

WEYL MODULES FOR CLASSICAL AND QUANTUM AFFINE ALGEBRAS

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ABSTRACT. We introduce and study the notion of a Weyl module for the classical affine algebras, these modules are universal finite-dimensional highest weight modules. We conjecture that the modules are the classical limit of a family of irreducible modules of the quantum affine algebra, and prove the conjecture in the case of sl_2 . The conjecture implies also that the Weyl modules are the classical limits of the standard modules introduced by Nakajima and further studied by Varagnolo and Vasserot.

0. INTRODUCTION

The study of the irreducible finite-dimensional representations of quantum affine algebras has been the subject of a number of papers, [AK], [CP3], [CP5], [FR], [FM], [GV], [KS] to name a few. However, the structure of these representations is still unknown except in certain special cases. In this paper, we approach the problem by studying the classical ($q \rightarrow 1$) limits of these representations. Standard results imply, for example, that if V is a finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$, its $q \rightarrow 1$ limit \bar{V} has the same character as a \mathfrak{g} -module as V has as a $U_q(\hat{\mathfrak{g}})$ -module.

We begin by studying an appropriate class of representations of the affine Lie algebra $\hat{\mathfrak{g}}$. The finite-dimensional irreducible representations of $\hat{\mathfrak{g}}$ were classified in [C], [CP1], where it was shown that such representations are highest weight in a suitable sense, the highest weight being an n -tuple of polynomials π , where n is the rank of \mathfrak{g} . We therefore study the class of all highest weight finite-dimensional representations of $\hat{\mathfrak{g}}$. In fact, we prove that corresponding to each irreducible finite-dimensional representation $V(\pi)$ there exists a unique (up to isomorphism) finite-dimensional highest weight module $W(\pi)$, such that *any* finite-dimensional highest weight module V with highest weight π is a quotient of $W(\pi)$. We call these modules the Weyl modules because of an analogy with the modular representation theory of \mathfrak{g} , which we now explain.

In [CP5], we showed that the irreducible representations of $U_q(\hat{\mathfrak{g}})$ are also highest weight and that their isomorphism classes are parametrized by n -tuples of polynomials π_q with coefficients in $\mathbf{C}(q)$. Under a natural condition on π_q , the corresponding representation $V_q(\pi_q)$ of $U_q(\hat{\mathfrak{g}})$ specializes as $q \rightarrow 1$ to a representation $\bar{V}_q(\pi_q)$ of $\hat{\mathfrak{g}}$ and is a quotient of $W(\pi)$, where π is obtained from π_q by setting $q = 1$. We conjecture that every $W(\pi)$ is the classical limit of an irreducible $U_q(\hat{\mathfrak{g}})$ -module. This is analogous to the fact that the Weyl modules for \mathfrak{g} in characteristic p are the mod p reductions of the irreducible modules in characteristic zero. We

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prove the conjecture in the case of $\mathfrak{g} = sl_2$ in this paper. The conjecture is also true for the fundamental representations of $U_q(\hat{\mathfrak{g}})$, but the details of that will appear elsewhere.

In Section 3, we prove a factorization property of Weyl modules analogous to the one for the irreducible modules proved in [CP1]. We also obtain a necessary and sufficient condition for the Weyl modules to be irreducible: the interesting feature of this proof is that it uses the fact that the specialized irreducible modules for the quantum algebra are quotients of the Weyl module. Further, the condition for irreducibility of the Weyl modules is the same as a condition that appears first in the work of Drinfeld on the closely related Yangians, [Dr1].

The Weyl modules we define are quotients of a family of level zero integrable modules for the extended affine Lie algebra, one corresponding to each dominant integral weight of \mathfrak{g} . We call these modules $W(\lambda)$. In Section 6, we explicitly construct the modules $W(\lambda)$ when \mathfrak{g} is of type sl_2 . The quantum analogues of these modules, $W_q(\lambda)$ are also defined in this paper. In [K], Kashiwara defined a family of integrable modules $V^{max}(\lambda)$ for each dominant integral weight of $\hat{\mathfrak{g}}$. According to unpublished work of Kashiwara, when λ has level zero, the module $V^{max}(\lambda)$ is isomorphic to $W_q(\lambda)$. In addition, Kashiwara has a number of conjectures for the modules $W_q(\lambda)$. In [CP7], we use the results of this paper to explicitly construct $V^{max}(\lambda)$ when \mathfrak{g} is of type sl_2 and this establishes one of his conjectures in this case.

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1. PRELIMINARIES AND SOME IDENTITIES

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra of rank n , \mathfrak{h} a Cartan subalgebra of \mathfrak{g} and R the set of roots of \mathfrak{g} with respect to \mathfrak{h} . Let $I = \{1, 2, \dots, n\}$, fix a set of simple roots α_i ($i \in I$), and let R^+ be the corresponding set of positive roots. Let $\theta \in R^+$ be the highest root in R^+ . For $\alpha \in R^+$, fix non-zero elements $x_\alpha^\pm \in \mathfrak{g}$, $h_\alpha \in \mathfrak{h}$ such that

$$[x_\alpha^+, x_\alpha^-] = h_\alpha, \quad [h_\alpha, x_\alpha^\pm] = \pm 2x_\alpha^\pm.$$

Let $Q = \bigoplus_{i=1}^n \mathbf{Z}\alpha_i$ (resp. $Q_+ = \bigoplus_{i=1}^n \mathbf{N}\alpha_i$) denote the root (resp. positive root) lattice of \mathfrak{g} . For $\eta \in Q^+$, $\eta = \sum_i r_i \alpha_i$, we set $\text{ht } \eta = \sum_i r_i$. The lattice P (resp. P_+) of integral (resp. dominant integral) weights is the set of elements $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_\alpha) \in \mathbf{Z}$ for all $\alpha \in \mathbf{N}$ (resp. $\lambda(h_\alpha) \geq 0$ for all $\alpha \in R^+$). For $i \in I$, the fundamental weight ω_i of \mathfrak{g} is given by $\omega_i(\alpha_j) = \delta_{ij}$. Let $\langle \cdot, \cdot \rangle$ be the bilinear pairing on P such that $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$. Set $a_{ij} = \langle \alpha_i, \alpha_j \rangle$ for $i, j \in I$. The bilinear form induces an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ such that, if $\beta = \sum_i r_i \alpha_i \in R^+$, then

$$h_\beta = \sum_j \frac{d_j r_j}{d_\beta} h_j,$$

where for a root $\alpha \in R$, we set $d_\alpha = \frac{1}{2} \langle \alpha, \alpha \rangle$. Let $W \subset \text{Aut}(\mathfrak{h}^*)$ be the Weyl group of \mathfrak{g} ; it is well known that W is generated by simple reflections s_i ($i \in I$).

The extended loop algebra of \mathfrak{g} is the Lie algebra

$$L^e(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}d,$$

with commutator given by

$$[d, x \otimes t^r] = rx \otimes t^r, \quad [x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$$

for $x, y \in \mathfrak{g}$, $r, s \in \mathbf{Z}$. The loop algebra $L(\mathfrak{g})$ is the subalgebra $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ of $L^e(\mathfrak{g})$. Let $\mathfrak{h}^e = \mathfrak{h} \oplus \mathbf{C}d$. Define $\delta \in (\mathfrak{h}^e)^*$ by

$$\delta(\mathfrak{h}) = 0, \quad \delta(d) = 1.$$

Extend $\lambda \in \mathfrak{h}^*$ to an element of $(\mathfrak{h}^e)^*$ by setting $\lambda(d) = 0$. Set $P^e = \bigoplus_{i=1}^n \mathbf{Z}\omega_i \oplus \mathbf{Z}\delta$, and define P_+^e in the obvious way. We regard W as acting on $(\mathfrak{h}^e)^*$ by setting $w(\delta) = \delta$ for all $w \in W$.

For any $x \in \mathfrak{g}$, $m \in \mathbf{Z}$, we denote by x_m the element $x \otimes t^m \in L^e(\mathfrak{g})$. Set $e_i^\pm = x_{\alpha_i}^\pm \otimes 1$ and $e_0^\pm = x_\beta^\mp \otimes t^{\pm 1}$. Then, the elements e_i^\pm ($i = 0, \dots, n$) and d generate $L^e(\mathfrak{g})$.

For any Lie algebra \mathfrak{a} , the universal enveloping algebra of \mathfrak{a} is denoted by $\mathbf{U}(\mathfrak{a})$. We set

$$\mathbf{U}(L^e(\mathfrak{g})) = \mathbf{U}^e, \quad \mathbf{U}(L(\mathfrak{g})) = \mathbf{U}, \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}^{fin}.$$

Let $\mathbf{U}(<)$ (resp. $\mathbf{U}(>)$) be the subalgebra of \mathbf{U} generated by the $x_{\alpha_i, m}^-$ (resp. $x_{\alpha_i, m}^+$) for $i \in I$, $m \in \mathbf{Z}$. Clearly, $x_{\alpha, m}^- \in \mathbf{U}(<)$ (resp. $x_{\alpha, m}^+ \in \mathbf{U}(>)$) for all $\alpha \in R^+$, $m \in \mathbf{Z}$. Set $\mathbf{U}^{fin}(<) = \mathbf{U}(<) \cap \mathbf{U}^{fin}$ and define $\mathbf{U}^{fin}(>)$ similarly. Finally, let $\mathbf{U}(0)$ be the subalgebra of \mathbf{U} generated by $h_{\alpha, m}$ for $\alpha \in R$, $m \in \mathbf{Z}$, $m \neq 0$. We have

$$\begin{aligned} \mathbf{U}^{fin} &= \mathbf{U}^{fin}(<)\mathbf{U}(\mathfrak{h})\mathbf{U}^{fin}(>), \\ \mathbf{U}^e &= \mathbf{U}(<)\mathbf{U}(0)\mathbf{U}(\mathfrak{h}^e)\mathbf{U}(>). \end{aligned}$$

Lemma 1.1. *The assignment $T(x_{\alpha_i, m}^\pm) = x_{\alpha_i, m \pm 1}^\pm$, for $i \in I$, $m \in \mathbf{Z}$, defines an algebra automorphism of \mathbf{U} . \square*

Next we recall some identities in \mathbf{U}^e , which are most conveniently stated using the generating series. Thus, for any $\beta \in R^+$, we introduce the following power series in an indeterminate u :

$$\begin{aligned} \tilde{X}_\beta^-(u) &= \sum_{m=-\infty}^{\infty} x_{\beta, m}^- u^{m+1}, & X_\beta^-(u) &= \sum_{m=1}^{\infty} x_{\beta, m}^- u^m, \\ X_\beta^+(u) &= \sum_{m=0}^{\infty} x_{\beta, m}^+ u^m, & X_{\beta, 0}^-(u) &= \sum_{m=0}^{\infty} x_{\beta, m}^- u^{m+1}, \\ \tilde{H}_\beta(u) &= \sum_{m=-\infty}^{\infty} h_{\beta, m} u^{m+1}, & \Lambda_\beta^\pm(u) &= \sum_{m=0}^{\infty} \Lambda_{\beta, \pm m} u^m = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{\beta, \pm k}}{k} u^k\right). \end{aligned}$$

Set $x_{\alpha_i}^\pm = x_i^\pm$, $x_{\alpha_i, m}^\pm = x_{i, m}^\pm$, and define h_i , Λ_i^\pm , etc., similarly.

The next lemma follows easily from the definition of the $\Lambda_{i, m}$ ($= \Lambda_{\alpha_i, m}$).

Lemma 1.2. *The subalgebra $\mathbf{U}(0)$ of \mathbf{U} is generated by the elements $\Lambda_{i, m}$, for $i \in I$, $m \in \mathbf{Z}$. \square*

For any power series f in u with coefficients in an algebra A , let f_m be the coefficient of u^m ($m \in \mathbf{Z}$) and let f' denote the derivative of f with respect to u . For $x \in \mathbf{U}$, $r \in \mathbf{Z}^+$, set

$$x^{(r)} = \frac{x^r}{r!}.$$

For an algebra A , let A_+ denote the augmentation ideal. The next result is a reformulation of a result of Garland [G].

Lemma 1.3. *Let $s \geq r \geq 1$, $\beta \in R^+$.*

(i)

$$(x_{\beta,0}^+)^{(r)}(x_{\beta,1}^-)^{(s)} = (-1)^r (X_{\beta}^-(u)^{(s-r)} \Lambda_{\beta}^+(u))_s \pmod{\mathbf{UU}(>)_+}.$$

(ii)

$$(x_{\beta,1}^+)^{(r)}(x_{\beta,0}^-)^{(s)} = (-1)^r (X_{\beta,0}^-(u)^{(s-r)} \Lambda_{\beta}^+(u))_s \pmod{\mathbf{UU}(>)_+}.$$

Proof. We use the identity in [G, Lemma 7.1], in the form given in [CP6, Lemma 5.1] for its quantum version, namely

$$\begin{aligned} & (x_{\beta,0}^+)^{(r)}(x_{\beta,1}^-)^{(s)} \\ &= \sum_{t=0}^r \sum_{m=0}^t \sum_{k=0}^m (-1)^t \left(u^{-s+t} X_{\beta}^-(u)^{(s-t)} \right)_{t-m} \Lambda_{\beta,k} \left(X_{\beta}^+(u)^{(r-t)} \right)_{m-k}. \end{aligned}$$

In the sum on the right-hand side of the equality, we get an element of $\mathbf{UU}(>)_+$ unless $t = r$ and $m = k$, and so we have

$$\begin{aligned} (x_{\beta,0}^+)^{(r)}(x_{\beta,1}^-)^{(s)} &= (-1)^r \sum_{m=0}^r \left(u^{-s+r} X_{\beta}^-(u)^{(s-r)} \right)_{r-m} \Lambda_{\beta,m} \pmod{\mathbf{UU}(>)_+} \\ &= \sum_{r=0}^m (-1)^r \left(X_{\beta}^-(u)^{(s-r)} \right)_{s-m} \Lambda_{\beta}^+(u)_m \pmod{\mathbf{UU}(>)_+} \\ &= (-1)^r \left(X_{\beta}^-(u)^{(s-r)} \Lambda_{\beta}^+(u) \right)_s \pmod{\mathbf{UU}(>)_+}. \end{aligned}$$

The identity in (ii) follows from (i) by applying the automorphism $T : \mathbf{U} \rightarrow \mathbf{U}$. \square

We conclude this section with some elementary properties of integrable $L^e(\mathfrak{g})$ -modules.

A representation V of \mathbf{U}^e is called integrable if the Chevalley generators e_i^{\pm} , for $i = 0, 1, \dots, n$, act locally nilpotently on V and

$$V = \bigoplus_{\lambda \in (\mathfrak{h}^e)^*} V_{\lambda},$$

where

$$V_{\lambda} = \{v \in V : h.v = \lambda(h)v \ \forall \ h \in \mathfrak{h}^e\}.$$

It is well known that this implies that the elements $x_{\beta,m}^{\pm}$ act locally nilpotently on V for all $\beta \in R^+$ and $m \in \mathbf{Z}$. Set

$$V_{\lambda}^+ = \{v \in V_{\lambda} : x_{\beta,m}^+ . v = 0 \ \forall \ \beta \in R^+, m \in \mathbf{Z}\}.$$

It is easy to see that, if V is integrable, then $V_{\lambda} \neq 0$ (resp. $V_{\lambda}^+ \neq 0$) only if $\lambda \in P^e$ (resp. $\lambda \in P_+^e$). Further, if $v \in V_{\lambda}^+$, then

$$(1.1) \quad (x_{\beta,m}^-)^{\lambda(h_{\beta})+1} . v = 0,$$

$$(1.2) \quad V_{\lambda} \neq 0 \implies V_{w\lambda} \neq 0 \ \forall \ w \in W.$$

If $\lambda \in P_+$, let $V^{fin}(\lambda)$ be the finite-dimensional irreducible \mathbf{U}^{fin} -module with highest weight λ . If $\lambda \in P_+^e$, the restriction of λ to \mathfrak{h}^* , also denoted by λ , is in P_+ .

If V is an integrable \mathbf{U}^e -module and $0 \neq v \in V_\lambda^+$, then $\mathbf{U}^{fin}.v$ is a \mathbf{U}^{fin} -submodule of V isomorphic to $V^{fin}(\lambda)$.

