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RELATIVE KAZHDAN–LUSZTIG CELLS

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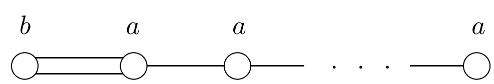
Abstract. In this paper, we study the Kazhdan–Lusztig cells of a Coxeter group W in a "relative" setting, with respect to a parabolic subgroup $W_I \subseteq W$. This relies on a factorization of the Kazhdan–Lusztig basis $\{\mathbf{C}_w\}$ of the corresponding (multi-parameter) Iwahori–Hecke algebra with respect to W_I. We obtain two applications to the "asymptotic case" in type B_n, as introduced by Bonnafé and Iancu: we show that $\{\mathbf{C}_w\}$ is a "cellular basis" in the sense of Graham and Lehrer, and we construct an analogue of Lusztig's canonical isomorphism from the Iwahori–Hecke algebra to the group algebra of the underlying Weyl group of type B_n.

1. Introduction

Let W be a Coxeter group and $L\colon W \to \mathbb{Z}_{\geqslant 0}$ a weight function, in the sense of Lusztig [18]. This gives rise to various pre-order relations on W, usually denoted by $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$. Let $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ be the corresponding equivalence relations. The equivalence classes are called the left, right and two-sided cells of W, respectively. They were first defined by Kazhdan and Lusztig [13] in the case where L is the length function on W (the "equal parameter case"), and by Lusztig [15] in general. They play a fundamental role, for example, in the representation theory of finite or p-adic groups of Lie type; see Lusztig [16], [17] and the survey in [18, Chap. 0].

The definition of the above relations relies on the construction of the Kazhdan–Lusztig basis $\{\mathbf{C}_w \mid w \in W\}$ in the associated Iwahori–Hecke algebra $\mathcal{H}$. This paper arose from an attempt to show that the basis $\{\mathbf{C}_w\}$ is a "cellular basis" in the sense of Graham and Lehrer [12], in the case where $W = W_n$ is of type B_n with diagram and weight function given by

B_n 

where a, b are positive integers such that b/a is "large" with respect to n. This is the "asymptotic case" studied by Bonnafé and Iancu [3].

After a number of intermediate results, this goal will be achieved in Section 6. Those intermediate results concern properties of left, right and two-sided cells which are important in their own right. In fact, combining the results in this paper with the results of Bonnafé and Iancu [3], Bonnafé [4], and Geck and Iancu [10], we have that **(P1)**–**(P14)** from Lusztig's list of conjectures in [18, Chap. 14], as well as a weak version of **(P15)**, hold in the "asymptotic case" in type B_n. The weak version

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of **(P15)** is sufficient, for example, to establish the existence of an analogue of Lusztig's canonical isomorphism [14] for the two-parameter Iwahori–Hecke algebra of type B_n. (These things will be discussed at the end of this paper, in Section 7.)

The main and unifying idea of this paper is to combine the existing theory (due to Lusztig in general, and to Bonnafé and Iancu as far as type B_n is concerned) with a detailed analysis of the decomposition of the Kazhdan–Lusztig basis of a Coxeter group with respect to a parabolic subgroup, based on the author's article [8].

Here is the first property that we consider in this paper. It has been conjectured by Lusztig [18, 14.2] that we always have the following implication for elements x, y in a Coxeter group W:

$$(\spadesuit) \qquad x \leqslant_{\mathcal{L}} y \quad \text{and} \quad x \sim_{\mathcal{LR}} y \quad \Rightarrow \quad x \sim_{\mathcal{L}} y.$$

This is known to hold in the equal parameter case when W is a finite or affine Weyl group[1]; see Lusztig [17]. However, although all the notions involved in the above statement are completely elementary, the proof is surprisingly complicated: it relies on a geometric interpretation of the Kazhdan–Lusztig basis of $\mathcal{H}$ and some deep results from algebraic geometry; see Springer [20] and Lusztig [17]. A somewhat different proof is given by Lusztig [14] for finite Weyl groups (relying on the connection between cells and primitive ideals in universal enveloping algebras via the main conjecture in [13]); in that article, ($\spadesuit$) is used to construct a canonical isomorphism from $\mathcal{H}$ to the group algebra of W. The property ($\spadesuit$) also plays an important role in Lusztig's study [16] of representations of reductive groups over finite fields.

In Section 4, we develop the formulation of a relative version of ($\spadesuit$), taking into account the presence of a parabolic subgroup $W_I \subseteq W$. (The original version of ($\spadesuit$) corresponds to the case where $W_I = W$.) The tools for dealing with this relative setting are provided by [8]; we recall the basic ingredients, with some refinements, in Section 3. We conjecture that the relative version of ($\spadesuit$) holds for all W, L and all choices of $W_I \subseteq W$. In Section 4, we prove our conjecture for finite and for affine Weyl groups in the equal parameter case. The method is inspired by Lusztig's proof of ($\spadesuit$) in [18, Chap. 15]. The additional complication arising from the presence of W_I is dealt with by Lemma 4.7, which reduces to a triviality if $W_I = W$.

A priori, we do not have any geometric interpretation of the Kazhdan–Lusztig basis in the general case of unequal parameters. (Note, however, that there is a conjectural geometrical interpretation by Lusztig [18, Chap. 27] for certain values of the parameters.) So the above methods and results will not apply in type B_n with parameters as specified as above. In Theorem 5.13, we do prove ($\spadesuit$) in this case, by reduction to the relative version of ($\spadesuit$) for the symmetric group $\mathfrak{S}_n$. Thus, eventually, the proof of ($\spadesuit$) in type B_n also rests on the geometric interpretation of the Kazhdan–Lusztig basis for $\mathfrak{S}_n$. The proof of that reduction argument occupies almost all of Section 5; this relies once more on the results in [8], and on the results of Bonnafé and Iancu [3] and Bonnafé [4] on the left cells and two-sided cells, respectively. At one point in the proof, we also use an idea of Dipper, James, and Murphy [6] to deal with the action of the generator with parameter b in the above diagram.

[1] Other situations where ($\spadesuit$) is known to hold include the quasi-split case discussed in [18, Chap. 16] (which is derived from the equal parameter case), and explicitly worked examples like the infinite dihedral group in [18, Chap. 17] or type F_4 in [9].

In Section 6, we go on to study the representations carried by the left cells in the "asymptotic case" in type B_n. The main result, Theorem 6.3, shows that two left cells which afford the same character actually give rise to exactly the same representation (and not only equivalent ones). Again, the proof relies on the techniques in [8], concerning the "induction" of cells. An analogous result for the left cell representations of the symmetric group has already been obtained by Kazhdan and Lusztig in their original article [13] where they introduced left cells and the corresponding representations.

Combining the main results of Bonnafé and Iancu [3] and Bonnafé [4] with Theorem 5.13 and Theorem 6.3 in this paper, we immediately get that $\{\mathbf{C}_w\}$ is a "cellular basis" in the "asymptotic case" in type B_n; see Corollary 6.4.

As a further application of our results, we can exhibit a new basis in the Iwahori–Hecke algebra of type B_n whose structure constants are integers. (In fact, the structure constants are 0, 1.) This uses an idea of Neunhöffer [19] concerning an explicit Wedderburn decomposition in terms of the Kazhdan–Lusztig basis. We show that the subring generated by that basis is nothing but Lusztig's ring J; we also obtain an analogue of Lusztig's homomorphism from the Iwahori–Hecke algebra into J; see Section 7. As an application, this gives rise to a "canonical" homomorphism from the generic Iwahori–Hecke algebra of type B_n into the group algebra of the underlying Weyl group. An explicit example is worked out in Example 7.9. In the equal parameter case, such a homomorphism was first constructed by Lusztig [14].

We close this introduction with the remark that the results in Sections 2–4 hold for general Coxeter groups and may be of independent interest. The applications to type B_n, to be found in Section 5–7, depend on the two articles by Bonnafé and Iancu [3] and Bonnafé [4] (where the left cells and the two-sided cells are determined), but are otherwise self-contained.

2. The basic set-up

We begin by recalling the basic definitions concerning Kazhdan–Lusztig cells in the general multi-parameter case. Let W be a Coxeter group, with generating set S. (We assume that S is a finite set, but the group W may be finite or infinite.) In [18], the parameters of the corresponding Iwahori–Hecke algebra are specified by an integer-valued weight function. Following a suggestion of Bonnafé [4], we can slightly modify Lusztig's definition so as to include the more general setting in [15] as well (where the parameters may be contained in a totally ordered abelian group). So let Γ be an abelian group (written additively) and assume that there is a total order $\leqslant$ on Γ compatible with the group structure. (In the setting of [18], we take $\Gamma = \mathbb{Z}$ with the natural order.)

