\mathcal{A}^{\mathfrak{z}})$. Therefore, the multisets $\{ d_1, \dots, d_s \}$ and $\{ n_1^{\mathfrak{z}}, \dots, n_r^{\mathfrak{z}} \}$ are equal. □

References

[1] W. Borho. Über Schichten halbeinfacher Lie-Algebren. *Invent. Math.*, 65:283–317, 1981. MR **83b:**17006

[2] A. Broer. The sum of generalized exponents and Chevalley's restriction theorem for modules of covariants. *Indag. Math.*, 6:385–396, 1995. MR **96j:**20058

[3] A. Broer. Lectures on decomposition classes. In *Representation Theory and Algebraic Geometry, Proceedings of the NATO Advanced Study Institute, Montreal, Canada, July 28 – August 8, 1997* (A. Broer, editor). Kluwer, 1998. MR **99:**03

[4] A. Broer. Hyperplane arrangements, Springer representations, and exponents. In *Advances in Geometry*, volume 172 of *Progress in Mathematics* (J. Brylinski, editor). Birkhäuser, 1999. MR **99:**07

[5] B. Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963. MR **28:**1252

[6] P. Orlik and L. Solomon. Coxeter arrangements. In *Singularities*, volume 40 of *Proc. Symp. Pure Math.*, pages 269–292, Amer. Math. Soc., 1983. MR **85b:**32016

[7] P. Orlik and H. Terao. *Arrangements of Hyperplanes*. Springer-Verlag, 1992. MR **94e:**52014

[8] P. Orlik and H. Terao. Coxeter arrangements are hereditarily free. *Tôhoku Math. J.*, 45:369–383, 1993. MR **94h:**52023

[9] E. Sommers and P. Trapa. The adjoint representation in rings of functions. *Represent. Theory*, 1:182–189, 1997. MR **98i:**22020

Department of Mathematics, University of North Texas, Denton, Texas 76203
E-mail address: douglass@unt.edu