Proposition 1.1. *Let V be an integrable \mathbf{U}^e -module. Let $\lambda \in P_+^e$, let $0 \neq v \in V_\lambda^+$ and let $\beta \in R^+$. Then*

- (i) $\Lambda_{\beta,m}.v = 0$ for $m > \lambda(h_\beta)$;
- (ii) for $r \geq 1$, $s > \lambda(h_\beta)$,

$$\left(X_\beta^-(u)^r \Lambda_\beta^+(u)\right)_s .v = 0, \quad \left(X_{\beta,0}^-(u)^r \Lambda_\beta^+(u)\right)_s .v = 0;$$

- (iii) for all $s \in \mathbf{Z}$,

$$\left(\tilde{X}_\beta^-(u) \Lambda_\beta^+(u)\right)_s .v = 0, \quad \left(\tilde{H}_\beta(u) \Lambda_\beta^+(u)\right)_s .v = 0;$$

- (iv) $\Lambda_{\beta,-m}.v = 0$ for all $m > \lambda(h_\beta)$;
- (v) for $0 \leq m \leq \lambda(h_\beta)$,

$$\Lambda_{\beta,\lambda(h_\beta)} \Lambda_{\beta,-m}.v = \Lambda_{\beta,\lambda(h_\beta)-m}.v.$$

Proof. (i) This follows by taking $r = s > \lambda(h_\beta)$ in Lemma 1.3(i) and using (1.1).

(ii) This follows from Lemma 1.3 by replacing r by $s - r$ and using (1.1).

(iii) Taking $r = \lambda(h_\beta)$, $s = \lambda(h_\beta) + 1$ in Lemma 1.3(i) gives

$$(1.3) \quad \sum_{m=0}^{\lambda(h_\beta)} x_{\beta,m+1}^- \Lambda_{\beta,\lambda(h_\beta)-m}.v = 0.$$

Applying $h_{\beta,k}$, for any $k \in \mathbf{Z}$, to the above equation and noting that $h_{\beta,k}.v \in V_\lambda^+$, we get

$$\sum_{m=0}^{\lambda(h_\beta)} x_{\beta,k+m+1}^- \Lambda_{\beta,\lambda(h_\beta)-m}.v = 0,$$

which can be written as

$$\left(\tilde{X}_\beta^-(u) \Lambda_\beta^+(u)\right)_{k+1} .v = 0.$$

Applying $x_{\beta,-s-1}^+$, for $s \in \mathbf{Z}$, to both sides of equation (1.3) gives

$$\sum_{m=0}^{\lambda(\beta)} h_{\beta,m-s} \Lambda_{\beta,\lambda(h_\beta)-m}.v = 0,$$

i.e.,

$$\sum_{m=-s}^{\lambda(h_\beta)-s} h_{\beta,m} \Lambda_{\beta,\lambda(h_\beta)-s-m}.v = 0.$$

Replacing s by $\lambda(h_\beta) - s + 1$ and using part (i) of the lemma, one sees that this identity is equivalent to the second identity in (iii).

(iv) and (v). During the remainder of this proof, write $\Lambda_\beta = \Lambda_\beta^+$, $\tilde{\Lambda}_\beta(u) = \Lambda_\beta^-(u^{-1})$, so that

$$\tilde{\Lambda}_\beta(u) = \exp\left(-\sum_{k=1}^{\infty} \frac{h_{\beta,-k}}{k} u^{-k}\right).$$

Note that, as operators on V_λ^+ ,

$$\left(\lambda(h_\beta) - u \frac{\Lambda'_\beta}{\Lambda_\beta} + u \frac{\tilde{\Lambda}'_\beta}{\tilde{\Lambda}_\beta} \right) = \tilde{H}_\beta.$$

By (iii), we have

$$\Lambda_\beta(u) \left(\lambda(h_\beta) - u \frac{\Lambda'_\beta}{\Lambda_\beta} + u \frac{\tilde{\Lambda}'_\beta}{\tilde{\Lambda}_\beta} \right) = \tilde{H}_\beta \Lambda_\beta = 0,$$

as operators on V_λ^+ , so, since $\Lambda_\beta(u)$ is invertible,

$$\lambda(h_\beta) \Lambda_\beta(u) \tilde{\Lambda}_\beta(u) = u(\Lambda'_\beta \tilde{\Lambda}_\beta - \Lambda_\beta \tilde{\Lambda}'_\beta).$$

Note that both sides of this equation make sense as a power series in u since $\Lambda_\beta(u)$ is already known by (i) to involve only finitely many positive powers of u . Hence, as a series with only finitely many positive powers (but possibly infinitely many negative powers), we have

$$\left(\frac{\Lambda_\beta}{\tilde{\Lambda}_\beta} \right)' = \lambda(h_\beta) u^{-1} \left(\frac{\Lambda_\beta}{\tilde{\Lambda}_\beta} \right),$$

and so

$$\frac{\Lambda_\beta(u)}{\tilde{\Lambda}_\beta(u)} = A_\beta u^{\lambda(h_\beta)},$$

where A_β is an operator on V_λ^+ independent of u . Equating coefficients of $u^{\lambda(h_\beta)}$ shows that $A_\beta = \Lambda_{\beta, \lambda(h_\beta)}$ and then the equation (of operators on V_λ^+)

$$\Lambda_{\beta, \lambda(h_\beta)}(u) \tilde{\Lambda}_\beta(u) = u^{-\lambda(h_\beta)} \Lambda_\beta(u),$$

proves both (iv) and (v). \square

Fix a total order \leq on R^+ .

Proposition 1.2. *Let V be an integrable \mathbf{U}^e -module, let $\lambda \in P_+^e$ and let $0 \neq v \in V_\lambda^+$ be such that $V = \mathbf{U}^e.v$.*

- (i) *If $V_\mu \neq 0$, then $\mu = \lambda - \eta + r\delta$ for some $\eta \in Q_+$, $r \in \mathbf{Z}$ such that $V^{fin}(\lambda)_{\lambda-\eta} \neq 0$.*
- (ii) *V is spanned by the elements*

$$x_{\beta_1, r_1}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- \mathbf{U}(0).v,$$

for $0 \leq r_t < \lambda(h_{\beta_t})$, $0 \leq t \leq s$, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_s \in R^+$.

Proof. Since $\mathbf{U}(>)_+.v = 0$, it follows that $V = \mathbf{U}(<)\mathbf{U}(0).v$ and hence

$$V_\mu \neq 0 \implies \mu = \lambda - \eta + r\delta,$$

for some $\eta \in Q^+$ and $r \in \mathbf{Z}$. Choose $\sigma \in W$ such that $\sigma(\lambda - \eta) \in P_+$. Since V is integrable, $V_{\sigma(\mu)} \neq 0$, hence $\sigma(\lambda - \eta) = \lambda - \eta'$ for some $\eta' \in Q_+$. This implies that $V^{fin}(\lambda)_{\sigma(\lambda-\eta)} \neq 0$, and hence that $V^{fin}(\lambda)_{\lambda-\eta} \neq 0$.

To prove (ii), note first that it is clear that elements of the form

$$x_{\beta_1, r_1}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- \mathbf{U}(0).v, \quad (0 \leq t \leq s, \beta_t \in R^+, r_t \in \mathbf{Z})$$

span V . We prove by induction on s that any such element is in the span of the elements

$$x_{\gamma_1, k_1}^- x_{\gamma_2, k_2}^- \cdots x_{\gamma_m, k_m}^- \mathbf{U}(0).v, \quad (0 \leq k_t < \lambda(h_{\gamma_t}), \quad \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m, \quad 0 \leq t \leq m).$$

For $s = 1$, $r_1 \geq \lambda(h_{\beta_1})$, we have, by Proposition 1.1(iii), that

$$\sum_{r=0}^{r_1} x_{\beta_1, r}^- \Lambda_{\beta_1, r_1-r} \mathbf{U}(0).v = 0.$$

Since $\Lambda_{\beta_1, 0} = 1$, this implies that $x_{\beta_1, r_1}^- .v$ is in the span of the elements $x_{\beta_1, r}^- \mathbf{U}(0).v$ with $0 \leq r < r_1$, from which the assertion follows. If $r_1 < 0$, we use

$$\sum_{r=0}^{\lambda(h_{\beta_1})} x_{\beta_1, r+r_1}^- \Lambda_{\beta_1, \lambda(h_{\beta_1})-r} .v = 0,$$

which follows from parts (i) and (ii) of Proposition 1.1. Since, by Proposition 1.1(v), $\Lambda_{\beta, \lambda(h_{\beta})}$ is invertible, it follows that $x_{\beta_1, r_1}^- .v$ is in the span of the elements $x_{\beta_1, r}^- .v$ ($0 \geq r > r_1$). An obvious induction now gives the result.

Suppose that we know the result for some $s \geq 1$. Let $\beta_t \in R^+$ and $r_t \in \mathbf{Z}$ for $0 \leq t \leq s$ be such that $0 \leq r_t < \lambda(h_{\beta_t})$ for all t . We have

$$\begin{aligned} x_{\beta_0, r_0}^- x_{\beta_1, r_1}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- .v &= x_{\beta_1, r_1}^- x_{\beta_0, r_0}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- .v \\ &\quad + [x_{\beta_0, r_0}^-, x_{\beta_1, r_1}^-] x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- .v. \end{aligned}$$

Since $[x_{\beta_0, r_0}^-, x_{\beta_1, r_1}^-] \in \mathfrak{g} \otimes t^{r_0+r_1}$, the induction hypothesis applies to the second term on the right-hand side of the equality. As for the first term, notice that the induction hypothesis applied to $x_{\beta_0, r_0}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- .v$ implies that this is in the span of the elements obtained by applying ordered monomials in the $x_{\gamma, k}^-$ to v , for $\gamma \in R^+$ and $0 \leq k < \lambda(h_{\gamma})$. Thus, we must show that every element of the form

$$x_{\beta_1, r_1}^- x_{\gamma_1, k_1}^- x_{\gamma_2, k_2}^- \cdots x_{\gamma_m, k_m}^- \mathbf{U}(0).v,$$

where $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m$ and $0 \leq k_t < \lambda(h_{\gamma_t})$ for $1 \leq t \leq m$, can be rewritten in the desired form. If $\beta_1 \leq \gamma_1$, there is nothing to prove. Otherwise, we have

$$[x_{\beta_1, r_1}^-, x_{\gamma_1, k_1}^-] \in \mathfrak{g}_{\beta_1+\gamma_1} \otimes t^{r_1+k_1}$$

and $\lambda(h_{\beta_1+\gamma_1}) \leq r_1 + k_1$. Using the induction hypothesis gives the result. \square

2. MAXIMAL INTEGRABLE AND MAXIMAL FINITE-DIMENSIONAL MODULES

In this section we define, for every $\lambda \in P_+^e$ an integrable \mathbf{U}^e -module $W(\lambda)$. Further, for any n -tuple $\boldsymbol{\pi} = (\pi_1(u), \pi_2(u), \cdots, \pi_n(u))$ of polynomials $\pi_i(u)$ in an indeterminate u with constant term 1 and degree $\lambda(h_i)$, we define a finite-dimensional quotient \mathbf{U} -module $W(\boldsymbol{\pi})$ of $W(\lambda)$.

For $\lambda \in P_+^e$, let I_λ be the left ideal in the subalgebra $\mathbf{U}(<)\mathbf{U}(0)\mathbf{U}(\mathfrak{h})$ of \mathbf{U}^e generated by the elements

$$\begin{aligned} &h - \lambda(h) \quad (h \in \mathfrak{h}), \quad \Lambda_{i, m} \quad (i \in I, \quad |m| > \lambda(h_i)), \\ &\Lambda_{i, \lambda(h_i)} \Lambda_{i, -m} - \Lambda_{i, \lambda(h_i)-m} \quad (i \in I, \quad 1 \leq m \leq \lambda(h_i)), \\ &\left(\tilde{X}_i^-(u) \Lambda_i^+(u) \right)_m \mathbf{U}(0) \quad (i \in I, \quad m \in \mathbf{Z}), \\ &(X_{i, 0}^-(u)^r \Lambda_i^+(u))_m \mathbf{U}(0) \quad (i \in I, \quad r \geq 1, \quad |m| > \lambda(h_i)). \end{aligned}$$

Let \tilde{I}_λ be the left ideal in \mathbf{U} generated by I_λ and the $x_{i,m}^+$ for all $i \in I$, $m \in \mathbf{Z}$, and let \tilde{I}_λ^e be the left ideal in \mathbf{U}^e generated by \tilde{I}_λ and $d - \lambda(d)$. Set

$$W(\lambda) = \mathbf{U}^e / \tilde{I}_\lambda^e = \mathbf{U} / \tilde{I}_\lambda.$$

Clearly, $W(\lambda)$ is a left \mathbf{U}^e -module (and a left \mathbf{U} -module) through left multiplication. Let w_λ be the image of 1 in $W(\lambda)$. Then,

$$\mathbf{U}(>)_+ . w_\lambda = 0, \quad W(\lambda) = \mathbf{U}^e . w_\lambda = \mathbf{U} . w_\lambda.$$

Since $\tilde{I}_\lambda \mathbf{U}(0) \subset \tilde{I}_\lambda$, we can and do regard $W(\lambda)$ as a right $\mathbf{U}(0)$ -module as well. For $\eta \in Q^+$, we set

$$W(\lambda)[\eta] = \bigoplus_{r \in \mathbf{Z}} W(\lambda)_{\lambda - \eta + r\delta}.$$

Clearly, $W(\lambda)[\eta]$ is a right $\mathbf{U}(0)$ -module for all $\eta \in Q^+$ and we have

$$W(\lambda) = \bigoplus_{\eta \in Q^+} W(\lambda)[\eta]$$

as right $\mathbf{U}(0)$ -modules.

Let $I_\lambda(0) = I_\lambda \cap \mathbf{U}(0)$. By the PBW theorem, it is easy to see that

$$(*) \quad \mathbf{U}(0) / I_\lambda(0) \cong \mathbf{C}[\Lambda_{i,m}, \Lambda_{i,\lambda(h_i)}^{-1} : i \in I, 1 \leq m \leq \lambda(h_i)].$$

In particular,

$$W(\lambda)[0] \cong \mathbf{C}[\Lambda_{i,m}, \Lambda_{i,\lambda(h_i)}^{-1} : i \in I, 1 \leq m \leq \lambda(h_i)]$$

as right $\mathbf{U}(0)$ -modules. It follows immediately from Proposition 1.2(ii) that, for all $\eta \in Q^+$, $W(\lambda)[\eta]$ is a finitely-generated $\mathbf{U}(0)$ -module.

Next, let $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ be an n -tuple of polynomials in an indeterminate u with constant term 1, and define, for $a \in \mathbf{C}$, an element $\lambda_{\boldsymbol{\pi},a} \in (\mathfrak{h}^e)^*$ by setting $\lambda_{\boldsymbol{\pi},a}(h_i) = \deg \pi_i$ ($i \in I$) and $\lambda_{\boldsymbol{\pi},a}(d) = a$. Set

$$(2.1) \quad \pi_i^+ = \pi_i, \quad \pi_i^-(u) = \frac{u^{\deg \pi_i} \pi_i(u^{-1})}{u^{\deg \pi_i} \pi_i(u^{-1})|_{u=0}}.$$

Let $I_{\boldsymbol{\pi}}(0)$ be the maximal ideal in $\mathbf{U}(0)$ generated by

$$(\Lambda_i^\pm(u) - \pi_i^\pm(u))_s \quad (i \in I, s \geq 0),$$

and let $\mathbf{C}_{\boldsymbol{\pi}} = \mathbf{U}(0) / I_{\boldsymbol{\pi}}(0)$ be the one-dimensional $\mathbf{U}(0)$ -module. Set

$$W(\boldsymbol{\pi}) = W(\lambda_{\boldsymbol{\pi},a}) \otimes_{\mathbf{U}(0)} \mathbf{C}_{\boldsymbol{\pi}}.$$

Then, $W(\boldsymbol{\pi})$ is a left \mathbf{U} -module (but not a \mathbf{U}^e -module) with $x \in \mathbf{U}$ acting as $x \otimes 1$. Let $w_{\boldsymbol{\pi}}$ be the image of 1 in $W(\boldsymbol{\pi})$. Note that $\Lambda_i^\pm(u) . w_{\boldsymbol{\pi}} = \pi_i^\pm(u) w_{\boldsymbol{\pi}}$ ($i \in I$). The assignment $w_{\lambda_{\boldsymbol{\pi},a}} \mapsto w_{\boldsymbol{\pi}}$ extends to a surjective \mathbf{U} -module homomorphism $W(\lambda_{\boldsymbol{\pi},a}) \rightarrow W(\boldsymbol{\pi})$.

Recall that the affine Lie algebra $\hat{\mathfrak{g}}$ is an extension of $L^e(\mathfrak{g})$ by a 1-dimensional central subalgebra $\mathbf{C}c$. Any representation of $L^e(\mathfrak{g})$ is a representation of $\hat{\mathfrak{g}}$ by making c act as zero. Set $h_0 = [e_0^+, e_0^-]$, the bracket being evaluated in $\hat{\mathfrak{g}}$. Then, $h_0 = c - h_\theta$.

The following result is proved in [CP2].