Let $A = \mathbb{Z}[\Gamma]$ be the free abelian group with basis $\{e^\gamma \mid \gamma \in \Gamma\}$. There is a well-defined ring structure on A such that $e^\gamma e^{\gamma'} = e^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. (Hence, if $\Gamma = \mathbb{Z}$, then A is nothing but the ring of Laurent polynomials in an indeterminate e.) We write $1 = e^0 \in A$. Given $a \in A$ we denote by a_γ the coefficient of e^γ, so that $a = \sum_{\gamma\in\Gamma} a_\gamma e^\gamma$. We let $A_{\geqslant 0} := \langle e^\gamma \mid \gamma \geqslant 0\rangle_{\mathbb{Z}}$; similarly, we define $A_{>0}$, $A_{\leqslant 0}$ and $A_{<0}$. We say that a function

$$L\colon W \to \Gamma$$

is a weight function if $L(ww') = L(w) + L(w')$ whenever we have $l(ww') = l(w) + l(w')$ where $l\colon W \to \mathbb{N}$ is the usual length function. (We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$.) We assume throughout that $L(s) > 0$ for all $s \in S$. Let $\mathcal{H} = \mathcal{H}(W, S, L)$ be the generic Iwahori–Hecke algebra over A with parameters $\{q_s \mid s \in S\}$ where $q_s := e^{L(s)}$ for $s \in S$. The algebra $\mathcal{H}$ is free over A with basis $\{T_w \mid w \in W\}$, and the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (q_s - q_s^{-1}) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. (Note that the above elements T_w are denoted $\tilde{T}_w$ in [15].)

For any $a \in A$, we define $\bar{a} := \sum_{\gamma \in \Gamma} a_\gamma e^{-\gamma}$. We extend the map $a \mapsto \bar{a}$ to a ring involution $\mathcal{H} \to \mathcal{H}$, $h \mapsto \overline{h}$, by the formula

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1} \qquad (a_w \in A).$$

Now we have a corresponding *Kazhdan–Lusztig basis* of $\mathcal{H}$, which we denote by $\{\mathbf{C}_w \mid w \in W\}$[2]. The basis element $\mathbf{C}_w$ is uniquely determined by the conditions that

$$\overline{\mathbf{C}}_w = \mathbf{C}_w \qquad \text{and} \qquad \mathbf{C}_w \equiv T_w \mod \mathcal{H}_{<0},$$

where $\mathcal{H}_{<0} := \sum_{w \in W} A_{<0}\, T_w$; see [15, Prop. 2] and [18, Theorem 5.2].

2.1. **Multiplication rules.** For any $x, y \in W$, we write

$$\mathbf{C}_x\, \mathbf{C}_y = \sum_{z \in W} h_{x,y,z}\, \mathbf{C}_z \qquad \text{where } h_{x,y,z} \in A \text{ for all } x, y, z \in W.$$

An easy induction on $l(x)$ shows that $T_x T_y$ is a linear combination of basis elements T_z where $l(z) \leqslant l(x) + l(y)$. This also implies that

$$h_{x,y,z} \neq 0 \qquad \Rightarrow \qquad l(z) \leqslant l(x) + l(y).$$

We have the following more explicit formula for $s \in S$, $y \in W$ (see [15, §6]):

$$\mathbf{C}_s\, \mathbf{C}_y = \begin{cases} \mathbf{C}_{sy} + \displaystyle\sum_{\substack{z \in W \\ sz < z < y}} M_{z,y}^s \mathbf{C}_z & \text{if } sy > y, \\ (q_s + q_s^{-1})\, \mathbf{C}_y & \text{if } sy < y, \end{cases}$$

where $M_{z,y}^s = \overline{M}_{z,y}^s \in A$ is determined as in [15, §3] and $\leqslant$ denotes the Bruhat–Chevalley order. In particular, we have

$$h_{s,y,z} \neq 0 \quad \Rightarrow \quad z = y > sy \quad \text{or} \quad z = sy > y \quad \text{or} \quad sz < z < y < sy.$$

2.2. **The Kazhdan–Lusztig pre-orders.** As in [18, §8], we write $x \leftarrow_{\mathcal{L}} y$ if there exists some $s \in S$ such that $h_{s,y,x} \neq 0$, that is, $\mathbf{C}_x$ occurs in $\mathbf{C}_s\, \mathbf{C}_y$ (when expressed in the $\mathbf{C}$-basis). The Kazhdan–Lusztig left pre-order $\leqslant_{\mathcal{L}}$ is the relation on W generated by $\leftarrow_{\mathcal{L}}$, that is, we have $x \leqslant_{\mathcal{L}} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ of elements in W such that $x_{i-1} \leftarrow_{\mathcal{L}} x_i$ for all i. The equivalence relation associated with $\leqslant_{\mathcal{L}}$ will be denoted by $\sim_{\mathcal{L}}$ and the corresponding equivalence classes are called the *left cells* of W.

Similarly, we can define a pre-order $\leqslant_{\mathcal{R}}$ by considering multiplication by $\mathbf{C}_s$ on the right in the defining relation. The equivalence relation associated with $\leqslant_{\mathcal{R}}$ will be denoted by $\sim_{\mathcal{R}}$ and the corresponding equivalence classes are called the *right cells* of W. We have

$$x \leqslant_{\mathcal{R}} y \quad \Leftrightarrow \quad x^{-1} \leqslant_{\mathcal{L}} y^{-1}.$$

[2]Note that this basis is denoted by C'_w in [15] and by c_w in [18].

This follows by using the antiautomorphism $\flat\colon \mathcal{H} \to \mathcal{H}$ given by $T_w^\flat = T_{w^{-1}}$; we have $\mathbf{C}_w^\flat = \mathbf{C}_{w^{-1}}$ for all $w \in W$; see [18, 5.6]. Thus, any statement concerning the left pre-order relation $\leqslant_{\mathcal{L}}$ has an equivalent version for the right pre-order relation $\leqslant_{\mathcal{R}}$, via $\flat$. Finally, we define a pre-order $\leqslant_{\mathcal{LR}}$ by the condition that $x \leqslant_{\mathcal{LR}} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leqslant_{\mathcal{L}} x_i$ or $x_{i-1} \leqslant_{\mathcal{R}} x_i$. The equivalence relation associated with $\leqslant_{\mathcal{LR}}$ will be denoted by $\sim_{\mathcal{LR}}$ and the corresponding equivalence classes are called the *two-sided cells* of W.

2.3. **Left cell representations.** Let $\mathfrak{C}$ be a left cell or, more generally, a union of left cells of W. We define an $\mathcal{H}$-module by $[\mathfrak{C}]_A := \mathfrak{I}_{\mathfrak{C}}/\hat{\mathfrak{I}}_{\mathfrak{C}}$, where

$$\mathfrak{I}_{\mathfrak{C}} := \langle \mathbf{C}_w \mid w \leqslant_{\mathcal{L}} z \text{ for some } z \in \mathfrak{C}\rangle_A,$$
$$\hat{\mathfrak{I}}_{\mathfrak{C}} := \langle \mathbf{C}_w \mid w \notin \mathfrak{C}, w \leqslant_{\mathcal{L}} z \text{ for some } z \in \mathfrak{C}\rangle_A.$$

Note that, by the definition of the pre-order relation $\leqslant_{\mathcal{L}}$, these are left ideals in $\mathcal{H}$. Now denote by c_x ($x \in \mathfrak{C}$) the residue class of $\mathbf{C}_x$ in $[\mathfrak{C}]_A$. Then the elements $\{c_x \mid x \in \mathfrak{C}\}$ form an A-basis of $[\mathfrak{C}]_A$ and the action of $\mathbf{C}_w$ ($w \in W$) is given by the formula

$$\mathbf{C}_w.c_x = \sum_{y \in \mathfrak{C}} h_{w,x,y}\, c_y.$$

Assume now that $\mathfrak{C}$ is a finite set and write $\mathfrak{C} = \{x_1, \ldots, x_d\}$. Let $\{c_1, \ldots, c_d\}$ be the corresponding standard basis of $[\mathfrak{C}]_A$, where $c_i = c_{x_i}$ for all i. Then we obtain a matrix representation

$$\mathfrak{X}_{\mathfrak{C}}\colon \mathcal{H} \to M_d(A) \quad \text{where} \quad \mathfrak{X}_{\mathfrak{C}}(\mathbf{C}_w) = \big(h_{w,x_j,x_i}\big)_{1 \leqslant i,j \leqslant d}$$

for any $w \in W$. Thus, h_{w,x_j,x_i} is the (i,j)-coefficient of the matrix $\mathfrak{X}_{\mathfrak{C}}(\mathbf{C}_w)$.

Although this will not play a role in this paper, we mention that, for various reasons, it is sometimes more convenient[3] to twist the action of $\mathcal{H}$ on $[\mathfrak{C}]_A$ by the A-algebra automorphism

$$\delta\colon \mathcal{H} \to \mathcal{H}, \qquad T_s \mapsto -T_s^{-1} \quad (s \in S).$$

We shall often write h^δ instead of $\delta(h)$. As in [18, 21.1], we define a new $\mathcal{H}$-module by taking the same underlying A-module as before, but where the action is given by the formulas

$$\mathbf{C}_w^\delta.c_x = \sum_{y \in \mathfrak{C}} h_{w,x,y}\, c_y \qquad (w \in W, x \in \mathfrak{C}).$$

We denote this new $\mathcal{H}$-module by $[\mathfrak{C}]_A^\delta$. It is readily checked that $[\mathfrak{C}]_A^\delta = \delta(\mathfrak{I}_{\mathfrak{C}})/\delta(\hat{\mathfrak{I}}_{\mathfrak{C}})$.