Lemma 2.1. *Let V be a $\widehat{\mathfrak{g}}$ -module generated by an element $v \in V_\lambda$ ($\lambda \in P_+^e$) such that*

$$\begin{aligned} (e_i^+)^{-\lambda(h_i)+1}.v &= 0 & \text{if } \lambda(h_i) \leq 0, \\ (e_i^-)^{\lambda(h_i)+1}.v &= 0 & \text{if } \lambda(h_i) \geq 0, \end{aligned}$$

for $i \in I$. Then, V is an integrable $\widehat{\mathfrak{g}}$ -module. □

Theorem 1. (i) *Let $\lambda \in P_+^e$. Then, $W(\lambda)$ is an integrable \mathbf{U}^e -module.*
(ii) *Let $\boldsymbol{\pi}$ be an n -tuple of polynomials with constant term one. Then, $W(\boldsymbol{\pi})$ is a finite-dimensional \mathbf{U} -module.*

Proof. To prove (i), note that by Lemma 2.1 it suffices to show that

$$\begin{aligned} e_i^+.w_\lambda = 0, & \quad (e_i^-)^{\lambda(h_i)+1}.w_\lambda = 0 \quad (i \in I), \\ e_0^-.w_\lambda = 0, & \quad (e_0^+)^{\lambda(h_\theta)+1}.w_\lambda = 0. \end{aligned}$$

Suppose that $i \in I$. Then, $e_i^\pm = x_{i,0}^\pm$ and it follows from the definition of $W(\lambda)$ that $e_i^+.w_\lambda = 0$. By Lemma 1.3(ii),

$$(x_{i,0}^-)^{\lambda(h_i)+1}.w_\lambda = \left(X_{i,0}^-(u)^{\lambda(h_i)+1} \Lambda_i^+(u) \right)_{\lambda(h_i)+1}.w_\lambda = 0.$$

In particular, this proves that $\mathbf{U}^{fin}.w_\lambda$ is a finite-dimensional \mathfrak{g} -module.

Turning to the case $i = 0$, notice that $e_0^- = x_{\theta,-1}^+$ is a linear combination of products of the $x_{i,m}^+$ ($i \in I, m \in \mathbf{Z}$). Hence, $x_{\theta,-1}^+.w_\lambda = 0$.

For any $m \geq 0$, let $w_m = (e_0^+)^{m+1}.w_\lambda$. Suppose that $w_m \neq 0$. Since $[e_0^+, x_{\beta,0}^-] = 0$, it follows that

$$\mathbf{U}^{fin}(<).w_m = e_0^{m+1} \mathbf{U}^{fin}(<).w_\lambda$$

is finite-dimensional. Since $W(\lambda)[\eta] = 0$ for all but finitely many $\eta \in Q^+$, it follows that

$$W_m = \mathbf{U}^{fin}.w_m = \mathbf{U}^{fin}(>)\mathbf{U}(\mathfrak{h})\mathbf{U}^{fin}(<).w_m$$

is a finite-dimensional \mathfrak{g} -module. Hence, for all $\sigma \in W$ (the Weyl group of \mathfrak{g}), we have

$$(W_m)_{\sigma(\lambda-(m+1)\theta+(m+1)\delta)} \neq 0.$$

Choosing σ so that $\sigma(\theta) = -\alpha_i$ for some $i \in I$, we get

$$W(\lambda)_{\sigma(\lambda)+(m+1)\alpha_i+(m+1)\delta} \neq 0.$$

But this can only happen for finitely many values of m .

This proves that $w_m = 0$ for all but finitely many m . The Lie subalgebra of $L(\mathfrak{g})$ generated by e_0^\pm and h_θ is isomorphic to sl_2 , and we have just shown that the corresponding sl_2 -submodule generated by w_λ is finite-dimensional. It follows from standard results that $(e_0^-)^{\lambda(h_\theta)+1}.w_\lambda = 0$.

To prove (ii), it suffices now to notice (using Proposition 1.2(ii)) that $W(\boldsymbol{\pi})$ is spanned by the elements

$$x_{\beta_1, r_1}^- x_{\beta_2, r_2}^- \cdots x_{\beta_s, r_s}^- \otimes 1$$

for $s \geq 0, 0 \leq r_t < \lambda(h_{\beta_t}), \beta_t \in R^+$. □

The modules $W(\lambda)$ and $W(\boldsymbol{\pi})$ have certain universal properties.

- Proposition 2.1.** (i) Let V be any integrable \mathbf{U}^e -module generated by a non-zero element $v \in V_\lambda^+$. Then, V is a quotient of $W(\lambda)$.
- (ii) Let V be a finite-dimensional quotient \mathbf{U} -module of $W(\lambda)$, and assume that $\dim V_\lambda = 1$. Then, V is a quotient of $W(\boldsymbol{\pi})$ for some choice of $\boldsymbol{\pi}$.
- (iii) Let V be finite-dimensional \mathbf{U} -module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then, V is a quotient of $W(\boldsymbol{\pi})$ for some $\boldsymbol{\pi}$.

Proof. Part (i) is immediate from Proposition 1.1 and the definition of $W(\lambda)$.

To prove (ii), let $v \neq 0$ be the image of w_λ in V . Notice that $\dim V_\lambda = 1$ implies that

$$\Lambda_{\beta,m} \cdot v = \pi_{\beta,m} v,$$

for some scalars $\pi_{\beta,m} \in \mathbf{C}$. By Proposition 1.1(i), it follows that

$$\pi_{\beta,m} = 0 \quad (|m| > \lambda(h_\beta)).$$

For $i \in I$, set $\pi_i(u) = \sum_{k=0}^{\lambda(h_i)} \pi_{\alpha_i,k} u^k$. The $\pi_i(u)$ are polynomials with constant term 1 and Proposition 1.1(v) shows that

$$\Lambda_i^\pm(u) \cdot v = \pi_i^\pm(u) v.$$

This shows that V is a quotient of $W(\boldsymbol{\pi})$, where $\boldsymbol{\pi}$ is the n -tuple of polynomials defined above.

The proof of (iii) is identical. \square

3. A TENSOR PRODUCT THEOREM FOR $W(\boldsymbol{\pi})$

The main result of this section is the following theorem.

Theorem 2. Let $\boldsymbol{\pi} = (\pi_1(u), \pi_2(u), \dots, \pi_n(u))$ and $\tilde{\boldsymbol{\pi}} = (\tilde{\pi}_1(u), \tilde{\pi}_2(u), \dots, \tilde{\pi}_n(u))$ be n -tuples of polynomials in u with constant term 1, such that π_i and $\tilde{\pi}_j$ are coprime for all $1 \leq i, j \leq n$. Then,

$$W(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}) \cong W(\boldsymbol{\pi}) \otimes W(\tilde{\boldsymbol{\pi}})$$

as \mathbf{U} -modules, where $\boldsymbol{\pi} \tilde{\boldsymbol{\pi}} = (\pi_1 \tilde{\pi}_1, \pi_2 \tilde{\pi}_2, \dots, \pi_n \tilde{\pi}_n)$.

For $\beta \in R^+$ and $\boldsymbol{\pi}$ as in Theorem 2, define $\pi_\beta(u)$ by

$$(3.1) \quad \Lambda_\beta^+(u) \cdot w_{\boldsymbol{\pi}} = \pi_\beta(u) w_{\boldsymbol{\pi}}.$$

Lemma 3.1. Let $\beta = \sum_{i=1}^n r_i \alpha_i \in R^+$. Then,

$$\pi_\beta(u) = \prod_{i=1}^n \pi_i^{r_i d_i / d_\beta}(u).$$

If $\theta_s \in R^+$ is the highest short root of \mathfrak{g} , then π_β divides π_{θ_s} for all $\beta \in R^+$.

Proof. The first statement is immediate from the formula for h_β in terms of the h_i given in Section 1 and the definition of the Λ_β^+ . The second statement is proved case by case, using the explicit formula for θ_s (there is a simple uniform proof when \mathfrak{g} is simply-laced). \square

From now on, we shall assume that, given $\beta \in R^+$ and an n -tuple of polynomials $\boldsymbol{\pi}$, we have defined a polynomial $\pi_\beta(u)$ by the formula given in (3.1). The polynomials $\pi_\beta^\pm(u)$ are defined as in (2.1).

Definition 3.1. Let $\boldsymbol{\pi} = \{\pi_1, \dots, \pi_n\}$ be an n -tuple of polynomials in u with constant term 1. Define $M(\boldsymbol{\pi})$ to be the left \mathbf{U} -module obtained by taking the quotient of \mathbf{U} by the left ideal generated by the following:

$$\begin{aligned} &x_{i,k}^+, \quad h_{i,0} - \deg \pi_i \quad (i \in I, k \in \mathbf{Z}), \\ &(\Lambda_i^\pm(u) - \pi_i^\pm(u))_s \quad (i \in I, s \geq 0), \\ &(\pi_\beta(u) \tilde{X}_\beta^-(u))_s \quad (\beta \in R^+, s \in \mathbf{Z}). \end{aligned}$$

Let $m_{\boldsymbol{\pi}}$ be the image of 1 in $M(\boldsymbol{\pi})$. It is clear from equation (3.1) and Proposition 1.1(iii) that, for all $\beta \in R^+$ and all $s \in \mathbf{Z}$,

$$\begin{aligned} &(\Lambda_\beta^\pm(u) - \pi_\beta^\pm(u))_s .m_{\boldsymbol{\pi}} = 0, \\ &(\pi_\beta(u) \tilde{H}_\beta(u))_s .m_{\boldsymbol{\pi}} = 0. \end{aligned}$$

Set $\lambda_{\boldsymbol{\pi}}(h_i) = \deg \pi_i$.

Lemma 3.2. *We have*

$$M(\boldsymbol{\pi}) = \bigoplus_{\eta \in Q^+} M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}} - \eta},$$

and $\dim M(\boldsymbol{\pi})_{\lambda - \eta} < \infty$. Further, for all $\beta \in R^+$, $s \in \mathbf{Z}$, we have

$$(\pi_{\theta_s}(u) \tilde{X}_\beta^-(u))_s .M(\boldsymbol{\pi}) = 0.$$

Proof. The set

$$\{x_{\beta,r}^- : \beta \in R^+, 0 \leq r < \lambda(h_\beta)\} \cup \{x_\beta^- \otimes t^m \pi_\beta(t) : \beta \in R^+, m \in \mathbf{Z}\}$$

is a basis of

$$\left(\bigoplus_{\beta \in R^+} \mathbf{C}x_\beta^- \right) \otimes \mathbf{C}[t, t^{-1}].$$

By the PBW basis theorem, we can write

$$\mathbf{U}(<) = \mathbf{U}_{\boldsymbol{\pi}} \mathbf{U}^{\boldsymbol{\pi}}$$

where $\mathbf{U}_{\boldsymbol{\pi}}$ (resp. $\mathbf{U}^{\boldsymbol{\pi}}$) consists of ordered monomials from the first (resp. second) set. The relation

$$(\pi_\beta(u) \tilde{X}_\beta^-(u))_s .m_{\boldsymbol{\pi}} = 0$$

for all $s \in \mathbf{Z}$ implies that $(\mathbf{U}^{\boldsymbol{\pi}})_+ .m_{\boldsymbol{\pi}} = 0$. A further use of the PBW theorem now shows that

$$M(\boldsymbol{\pi}) \cong \mathbf{U}_{\boldsymbol{\pi}}$$

as vector spaces. Moreover, this isomorphism takes $M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}} - \eta}$ to

$$\mathbf{U}_{\boldsymbol{\pi}}(\eta) = \{x \in \mathbf{U}_{\boldsymbol{\pi}} : [h, x] = \eta(h)x \quad \forall h \in \mathfrak{h}\}.$$

Since this space is clearly finite-dimensional, the first statement of the lemma follows.

For the second statement, we show by induction on $\text{ht } \eta$ that

$$(\pi_\theta(u) \tilde{X}_\beta^-(u))_s .M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}} - \eta} = 0 \quad (\beta \in R^+, s \in \mathbf{Z}).$$

If $\eta = 0$, the result is immediate from the definition of $M(\boldsymbol{\pi})$ and the last part of Lemma 3.1. In general, let $x_{\beta_1, r_1}^- \cdots x_{\beta_k, r_k}^- \cdot m\boldsymbol{\pi} \in M(\boldsymbol{\pi})_{\lambda\boldsymbol{\pi} - \eta}$. Then,

$$\begin{aligned} \left(\pi_{\theta_s}(u) \tilde{X}_{\beta}^-(u) \right)_s x_{\beta_1, r_1}^- \cdots x_{\beta_k, r_k}^- \cdot m\boldsymbol{\pi} &= x_{\beta_1, r_1}^- \left(\pi_{\theta}(u) \tilde{X}_{\beta}^-(u) \right)_s x_{\beta_2, r_2}^- \cdots x_{\beta_k, r_k}^- \cdot m\boldsymbol{\pi} \\ &+ \left(\pi_{\theta}(u) \tilde{X}_{\beta+\beta_1}^-(u) \right)_{s+r_1} x_{\beta_2, r_2}^- \cdots x_{\beta_k, r_k}^- \cdot m\boldsymbol{\pi}, \end{aligned}$$

where we understand that $\tilde{X}_{\beta+\beta_1}^-(u) = 0$ if $\beta + \beta_1 \notin R^+$. The right-hand side is zero by the induction hypothesis, and the inductive step is complete. \square

Lemma 3.3. *The \mathbf{U} -module $W(\boldsymbol{\pi})$ is a quotient of $M(\boldsymbol{\pi})$, and any finite-dimensional quotient of $M(\boldsymbol{\pi})$ is a quotient of $W(\boldsymbol{\pi})$.*

Proof. Let V be a finite-dimensional quotient of $M(\boldsymbol{\pi})$, let $v \in V$ be the image of $m\boldsymbol{\pi}$, and let $\lambda = \lambda\boldsymbol{\pi}$. Then, $\dim V_{\lambda} = \dim M(\boldsymbol{\pi})_{\lambda} = 1$, so by Proposition 2.1(ii), V is a quotient of some $W(\tilde{\boldsymbol{\pi}})$ with $\lambda = \lambda\tilde{\boldsymbol{\pi}}$. Since $\dim W(\tilde{\boldsymbol{\pi}})_{\lambda} = 1$, v is a scalar multiple of the image of $w_{\tilde{\boldsymbol{\pi}}}$. But then by comparing the action of $\Lambda_i^{\pm}(u)$ on $w_{\tilde{\boldsymbol{\pi}}}$ and on $m\boldsymbol{\pi}$, we see that $\boldsymbol{\pi} = \tilde{\boldsymbol{\pi}}$.

To show that $W(\boldsymbol{\pi})$ is a quotient of $M(\boldsymbol{\pi})$, it is clear from the definitions of these modules that we only need to show that

$$\left(\pi_{\beta}(u) \tilde{X}_{\beta}^-(u) \right)_s \cdot w\boldsymbol{\pi} = 0 \quad (s \in \mathbf{Z}).$$

Since $W(\boldsymbol{\pi})$ is a quotient of $W(\lambda\boldsymbol{\pi}, 0)$, this follows from Proposition 1.1, thus completing the proof of the lemma. \square

Denote by $\Delta : \mathbf{U} \rightarrow \mathbf{U} \otimes \mathbf{U}$ the comultiplication of \mathbf{U} defined by extending to an algebra homomorphism the assignment

$$x \rightarrow x \otimes 1 + 1 \otimes x,$$

for all $x \in L(\mathfrak{g})$. The following lemma is proved in [G].

Lemma 3.4. *For all $\beta \in R^+$,*

$$\Delta(\Lambda_{\beta}^{\pm}) = \Lambda_{\beta}^{\pm} \otimes \Lambda_{\beta}^{\pm},$$

where

$$\Lambda_{\beta}^{\pm} \otimes \Lambda_{\beta}^{\pm} = \sum_{k, m \geq 0} (\Lambda_{\beta, \pm k} \otimes \Lambda_{\beta, \pm m}) u^{k+m}.$$

\square

Theorem 2 is now clearly a consequence of the following proposition.

Proposition 3.1. *Assume that $\boldsymbol{\pi} = \{\pi_1, \pi_2, \dots, \pi_n\}$ and $\tilde{\boldsymbol{\pi}} = \{\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_n\}$ are n -tuples of polynomials with constant term 1, such that π_i and $\tilde{\pi}_j$ are coprime for all $1 \leq i, j \leq n$. Then*

- (i) $M(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}) \cong M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$;
- (ii) every finite-dimensional \mathbf{U} -module of $M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$ is a quotient of $W(\boldsymbol{\pi}) \otimes W(\tilde{\boldsymbol{\pi}})$.

Proof. Set $\lambda = \lambda\boldsymbol{\pi} + \lambda\tilde{\boldsymbol{\pi}}$. It is clear from the proof of Lemma 3.2 that, for all $\eta \in Q_+$, we have

$$M(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}})_{\lambda - \eta} \cong (M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}}))_{\lambda - \eta}$$

as (finite-dimensional) vector spaces. To prove (i), it therefore suffices to prove that there exists a surjective homomorphism of \mathbf{U} -modules $M(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}) \rightarrow M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$. It is easy to see, using Lemma 3.4, that the element $m_{\boldsymbol{\pi}} \otimes m_{\tilde{\boldsymbol{\pi}}}$ satisfies the defining relations of $M(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}})$, so there exists a \mathbf{U} -module map $M(\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}) \rightarrow M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$ that sends $m_{\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}}$ to $m_{\boldsymbol{\pi}} \otimes m_{\tilde{\boldsymbol{\pi}}}$. Thus, to prove (i), we must show that, if π_i and $\tilde{\pi}_j$ have no roots in common, the element $m_{\boldsymbol{\pi}} \otimes m_{\tilde{\boldsymbol{\pi}}}$ generates $M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$ as a \mathbf{U} -module.

Set

$$N = \mathbf{U} \cdot (m_{\boldsymbol{\pi}} \otimes m_{\tilde{\boldsymbol{\pi}}}).$$

Assume that, for all $\eta = \sum_i r_i \alpha_i$, $\tilde{\eta} = \sum_i \tilde{r}_i \alpha_i$, with $\text{ht } \eta = \sum_i r_i < s$, $\text{ht } \tilde{\eta} = \sum_i \tilde{r}_i < s$, we have

$$M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}-\eta}} \otimes M(\tilde{\boldsymbol{\pi}})_{\lambda_{\tilde{\boldsymbol{\pi}}-\tilde{\eta}}} \subset N.$$

We shall prove that

$$\begin{aligned} (x_{i,m}^- \cdot M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}-\eta}}) \otimes M(\tilde{\boldsymbol{\pi}})_{\lambda_{\tilde{\boldsymbol{\pi}}-\tilde{\eta}}} &\subset N, \\ M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}-\eta}} \otimes x_{i,m}^- \cdot (M(\tilde{\boldsymbol{\pi}})_{\lambda_{\tilde{\boldsymbol{\pi}}-\tilde{\eta}}}) &\subset N \end{aligned}$$

for all $i \in I$, $m \in \mathbf{Z}$. This will prove that

$$M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}-\eta}} \otimes M(\tilde{\boldsymbol{\pi}})_{\lambda_{\tilde{\boldsymbol{\pi}}-\tilde{\eta}}} \subset N,$$

when $\text{ht } \eta \leq s$, $\text{ht } \tilde{\eta} \leq s$, and hence, by induction on s , that $N = M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$.

Since π_{θ_s} and $\tilde{\pi}_{\theta_s}$ are coprime, we can choose polynomials $R(u)$, $\tilde{R}(u)$ such that

$$R\pi_{\theta_s} + \tilde{R}\tilde{\pi}_{\theta_s} = 1.$$

By the second part of Lemma 3.2,

$$\begin{aligned} \left(R\pi_{\theta_s} \tilde{X}_i^-(u) \right)_m \cdot w &= 0, \\ \left(\tilde{R}\tilde{\pi}_{\theta_s} \tilde{X}_i^-(u) \right)_m \cdot \tilde{w} &= 0, \end{aligned}$$

for all $i \in I$, $m \in \mathbf{Z}$, $w \in M(\boldsymbol{\pi})$, $\tilde{w} \in M(\tilde{\boldsymbol{\pi}})$. Hence,

$$\begin{aligned} \left(R\pi_{\theta_s} \tilde{X}_i^-(u) \right)_m \cdot (w \otimes \tilde{w}) &= \left(R\pi_{\theta_s} \tilde{X}_i^-(u) \right)_m \cdot w \otimes \tilde{w} + w \otimes \left(R\pi_{\theta_s} \tilde{X}_i^-(u) \right)_m \cdot \tilde{w} \\ &= w \otimes \left((1 - \tilde{R}\tilde{\pi}_{\theta_s}) \tilde{X}_i^-(u) \right)_m \cdot \tilde{w} \\ &= w \otimes x_{i,m}^- \cdot \tilde{w}. \end{aligned}$$

Taking $w \in M(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}-\eta}}$ and $\tilde{w} \in M(\tilde{\boldsymbol{\pi}})_{\lambda_{\tilde{\boldsymbol{\pi}}-\tilde{\eta}}}$, so that $w \otimes \tilde{w} \in N$, it follows that $w \otimes x_{i,m}^- \cdot \tilde{w} \in N$ for all $i \in I$, $m \in \mathbf{Z}$. The other inclusion is proved similarly, and the proof of part (i) is complete.

Suppose that V is a finite-dimensional quotient of $M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}})$ with kernel K . We shall prove that, for all $r \geq 1$, $i \in I$, $s \geq \lambda_{\boldsymbol{\pi}}(h_i) + 1$, $\tilde{s} \geq \lambda_{\tilde{\boldsymbol{\pi}}}(h_i) + 1$,

$$(3.2) \quad (\pi_i(u) X_i^-(u)^r)_s \cdot m_{\boldsymbol{\pi}} \otimes M(\tilde{\boldsymbol{\pi}}) \subset K,$$

$$(3.3) \quad M(\boldsymbol{\pi}) \otimes (\pi_i(u) X_i^-(u)^r)_s \cdot m_{\tilde{\boldsymbol{\pi}}} \subset K.$$

Since the sum of these subspaces is the kernel of the quotient map

$$M(\boldsymbol{\pi}) \otimes M(\tilde{\boldsymbol{\pi}}) \rightarrow W(\boldsymbol{\pi}) \otimes W(\tilde{\boldsymbol{\pi}}),$$

it follows that V is a quotient of $W(\boldsymbol{\pi}) \otimes W(\tilde{\boldsymbol{\pi}})$, which proves part (ii).

To prove equation (3.2), it suffices (by the proof of Proposition 1.1(ii)) to prove that, for all $i \in I$, $m \in \mathbf{Z}$,

$$(x_{i,m}^-)^{r_i+1} \cdot m\boldsymbol{\pi} \otimes m\tilde{\boldsymbol{\pi}} \in K,$$

where $r_i = \deg \pi_i = \lambda_{\boldsymbol{\pi}}(h_i)$. Since V is finite-dimensional, the element $(x_{i,m}^-)^r \cdot m\boldsymbol{\pi} \otimes m\tilde{\boldsymbol{\pi}} \in K$ for some $r \geq 0$. Let r_0 be the smallest value of r with this property. Since

$$x_{i,-m}^+ (x_{i,m}^-)^{r_0} \cdot m\boldsymbol{\pi} \otimes m\tilde{\boldsymbol{\pi}} = (r_i - r_0 + 1)(x_{i,m}^-)^{r_0-1} \cdot m\boldsymbol{\pi} \otimes m\tilde{\boldsymbol{\pi}},$$

it follows by the minimality of r_0 that $r_i + 1 = r_0$.

Equation (3.3) is proved similarly, and we are done. □

Note that, since $\dim W(\boldsymbol{\pi})_{\lambda_{\boldsymbol{\pi}}} = 1$, it follows that $W(\boldsymbol{\pi})$ has a unique irreducible quotient $V(\boldsymbol{\pi})$. Write $\boldsymbol{\pi}$ as a product

$$\boldsymbol{\pi} = \boldsymbol{\pi}^{(1)} \boldsymbol{\pi}^{(2)} \dots \boldsymbol{\pi}^{(k)},$$

where $\boldsymbol{\pi}^{(j)}$ is such that

$$\pi_{\theta_s}^{(j)} = (1 - a_j u)^{m_j},$$

for some $m_j > 0$ and $a_j \in \mathbf{C}^\times$ with $a_j \neq a_k$ if $j \neq k$. The following result was proved in [CP1].

Proposition 3.2. *With the above notation,*

$$V(\boldsymbol{\pi}) \cong V(\boldsymbol{\pi}^{(1)}) \otimes V(\boldsymbol{\pi}^{(2)}) \otimes \dots \otimes V(\boldsymbol{\pi}^{(k)})$$

as $L(\mathfrak{g})$ -modules. Further,

$$V(\boldsymbol{\pi}^{(j)}) \cong V^{fin}(\lambda_{\boldsymbol{\pi}^{(j)}})$$

as \mathfrak{g} -modules. □

4. THE QUANTUM CASE

In the remainder of this paper, we shall assume that \mathfrak{g} is simply-laced.

Let q be an indeterminate, let $\mathbf{C}(q)$ be the field of rational functions in q with complex coefficients, and let $\mathbf{A} = \mathbf{C}[q, q^{-1}]$ be the subring of Laurent polynomials. For $r, m \in \mathbf{N}$, $m \geq r$, define

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = [m][m-1] \dots [2][1], \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!}.$$

Then, $\begin{bmatrix} m \\ r \end{bmatrix} \in \mathbf{A}$.

Let \mathbf{U}_q^e be the quantized enveloping algebra over $\mathbf{C}(q)$ associated to $L^e(\mathfrak{g})$. Thus, \mathbf{U}_q^e is the quotient of the quantum affine algebra obtained by setting the central generator equal to 1. It follows from [Dr2], [B], [J] that \mathbf{U}_q^e is the algebra with

generators $\mathbf{x}_{i,r}^\pm$ ($i \in I, r \in \mathbf{Z}$), $K_i^{\pm 1}$ ($i \in I$), $\mathbf{h}_{i,r}$ ($i \in I, r \in \mathbf{Z} \setminus \{0\}$), $D^{\pm 1}$, and the following defining relations:

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, & D D^{-1} &= D^{-1} D = 1, \\ K_i K_j &= K_j K_i, & K_i \mathbf{h}_{j,r} &= \mathbf{h}_{j,r} K_i, \\ K_i \mathbf{x}_{j,r}^\pm K_i^{-1} &= q^{\pm a_{ij}} \mathbf{x}_{j,r}^\pm, & D \mathbf{x}_{j,r}^\pm D^{-1} &= q^r \mathbf{x}_{j,r}^\pm, \\ [\mathbf{h}_{i,r}, \mathbf{h}_{j,s}] &= 0, & [\mathbf{h}_{i,r}, \mathbf{x}_{j,s}^\pm] &= \pm \frac{1}{r} [ra_{ij}] \mathbf{x}_{j,r+s}^\pm, \\ \mathbf{x}_{i,r+1}^\pm \mathbf{x}_{j,s}^\pm - q^{\pm a_{ij}} \mathbf{x}_{j,s}^\pm \mathbf{x}_{i,r+1}^\pm &= q^{\pm a_{ij}} \mathbf{x}_{i,r}^\pm \mathbf{x}_{j,s+1}^\pm - \mathbf{x}_{j,s+1}^\pm \mathbf{x}_{i,r}^\pm, \\ [\mathbf{x}_{i,r}^+, \mathbf{x}_{j,s}^-] &= \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q - q^{-1}}, \end{aligned}$$

$$\sum_{\pi \in \Sigma_m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \mathbf{x}_{i,r_{\pi(1)}}^\pm \cdots \mathbf{x}_{i,r_{\pi(k)}}^\pm \mathbf{x}_{j,s}^\pm \mathbf{x}_{i,r_{\pi(k+1)}}^\pm \cdots \mathbf{x}_{i,r_{\pi(m)}}^\pm = 0, \text{ if } i \neq j,$$

for all sequences of integers r_1, \dots, r_m , where $m = 1 - a_{ij}$, Σ_m is the symmetric group on m letters, and the $\psi_{i,r}^\pm$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty} \psi_{i,\pm r}^\pm u^{\pm r} = K_i^{\pm 1} \exp \left(\pm (q - q^{-1}) \sum_{s=1}^{\infty} \mathbf{h}_{i,\pm s} u^{\pm s} \right).$$

Define the q -divided powers

$$(\mathbf{x}_{i,k}^\pm)^{(r)} = \frac{(\mathbf{x}_{i,k}^\pm)^r}{[r]!},$$

for all $i \in I, k \in \mathbf{Z}, r \geq 0$.

Suppose that $a_{ij} = -1$. Then, a special case of the above relations is

$$(\mathbf{x}_{i,s}^-)^2 \mathbf{x}_{j,r}^- - (q + q^{-1}) \mathbf{x}_{i,s}^- \mathbf{x}_{j,r}^- \mathbf{x}_{i,s}^- + \mathbf{x}_{j,r}^- (\mathbf{x}_{i,s}^-)^2 = 0.$$

Set $\gamma_{s,r}^{i,j} = \mathbf{x}_{i,s}^- \mathbf{x}_{j,r}^- - q \mathbf{x}_{j,r}^- \mathbf{x}_{i,s}^-$. Again, the relations in \mathbf{U}_q^e imply that

$$\gamma_{s,r}^{i,j} = -\gamma_{r+1,s-1}^{j,i}.$$

The following result is proved in [L2].

Proposition 4.1. *For $i, j \in I$ with $a_{ij} = -1, r, s, l, m \in \mathbf{Z}$, and $l, m \geq 0$, there exist $f_p \in \mathbf{A}$, for $0 \leq p \leq \min(l, m)$, such that*

$$(\mathbf{x}_{i,s}^-)^{(l)} (\mathbf{x}_{j,r}^-)^{(m)} = \sum_p f_p (\mathbf{x}_{j,r}^-)^{(m-p)} (\gamma_{s,r}^{i,j})^{(p)} (\mathbf{x}_{i,s}^-)^{(l-p)}.$$

Further, there exists $g_p \in \mathbf{A}$, for $0 \leq p \leq m$, such that

$$(\gamma_{s,r}^{i,j})^{(m)} = \sum_p g_p (\mathbf{x}_{j,r}^-)^{(p)} (\mathbf{x}_{i,s}^-)^{(m)} (\mathbf{x}_{j,r}^-)^{(m-p)}.$$

□

Define

$$\Lambda_i^\pm(u) = \sum_{m=0}^{\infty} \Lambda_{i,\pm m} u^m = \exp \left(- \sum_{k=1}^{\infty} \frac{\mathbf{h}_{i,\pm k}}{[k]} u^k \right).$$

The subalgebras \mathbf{U}_q , \mathbf{U}_q^{fin} , $\mathbf{U}_q(<)$, etc., are defined in the obvious way. Let $\mathbf{U}_q(\mathfrak{h}^e)$ be the subalgebra of \mathbf{U}_q^e generated by $K_i^{\pm 1}$ ($i \in I$) and $D^{\pm 1}$. Let $\mathbf{U}_q(0)$ be the subalgebra of \mathbf{U}_q^e generated by the elements $\Lambda_{i,m}$ ($i \in I, m \in \mathbf{Z}$). The following result is a simple corollary of the PBW theorem for \mathbf{U}_q^e [B].

Lemma 4.1. $\mathbf{U}_q^e \cong \mathbf{U}_q(<)\mathbf{U}_q(0)\mathbf{U}_q(\mathfrak{h}^e)\mathbf{U}_q(>)$. □

For any invertible element $x \in \mathbf{U}_q^e$, define

$$\begin{bmatrix} x \\ r \end{bmatrix} = \frac{xq^r - x^{-1}q^{-r}}{q - q^{-1}}.$$

Let $\mathbf{U}_{\mathbf{A}}^e$ be the \mathbf{A} -subalgebra of \mathbf{U}_q^e generated by the $K_i^{\pm 1}$, $(\mathbf{x}_{i,k}^{\pm})^{(r)}$ ($i \in I, k \in \mathbf{Z}, r \geq 0$), $D^{\pm 1}$, and $\begin{bmatrix} D \\ r \end{bmatrix}$ ($r \geq 1$). Then, [L2], [BCP],

$$\mathbf{U}_q^e = \mathbf{C}(q) \otimes_{\mathbf{A}} \mathbf{U}_{\mathbf{A}}^e.$$

Define $\mathbf{U}_{\mathbf{A}}(<)$, $\mathbf{U}_{\mathbf{A}}(0)$ and $\mathbf{U}_{\mathbf{A}}(>)$ in the obvious way. Let $\mathbf{U}_{\mathbf{A}}(\mathfrak{h}^e)$ be the \mathbf{A} -subalgebra of $\mathbf{U}_{\mathbf{A}}^e$ generated by the elements $K_i^{\pm 1}$, $D^{\pm 1}$, $\begin{bmatrix} K_i \\ r \end{bmatrix}$ and $\begin{bmatrix} D \\ r \end{bmatrix}$ ($i \in I, r \in \mathbf{Z}$). The following is proved as in Proposition 2.7 in [BCP].

Proposition 4.2. $\mathbf{U}_{\mathbf{A}}^e = \mathbf{U}_{\mathbf{A}}(<)\mathbf{U}_{\mathbf{A}}(0)\mathbf{U}_{\mathbf{A}}(\mathfrak{h}^e)\mathbf{U}_{\mathbf{A}}(>)$. □

The next lemma is easily checked.