Remark 2.4. We have a unique ring involution $j\colon \mathcal{H} \to \mathcal{H}$ such that $j(e^\gamma) = e^{-\gamma}$ for $\gamma \in \Gamma$ and $j(T_w) = (-1)^{l(w)}T_w$ for $w \in W$. Then j commutes with δ and the composition $j \circ \delta$ is nothing but the involution $h \mapsto \overline{h}$ on $\mathcal{H}$; see [15, §6]. Thus, we have

$$\delta(\mathbf{C}_w) = \delta(\overline{\mathbf{C}}_w) = j(\mathbf{C}_w) \qquad \text{for any } w \in W.$$

[3] Here is a simple example to illustrate this point: Let $\mathfrak{C} = \{1\}$ be the left cell consisting of the identity element of W. Then $[\mathfrak{C}]_A$ affords the representation $T_s \mapsto -q_s^{-1}$ ($s \in S$) and $[\mathfrak{C}]_A^\delta$ affords the representation $T_s \mapsto q_s$ ($s \in S$). Specializing $q_s \mapsto 1$, we obtain the sign and the unit representation of W, respectively. It is sometimes more natural to associate the unit representation with the left cell $\{1\}$; so one should work with $[\mathfrak{C}]_A^\delta$ in this case. Especially, this can be seen in [18, Chap. 21] where Lusztig works with $[\mathfrak{C}]_A^\delta$ throughout.

This observation can be used to obtain formulas for $\delta(h)$ ($h \in \mathcal{H}$) which would be difficult to compute using the definition of δ. For example, we obtain

$$\delta(\mathbf{C}_w) = j(\mathbf{C}_w) = (-1)^{l(w)} T_w + \sum_{\substack{y \in W \\ y < w}} (-1)^{l(y)}\, \overline{P}^{\,*}_{y,w}\, T_y$$

for any $w \in W$.

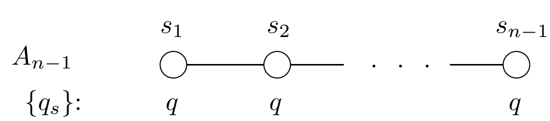
We shall be interested in the following property.

Definition 2.5. Let $\mathfrak{C}$ and $\mathfrak{C}_1$ be left cells or, more generally, be unions of left cells of W. We write $\mathfrak{C} \approx \mathfrak{C}_1$, if there exists a bijection $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, such that the following condition is satisfied:

$$(\heartsuit) \qquad h_{w,x,y} = h_{w,x_1,y_1} \qquad \text{for all } w \in W \text{ and all } x, y \in \mathfrak{C}.$$

This means that the $\mathcal{H}$-modules $[\mathfrak{C}]_A$ and $[\mathfrak{C}_1]_A$ are not only isomorphic, but even the action of any $\mathbf{C}_w$ ($w \in W$) is given by exactly the same formulas with respect to the standard bases of $[\mathfrak{C}]_A$ and $[\mathfrak{C}]_A$, respectively. A similar remark applies, of course, to the $\mathcal{H}$-modules $[\mathfrak{C}]_A^\delta$ and $[\mathfrak{C}_1]_A^\delta$. Note that, in order to verify that ($\heartsuit$) holds, it is enough to consider the case where $w = s \in S$ (since the elements $\mathbf{C}_s$, $s \in S$, generate $\mathcal{H}$ as an A-algebra).

Example 2.6. Let $W = \mathfrak{S}_n$ be the symmetric group, with generating set $S = \{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. Let $\Gamma = \mathbb{Z}$ with its natural order, and set $q := e^1$. Then $A = \mathbb{Z}[\Gamma] = \mathbb{Z}[q, q^{-1}]$ is the ring of Laurent polynomials in an indeterminate q. Let $L\colon \mathfrak{S}_n \to \mathbb{Z}$ be the weight function given by $L(s_i) = 1$ for $1 \leqslant i \leqslant n-1$, and denote by $H(\mathfrak{S}_n)$ the corresponding Iwahori–Hecke algebra over A. Thus, we have the following diagram specifying the generators, relations and parameters:



The classical Robinson–Schensted correspondence associates with each element $\sigma \in \mathfrak{S}_n$ a pair of standard tableaux $(A(\sigma), B(\sigma))$ of the same shape. For any partition ν of n, we set

$$\mathfrak{R}_\nu := \{\sigma \in \mathfrak{S}_n \mid \ A(\sigma),\, B(\sigma) \text{ have shape } \nu\}.$$

Thus, we have $\mathfrak{S}_n = \coprod_\nu \mathfrak{R}_\nu$ where ν runs over all partitions of n. Then the following hold.

(a) *For a fixed standard tableau T, the set $\{\sigma \in \mathfrak{S}_n \mid B(\sigma) = T\}$ is a left cell of $\mathfrak{S}_n$ and $\{\sigma \in \mathfrak{S}_n \mid A(\sigma) = T\}$ is a right cell of $\mathfrak{S}_n$. Furthermore, all left cells and all right cells arise in this way.*

(b) *Let $\mathfrak{C}, \mathfrak{C}_1$ be left cells and assume that $\mathfrak{C} \subseteq \mathfrak{R}_\nu$, $\mathfrak{C}_1 \subseteq \mathfrak{R}_{\nu_1}$. Then we have $\mathfrak{C} \approx \mathfrak{C}_1$ if and only if $\nu = \nu_1$. The required bijection from $\mathfrak{C}$ onto $\mathfrak{C}_1$ can be explicitly described in terms of the "star" operation defined in* [13, §4].

These statements were first proved by Kazhdan–Lusztig [13, §5]. (See also Ariki [1].) It is actually shown there that the bijection $x \mapsto x_1$ is determined by the condition that $x \in \mathfrak{C}$ and $x_1 \in \mathfrak{C}_1$ lie in the same right cell. In Proposition 2.13, we will see that this property automatically follows from some general principles.

Lemma 2.7. *Let $\varphi\colon W \to W$ be a group automorphism such that $\varphi(S) = S$ and $q_{\varphi(s)} = q_s$ for all $s \in S$. Let $\mathfrak{C}, \mathfrak{C}_1$ be left cells of W. Then $\varphi(\mathfrak{C}), \varphi(\mathfrak{C}_1)$ are left cells and we have $\mathfrak{C} \approx \mathfrak{C}_1$ if and only if $\varphi(\mathfrak{C}) \approx \varphi(\mathfrak{C}_1)$.*

Proof. Our assumptions imply that φ induces an A-algebra automorphism $\tilde{\varphi}\colon \mathcal{H} \to \mathcal{H}$ such that $\tilde{\varphi}(T_w) = T_{\varphi(w)}$ for all $w \in W$. It is readily checked that $\tilde{\varphi}$ commutes with the involution $h \mapsto \overline{h}$ of $\mathcal{H}$. This implies that

$$\tilde{\varphi}(\mathbf{C}_w) = \mathbf{C}_{\varphi(w)} \qquad \text{for all } w \in W.$$

Consequently, we also have $h_{x,y,z} = h_{\varphi(x),\varphi(y),\varphi(z)}$ for all $x, y, z \in W$. By the definition of left cells, this yields that φ preserves the partition of W into left cells and that we have $\mathfrak{C} \approx \mathfrak{C}_1$ if and only if $\varphi(\mathfrak{C}) \approx \varphi(\mathfrak{C}_1)$. □

Lemma 2.8. *Assume that W is finite and let $w_0 \in W$ be the unique element of maximal length. Let $\mathfrak{C}$ and $\mathfrak{C}_1$ be left cells such that $\mathfrak{C} \approx \mathfrak{C}_1$. Then we also have $\mathfrak{C}w_0 \approx \mathfrak{C}_1 w_0$ and $w_0\mathfrak{C} \approx w_0\mathfrak{C}_1$. (Note that $\mathfrak{C}w_0$, $\mathfrak{C}_1w_0$ and $w_0\mathfrak{C}$, $w_0\mathfrak{C}_1$ are left cells; see* [18, Cor. 11.7]*.)*

Proof. First we prove that $\mathfrak{C}w_0 \approx \mathfrak{C}_1w_0$. Let $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, be a bijection such that (♡) holds; see Definition 2.5. In particular, this means that $h_{s,x,y} = h_{s,x_1,y_1}$ for all $s \in S$ and $x, y \in \mathfrak{C}$.

Now recall the formula for multiplication by $\mathbf{C}_s$ from (2.1). That formula shows that, for any $s \in S$ and any $x \in \mathfrak{C}$, we have $sx < x$ if and only if $sx_1 < x_1$. Furthermore, by [18, Prop. 11.6], we have

$$M^s_{xw_0,yw_0} = -(-1)^{l(x)+l(y)} M^s_{y,x} \qquad \text{if } sy < y < x < sx.$$

Hence we obtain

$$h_{s,xw_0,yw_0} = h_{s,x_1w_0,y_1w_0} \quad \text{for all } s \in S \text{ and } x, y \in \mathfrak{C}.$$

Consequently, (♡) holds for the bijection $\mathfrak{C}w_0 \to \mathfrak{C}_1w_0$, $xw_0 \mapsto x_1w_0$. Now consider the group automorphism $\varphi\colon W \to W$ given by $\varphi(w) = w_0ww_0$. It is well-known that $\varphi(S) = S$. Furthermore, since $s \in S$ and $\varphi(s)$ are conjugate, we have $q_s = q_{\varphi(s)}$. Hence, Lemma 2.7 shows that $w_0\mathfrak{C}w_0$ and $w_0\mathfrak{C}_1w_0$ are left cells such that $w_0\mathfrak{C}w_0 \approx w_0\mathfrak{C}_1w_0$. Hence the previous argument shows that $w_0\mathfrak{C} = (w_0\mathfrak{C}w_0)w_0 \approx (w_0\mathfrak{C}_1w_0)w_0 = w_0\mathfrak{C}_1$. □

We close this section with some results which show that, under suitable hypotheses, a bijection $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, satisfying (♡) automatically respects the right cells of W. (These results will also play an important role in Section 7.) Let us assume throughout that W is a finite group. Since the group Γ is totally ordered, $A = \mathbb{Z}[\Gamma]$ is easily seen to be an integral domain. Let K be the field of fractions of $\mathbb{R}[\Gamma] \supseteq A$. By extension of scalars, we obtain a K-algebra $\mathcal{H}_K = K \otimes_A \mathcal{H}$.

Remark 2.9. The algebra $\mathcal{H}_K$ is split semisimple.

Proof. The fact that $\mathcal{H}_K$ is semisimple relies on two ingredients: first, $\mathbb{R}W$ (the group algebra of W over $\mathbb{R}$) is known to be split semisimple and, second, $\mathbb{R}[\Gamma] \otimes_A \mathcal{H}$ specializes to $\mathbb{R}W$, via the ring homomorphism $\theta\colon \mathbb{R}[\Gamma] \to \mathbb{R}$ such that $\theta(e^\gamma) = 1$ for all $\gamma \in \Gamma$. Then it remains to use known results on splitting fields; see [11, §9] and the references there. For more details, see [10, Remark 3.1]. □

Let $\mathrm{Irr}(\mathcal{H}_K)$ be the set of irreducible characters of $\mathcal{H}_K$. We write this set in the form

$$\mathrm{Irr}(\mathcal{H}_K) = \{\chi_\lambda \mid \lambda \in \Lambda\},$$

where Λ is some finite indexing set. The algebra $\mathcal{H}_K$ is *symmetric* with respect to the trace function $\tau\colon \mathcal{H}_K \to K$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$; see [11, §8.1]. The fact that $\mathcal{H}_K$ is split semisimple yields that

$$\tau = \sum_{\lambda \in \Lambda} \frac{1}{c_\lambda} \chi_\lambda \qquad \text{where } 0 \neq c_\lambda \in \mathbb{R}[\Gamma];$$

see [11, §7.2 and 9.3.5]. The elements c_λ are called the *Schur elements*.

For any $\lambda \in \Lambda$, let us denote by $\mathfrak{X}_\lambda\colon H_K \to M_{d_\lambda}(K)$ a matrix representation with character χ_λ. Let $\mathfrak{X}_\lambda^{ij}(h)$ denote the (i,j)-coefficient of $\mathfrak{X}_\lambda(h)$ for any $h \in \mathcal{H}_K$. By Wedderburn's theorem, the algebra $\mathcal{H}_K$ is abstractly isomorphic to the direct sum of the matrix algebras $M_{d_\lambda}(K)$ $(\lambda \in \Lambda)$. Since $\mathcal{H}_K$ is symmetric, this isomorphism can be described explicitly:

Proposition 2.10 (Explicit Wedderburn decomposition)**.** *Let $\mathcal{B}$ be any basis of $\mathcal{H}_K$ and $\mathcal{B}^\vee = \{b^\vee \mid b \in \mathcal{B}\}$ the dual basis with respect to τ. We set*

$$E_\lambda^{ij} = \frac{1}{c_\lambda} \sum_{b \in \mathcal{B}} \mathfrak{X}_\lambda^{ji}(b)\, b^\vee \qquad \text{for any } \lambda \in \Lambda,\ 1 \leqslant i,j \leqslant d_\lambda.$$

Then $\mathfrak{X}_\lambda(E_\lambda^{ij}) \in M_{d_\lambda}(K)$ is the matrix with (i,j)-coefficient 1 and coefficient 0 otherwise. Furthermore, if $\mu \neq \lambda$, we have $\mathfrak{X}_\mu^{kl}(E_\lambda^{ij}) = 0$ for all $1 \leqslant k,l \leqslant d_\mu$. In particular, the elements

$$\{E_\lambda^{ij} \mid \lambda \in \Lambda, 1 \leqslant i,j \leqslant d_\lambda\}$$

form a basis of $\mathcal{H}_K$.

(For a proof, see [11, Prop. 7.2.7], for example.)

We want to apply the above result to the Kazhdan–Lusztig basis $\mathcal{B} := \{\mathbf{C}_w \mid w \in W\}$. The dual basis can be described as follows. We set

$$\mathbf{D}_{z^{-1}} := (-1)^{l(z)+l(w_0)} \mathbf{C}_{zw_0}^\delta T_{w_0} \qquad \text{for any } z \in W.$$

where $w_0 \in W$ is the unique element of maximal length. Then we have

$$\tau(\mathbf{C}_w \mathbf{D}_{z^{-1}}) = \begin{cases} 1 & \text{if } w = z, \\ 0 & \text{if } w \neq z; \end{cases}$$

see [18, Prop. 11.5]. Hence we have $\mathbf{C}_w^\vee = \mathbf{D}_{w^{-1}}$ for all $w \in W$. In particular, the structure constants of $\mathcal{H}$ can be expressed by

$$h_{x,y,z} = \tau(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_{z^{-1}}) \qquad \text{for all } x, y, z \in W.$$

This immediately yields that

$$\mathbf{C}_x \mathbf{D}_{y^{-1}} = \sum_{w \in W} h_{w,x,y}\, \mathbf{D}_{w^{-1}} \qquad \text{for any } x, y \in W.$$

The following two results were observed by Neunhöffer in his thesis [19, Kap. VI, §4]. For any left cell $\mathfrak{C}$, denote by $\chi_{\mathfrak{C}}$ the character afforded by the left cell module $[\mathfrak{C}]_K := K \otimes_A [\mathfrak{C}]_A$ of $\mathcal{H}_K$.

Lemma 2.11 (Neunhöffer). *Let $\mathfrak{C}$ be a left cell such that $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_K)$. Writing $\mathfrak{C} = \{x_1, \ldots, x_d\}$ and using the notation in* (2.3)*, we have*

$$E_\lambda^{ij} = \frac{1}{c_\lambda} \mathbf{C}_{x_i} \, \mathbf{D}_{x_j^{-1}} \qquad \textit{for } 1 \leqslant i, j \leqslant d,$$

where $\lambda \in \Lambda$ is such that $\chi_\lambda = \chi_{\mathfrak{C}}$ and where we take $\mathfrak{X}_\lambda = \mathfrak{X}_{\mathfrak{C}}$. In particular, we have $\mathfrak{X}_\lambda(\mathbf{C}_{x_i}\mathbf{D}_{x_j}) \neq 0$ and $\mathfrak{X}_\mu(\mathbf{C}_{x_i}\mathbf{D}_{x_j}) = 0$ for any $\mu \in \Lambda \setminus \{\lambda\}$.

Proof. Since Neunhöffer only considers the case of the symmetric group, we give a proof here. By the formula in Proposition 2.10, we have

$$E_\lambda^{ij} = \frac{1}{c_\lambda} \sum_{w \in W} \mathfrak{X}_\lambda^{ji}(\mathbf{C}_w)\mathbf{D}_{w^{-1}} \qquad \text{where } \mathfrak{X}_\lambda := \mathfrak{X}_{\mathfrak{C}}.$$

We have observed in (2.3) that $\mathfrak{X}_\lambda^{ji}(\mathbf{C}_w) = h_{w,x_i,x_j}$. Hence we have

$$E_\lambda^{ij} = \frac{1}{c_\lambda} \sum_{w \in W} h_{w,x_i,x_j} \, \mathbf{D}_{w^{-1}} = \frac{1}{c_\lambda} \mathbf{C}_{x_i} \, \mathbf{D}_{x_j^{-1}},$$

as desired. The remaining statements are clear by Proposition 2.10. □

Lemma 2.12 (Neunhöffer). *Let $\mathfrak{C}, \mathfrak{C}_1$ be two left cells of W such that $\mathfrak{C} \approx \mathfrak{C}_1$. Let $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, be a bijection such that condition* (♡) *in Definition* 2.5 *holds. Then we have*

$$\mathbf{C}_x\mathbf{D}_{y^{-1}} = \mathbf{C}_{x_1}\mathbf{D}_{y_1^{-1}} \qquad \textit{for all } x, y \in \mathfrak{C}.$$