Lemma 4.2. (i) *There is a unique \mathbf{C} -linear anti-automorphism Ψ of \mathbf{U}_q^e such that $\Psi(q) = q^{-1}$ and*

$$\begin{aligned} \Psi(K_i) &= K_i, & \Psi(D) &= D, \\ \Psi(x_{i,r}^{\pm}) &= x_{i,r}^{\pm}, & \Psi(h_{i,r}) &= -h_{i,r}, \end{aligned}$$

for all $i \in I, r \in \mathbf{Z}$.

(ii) *There is a unique algebra automorphism Φ of \mathbf{U}_q^e over $\mathbf{C}(q)$ such that $\Phi(\mathbf{x}_{i,r}^{\pm}) = \mathbf{x}_{i,-r}^{\pm}$, $\Phi(\Lambda_i^+(u)) = \Lambda_i^-(u)$.*

The first part of the following result is proved in [BCP], and the second part follows from it by applying Ψ .

Lemma 4.3. (i) *Let $s > s', m, m' \geq 0, i \in I$. Then, $(\mathbf{x}_{i,s}^-)^{(m)}(\mathbf{x}_{i,s'}^-)^{(m')}$ is in the span of the elements*

$$(\mathbf{x}_{i,r_1}^-)^{(k_1)}(\mathbf{x}_{i,r_2}^-)^{(k_2)} \dots (\mathbf{x}_{i,r_l}^-)^{(k_l)},$$

where $s' \leq r_1 < r_2 < \dots < r_l \leq s$, $\sum_p k_p = m + m'$ and $\sum_p k_p r_p = ms + m's'$.

(ii) *Let $s < s', m, m' \geq 0, i \in I$. Then, $(\mathbf{x}_{i,s}^-)^{(m)}(\mathbf{x}_{i,s'}^-)^{(m')}$ is in the span of the elements*

$$(\mathbf{x}_{i,r_1}^-)^{(k_1)}(\mathbf{x}_{i,r_2}^-)^{(k_2)} \dots (\mathbf{x}_{i,r_l}^-)^{(k_l)},$$

where $s' \geq r_1 > r_2 > \dots > r_l \geq s$, $\sum_p k_p = m + m'$ and $\sum_p k_p r_p = ms + m's'$. □

For $i \in I$, let $\tilde{\mathbf{X}}_i^-(u)$, $\mathbf{X}_i^-(u)$, $\mathbf{X}_{i,0}^-(u)$ be the formal power series with coefficients in \mathbf{U}_q given by the same formulas as the $\tilde{X}_i^-(u)$, etc., in Section 1. The next result is a reformulation of results in [CP6, Section 5].

Lemma 4.4. *Let $s \geq r \geq 1$, $i \in I$.*

(i)

$$(\mathbf{x}_{i,0}^+)^{(r)}(\mathbf{x}_{i,1}^-)^{(s)} = (-1)^r (\mathbf{X}_i^-(u)^{(s-r)} \mathbf{\Lambda}_i^+(u))_s \pmod{\mathbf{U}_q \mathbf{U}_q(>)_+}.$$

(ii)

$$(\mathbf{x}_{i,1}^+)^{(r)}(\mathbf{x}_{i,0}^-)^{(s)} = (-1)^r (\mathbf{X}_{i,0}^-(u)^{(s-r)} \mathbf{\Lambda}_i^+(u))_s \pmod{\mathbf{U}_q \mathbf{U}_q(>)_+}.$$

(iii)

$$\left(\mathbf{X}_i^-(u)^{(r)} \right)_{s+r} = \sum \mu(s_0, s_1, \dots) (\mathbf{x}_{i,0}^-)^{(s_0)} (\mathbf{x}_{i,1}^-)^{(s_1)} \dots,$$

where the sum is over those non-negative integers s_0, s_1, \dots such that $\sum_t s_t = r$ and $\sum_t t s_t = s + r$, and the coefficients $\mu(s_0, s_1, \dots) \in \mathbf{A}$. In particular, the coefficient of $(\mathbf{x}_{i,s+1}^-)^{(r)}$ in $(\mathbf{X}_i^-(u)^{(r)})_{(s+1)r}$ is $q^{sr(r-1)}$. \square

Definition 4.1. A \mathbf{U}_q^e -module V_q is said to be of type 1 if

$$V_q = \bigoplus_{\lambda \in P^e} (V_q)_\lambda,$$

where

$$(V_q)_\lambda = \{v \in V_q : K_i.v = q^{\lambda(h_i)}v \ \forall \ i \in I, \ D.v = q^{\lambda(d)}.v\}.$$

The subspaces $(V_q)_\lambda^+$ are defined in the obvious way. We say that a type 1 module is integrable if the elements $\mathbf{x}_{i,k}^\pm$ act locally nilpotently on V_q for all $i \in I$ and $k \in \mathbf{Z}$.

As in the classical case, one shows [L2] that, if V_q is integrable, then

$$(V_q)_\lambda \neq 0 \implies (V_q)_{\sigma(\lambda)} \neq 0 \ \forall \sigma \in W.$$

The type 1 \mathbf{U}_q^{fin} -modules and their weight spaces are defined analogously. If $\lambda \in P_+$, there is a unique finite-dimensional irreducible \mathbf{U}_q^{fin} -module $V_q^{fin}(\lambda)$ generated by a vector v such that

$$k_i.v = q^{\lambda(h_i)}v, \quad x_{i,0}^+.v = 0, \quad (x_{i,0}^-)^{\lambda(h_i)+1}.v = 0,$$

for all $i \in I$. Further,

$$\dim_{\mathbf{C}(q)}(V_q^{fin}(\lambda)_\mu) = \dim_{\mathbf{C}}(V^{fin}(\lambda)_\mu)$$

for all $\mu \in P$.

From now on, we shall only consider modules of type 1. The next result is the quantum analogue of Proposition 1.1 and is proved in exactly the same way.

Proposition 4.3. *Let V_q be an integrable \mathbf{U}_q^e -module. Let $\lambda \in P_+^e$ and assume that $0 \neq v \in (V_q)_\lambda^+$.*

(i) $\mathbf{\Lambda}_{i,m}.v = 0$ for all $i \in I$ and $|m| > \lambda(h_i)$.

(ii) $\mathbf{\Lambda}_{i,\lambda(h_i)}\mathbf{\Lambda}_{i,-m}.v = \mathbf{\Lambda}_{i,\lambda(h_i)-m}.v$ for all $i \in I$ and $0 \leq m \leq \lambda(h_i)$.

(iii) For $r \geq 1$, $s > \lambda(h_i)$, $m \in \mathbf{Z}$, $i \in I$,

$$\left(\mathbf{X}_i^-(u)^r \mathbf{\Lambda}_i^+(u) \right)_s .v = 0, \quad \left(\mathbf{X}_{i,0}^-(u)^r \mathbf{\Lambda}_i^+(u) \right)_s .v = 0,$$

$$\left(\tilde{\mathbf{X}}_i^-(u) \mathbf{\Lambda}_i^+(u) \right)_m .v = 0, \quad \left(\tilde{\mathbf{H}}_i(u) \mathbf{\Lambda}_i^+(u) \right)_m .v = 0.$$

(iv)

$$\left(\Phi(\mathbf{X}_i^-(u)^r) \mathbf{\Lambda}_i^-(u) \right)_s .v = 0, \quad \left(\Phi(\mathbf{X}_{i,0}^-(u)^r) \mathbf{\Lambda}_i^-(u) \right)_s .v = 0.$$

□

Proposition 4.4. *Let V_q be an integrable type 1 \mathbf{U}_q^e -module and assume that $\lambda \in P_+^e$, $0 \neq v \in (V_q)_\lambda$ is such that $V_q = \mathbf{U}_q^e.v$ and*

$$\mathbf{x}_{i,k}^+.v = 0 \quad \forall i \in I, k \in \mathbf{Z}.$$

Then, there exists $s_\lambda \geq 0$ such that V_q is spanned by the elements

$$(\mathbf{x}_{i_1, s_1}^-)^{(l_1)} (\mathbf{x}_{i_2, s_2}^-)^{(l_2)} \cdots (\mathbf{x}_{i_k, s_k}^-)^{(l_k)} \mathbf{U}_\mathbf{A}(0).v$$

for $0 \leq j \leq k$, $l_j \geq 0$, $i_j \in I$, $0 \leq s_j \leq s_\lambda$.

Proof. For any $N \geq 0$, let V_N be the $\mathbf{C}(q)$ -subspace of V_q spanned by the elements

$$(\mathbf{x}_{i_1, s_1}^-)^{(l_1)} (\mathbf{x}_{i_2, s_2}^-)^{(l_2)} \cdots (\mathbf{x}_{i_k, s_k}^-)^{(l_k)} \mathbf{U}_\mathbf{A}(0).v$$

for $0 \leq j \leq k$, $0 \leq s_j \leq N$, $l_j \geq 0$.

By Lemma 4.1, we have $V_q = \mathbf{U}_q(<) \mathbf{U}_q(0).v$ and hence

$$V_q = \bigoplus_{\eta, m} (V_q)_{\lambda - \eta + m\delta},$$

where $\eta \in Q^+$, $m \in \mathbf{Z}$. The argument given in the proof of Proposition 1.2(i) (but replacing the modules by their quantum analogues) shows that $(V_q)_{\lambda - \eta + m\delta} \neq 0$ for only finitely many $\eta \in Q^+$. Hence, it suffices to prove that:

*for each $\eta \in Q^+$, there exists $N(\eta) \geq 0$ such that $(V_q)_{\lambda - \eta + m\delta} \subset V_{N(\eta)}$
for all $m \in \mathbf{Z}$.*

We proceed by induction on $\text{ht } \eta$. By Proposition 4.3, we see that, if $s > \lambda(h_i)$, $p \geq 1$,

$$\left(\mathbf{X}_i^-(u)^{(p)} \Lambda_i^+(u) \right)_{ps} \mathbf{U}_\mathbf{A}(0).v_\lambda = 0,$$

or equivalently that

$$(4.1) \quad \sum_{l=0}^{ps} \left(\mathbf{X}_i^-(u)^{(p)} \right)_l \Lambda_i^+(u)_{ps-l} \mathbf{U}_\mathbf{A}(0).v_\lambda = 0.$$

If $p = 1$, it follows easily from equation (4.1) by induction on s that

$$\mathbf{x}_{i,s}^-. (V_q)_{\lambda + m\delta} \subset V_{\lambda(h_i)}$$

if $s > \lambda(h_i)$. To deal with the case $s \leq 0$, we apply $h_{i,s}$ to both sides of equation (4.1), and as in the proof of Proposition 1.2 use the fact that $\Lambda_{i,\lambda(h_i)}$ is invertible. Thus, the induction begins with $N(\alpha_i) = \lambda(h_i)$.

Assume that we have proved the italicized statement above for all $\eta \in Q^+$ with $\text{ht } \eta < p$. We deal first with the case $\eta = p\alpha_i$, for some $i \in I$, $p \geq 1$. We show that we can take $N(p\alpha_i) = \lambda(h_i)$. For this, it suffices to prove that

$$(4.2) \quad (\mathbf{x}_{i,s}^-)^{(l)} (V_q)_{\lambda + (m-sl)\delta - (p-l)\alpha_i} \subset V_{\lambda(h_i)}$$

for all $m, s \in \mathbf{Z}$, $p \geq l > 0$. We prove this by induction on s (for p fixed), assuming first that $s \geq 0$. The induction begins since there is nothing to prove if $0 \leq s \leq \lambda(h_i)$. Assume that $s > \lambda(h_i)$ and that the result holds for all smaller values of $s \geq 0$.

If $p > l > 0$ then, by the induction on p , we have

$$(V_q)_{\lambda-(p-l)\alpha_i+(m-sl)\delta} \subset \sum_{s',l'} (\mathbf{x}_{i,s'}^-)^{(l')} V_{\lambda-(p-l-l')\alpha_i+(m-sl-s'l')\delta},$$

where $0 \leq s' \leq \lambda(h_i)$, $l' > 0$. By Lemma 4.3(i), we have

$$(\mathbf{x}_{i,s}^-)^{(l)} (\mathbf{x}_{i,s'}^-)^{(l')} V_{\lambda-(p-l-l')\alpha_i+(m-sl-s'l')\delta} \subset \sum (\mathbf{x}_{i,s''}^-)^{(l'')} V_{\lambda-(p-l'')\alpha_i+(m-l''s'')\delta},$$

where $s' \leq s'' < s$. It follows by the induction hypothesis on s that

$$\sum (\mathbf{x}_{i,s''}^-)^{(l'')} V_{\lambda-(p-l'')\alpha_i+(m-l''s'')\delta} \subset V_{\lambda(h_i)},$$

which proves equation (4.2).

In the case $p = l$, we must show that

$$(\mathbf{x}_{i,s}^-)^{(p)} \mathbf{U}_{\mathbf{A}}(0).v \subset V_{\lambda(h_i)}.$$

Since $ps > \lambda(h_i)$, by Proposition 4.3 we see that

$$(\mathbf{X}_i^-(u)^{(p)})_{ps} \mathbf{U}_{\mathbf{A}}(0).v + \sum_{s' < ps} (\mathbf{X}_i^-(u)^{(p)})_{s'} \mathbf{\Lambda}_{i,ps-s'} \mathbf{U}_{\mathbf{A}}(0).v = 0.$$

Now, by Lemma 4.4(iii), we see that for $s' < ps$ the element $(\mathbf{X}_i^-(u)^{(p)})_{s'}$ is a sum of terms of the form

$$(4.3) \quad (\mathbf{x}_{i,s_1}^-)^{(r_1)} (\mathbf{x}_{i,s_2}^-)^{(r_2)} \dots$$

where $0 \leq s_1 \leq s_2 \leq \dots$, $r_1, r_2, \dots > 0$, and $s_1 < s$. The induction hypothesis on p and s proves that $(\mathbf{X}_i^-(u)^{(p)})_{s'} \mathbf{U}_{\mathbf{A}}(0).v \subset V_{\lambda(h_i)}$. Finally, again by Lemma 4.4(iii), we have

$$\left(\mathbf{X}_i^-(u)^{(p)} \right)_{ps} = (\mathbf{x}_{i,s}^-)^{(p)} + A,$$

where A is a sum of terms of the form in (4.3) where either $s_1 < s$ or $s_1 = s$ and $r_1 < p$. If $s_1 < s$ it follows as before that $(\mathbf{X}_i^-(u)^{(p)})_{s'} \mathbf{U}_{\mathbf{A}}(0).v \subset V_{\lambda(h_i)}$, and if $s_1 = s$ it follows by the case $l < p$ of equation (4.2) proved above. Thus, the induction on s is completed in the case $p = l$ also.

This completes the proof of (4.2) when $s \geq 0$. Next, consider the case when $s \leq 0$. The case $p = 1$ was proved above. For $p < l$, the same method used for $s \geq 0$ works, this time using Lemma 4.2(ii). Finally, for the case $p = l$, we use the relation

$$(\Phi(\mathbf{X}_i^-(u)^{(p)} \mathbf{\Lambda}_i^-(u)))_{ps} .v = 0$$

and parts (i) and (ii) of Proposition 4.3, and proceed as in the case $s \geq 0$. We omit the details.

This completes the proof of the italicized statement when η is a multiple of α_i .

We now turn to the case of arbitrary $\eta = \sum r_i \alpha_i$ of height p . Choose M so that if $\sum r_i < p$, then $(V_q)_{\lambda-\eta+m\delta} \subset V_M$. As in the special case $\eta = p\alpha_i$, to complete the induction on p it suffices to prove that there exists $N \geq 0$ such that

$$(\mathbf{x}_{i,s}^-)^{(l)} (V_q)_{\lambda-(\eta-l\alpha_i)+(m-ls)\delta} \subset V_N$$

for all $i \in I$, $s \in \mathbf{Z}$, $l > 0$. Since $\text{ht}(\eta - l\alpha_i) < p$, it follows that

$$(V_q)_{\lambda-(\eta-l\alpha_i)+(m-ls)\delta} \subset \sum_{j,s_j,l_j} (\mathbf{x}_{j,s_j}^-)^{(l_j)} (V_q)_{\lambda-(\eta-l\alpha_i-l_j\alpha_j)+(m-ls-l_j s_j)\delta}$$

where $0 \leq s_j \leq M$. Thus, we must show that there exists N such that

$$(\mathbf{x}_{i,s}^-)^{(l)} (\mathbf{x}_{j,s_j}^-)^{(l_j)} (V_q)_{\lambda - (\eta - l\alpha_i - l_j\alpha_j) + (m - ls - l_j s_j)\delta} \subset V_N,$$

for all $i, j \in I$, $s, s_j \in \mathbf{Z}$, $0 \leq s_j \leq M$. Assume that $s \geq 0$ (the case $s \leq 0$ is similar). If $s \leq M$, there is nothing to prove. Assume that we know the result for all i , and all smaller positive values of s . If $i = j$, then we prove exactly as in the case $\eta = r\alpha_i$ that we can take $N = M$.