Proof. Condition (♡) means that $h_{w,x,y} = h_{w,x_1,y_1}$ for all $w \in W$ and all $x, y \in \mathfrak{C}$. Hence we also have

$$\mathbf{C}_x\mathbf{D}_{y^{-1}} = \sum_{w \in W} h_{w,x,y} \, \mathbf{D}_{w^{-1}} = \sum_{w \in W} h_{w,x_1,y_1} \, \mathbf{D}_{w^{-1}} = \mathbf{C}_{x_1}\mathbf{D}_{y_1^{-1}},$$

as required. □

Proposition 2.13. *In the above setting, let $\mathfrak{C}, \mathfrak{C}_1$ be left cells such that $\chi_{\mathfrak{C}} = \chi_{\mathfrak{C}_1} \in \mathrm{Irr}(\mathcal{H}_K)$ and $\mathfrak{C} \approx \mathfrak{C}_1$. Let $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, be a bijection such that condition* (♡) *in Definition* 2.5 *holds. Then we have $x \sim_{\mathcal{R}} x_1$ for any $x \in \mathfrak{C}$.*

Proof. Let $x \in \mathfrak{C}$. We show that $x_1 \leqslant_{\mathcal{R}} x$. To see this, we argue as follows. Choose an enumeration of the elements in $\mathfrak{C}$ where x is the first element. Consider the corresponding matrix representation $\mathfrak{X}_{\mathfrak{C}}$. By Lemma 2.11, $\mathfrak{X}_{\mathfrak{C}}(\mathbf{C}_x\mathbf{D}_{x^{-1}})$ is a matrix with a non-zero coefficient at position $(1, 1)$ and coefficient 0 otherwise. Consequently, some coefficient in the first row of $\mathfrak{X}_{\mathfrak{C}}(\mathbf{C}_x)$ must be non-zero. Using (2.3) we see that there exists some $y \in \mathfrak{C}$ such that $h_{x,y,x} \neq 0$. Then, by (♡), we have $h_{x,y_1,x_1} = h_{x,y,x} \neq 0$ and so $x_1 \leqslant_{\mathcal{R}} x$, as claimed.

We now apply a similar discussion to the left cell $\mathfrak{C}_1$ and the element x_1. Working with the representation $\mathfrak{X}_{\mathfrak{C}_1}$, we see that there exists some $z_1 \in \mathfrak{C}_1$ such that $h_{x_1,z_1,x_1} \neq 0$. But then we have $h_{x_1,z,x} = h_{x_1,z_1,x_1} \neq 0$ and so $x \leqslant_{\mathcal{R}} x_1$. Hence we conclude that $x \sim_{\mathcal{R}} x_1$. □

Example 2.14. Let us consider once more the case where $W = \mathfrak{S}_n$, as in Example 2.6. It is shown by Kazhdan and Lusztig [13] that

$$\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}(\mathfrak{S}_n)_K) \qquad \text{for any left cell } \mathfrak{C} \subseteq \mathfrak{S}_n.$$

Now let $\mathfrak{C}, \mathfrak{C}_1$ be left cells such that $\mathfrak{C} \approx \mathfrak{C}_1$; see Example 2.6(b) for a characterisation of this condition. Let $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, be a bijection such that condition (♡) in

Definition 2.5 holds. Then, by Proposition 2.13, we have $x \sim_{\mathcal{R}} x_1$ for any $x \in \mathfrak{C}$. However, the Robinson–Schensted correspondence shows that two elements which lie in the same left cell and in the same right cell must be equal. Hence the element $x_1 \in \mathfrak{C}_1$ is uniquely determined by the condition that $x \sim_{\mathcal{R}} x_1$.

3. On the induction of Kazhdan–Lusztig cells

In [8], it is shown that the Kazhdan–Lusztig basis of $\mathcal{H}$ behaves well with respect to parabolic subalgebras. One of the aims of this section is to show that the relation "$\approx$" in Definition 2.5 also behaves well. Corollary 3.10 (obtained at the end of this section) will play a crucial role in the proof of Theorem 6.3. In a different direction, the techniques developed in this section lay the foundations for the discussion of the relative version of (♠).

We keep the basic set-up of the previous section. Let us fix a subset $I \subseteq S$ and consider the corresponding parabolic subgroup $W_I = \langle I \rangle \subseteq W$. Let $\mathcal{H}_I = \langle T_w \mid w \in W_I \rangle_A$ be the parabolic subalgebra corresponding to W_I. It is clear by the definition that, for any $w \in W_I$, we have that $\mathbf{C}_w$ computed inside $\mathcal{H}_I$ is the same as $\mathbf{C}_w$ computed in $\mathcal{H}$.

The following definitions already appear, in a somewhat different form, in the work of Barbasch and Vogan [2, §3].

3.1. **Relative Kazhdan–Lusztig pre-orders.** Given $x, y \in W$, we write $x \leftarrow_{\mathcal{L},I} y$ if there exists some $s \in I$ such that $h_{s,y,x} \neq 0$, that is, $\mathbf{C}_x$ occurs in $\mathbf{C}_s\,\mathbf{C}_y$ (when expressed in the $\mathbf{C}$-basis). Let $\leqslant_{\mathcal{L},I}$ be the pre-order relation on W generated by $\leftarrow_{\mathcal{L},I}$, that is, we have $x \leqslant_{\mathcal{L},I} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ of all elements in W such that $x_{i-1} \leftarrow_{\mathcal{L},I} x_i$ for all i. The equivalence relation associated with $\leqslant_{\mathcal{L},I}$ will be denoted by $\sim_{\mathcal{L},I}$ and the corresponding equivalence classes are called the *relative left cells* of W with respect to I. Note that the restriction of $\leqslant_{\mathcal{L},I}$ to W_I is nothing but the usual left pre-order on W_I.

Similarly, we can define a pre-order $\leqslant_{\mathcal{R},I}$ by considering multiplication by $\mathbf{C}_s$ ($s \in I$) on the right in the defining condition. The equivalence relation associated with $\leqslant_{\mathcal{R},I}$ will be denoted by $\sim_{\mathcal{R},I}$ and the corresponding equivalence classes are called the *relative right cells* of W (with respect to I). We have

$$x \leqslant_{\mathcal{R},I} y \quad \Leftrightarrow \quad x^{-1} \leqslant_{\mathcal{L},I} y^{-1}.$$

This follows, as before, by using the antiautomorphism $\flat\colon \mathcal{H} \to \mathcal{H}$ given by $T_w^\flat = T_{w^{-1}}$. Finally, we define a pre-order $\leqslant_{\mathcal{LR},I}$ by the condition that $x \leqslant_{\mathcal{LR},I} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leqslant_{\mathcal{L},I} x_i$ or $x_{i-1} \leqslant_{\mathcal{R},I} \leqslant x_i$. The equivalence relation associated with $\leqslant_{\mathcal{LR},I}$ will be denoted by $\sim_{\mathcal{LR},I}$ and the corresponding equivalence classes are called the *relative two-sided cells* of W.

Let X_I be the set of distinguished left coset representatives; we have

$$X_I = \{w \in W \mid w \text{ has minimal length in } wW_I\}.$$

Furthermore, the map $X_I \times W_I \to W$, $(x, u) \mapsto xu$, is a bijection and we have $l(xu) = l(x) + l(u)$ for all $u \in W_I$ and all $x \in X_I$. We define a relation "$\sqsubseteq$" as follows. Let $x, y \in X_I$ and $u, v \in W_I$. We write $xu \sqsubset yv$ if $x < y$ (Bruhat–Chevalley order) and $u \leqslant_{\mathcal{L},I} v$ (Kazhdan–Lusztig pre-order). We write $xu \sqsubseteq yv$ if $xu \sqsubset yv$ or $x = y$ and $u = v$. With this notation, we have the following result.

Proposition 3.2 (See [8, Prop. 3.3])**.** *For any $y \in X_I$, $v \in W_I$ we have*

$$\mathbf{C}_{yv} = \sum_{\substack{x \in X_I, u \in W_I \\ xu \sqsubseteq yv}} p^*_{xu,yv}\, T_x\, \mathbf{C}_u$$

where $p^*_{yv,yv} = 1$ and $p^*_{xu,yv} \in A_{<0}$ for $ux \sqsubset yv$.

For later use, we have to recall the basic ingredients in the construction of the polynomials $p^*_{xu,yv}$; we also prove some refinements of the results in [8, §3]. Let $y \in X_I$ and $v \in W_I$. Then we can write uniquely

$$\overline{T_y \mathbf{C}_v} = T_{y^{-1}}^{-1}\, \mathbf{C}_v = \sum_{\substack{x \in X_I \\ u \in W_I}} \overline{r}_{xu,yv}\, T_x\, \mathbf{C}_u \qquad \text{where } r_{xu,yv} \in A$$

and where only finitely many terms $r_{xu,yv}$ are non-zero.

Lemma 3.3. *Let $x, y \in X_I$ and $u, v \in W_I$. Then we have $r_{xu,yv} = 0$ unless $l(xu) < l(yv)$ or $xu = yv$. Furthermore, we have*

$$\overline{r}_{xu,yv} = \sum_{\substack{w \in W_I \\ xw \leqslant y}} \sum_{\substack{w' \in W_I \\ w' \leqslant w}} \overline{R}^*_{xw,y}\, \tilde{p}_{w',w}\, h_{w',v,u}$$

where $\tilde{p}_{w',w} \in A$ are independent of x, y, u, v and the $R^*_{z,y} \in A$ are the "absolute" R-polynomials defined in [15, §1].

Proof. First we establish the above identity. Let us fix $y \in X_I$ and $v \in W_I$. We can write

$$T_{y^{-1}}^{-1} = \sum_{\substack{z \in W \\ z \leqslant y}} \overline{R}^*_{z,y}\, T_z \qquad (R^*_{y,y} = 1).$$

Now let $z \in W$ be such that T_z occurs in the above expression. Then we can write $z = xw$ where $x \in X_I$ and $w \in W_I$; note that $x \leqslant z \leqslant y$. Since $l(xw) = l(x) + l(w)$, we have $T_z = T_x\, T_w$ and so

$$T_{y^{-1}}^{-1}\, \mathbf{C}_v = \sum_{\substack{x \in X_I \\ x \leqslant y}} \sum_{\substack{w \in W_I \\ xw \leqslant y}} \overline{R}^*_{xw,y}\, T_x\, T_w\, \mathbf{C}_v.$$

Now, by [18, Theorem 5.2], $\mathbf{C}_w$ is a linear combination of terms $T_{w'}$ where $w' \leqslant w$ and the coefficient of T_w is 1. Hence we can also write $T_w = \sum_{w'} \tilde{p}_{w',w} \mathbf{C}_{w'}$ where $\tilde{p}_{w,w} = 1$ and $\tilde{p}_{w',w} = 0$ unless $w' \leqslant w$. Thus, we have

$$\begin{aligned} T_{y^{-1}}^{-1}\, \mathbf{C}_v &= \sum_{\substack{x \in X_I \\ x \leqslant y}} \sum_{\substack{w \in W_I \\ xw \leqslant y}} \overline{R}^*_{xw,y} \sum_{\substack{w' \in W_I \\ w' \leqslant w}} \tilde{p}_{w',w}\, T_x\, \mathbf{C}_{w'}\, \mathbf{C}_v \\ &= \sum_{\substack{x \in X_I \\ x \leqslant y}} \sum_{u \in W_I} \Big(\sum_{\substack{w \in W_I \\ xw \leqslant y}} \sum_{\substack{w' \in W_I \\ w' \leqslant w}} \overline{R}^*_{xw,y}\, \tilde{p}_{w',w}\, h_{w',v,u} \Big) T_x\, \mathbf{C}_u. \end{aligned}$$

This yields the desired identity. Now assume that $r_{xu,yv} \neq 0$. Then there exist $w, w' \in W_I$ such that $w' \leqslant w$, $xw \leqslant y$ and $h_{w',v,u} \neq 0$. The latter condition certainly implies that $l(u) \leqslant l(w') + l(v)$; see (2.1). Combining this with the inequalities $l(w') \leqslant l(w)$ and $l(xw) \leqslant l(y)$, we obtain $l(xu) \leqslant l(yv)$, as desired. Furthermore, if equality holds, then equality holds in all intermediate inequalities, and so we must have $w' = w$, $xw = y$ and, hence, $w' = w = 1$. Since $h_{1,v,u} \neq 0$, this also yields $u = v$, as desired. $\square$

Now the arguments in the proofs of Lemma 3.2 and Proposition 3.3 in [8] (which themselves are an adaptation of the proof of Lusztig [15, Prop. 2]) show that the family of elements

$$\{p^*_{xu,yv} \mid x,y \in X_I, u,v \in W_I, xu \sqsubseteq yv\}$$

is uniquely determined by the following three conditions:

$$p^*_{yv,yv} = 1, \tag{KL1}$$

$$p^*_{xu,yv} \in A_{<0} \qquad \text{if } xu \sqsubset yv, \tag{KL2}$$

$$\overline{p}^*_{xu,yv} - p^*_{xu,yv} = \sum_{\substack{z \in X_I, w \in W_I \\ xu \sqsubset zw \sqsubseteq yv}} r_{xu,zw}\, p^*_{zw,yv} \qquad \text{if } xu \sqsubset yv. \tag{KL3}$$

The arguments in [*loc. cit.*] provide an inductive procedure for solving the above system of equations.

The following result yields a further property of the elements $p^*_{xu,yv}$.

Lemma 3.4. *Let $x,y \in X_I$ and $u,v \in W_I$. Then $p^*_{xu,yv} = 0$ unless $xu \leqslant yv$ (Bruhat–Chevalley order).*

Proof. First we claim that $p^*_{xu,yv} = 0$ unless $xu = yv$ or $l(xu) < l(yv)$. To prove this, we argue as follows. We have seen in Lemma 3.3 that $r_{xu,yv} = 0$ unless $xu = yv$ or $l(xu) < l(yv)$. Following the inductive procedure for solving the system of equations given by (KL1)–(KL3) above, we see that we also must have $p^*_{xu,yv} = 0$ unless $xu = yv$ or $l(xu) < l(yv)$.

Now let $x,y \in X_I$ and $u,v \in W_I$ be such that $l(xu) \leqslant l(yv)$ (with equality only for $xu = yv$). We want to prove that $p^*_{xu,yv} = 0$ unless $xu \leqslant yv$. We proceed by induction on $l(yv) - l(xu)$. If $l(xu) = l(yv)$, then $xu = yv$ and $p^*_{yv,yv} = 1$. Now assume that $l(xu) < l(yv)$ and $p^*_{xu,yv} \neq 0$. By the proof of [8, Prop. 3.3], we have

$$0 \neq p^*_{xu,yv} = P^*_{xu,yv} - \sum_{u<u_1} p^*_{xu_1,yv}\, P^*_{u,u_1}.$$

Now, if $P^*_{xu,yv} \neq 0$, then it is well-known that $xu \leqslant yv$, as required. On the other hand, if there is some $u_1 \in W$ such that $u < u_1$ and $p^*_{xu_1,yv}\, P^*_{u,u_1} \neq 0$, then we have $xu_1 \leqslant yv$ by induction, and so $xu \leqslant xu_1 \leqslant yv$. □

Corollary 3.5. *Let $y \in X_I$ and $v \in W_I$.*

(a) *$\mathbf{C}_{yv}$ is a linear combination of $T_y\,\mathbf{C}_v$ and terms $T_x\,\mathbf{C}_u$ where $x \in X_I$ and $u \in W_I$ are such that $x < y$, $u \leqslant_{\mathcal{L},I} v$ and $xu < yv$.*

(b) *Conversely, $T_y\,\mathbf{C}_v$ is a linear combination of $\mathbf{C}_{yv}$ and terms $\mathbf{C}_{xu}$ where $x \in X_I$ and $u \in W_I$ are such that $x < y$, $u \leqslant_{\mathcal{L},I} v$ and $xu < yv$.*

Proof. (a) This is just a restatement of Proposition 3.2, taking into account the additional information in Lemma 3.4.

(b) Let $w \in W$ and set $B_w := T_y\mathbf{C}_v$ where $y \in X_I$ and $v \in W_I$ are such that $w = yv$. Then $\{B_w \mid w \in W\}$ is a basis of $\mathcal{H}$ and the formula in Proposition 3.2 describes the base change from the $\mathbf{C}_w$-basis to the B_w-basis. By an easy induction on $l(w)$, we can invert these formulas. (Note that the base change takes place inside the finite sets $\{w \in W \mid l(w) \leqslant n\}$ for $n = 0,1,2,\ldots$.) Hence we obtain expressions for the elements in the B_w-basis in terms of the $\mathbf{C}_w$-basis. The terms arising in these expressions must satisfy conditions which are analogous to those in (a). □

Recall that, if V is any $\mathcal{H}_I$-module, then

$$\mathrm{Ind}_I^S(V) := \mathcal{H} \otimes_{\mathcal{H}_I} V$$

is an $\mathcal{H}$-module, called the *induced module*; see, for example, [11, §9.1]. If V is free over A with basis $\{v_\alpha \mid \alpha \in \mathcal{A}\}$, then $\mathrm{Ind}_I^S(V)$ is free with basis $\{T_x \otimes v_\alpha \mid x \in X_I, \alpha \in \mathcal{A}\}$.

Theorem 3.6 (See [8, Theorem 1])**.** *Let $\mathfrak{C}$ be a left cell of W_I. Then the set $X_I\mathfrak{C}$ is a union of left cells of W. We have an isomorphism of $\mathcal{H}$-modules*

$$[X_I\mathfrak{C}]_A \xrightarrow{\sim} \mathrm{Ind}_I^S([\mathfrak{C}]_A), \qquad c_{yv} \mapsto \sum_{\substack{x\in X_I, u\in\mathfrak{C}\\ xu\sqsubseteq yv}} p^*_{xu,yv}\,\big(T_x \otimes c_u\big),$$

where $\{c_{yv} \mid y \in X_I, v \in \mathfrak{C}\}$ is the standard basis of $[X_I\mathfrak{C}]_A$ and $\{c_u \mid u \in \mathfrak{C}\}$ is the standard basis of $[\mathfrak{C}]_A$.