If $a_{ij} = 0$, the result is obvious. Assume now that $a_{ij} = -1$. Then, with the notation in Proposition 4.1, we see that

$$(\gamma_{s,s_j}^{i,j})^{(m)} = (-1)^m (\gamma_{s_j+1,s-1}^{j,i})^{(m)} = \sum_{p'=0}^m g_{p'} (\mathbf{x}_{i,s-1}^-)^{(p')} (\mathbf{x}_{j,s_j+1}^-)^{(m)} (\mathbf{x}_{i,s-1}^-)^{(m-p')},$$

where the $g_{p'} \in \mathbf{A}$. Using the induction hypothesis on s , we get that

$$(\gamma_{s,s_j}^{i,j})^{(m)} \cdot (V_q)_{\lambda - (\eta - m\alpha_i - m\alpha_j) + m'\delta} \subset V_{M+1}$$

for all $m' \in \mathbf{Z}$. Now, using Proposition 4.1 again, we see that

$$(\mathbf{x}_{i,s}^-)^{(l)} (\mathbf{x}_{j,s_j}^-)^{(l_j)} (V_q)_{\lambda - (\eta - l\alpha_i - l_j\alpha_j) + (m - ls - l_j s_j)\delta} \subset V_{M+1}.$$

This proves the result. \square

Let $I_q(0)$ be the left ideal in \mathbf{U}_q^e generated by the following elements:

$$\begin{aligned} & \mathbf{\Lambda}_{i,m} \quad (i \in I, |m| > \lambda(h_i)), \\ & \mathbf{\Lambda}_{i,\lambda(h_i)} \mathbf{\Lambda}_{i,-m} - \mathbf{\Lambda}_{i,\lambda(h_i)-m} \quad (i \in I, 1 \leq m \leq \lambda(h_i)), \\ & \left(\tilde{\mathbf{X}}_i^-(u) \mathbf{\Lambda}_i^+(u) \right)_m \mathbf{U}_q(0) \quad (i \in I, m \in \mathbf{Z}), \\ & (\mathbf{X}_{i,0}^-(u))^r \mathbf{\Lambda}_i^+(u) \Big|_m \mathbf{U}_q(0) \quad (i \in I, r \geq 1, |m| > \lambda(h_i)). \end{aligned}$$

Let $I_q^e(\lambda)$ be the left ideal in \mathbf{U}_q^e generated by $I_q(0)$, $\mathbf{U}_q(>)$, the elements $K_i^{\pm 1} - q^{\pm \lambda(h_i)}$ ($i \in I$), and $D^{\pm 1} - q^{\pm \lambda(d)}$. The ideal $I_q(\lambda)$ in \mathbf{U}_q is defined in the obvious way. Let

$$W_q(\lambda) = \mathbf{U}_q^e / I_q^e(\lambda) = \mathbf{U}_q / I_q(\lambda)$$

be the corresponding left \mathbf{U}_q^e -module. Let w_λ be the image of 1 in $W_q(\lambda)$. We have

$$\mathbf{U}_q(0) \cdot w_\lambda \cong \mathbf{U}_q(0) / I_q(0)$$

as $\mathbf{U}_q(0)$ -modules.

Proposition 4.5. *The \mathbf{U}_q^e -module $W_q(\lambda)$ is integrable.*

Proof. In [K], Kashiwara defines an integrable \mathbf{U}_q^e -module $V^{max}(\lambda)$ for all $\lambda \in P_+^e$. In fact, according to unpublished work of Kashiwara,

$$V^{max}(\lambda) \cong W_q(\lambda),$$

and hence $W_q(\lambda)$ is an integrable \mathbf{U}_q^e -module. One can also give a direct proof that $W_q(\lambda)$ is integrable along the lines of the proof of Theorem 1. One works with the presentation of \mathbf{U}_q^e in terms of Chevalley generators and the quantum version of Lemma 2.1 proved in [K], [L2]. We omit the details. \square

Given any \mathbf{U}_q^e -module V_q and a $\mathbf{U}_\mathbf{A}$ -submodule $V_\mathbf{A}$ of V_q such that

$$V_q \cong \mathbf{C}(q) \otimes_{\mathbf{A}} V_\mathbf{A},$$

we set

$$\overline{V}_q \cong \mathbf{C}_1 \otimes_{\mathbf{A}} V_\mathbf{A},$$

where we regard \mathbf{C}_1 as an \mathbf{A} -module by letting q act as 1. The algebra \mathbf{U}^e is a quotient of $\mathbf{C}_1 \otimes_{\mathbf{A}} \mathbf{U}_\mathbf{A}$ by the ideal generated by $K_i - 1$, $i \in I$. In particular, this means that we can regard \overline{V}_q as a \mathbf{U}^e -module. Similar results hold for \mathbf{U}_q -modules.

Let π_q be an n -tuple of polynomials with constant term 1 and coefficients in $\mathbf{C}(q)$. Define $\lambda_{\pi_q} \in \mathcal{P}_+^e$ and $\pi_q^\pm(u)$ as in Section 2. Let $I_q(\pi_q)$ be the left ideal in \mathbf{U}_q generated by $I_q(\lambda_{\pi_q})$ and the elements

$$(\mathbf{A}_i^\pm(u) - \pi_i^\pm(u))_s \quad (i \in I, s \geq 0).$$

Set $W_q(\pi_q) = \mathbf{U}_q/I_q(\pi_q)$.

Lemma 4.5. $W_q(\pi_q)$ is a finite-dimensional \mathbf{U}_q -module.

Proof. This is proved in the same way as the corresponding result for \mathbf{U} . We use Proposition 4.4 instead of Proposition 1.2. \square

One has the following analogue of Proposition 2.1 for the modules $W_q(\lambda)$ and $W_q(\pi_q)$. We omit the proof, which is entirely similar to that of Proposition 2.1.

Proposition 4.6. (i) Let V_q be any integrable \mathbf{U}_q^e -module generated by an element of $(V_q)_\lambda^+$. Then, V_q is a quotient of $W_q(\lambda)$.
(ii) Let V_q be a finite-dimensional quotient \mathbf{U}_q -module of $W_q(\lambda)$, and assume that $\dim (V_q)_\lambda = 1$. Then, V_q is a quotient of $W_q(\pi_q)$ for some choice of π_q .
(iii) Let V_q be finite-dimensional \mathbf{U}_q -module generated by an element of $(V_q)_\lambda^+$ and such that $\dim (V_q)_\lambda = 1$. Then, V_q is a quotient of $W_q(\pi_q)$ for some π_q .

Definition 4.2. We call π_q integral if the polynomials $\pi_i^\pm(u)$ have coefficients in \mathbf{A} for all $i \in I$. Equivalently, for all $i \in I$, $\pi_i(u)$ has coefficients in \mathbf{A} and the coefficient of the highest power of u should lie in $\mathbf{C}^\times q^{\mathbf{Z}}$. Let $\overline{\pi}_q$ be the n -tuple of polynomials with coefficients in \mathbf{C} and the constant term one obtained from π_q by evaluating its coefficients at $q = 1$.

For any π_q , $W_\mathbf{A}(\pi_q) = \mathbf{U}_\mathbf{A}.w\pi_q$ is a $\mathbf{U}_\mathbf{A}$ -module.

Lemma 4.6. Assume that π_q is integral.

(i) $W_\mathbf{A}(\pi_q)$ is a free \mathbf{A} -module and we have

$$W_q(\pi_q) \cong \mathbf{C}(q) \otimes_{\mathbf{A}} W_\mathbf{A}(\pi_q).$$

(ii) The \mathbf{U} -module $\overline{W}_q(\pi_q)$ is a quotient of $W(\overline{\pi}_q)$.

Proof. Since $W_\mathbf{A}(\pi_q)$ is a quotient of $W_q(\lambda_{\pi_q})$, it follows from Proposition 4.4 that $W_\mathbf{A}(\pi_q)$ is a finitely-generated \mathbf{A} -module. Since it is clearly a torsion-free \mathbf{A} -module, part (i) follows.

To prove (ii), observe that the defining relations of $W_q(\pi_q)$ specialize to those of $W(\overline{\pi}_q)$. The result now follows from Proposition 2.1. \square

The \mathbf{U}_q -module $W_q(\boldsymbol{\pi}_q)$ has a unique irreducible quotient $V_q(\boldsymbol{\pi}_q)$. Let $v_{\boldsymbol{\pi}_q}$ be the image of $w_{\boldsymbol{\pi}_q}$ and set

$$V_{\mathbf{A}}(\boldsymbol{\pi}_q) = \mathbf{U}_{\mathbf{A}} \cdot v_{\boldsymbol{\pi}_q}.$$

If $\boldsymbol{\pi}_q$ is integral, $V_{\mathbf{A}}(\boldsymbol{\pi}_q)$ is a $\mathbf{U}_{\mathbf{A}}$ -module and is free as an \mathbf{A} -module (since it is torsion-free and the quotient of a finitely-generated \mathbf{A} -module). Let $V(\boldsymbol{\pi})$ be the unique irreducible quotient of the \mathbf{U} -module $W(\boldsymbol{\pi})$.

Lemma 4.7. *The \mathbf{U} -module $V(\overline{\boldsymbol{\pi}_q})$ is a quotient of $\overline{V_q(\boldsymbol{\pi}_q)}$.*

Proof. The module $\overline{V_q(\boldsymbol{\pi}_q)}$ has a unique irreducible quotient V . By Lemma 4.6(ii), V is a quotient of $W(\overline{\boldsymbol{\pi}_q})$ and hence by uniqueness $V \cong V(\overline{\boldsymbol{\pi}_q})$. \square

We have thus proved the following statement. Assume that $\boldsymbol{\pi}_q$ is integral. Then, we have a commutative diagram of surjective \mathbf{U} -module homomorphisms

$$\begin{array}{ccc} W(\overline{\boldsymbol{\pi}_q}) & \longrightarrow & \overline{W_q(\boldsymbol{\pi}_q)} \\ \downarrow & & \downarrow \\ V(\overline{\boldsymbol{\pi}_q}) & \longleftarrow & \overline{V_q(\boldsymbol{\pi}_q)}. \end{array}$$

Conjecture. *If $\boldsymbol{\pi}_q = \boldsymbol{\pi}$ has coefficients in \mathbf{C} , the natural map $W(\boldsymbol{\pi}) \rightarrow \overline{V_q(\boldsymbol{\pi})}$ is an isomorphism of \mathbf{U} -modules, and hence $W_q(\boldsymbol{\pi}) \cong V_q(\boldsymbol{\pi})$.* \square

In Section 6, we prove this conjecture when $\mathfrak{g} = sl_2$.

5. AN IRREDUCIBILITY CRITERION

In this section we return to the classical case and obtain a criterion for the irreducibility of the modules $W(\boldsymbol{\pi})$.

For any $a \in \mathbf{C}^\times$, and any \mathbf{U}^{fin} -module V , define a \mathbf{U} -module structure on V by

$$(x \otimes t^r) \cdot v = a^r x \cdot v$$

for $x \in \mathfrak{g}$, $r \in \mathbf{Z}$, $v \in V$. Let $V(a)$ denote the corresponding \mathbf{U} -module.

For $i \in I$, $a \in \mathbf{C}^\times$, we denote by $W(i, a)$ the \mathbf{U} -module corresponding to the n -tuple $\boldsymbol{\pi}$ of polynomials defined by

$$\pi_j(u) = 1 \text{ if } j \neq i, \quad \pi_i(u) = 1 - au,$$

and denote $w_{\boldsymbol{\pi}}$ by $w_{i,a}$. Clearly, $V^{fin}(\omega_i)(a)$ is the irreducible quotient of $W(i, a)$. We set $V^{fin}(\omega_i)(a) = V(i, a)$.

For $i \in I$ and $a \in \mathbf{C}(q)^\times$, the \mathbf{U}_q -modules $W_q(i, a)$ and $V_q(i, a)$ are defined similarly.

We need the following result, due to [CP3] for \mathfrak{g} of type sl_2 and due to [K2] and [FM] in general.

Proposition 5.1. *Let $r \geq 1$, $a_1, \dots, a_r \in \mathbf{C}(q)^\times$, $i_1, i_2, \dots, i_r \in I$. There is a finite set $S \subset \mathbf{C}(q)^\times$ (depending on i_1, \dots, i_r) such that the tensor product*

$$V_q(i_1, a_1) \otimes V_q(i_2, a_2) \otimes \cdots \otimes V_q(i_r, a_r),$$

is irreducible if $a_l/a_m \notin S$ for all $l, m = 1, 2, \dots, r$. If $\mathfrak{g} = sl_2$, $S = \{q^{\pm 2}\}$. \square

Proposition 5.2. *For $i \in I$, $a \in \mathbf{C}^\times$, $W(i, a) \cong V(i, a)$ if and only if $r_i = 1$.*

Proof. The proof rests on the following fact, which can be established by a case by case check: $r_i = 1$ if and only if there exists $\mu \in P_+$ with $0 \neq \omega_i - \mu \in Q_+$ such that $V^{fin}(\mu)$ occurs as a component of $V^{fin}(\theta) \otimes V^{fin}(\omega_i)$.

Suppose first that $r_i > 1$. Let $\mu \in P_+$ have the above property. For $x \in \mathfrak{g}$, $m \in \mathbf{Z}$, $v \in V^{fin}(\omega_i)$, $v' \in V^{fin}(\mu)$, define

$$x_m \cdot (v, v') = a^m(x.v, m \operatorname{pr}(x \otimes v) + x.v'),$$

where $\operatorname{pr} : V^{fin}(\theta) \otimes V^{fin}(\omega_i) \rightarrow V^{fin}(\mu)$ is the \mathfrak{g} -module projection. It is straightforward to check that this defines a \mathbf{U} -module structure on $V^{fin}(\omega_i) \oplus V^{fin}(\mu)$, and that this \mathbf{U} -module is generated by the highest weight vector in $V^{fin}(\omega_i)$. It is therefore a quotient of $W(i, a)$.

To prove the converse, notice that, as a \mathfrak{g} -module, $W(i, a)$ is completely reducible, and hence

$$W(i, a) \cong V^{fin}(\omega_i) \oplus \bigoplus_{\mu < \omega_i} V^{fin}(\mu)^{m_\mu},$$

where m_μ is the multiplicity with which $V^{fin}(\mu)$ occurs in $W(i, a)$. Consider the map $L(\mathfrak{g}) \otimes W(i, a) \rightarrow W(i, a)$ given by

$$x_n \otimes v \mapsto x_n \cdot v.$$

This is clearly a map of \mathfrak{g} -modules, where we regard $L(\mathfrak{g})$ as a module for \mathfrak{g} through the adjoint representation. For each $m \in \mathbf{Z}$, consider the restriction of this map to $(\mathfrak{g} \otimes t^m) \otimes V(\omega_i)$. Since $\mathfrak{g} \otimes t^m \cong V^{fin}(\theta)$ as \mathfrak{g} -modules, we have a \mathfrak{g} -module map $V^{fin}(\theta) \otimes V(i, a) \rightarrow W(i, a)$. By the fact stated above, this map takes $(\mathfrak{g} \otimes t^m) \otimes V(\omega_i)$ into the $\mathbf{U}(\mathfrak{g})$ -submodule $V^{fin}(\omega_i)$ of $W(i, a)$ for all $m \in \mathbf{Z}$. This proves that $V^{fin}(\omega_i)$ is a \mathbf{U} -submodule of $W(i, a)$ and hence (since $w_{i,a} \in V^{fin}(\omega_i)$) is equal to $W(i, a)$. \square

Remark 5.1. The same criterion $r_i = 1$ occurs, for the same reason, in Drinfeld's work on finite-dimensional representations of Yangians, [Dr1]. See also [CP4, Proposition 12.1.17].

We can now state the main result of this section.

Theorem 3. *Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$ be an n -tuple of polynomials in $\mathbf{C}[u]$ with constant coefficient one. Then, the \mathbf{U} -module $W(\boldsymbol{\pi})$ is irreducible if and only if π_θ has distinct roots.*

Proof. Assume that π_θ has distinct roots. By Lemma 3.1, it follows that $\pi_i = 1$ if $r_i \neq 1$. Let

$$I' = \{i \in I : r_i = 1\}.$$

If $i \in I'$, then π_i must have distinct roots, and for any $i, j \in I'$, $i \neq j$, the polynomials π_i and π_j must be relatively prime. Hence, by Theorem 2 and Proposition 5.2, it follows that

$$W(\boldsymbol{\pi}) \cong \bigotimes_{i \in I', a_{ij} \in \mathbf{C}^\times} W(i, a_{ij}) \cong \bigotimes_{i \in I', a_{ij} \in \mathbf{C}^\times} V(i, a_{ij}),$$

where the a_{ij} are all distinct. By Proposition 3.2, we see that the second tensor product in the preceding equation is an irreducible $L(\mathfrak{g})$ -module.

For the converse, suppose that π_θ has repeated roots. By Theorem 2, it follows that $W(\boldsymbol{\pi})$ is isomorphic to a tensor product of modules $W(\boldsymbol{\pi}^a)$, where $\boldsymbol{\pi}^a$ is an

n -tuple of polynomials such that $\pi_\theta^a = (1 - au)^m$ for some $a \in \mathbf{C}^\times$ and $m \geq 1$, and where $m > 1$ for at least one value of a . Thus, it suffices to prove the theorem in the case where $\pi_\theta(u) = (1 - au)^m$ with $a \in \mathbf{C}^\times$ and $m > 1$. From now on, we shall assume that we are in this case.

To prove that $W(\boldsymbol{\pi})$ is not isomorphic to $V(\boldsymbol{\pi})$ as $L(\mathfrak{g})$ -modules, recall that by Proposition 3.2, we have $V(\boldsymbol{\pi}) \cong V^{fin}(\lambda\boldsymbol{\pi})$ as \mathfrak{g} -modules. Hence, it suffices to prove that $W(\boldsymbol{\pi})$ is reducible as a \mathfrak{g} -module.

Let $\boldsymbol{\pi}_q$ be an n -tuple of polynomials with constant term 1 such that

$$\pi_i(u) = (1 - a_{i,1}u)(1 - a_{i,2}u) \cdots (1 - a_{i,m_i}u),$$

where $a_{i,j} = aq^{l_{ij}}$, for some $l_{ij} \in \mathbf{Z}$. Let $v_{i,j}$ be the highest weight vector in $V_q(i, a_j)$ and consider

$$\begin{aligned} \mathbf{V} &= V_q(1, a_{1,1}) \otimes V_q(1, a_{1,2}) \otimes \cdots \otimes V(1, a_{1,m_1}) \otimes \cdots \\ &\quad \otimes V_q(n, a_{n,1}) \otimes \cdots \otimes V_q(n, a_{n,m_n}). \end{aligned}$$

Let $\mathbf{v} = v_{1,1} \otimes v_{1,2} \otimes \cdots \otimes v_{n,m_n}$ and set

$$Z_q(\boldsymbol{\pi}_q) = \mathbf{U}_q \cdot \mathbf{v} \subset \mathbf{V}, \quad Z_{\mathbf{A}}(\boldsymbol{\pi}_q) = \mathbf{U}_{\mathbf{A}} \cdot \mathbf{v}.$$

Since $Z_q(\boldsymbol{\pi}_q)$ is a quotient of $W_q(\boldsymbol{\pi}_q)$, and $\boldsymbol{\pi}_q$ is integral, it follows that

$$Z_q(\boldsymbol{\pi}_q) \cong Z_{\mathbf{A}}(\boldsymbol{\pi}_q) \otimes_{\mathbf{A}} \mathbf{C}(q),$$

so we can define the \mathbf{U} -module $\overline{Z_q(\boldsymbol{\pi}_q)} = Z_{\mathbf{A}}(\boldsymbol{\pi}_q) \otimes_{\mathbf{A}} \mathbf{C}_1$. Clearly, $\overline{Z_q(\boldsymbol{\pi}_q)}$ is a quotient of $W(\boldsymbol{\pi})$ and hence it suffices to show that $\overline{Z_q(\boldsymbol{\pi}_q)}$ is reducible as a \mathfrak{g} -module.

Suppose first that $m_{i_0} \geq 2$ for some $i_0 \in I$. Take $l_{ij} = 0$ for all $i \in I$ and $j = 1, \dots, m_i$. Let $\mathbf{U}_q^{i_0}$ be the subalgebra of \mathbf{U}_q generated by $K_{i_0}^{\pm 1}$ and $x_{i_0,k}^{\pm}$ for $k \in \mathbf{Z}$. Consider the $\mathbf{U}_q^{i_0}$ -module

$$Z_q^{i_0}(\boldsymbol{\pi}_q) = \mathbf{U}_q^{i_0} \cdot \mathbf{v}.$$

Let ω be the fundamental weight of sl_2 . Then, by Proposition 5.1, we know that $V_q(\omega, a)^{\otimes m_{i_0}}$ is irreducible and hence is a quotient of $Z_q^{i_0}(\boldsymbol{\pi}_q)$. Clearly,

$$\dim(Z_q(\boldsymbol{\pi}_q)_{\lambda - \alpha_{i_0}}) \geq \dim(V(\omega, a)^{\otimes m_{i_0}})_{m_{i_0}\omega - \alpha} = m_{i_0}$$

hence

$$\dim(\overline{Z_q(\boldsymbol{\pi}_q)}_{\lambda - \alpha_{i_0}}) \geq m_{i_0} > 1.$$

On the other hand, $V(\boldsymbol{\pi})$ is a quotient \mathbf{U} -module of $\overline{Z_q(\boldsymbol{\pi}_q)}$, since $\boldsymbol{\pi}_q = \boldsymbol{\pi}$, and $V(\boldsymbol{\pi}) \cong V^{fin}(\lambda\boldsymbol{\pi})$ as \mathbf{U}^{fin} -modules from above. But

$$\dim(V^{fin}(\lambda\boldsymbol{\pi})_{\lambda\boldsymbol{\pi} - \alpha_{i_0}}) = 1.$$

Hence, $\overline{Z_q(\boldsymbol{\pi}_q)}$ is reducible as a \mathbf{U}^{fin} -module.

We can therefore assume that each $m_i = 0$ or 1, and that at least one $m_i = 1$. Consider first the case $m_{i_0} = m_{i_1} = 1$ with $i_0 < i_1$, and $m_j = 0$ for all $i_0 < j < i_1$. Set $J = \{i_0, i_0 + 1, \dots, i_1\}$. By a suitable choice of the numbering, we can assume that the corresponding diagram subalgebra \mathfrak{g}^J of \mathfrak{g} is of type $A_{|J|}$. Let \mathbf{U}_q^J be the subalgebra of \mathbf{U}_q generated by $K_i^{\pm 1}$ and $x_{i,k}^{\pm}$ for $k \in \mathbf{Z}$ and $i \in J$. Define

$$Z_q^J(\boldsymbol{\pi}_q) = \mathbf{U}_q^J \cdot \mathbf{v}.$$

By Proposition 5.2, the \mathbf{U}_q^J -module $V_q^J(i_0, a_{i_0}) \otimes V_q^J(i_1, a_{i_1})$ is irreducible except for finitely many values of the ratio a_{i_0}/a_{i_1} . Since each a_i can be chosen from the infinite set $\{aq^m : m \in \mathbf{Z}\}$, we can assume that a_{i_0} and a_{i_1} are chosen so that $V_q^J(i_0, a_{i_0}) \otimes V_q^J(i_1, a_{i_1})$ is an irreducible \mathbf{U}_q^J -module and hence a quotient of $Z_q^J(\pi_q)$. If θ_J is the highest root of the subdiagram J , then

$$\begin{aligned} \dim(Z_q(\pi_q)_{\lambda_{\pi_q} - \theta_J}) &\geq \dim(Z_q^J(\pi_q)_{\lambda_{\pi_q} - \theta_J}) \\ &\geq \dim((V_q^J(i_0, a_{i_0}) \otimes V_q^J(i_1, a_{i_1}))_{\lambda_{\pi_q} - \theta_J}) \\ &\geq \dim((V_q^{fin, J}(\omega_{i_0}) \otimes V_q^{fin, J}(\omega_{i_1}))_0) = |J| + 1 \end{aligned}$$

(in an obvious notation). On the other hand, $V(\pi)$ is a quotient \mathbf{U} -module of $\overline{Z_q(\pi_q)}$ and

$$V(\pi)_{\lambda_{\pi} - \theta_J} = V^J(\pi)_{\lambda_{\pi} - \theta_J} = V^{fin, J}(\lambda_{\pi})_{\lambda_{\pi} - \theta_J},$$

which has dimension $|J|$. Hence, $V(\pi)$ is a proper quotient of $\overline{Z_q(\pi_q)}$. This shows that $Z_q(\pi_q)$ is not isomorphic to $V_q(\lambda_{\pi_q})$ and hence is reducible as a $\mathbf{U}_q(\mathfrak{g})$ -module.

It remains to consider the case when exactly one $m_i = 1$, say $m_{i_0} = 1$ and all other $m_i = 0$. Since π_θ has repeated roots, this means that $r_{i_0} \neq 1$. By Proposition 5.2 we know that $W(\pi) = W(i, a)$ is reducible.

This completes the proof of the theorem. □

6. THE sl_2 CASE

In this section, $\mathfrak{g} = sl_2$. Let ω be the fundamental weight, α the positive root, and set $x^\pm = x_{\pm\alpha}$, $h = h_\alpha$. Let π be a polynomial with coefficients in \mathbf{C} and constant term 1. When $\pi = 1 - au$, denote $V_q(\pi)$ by $V_q(a)$, and define $V(a)$ similarly. Note that these modules are two-dimensional over $\mathbf{C}(q)$ and \mathbf{C} , respectively.

Set $V = V^{fin}(\omega)$ and let $L(V) = V \otimes \mathbf{C}[t, t^{-1}]$ be the obvious \mathbf{U}^e -module, given by

$$x_r \cdot (v \otimes t^s) = x \cdot v \otimes t^{r+s}, \quad d \cdot (v \otimes t^r) = rv \otimes t^r,$$

for $r, s \in \mathbf{Z}$, $x \in \mathfrak{g}$ and $v \in V$.

Set $\mathcal{P}_m = \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_m^{\pm 1}]$. Let Σ_m be the symmetric group on m letters and let \mathcal{P}^{Σ_m} be the subalgebra of polynomials invariant under the obvious action of Σ_m .

Let $S^m(L(V))$ be the symmetric part of the m -fold tensor product $T^m(L(V))$ of $L(V)$. Then, $T^m(L(V))$ is a \mathbf{U}^e -module in the obvious way, and $S^m(L(V))$ is a \mathbf{U}^e -submodule. Moreover, as vector spaces,

$$T^m(L(V)) \cong V^{\otimes m} \otimes \mathcal{P}_m,$$

and so $T^m(L(V))$ is a right module for \mathcal{P}_m by right multiplication. This induces a right $\mathcal{P}_m^{\Sigma_m}$ -action on $S^m(L(V))$.

The left \mathbf{U}^e -module $W(m\omega)$ is also a right $\mathcal{P}_m^{\Sigma_m}$ -module. In fact, by equation (*), $W(m\omega)$ is a right $\mathbf{U}(0)/I_{m\omega}(0)$ -module, i.e., a right module for the algebra $\mathbf{C}[\Lambda_1, \dots, \Lambda_m, \Lambda_m^{-1}]$. But this algebra is isomorphic to $\mathcal{P}_m^{\Sigma_m}$ by taking Λ_r to the r th elementary symmetric function of t_1, \dots, t_m .

In this section, we shall prove the following two theorems.

Theorem 4. *As left \mathbf{U}^e -modules and as right $\mathcal{P}_m^{\Sigma_m}$ -modules, we have*

$$W(m\omega) \cong S^m(L(V)).$$

To prove Theorem 4 we shall need

Theorem 5. *Let $\pi(u)$ be a polynomial with coefficients in \mathbf{C} and constant term 1. Then, the dimension of $W(\pi)$ is $2^{\deg \pi}$. In fact,*

$$W(\pi) \cong \overline{V_q(\pi)},$$

as \mathbf{U} -modules, and

$$V_q(\pi) \cong \bigotimes_a V_q(a)$$

where a^{-1} runs over the set of roots of π counted with multiplicity.

It does not follow from this result that $W(\pi) \cong \bigotimes_a V(a)$. In fact, this is false except when $\deg \pi = 1$. The point is that the \mathbf{A} -form of $\bigotimes_a V_q(a)$ is not the tensor product of the \mathbf{A} -forms of $V_q(a)$ (in fact, the former is a proper subset of the latter, unless $\deg \pi = 1$). We note the following corollary.

Corollary 6.1. *For any $\pi(u)$ as in Theorem 5, we have $W_q(\pi) \cong V_q(\pi)$ as \mathbf{U}_q -modules.*

Proof. Since $V_q(\pi)$ is a quotient of $W_q(\pi)$, it suffices to prove that

$$\dim_{\mathbf{C}(q)} W_q(\pi) \leq 2^{\deg \pi}.$$

But this is clear from Theorem 5, since $\overline{W_q(\pi)}$ is a quotient of $W(\pi)$, so

$$\dim_{\mathbf{C}(q)} W_q(\pi) = \dim_{\mathbf{C}} \overline{W_q(\pi)} \leq \dim_{\mathbf{C}} W(\pi) = 2^{\deg \pi}.$$

□

Assume Theorem 5 for the moment. To prove Theorem 4, we begin with the following trivial lemma.

Lemma 6.1. *The \mathbf{U}^e -module $S^m(L(V))$ is integrable.*

□

Let $\{v_+, v_-\}$ be the usual basis of V , so that

$$x^\pm.v_\pm = 0, \quad x^\pm.v_\mp = v_\pm, \quad hv_\pm = \pm v_\pm.$$

For $0 \leq r \leq m$, $l_1, \dots, l_m \in \mathbf{Z}$, define

$$\mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+1}, \dots, l_m)} = \sum_{\sigma \in \Sigma_m} v_{\sigma(1)} t^{l_{\sigma(1)}} \otimes \dots \otimes v_{\sigma(m)} t^{l_{\sigma(m)}},$$

where we set

$$\begin{aligned} v_s &= v_-, & \text{if } 1 \leq s \leq r, \\ v_s &= v_+, & \text{if } r+1 \leq s \leq m. \end{aligned}$$

Clearly the set

$$\{\mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+1}, \dots, l_m)} : 0 \leq r \leq m, l_1, \dots, l_m \in \mathbf{Z}\},$$

is a \mathbf{C} -basis of $S^m(L(V))$.

Lemma 6.2. *The $\mathcal{P}_m^{\Sigma_m}$ -module $S^m(L(V))$ is free of rank 2^m .*

Proof. For $0 \leq r \leq m$, let $S^m(L(V))_r$ be the subspace of $S^m(L(V))$ spanned by the elements

$$\{\mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+1}, \dots, l_m)} : l_1, \dots, l_m \in \mathbf{Z}\}.$$

Clearly, $S^m(L(V))_r$ is a right $\mathcal{P}_m^{\Sigma_m}$ -submodule of $S^m(L(V))$. It is easy to see that it is isomorphic to the $\mathcal{P}_m^{\Sigma_m}$ -module $\mathcal{P}_m^{\Sigma_r \times \Sigma_{m-r}}$, consisting of the elements in \mathcal{P}_m invariant under permutation of the first r and the last $m - r$ variables. But it is well known that the latter module is free of rank $\binom{m}{r}$. This proves the lemma. \square

Lemma 6.3. *The assignment $w_{m\omega} \rightarrow \mathbf{v}_+^{\otimes m}$ extends to a well defined surjective homomorphism $W(m\omega) \rightarrow S^m(L(V))$ of left \mathbf{U}^e -modules and right $\mathcal{P}_m^{\Sigma_m}$ -modules.*

Proof. It follows by Proposition 2.1(i) that there exists a \mathbf{U}^e -module homomorphism $\phi : W(m\omega) \rightarrow S^m(L(V))$ that takes $w_{m\omega}$ to $\mathbf{v} = v_+^{\otimes m}$. It is trivial to check that ϕ is also a map of right $\mathcal{P}_m^{\Sigma_m}$ -modules. To show that ϕ is surjective, it is enough to prove that

$$(6.1) \quad S^m(L(V)) = \mathbf{U}^e \cdot \mathbf{v}.$$

We prove by induction on r that

$$(6.2) \quad \mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+1}, \dots, l_m)} \in \mathbf{U}^e \cdot \mathbf{v}$$

for all $l_1, \dots, l_m \in \mathbf{Z}$. Consider the case $r = 0$. For any $k_1, k_2, \dots, k_m \in \mathbf{Z}$, we have

$$(x_0^+)^m x_{k_1}^- x_{k_2}^- \cdots x_{k_m}^- \cdot \mathbf{v} = \sum_{\sigma \in \Sigma_m} v_+ t^{k_{\sigma(1)}} \otimes v_+ t^{k_{\sigma(2)}} \otimes \cdots \otimes v_+ t^{k_{\sigma(m)}},$$

which proves (6.2) in this case. The case $r = m$ can be done similarly, since the element $\mathbf{v}^- = v_-^{\otimes m} = \frac{1}{m!} (x^-)^m \cdot \mathbf{v} \in \mathbf{U}^e \cdot \mathbf{v}$.