Proof. The fact that $X_I\mathfrak{C}$ is a union of left cells is proved in [8, §4]. Since the statement concerning $[X_I\mathfrak{C}]_A$ is not explicitly mentioned in [*loc. cit.*], let us give the details here. Recall that $[X_I\mathfrak{C}]_A = \mathfrak{I}_{X_I\mathfrak{C}}/\hat{\mathfrak{I}}_{X_I\mathfrak{C}}$ where

$$\mathfrak{I}_{X_I\mathfrak{C}} = \left\langle \mathbf{C}_{xu} \;\middle|\; \begin{array}{l} x \in X_I, u \in W_I, ux \leqslant_{\mathcal{L}} vy, \\ \text{for some } y \in X_I, v \in \mathfrak{C} \end{array} \right\rangle_A,$$

$$\hat{\mathfrak{I}}_{X_I\mathfrak{C}} = \left\langle \mathbf{C}_{xu} \;\middle|\; \begin{array}{l} x \in X_I, u \in W_I, ux \not\in X_I\mathfrak{C}, ux \leqslant_{\mathcal{L}} vy, \\ \text{for some } y \in X_I, v \in \mathfrak{C} \end{array} \right\rangle_A.$$

Now, for any $x, y \in X_I$ and $u, v \in W_I$, we have the implication

$$xu \leqslant_{\mathcal{L}} yv \quad\Rightarrow\quad u \leqslant_{\mathcal{L},I} v;$$

see [8, §4]. On the other hand, we have $xu \leqslant_{\mathcal{L}} u$ for any $x \in X_I$ and $u \in W_I$ (since $l(xu) = l(x) + l(u)$). These two relations readily imply that we have

$$\mathfrak{I}_{X_I\mathfrak{C}} = \langle \mathbf{C}_{xu} \mid x \in X_I, u \in W_I, u \leqslant_{\mathcal{L},I} v \text{ for some } v \in \mathfrak{C}\rangle_A,$$

$$\hat{\mathfrak{I}}_{X_I\mathfrak{C}} = \langle \mathbf{C}_{xu} \mid x \in X_I, u \in W_I, u \not\in \mathfrak{C}, u \leqslant_{\mathcal{L},I} v \text{ for some } v \in \mathfrak{C}\rangle_A.$$

By [8, Cor. 3.4], this yields

$$\mathfrak{I}_{X_I\mathfrak{C}} = \langle T_x\mathbf{C}_u \mid x \in X_I, u \in W_I, u \leqslant_{\mathcal{L},I} v \text{ for some } v \in \mathfrak{C}\rangle_A,$$

$$\hat{\mathfrak{I}}_{X_I\mathfrak{C}} = \langle T_x\mathbf{C}_u \mid x \in X_I, u \in W_I, u \not\in \mathfrak{C}, u \leqslant_{\mathcal{L},I} v \text{ for some } v \in \mathfrak{C}\rangle_A.$$

Thus, we see that the $\mathcal{H}$-module $[X_I\mathfrak{C}]_A$ has two A-bases: firstly, the standard basis $\{c_{xu} \mid x \in X_I, u \in \mathfrak{C}\}$ where c_{xu} is the residue class of $\mathbf{C}_{xu}$ and, secondly, the basis $\{f_{xu} \mid x \in X_I, u \in \mathfrak{C}\}$ where f_{xu} denotes the residue class of $T_x\mathbf{C}_u$. The change of basis is given by the equations:

$$c_{yv} = \sum_{\substack{x\in X_I, u\in\mathfrak{C}\\ xu\sqsubseteq yv}} p^*_{xu,yv}\, f_{xu} \qquad \text{for any } y \in X_I,\, v \in \mathfrak{C}.$$

Furthermore, recalling the definition of f_{xu}, it is obvious that the map

$$\mathcal{H} \otimes_{\mathcal{H}_I} [\mathfrak{C}]_A \to [X_I\mathfrak{C}]_A, \qquad T_x \otimes c_u \mapsto f_{xu} \quad (x \in X_I, u \in \mathfrak{C}),$$

is an isomorphism of $\mathcal{H}$-modules, where $\{c_u \mid u \in \mathfrak{C}\}$ is the standard basis of $[\mathfrak{C}]_A$ as in (2.3). □

Remark 3.7. In the above setting, we also have an isomorphism of $\mathcal{H}$-modules

$$[X_I\mathfrak{C}]_A^\delta \xrightarrow{\sim} \operatorname{Ind}_I^S([\mathfrak{C}]_A^{\delta_I}), \qquad c_{yv} \mapsto \sum_{\substack{x\in X_I, u\in\mathfrak{C}\\ xu\sqsubseteq yv}} (-1)^{l(x)} p^*_{xu,yv}\big(T_x\otimes c_u\big),$$

where δ_I denotes the restriction of δ to $\mathcal{H}_I$. Indeed, applying Remark 2.4 to the formula in Proposition 3.2 yields

$$\begin{aligned}\delta(\mathbf{C}_{yv}) = j(\mathbf{C}_{yv}) &= \sum_{\substack{x\in X_I, u\in W_I\\ xu\sqsubseteq yv}} (-1)^{l(x)} p^*_{xu,yv}\, T_x\, j(\mathbf{C}_u)\\ &= \sum_{\substack{x\in X_I, u\in W_I\\ xu\sqsubseteq yv}} (-1)^{l(x)} p^*_{xu,yv}\, T_x\, \delta_I(\mathbf{C}_u)\end{aligned}$$

for any $y\in X_I$ and $v\in W_I$. We can now argue as in the above proof, using the fact that $[\mathfrak{C}]_A^{\delta_I} = \delta_I(\mathfrak{I}_\mathfrak{C})/\delta_I(\hat{\mathfrak{I}}_\mathfrak{C})$ and $[X_I\mathfrak{C}]_A^\delta = \delta(\mathfrak{I}_{X_I\mathfrak{C}})/\delta(\hat{\mathfrak{I}}_{X_I\mathfrak{C}})$.

Our aim is to show that the relation "$\approx$" in Definition 2.5 behaves well with respect to the induction of cells. We begin with the following result.

Lemma 3.8. *Assume that $\mathfrak{C},\mathfrak{C}_1$ are two left cells in W_I such that $\mathfrak{C}\approx\mathfrak{C}_1$. Let $\mathfrak{C}\xrightarrow{\sim}\mathfrak{C}_1$, $u\mapsto u_1$, be a bijection such that the property* ($\heartsuit$) *in Definition* 2.5 *holds. Then we have*

$$p^*_{xu,yv} = p^*_{xu_1,yv_1} \qquad \textit{for all } x,y\in X_I \textit{ and all } u,v\in\mathfrak{C}.$$

Proof. First we claim that

$$(*) \qquad r_{xu,yv} = r_{xu_1,yv_1} \qquad \text{for all } x,y\in X_I \text{ and all } u,v\in\mathfrak{C}.$$

To see this, consider the expression of $r_{xu,yv}$ in Lemma 3.3 and note that the coefficients $\overline{R}^*_{xw,y}$ and $\tilde{p}_{w',w}$ do not depend on u or v. Hence our assumption ($\heartsuit$) implies that $(*)$ holds. Now, following once more the inductive procedure for solving the system of equations given by (KL1)–(KL3) above, we see that we also have $p^*_{xu,yv} = p^*_{xu_1,yv_1}$ for all $x,y\in X_I$ and all $u,v\in C$. Just note that, for $u,v\in\mathfrak{C}$, the condition $xu\sqsubset zw\sqsubseteq vy$ implies that $u\leqslant_{\mathcal{L},I} w\leqslant_{\mathcal{L},I} v$ and so $w\in\mathfrak{C}$. $\square$

Proposition 3.9. *Let $\mathfrak{C},\mathfrak{C}_1$ be two left cells in W_I such that $\mathfrak{C}\approx\mathfrak{C}_1$. Then we also have $X_I\mathfrak{C}\approx X_I\mathfrak{C}_1$. More precisely, let $\mathfrak{C}\xrightarrow{\sim}\mathfrak{C}_1$, $u\mapsto u_1$, be a bijection satisfying* ($\heartsuit$)*. Then the bijection $X_I\mathfrak{C}\xrightarrow{\sim}X_I\mathfrak{C}_1$, $xu\mapsto xu_1$, satisfies* ($\heartsuit$)*.*

Proof. We have seen in Theorem 3.6 that there is an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H}\otimes_{\mathcal{H}_I}[\mathfrak{C}]_A \to [X_I\mathfrak{C}]_A, \qquad T_x\otimes c_u\mapsto f_{xu} \quad (x\in X_I, u\in\mathfrak{C}),$$

where $\{c_u \mid u\in\mathfrak{C}\}$ is the standard basis of $[\mathfrak{C}]_A$ as in (2.3) and f_{xu} denotes the residue class of $T_x\mathbf{C}_u$ in $[X_I\mathfrak{C}]_A$. The base change is given by the equations

$$c_{yv} = \sum_{\substack{x\in X_I, u\in\mathfrak{C}\\ xu\sqsubseteq yv}} p^*_{xu,yv}\, f_{xu} \qquad \text{for any } y\in X_I,\ v\in\mathfrak{C}.$$

Similarly, we have an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H}\otimes_{\mathcal{H}_I}[\mathfrak{C}_1]_A \to [X_I\mathfrak{C}_1]_A, \qquad T_x\otimes c_{u_1}\mapsto f_{xu_1} \quad (x\in X_I, u_1\in\mathfrak{C}_1),$$

where $\{c_{u_1} \mid u_1 \in \mathfrak{C}_1\}$ is the standard basis of $[\mathfrak{C}_1]_A$ as in (2.3) and f_{xu_1} denotes the residue class of $T_x\mathbf{C}_{u_1}$ in $[X_I\mathfrak{C}_1]_A$. The base change is given by the equations

$$c_{yv_1} = \sum_{\substack{x\in X_I, u_1\in\mathfrak{C}_1\\ xu_1\sqsubseteq yv_1}} p^*_{xu_1,yv_1}\, f_{xu_1} \qquad \text{for any } y \in X_I,\ v_1 \in \mathfrak{C}.$$