Assuming the result for r , we prove it for $r + 1$. For this we shall proceed by an induction on

$$N = \#\{j : j \geq r + 1, l_j \neq 0\}.$$

Now,

$$x_k^- \cdot \mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+1}, \dots, l_m)} = \sum_{s=r+1}^m \mathbf{v}_{(l_1, l_2, \dots, l_r, l_s+k), (l_{r+1}, \dots, \hat{l}_s, \dots, l_m)}.$$

Taking $l_s = 0$ for all $s > r$, we get that

$$\mathbf{v}_{(l_1, l_2, \dots, l_r, k), (0, \dots, 0)} \in \mathbf{U}^e \cdot \mathbf{v},$$

for all $k \in \mathbf{Z}$, proving our assertion when $N = 0$. Assume the result for $N - 1$. We have to show that

$$\mathbf{v}_{(l_1, l_2, \dots, l_r, l_{r+1}), (l_{r+2}, \dots, l_{r+N+1}, 0, \dots, 0)} \in \mathbf{U}^e \cdot \mathbf{v}.$$

Now

$$x_{l_{r+1}}^- \cdot \mathbf{v}_{(l_1, l_2, \dots, l_r), (l_{r+2}, \dots, l_{r+N+1}, 0, \dots, 0)}$$

is in $\mathbf{U}^e \cdot \mathbf{v}$ by the induction hypothesis on r , and is a sum of the term

$$(m - r) \mathbf{v}_{(l_1, \dots, l_{r+1}), (l_{r+2}, \dots, l_{r+N+1}, 0, \dots, 0)}$$

and terms of type

$$\mathbf{v}_{(l_1, \dots, l_r, l_s+l_{r+1}), (l_{r+2}, \dots, \hat{l}_s, \dots, l_{r+N+1}, 0, \dots, 0)},$$

for $r + 2 \leq s \leq r + N + 1$. Since, by the induction hypothesis on N , all the terms of the second type are in $\mathbf{U}^e \cdot \mathbf{v}$, it follows that

$$\mathbf{v}_{(l_1, \dots, l_{r+1}), (l_{r+2}, \dots, l_{r+N+1}, 0, \dots, 0)} \in \mathbf{U}^e \cdot \mathbf{v}.$$

This completes the proof that ϕ is surjective. \square

Proof of Theorem 4. Let K be the kernel of the homomorphism $W(m\omega) \rightarrow S^m(L(V))$ given by Lemma 6.3. Since $S^m(L(V))$ is a free, hence projective, right $\mathcal{P}_m^{\Sigma_m}$ -module by Lemma 6.2, it follows that

$$W(m\omega) = S^m(L(V)) \oplus K,$$

as right $\mathcal{P}_m^{\Sigma_m}$ -modules.

Let \mathfrak{m} be any maximal ideal in $\mathcal{P}_m^{\Sigma_m}$. Identifying $\mathcal{P}_m^{\Sigma_m}$ with $\mathbf{U}(0)/I_{m\omega}(0)$ as described earlier in this section, it is clear that

$$\mathfrak{m} = I_\pi(0)/I_{m\omega}(0),$$

for some polynomial π with constant term 1 such that $\deg \pi = m$. It follows that

$$W(m\omega)/W(m\omega)\mathfrak{m} \cong W(\pi)$$

as vector spaces over \mathbf{C} , and hence has dimension 2^m . On the other hand, by Lemma 6.2, $S^m(L(V))/S^m(L(V))\mathfrak{m}$ also has dimension 2^m over \mathbf{C} . It follows that $K/K\mathfrak{m} = 0$. Since this holds for all maximal ideals \mathfrak{m} , Nakayama's lemma implies that $K = 0$, proving the theorem. \square

The rest of the section is devoted to proving Theorem 5. First, observe that, in view of Theorem 2, it suffices to consider the case when $\pi(u) = (1 - au)^m$ for some $a \in \mathbf{C}^\times$. Since we have a surjective map $W(\pi) \rightarrow \overline{V_q(\pi)}$, it suffices to prove that

$$(6.3) \quad \dim_{\mathbf{C}} W(\pi) \leq \dim_{\mathbf{C}} \overline{V_q(\pi)} = 2^m.$$

For $a \in \mathbf{C}^\times$, let $\tau_a : \mathfrak{g} \otimes \mathbf{C}[t] \rightarrow \mathfrak{g} \otimes \mathbf{C}[t]$ be the Lie algebra automorphism obtained by extending the assignment

$$x \otimes t^k \rightarrow x \otimes (t - a)^k, \quad \forall x \in \mathfrak{g}, k \geq 0.$$

Set

$$X_a^-(u) = \tau_a(X_{\alpha,0}^-(u)), \quad \Lambda_a^+(u) = \tau_a(\Lambda_\alpha^+(u)) = \exp\left(-\sum_{k=1}^{\infty} \frac{h \otimes (t-a)^k}{k} u^k\right).$$

It is easy to see, using the relation between the Λ_m and h_m , that

$$(6.4) \quad h_k \cdot w\pi = ma^k w\pi,$$

or equivalently that

$$h \otimes (t-a)^k \cdot w\pi = 0,$$

for all $k \geq 0$. It follows that

$$(6.5) \quad (\Lambda_a^+)_k \cdot w\pi = 0, \quad \forall k > 0.$$

Further, using Lemma 1.3(ii) and observing that the identity there is actually an identity in $\mathbf{U}(\mathfrak{g} \otimes \mathbf{C}[t])$, we get by applying τ_a that

$$\tau_a(x_1^+)^{(r)}(x_0^-)^{(s)} = (-1)^r \left(X_a^-(u)^{(s-r)} \Lambda_a^+(u) \right)_s \quad \text{mod } \mathbf{U}\mathbf{U}(>)_+,$$

for $s \geq r \geq 1$. Together with (6.5), it follows that

$$(6.6) \quad \left(X_a^-(u)^{(s-r)}\right)_s \cdot w\boldsymbol{\pi} = 0 \quad \forall r \geq 1, s \geq m + 1.$$

In particular, this means that

$$(x^- \otimes (t - a)^s) \cdot w\boldsymbol{\pi} = 0 \quad \forall s \geq m.$$

Let $\mathbf{U}_a^+(\langle \rangle)$ be the commutative subalgebra of \mathbf{U} generated by the elements $\tau_a(x_k^-)$ for all $k \geq 0$, and let $I_a(m)$ be the ideal in $\mathbf{U}_a^+(\langle \rangle)$ generated by the elements $\left(X_a^-(u)^{(s-r)}\right)_s$ for all $r \geq 1, s \geq m + 1$.

Lemma 6.4. *The assignment $u \rightarrow u \cdot w\boldsymbol{\pi}$ induces a surjective map of vector spaces $\mathbf{U}_a^+(\langle \rangle)/I_a(m) \rightarrow W(\boldsymbol{\pi})$.*

Proof. The map is well defined by (6.6). It is obviously surjective because the polynomials $(t - a)^k$ for $k \geq 0$ are a basis of $\mathbf{C}[t]$. □

Thus, to prove (6.3) it suffices to show that the dimension of $\mathbf{U}_a^+(\langle \rangle)/I_a(m)$ is at most 2^m . It is convenient to reformulate the problem as follows.

Let $R_m = \mathbf{C}[z_0, \dots, z_{m-1}]$ be the polynomial algebra in m variables. For $0 \leq j < m$, set

$$Z_j(u) = \sum_{i=j}^{m-1} z_i u^{i-j+1} \in R_m[u].$$

Let J_m be the ideal in R_m generated by the elements $(Z_0(u)^r)_s$, for $r \geq 1, s \geq m + 1$. It is trivial to see that

$$R_m/J_m \cong \mathbf{U}_a^+(\langle \rangle)/I_a(m),$$

via the map $\tau(x_k^-) \rightarrow z_k$. It is clear that (6.3) is now a consequence of the following proposition and Lemma 6.4.

Proposition 6.1. *For $m \geq r > 0$, let*

$$\mathcal{B}_{m,r} = \{z_{i_1} z_{i_2} \cdots z_{i_r} : 0 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq m - r\}.$$

Let $\mathcal{B}_{m,0} = \{1\}$. The set

$$\mathcal{B}_m = \bigcup_{r=0}^m \mathcal{B}_{m,r}$$

spans R_m/J_m .

We prove Proposition 6.1 by induction on m . The case $m = 1$ is trivial, but for the inductive step, we need the following lemmas.

Set $J_m = J_{m,0}$ and, for $0 < j < m$, define ideals $J_{m,j}$ in R_m inductively by

$$J_{m,j} = J_{m,j-1} + \sum_{r=1}^j R_m (Z_1(u)^r)_{m-j} = J_m + \sum_{s \geq m-j} \sum_{1 \leq r \leq m-s} R_m (Z_1(u)^r)_s.$$

Lemma 6.5. *If $j \geq 1$, there is a unique homomorphism of algebras*

$$R_{m-j}/J_{m-j} \rightarrow R_m/J_{m,j-1}$$

such that $z_i \mapsto z_{i+1}$ for $0 \leq i < m - j$.

Proof. To establish the lemma, we must prove that

$$(6.7) \quad \left(\left(\sum_{i=1}^{m-j} z_i u^i \right)^r \right)_s \in J_{m,j-1},$$

for all $r \geq 1$ and $s \geq m - j + 1$.

We proceed by induction on j . For $j = 1$, we must show that

$$(Z_1(u)^r)_s \in J_m \quad \forall \quad s \geq m.$$

If $r = 1$, this is trivial from the definition of $Z_1(u)$. Assume the result for smaller values of r . Writing

$$Z_0(u) = u(z_0 + Z_1(u)),$$

we have

$$(Z_0(u)^r)_s = \left(\sum_{t=0}^r \binom{r}{t} z_0^t (Z_1(u)^{r-t})_{s-r} \right).$$

Take $s = n + r$ with $n \geq m$. Then, $(Z_0(u)^r)_{n+r} \in J_m$ by the definition of J_m , so

$$\sum_{t=0}^r \binom{r}{t} z_0^t (Z_1(u)^{r-t})_n \in J_m.$$

But, if $t > 0$, then $(Z_1(u)^{r-t})_n \in J_m$ for all $n \geq m$ by the induction hypothesis on r , so $(Z_1(u)^r)_n \in J_m$ for all $n \geq m$, thus completing the induction on r , and establishing (6.7) when $j = 1$.

So now assume that we know (6.7) for $j - 1$. Write

$$\sum_{i=1}^{m-j+1} z_i u^i = z_{m-j+1} u^{m-j+1} + \sum_{i=1}^{m-j} z_i u^i.$$

By the induction hypothesis on j ,

$$\left(\left(\sum_{i=1}^{m-j+1} z_i u^i \right)^r \right)_s \in J_{m,j-2}$$

for all $r \geq 1$, $s \geq m - j + 2$. Since $z_{m-j+1} \in J_{m,j-1}$, we can conclude, by using the binomial expansion, that

$$\left(\left(\sum_{i=1}^{m-j} z_i u^i \right)^r \right)_s \in J_{m,j-1},$$

if $r \geq 1$, $s > m - j + 1$. Thus, it suffices to prove that

$$(6.8) \quad \left(\left(\sum_{i=1}^{m-j} z_i u^i \right)^r \right)_{m-j+1} \in J_{m,j-1}$$

for all $r \geq 1$. If $r \leq j - 1$, we have $(Z_1(u)^r)_{m-j+1} \in J_{m,j-1}$ by definition. Further, the elements $z_{m-j+1}, \dots, z_{m-1} \in J_{m,j-1}$. Thus, writing

$$Z_1(u) = \sum_{i=1}^{m-j} z_i u^i + \sum_{i=m-j+1}^{m-1} z_i u^i,$$

and using the binomial expansion, we see that (6.8) follows.

If $r > j - 1$, then $r + m - j + 1 > m$, so

$$(Z_0(u)^r)_{r+m-j+1} \in J_m \subset J_{m,j-1}$$

by the definition of J_m , we have

$$((z_0 + Z_1(u))^r)_{m-j+1} \in J_{m,j-1}.$$

Now, $Z_1(u)_{m-j+1} = z_{m-j+1} \in J_{m,j-1}$ by definition, and so using the binomial expansion again and an induction on r , we conclude that

$$(Z_1(u)^r)_{m-j+1} \in J_{m,j-1}.$$

But now the proof is completed as in the case $r < j$. □

The proof of the following lemma is elementary.

Lemma 6.6. *Let $r \geq k \geq 0$. Then, the matrix*

$$\begin{bmatrix} \binom{k+1}{1} & \binom{k+1}{2} & \cdots & \binom{k+1}{r-k+1} \\ \binom{k+2}{1} & \binom{k+2}{2} & \cdots & \binom{k+2}{r-k+1} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \binom{r+1}{1} & \binom{r+1}{2} & \cdots & \binom{r+1}{r-k+1} \end{bmatrix}$$

has determinant $\binom{r+1}{k}$ and hence is invertible. □

Lemma 6.7. *For $r \geq t > 0$, $0 \leq s \leq r$, the element $z_0^{r-s}(Z_1(u)^{s+1})_{m-t}$ belongs to the span of*

$$\{z_0^{t-j}(Z_1(u)^{r-t+j+1})_{m-t} : 1 \leq j \leq t\}.$$

Proof. We assume that $m > 1$, otherwise there is nothing to prove. We consider the following equations in R_m/J_m :

$$z_0^{r-j}(Z_0(u)^{j+1})_{m+j+1-t} = 0, \quad 0 < t \leq j,$$

i.e.,

$$z_0^{r-j}((z_0 + Z_1(u))^{j+1})_{m-t} = 0, \quad 0 < t \leq j,$$

i.e.,

$$\sum_{i=0}^j \binom{j+1}{i} z_0^{r-j+i}(Z_1(u)^{j+1-i})_{m-t} = 0.$$

We must show that these equations, for $j = t, t + 1, \dots, r$, can be solved for the elements $z_0^s(Z_1(u)^{r+1-s})_{m-t}$ with $t \leq s \leq r$ in terms of those with $s < t$. But this follows from the preceding lemma. □

Proof of Proposition 6.1. The proposition is trivially true if $m = 1$. Assume now that we know the result for $m - 1$.

For $0 \leq k < r \leq m$, set

$$\mathcal{B}_{m,r,k} = \{z_{i_1} z_{i_2} \cdots z_{i_r} \in \mathcal{B}_{m,r} : i_{k+1} \geq 1\}.$$

The proposition obviously follows from

Claim. Let $r \geq k \geq 0$, and let $g \in R_m$ be a homogenous polynomial of degree $r - k$ in z_1, z_2, \dots, z_{m-1} . Then, $z_0^k g$ is in the span of $\mathcal{B}_{m,r,k}$ modulo J_m .

We proceed by induction on k . If $k = 0$, then by Lemma 6.5 we have a homomorphism $R_{m-1}/J_{m-1} \rightarrow R_m/J_m$ which sends $z_i \rightarrow z_{i+1}$. Clearly, g is in the image of this homomorphism and the induction hypothesis on m implies that $g \in \mathcal{B}_{m,r,0}$.

Assume the result for $k - 1$. Write

$$g = g_0 + g_1 z_{m-k} + g_2 z_{m-k+1} + \cdots + g_k z_{m-1},$$

where for $0 \leq j \leq k$, g_j is a polynomial in $z_1, z_2, \dots, z_{m-k+j-1}$. Now, for $j \geq 0$, we see by Lemma 6.7 that the element $z_0^k z_{m-k+j}$ is in the span of the sets $\mathcal{B}_{m,k+1,s}$ with $s < k$. Thus, the element $z_0^k z_{m-k+j} g_{j+1}$ can be written as a sum $\sum_{s < k} z_0^s h_{sj}$, where the h_{sj} are polynomials in z_1, \dots, z_{m-1} . Hence, by the induction on k ,

$$z_0^k z_{m-k+j} g_{j+1} \in \mathcal{B}_{m,r,k},$$

for $j \geq 0$. Finally observe that by Lemma 6.5, g_0 is in the image of the map $R_{m-k-1}/J_{m-k+j-1} \rightarrow R_m/J_m$ and hence, by using the induction on m , we get that

$$g_0 \in \text{span}(\mathcal{B}_{m,r-k,0}) \pmod{J_{m,k+1}}.$$

Thus, $z_0^k g_0$ is in the span of $\mathcal{B}_{m,r,k}$ provided that $z_0^k J_{m,k}$ is also in the span of $\mathcal{B}_{m,r,k}$, i.e if $z_0^k (Z_1(u)^r)_{m-k-1}$ is in the span of $\mathcal{B}_{m,r,k}$ for all $s \geq m - k$, $1 \leq r \leq m - s$. But this follows from Lemma 6.7 again, and the proof of the proposition is complete. \square

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