Now, the fact that ($\heartsuit$) holds for the bijection $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$ means that any $\mathbf{C}_s$ (where $s \in I$ is a generator of W_I) acts in the same way on the standard bases of $[\mathfrak{C}]_A$ and of $[\mathfrak{C}_1]_A$, respectively. Hence, by the definition of the induced module (see also the explicit formulas in [11, §9.1]), it is clear that any $\mathbf{C}_s$ (where $s \in S$ is a generator of W) will act in the same way on the bases $\{T_x \otimes c_u\}$ and $\{T_x \otimes c_{u_1}\}$ of $\mathcal{H} \otimes_{\mathcal{H}_I} [\mathfrak{C}]_A$ and $\mathcal{H} \otimes_{\mathcal{H}_I} [\mathfrak{C}_1]_A$, respectively. Then the above two isomorphisms show that any $\mathbf{C}_s$ ($s \in S$) acts in the same way on the bases $\{f_{xu}\}$ and $\{f_{xu_1}\}$ of $[X_I\mathfrak{C}]_A$ and of $[X_I\mathfrak{C}_1]_A$, respectively. Finally, by Lemma 3.8, the two base changes are performed by using exactly the same coefficients. Hence, any $\mathbf{C}_s$ ($s \in W$) will also act in the same way on the standard bases $\{c_{xu}\}$ and $\{c_{xu_1}\}$ of $[X_I\mathfrak{C}]_A$ and of $[X_I\mathfrak{C}_1]_A$, respectively. □

Corollary 3.10. *In the setting of Proposition* 3.9*, assume that the partitions of* $X_I\mathfrak{C}$ *and* $X_I\mathfrak{C}_1$ *into left cells of* W *are given by*

$$X_I\mathfrak{C} = \coprod_{\alpha\in\mathcal{A}} \mathfrak{C}^{(\alpha)} \qquad \textit{and} \qquad X_I\mathfrak{C}_1 = \coprod_{\beta\in\mathcal{B}} \mathfrak{C}_1^{(\beta)},$$

respectively, where $\mathcal{A}$ *and* $\mathcal{B}$ *are some indexing sets. Then there exists a bijection* $f\colon \mathcal{A} \to \mathcal{B}$ *such that* $\mathfrak{C}^{(\alpha)} \approx \mathfrak{C}_1^{(f(\alpha))}$ *for all* $\alpha \in \mathcal{A}$*.*

Proof. We have seen in Proposition 3.9 that the bijection $X_I\mathfrak{C} \xrightarrow{\sim} X_I\mathfrak{C}_1$, $xu \mapsto xu_1$, satisfies ($\heartsuit$), that is, we have

$$h_{s,xu,yv} = h_{s,xu_1,yv_1} \qquad \text{for } s \in S,\ x, y \in X_I \text{ and } u, v \in \mathfrak{C}.$$

By the definition of left cells, this immediately implies that the bijection $X_I\mathfrak{C} \xrightarrow{\sim} X_I\mathfrak{C}_1$ preserves the partition of the sets $X_I\mathfrak{C}$ and $X_I\mathfrak{C}_1$ into left cells, and that corresponding left cells are related by "$\approx$". □

4. Relative left, right and two-sided cells

We preserve the setting of the previous sections, where we consider a parabolic subgroup W_I. In this section, we pursue the study of the relative pre-orders $\leqslant_{\mathcal{L},I}$, $\leqslant_{\mathcal{R},I}$, etc., introduced in (3.1). Our Conjecture 4.5 predicts that we have an analogue of ($\spadesuit$) (see Section 1) in this relative setting. The main result of this section shows that the conjecture is true in the equal parameter case. This will play an essential role in our proof of property ($\spadesuit$) for groups of type B_n in the "asymptotic case".

Remark 4.1. Recall that X_I is the set of distinguished left coset representatives of W_I in W. Applying the anti-automorphism $\flat\colon \mathcal{H} \to \mathcal{H}$ such that $T_w^\flat = T_{w^{-1}}$ for all $w \in W$, we also obtain "right-handed" versions of the results in Section 3. First of all, the set $Y_I := X_I^{-1}$ is the set of distinguished right coset representatives of W_I in W. Thus, we can write any $w \in W$ uniquely in the form $w = ux$ where $u \in W_I$, $x \in Y_I$ and $l(ux) = l(u) + l(x)$. Since this will play a crucial role in the proof of Lemma 4.7, let us explicitly state the analogue of Corollary 3.5. Let $y \in Y_I$ and $v \in W_I$.

(a) $\mathbf{C}_{vy}$ is a linear combination of $\mathbf{C}_v T_y$ and terms $\mathbf{C}_u T_x$ where $x \in Y_I$ and $u \in W_I$ are such that $x < y$, $u \leqslant_{\mathcal{R},I} v$ and $ux < vy$. More precisely, by Proposition 3.2, we have

$$\mathbf{C}_{vy} = \sum_{x \in X_I, u \in W_I} a_{ux,vy}\, \mathbf{C}_u T_x$$

where the coefficients satisfy the following conditions:

$$\begin{aligned} a_{vy,vy} &= 1 && \text{if } ux = vy, \\ a_{ux,vy} &\in A_{<0} && \text{if } u \leqslant_{\mathcal{R},I} v \text{ and } x < y, \\ a_{ux,vy} &= 0 && \text{otherwise.} \end{aligned}$$

(We have $a_{ux,vy} = p^*_{(ux)^{-1},(vy)^{-1}}$ in the notation of Proposition 3.2.)

(b) Conversely, $\mathbf{C}_v T_y$ is a linear combination of $\mathbf{C}_{vy}$ and terms $\mathbf{C}_{ux}$ where $x \in X_I$ and $u \in W_I$ are such that $x < y$, $u \leqslant_{\mathcal{R},I} v$ and $ux < vy$. More precisely, arguing as in the proof of Corollary 3.5, we have

$$T_v \mathbf{C}_y = \sum_{x \in X_I, u \in W_I} b_{ux,vy}\, \mathbf{C}_{ux}$$

where the coefficients satisfy the following conditions:

$$\begin{aligned} b_{vy,vy} &= 1 && \text{if } ux = vy, \\ b_{ux,vy} &\in A_{<0} && \text{if } u \leqslant_{\mathcal{R},I} v \text{ and } x < y, \\ b_{ux,vy} &= 0 && \text{otherwise.} \end{aligned}$$

Using the above relations, we obtain the following formula.

Lemma 4.2. *Let $u, v, w \in W_I$ and $x, y \in Y_I$. Then we have*

$$h_{w,vy,ux} = \sum_{\substack{x_1 \in Y_I \\ u', u_1 \in W_I}} a_{u'x_1,vy}\, h_{w,u',u_1}\, b_{ux,u_1x_1}.$$

In the above sum, we can assume that $u \leqslant_{\mathcal{LR}} u_1 \leqslant_{\mathcal{LR}} u' \leqslant_{\mathcal{LR}} v$ and $x \leqslant x_1 \leqslant y$.

Proof. Using the formulas in Remark 4.1, we compute:

$$\begin{aligned} \mathbf{C}_w \mathbf{C}_{vy} &= \sum_{x_1 \in Y_I, u' \in W_I} a_{u'x_1,vy} \mathbf{C}_w \mathbf{C}_{u'} T_{x_1} \\ &= \sum_{\substack{x_1 \in Y_I \\ u', u_1 \in W_I}} a_{u'x_1,vy}\, h_{w,u',u_1}\, \mathbf{C}_{u_1} T_{x_1} \\ &= \sum_{\substack{x, x_1 \in Y_I \\ u, u', u_1 \in W_I}} a_{u'x_1,vy}\, h_{w,u',u_1}\, b_{ux,u_1x_1}\, \mathbf{C}_{ux}. \end{aligned}$$

This yields the above formula. Now let $x_1 \in Y_I$ and $u', u_1 \in W_I$ be such that the corresponding term in the expression for $h_{w,vy,ux}$ is non-zero. Then $a_{u'x_1,vy} \neq 0$ and $b_{ux,u_1x_1} \neq 0$. This implies $x \leqslant x_1 \leqslant y$, $u \leqslant_{\mathcal{R},I} u_1$ and $u' \leqslant_{\mathcal{R},I} v$; see the conditions in Remark 4.1. Furthermore, if $h_{w,u',u_1} \neq 0$, then $u_1 \leqslant_{\mathcal{L},I} u'$. In particular, we have $u \leqslant_{\mathcal{LR}} u_1 \leqslant_{\mathcal{LR}} u' \leqslant_{\mathcal{LR}} v$. $\square$

Lemma 4.3. *Let $u, v \in W_I$ and $y \in Y_I$. Then we have*

$$uy \leqslant_{\mathcal{L},I} vy \qquad \Leftrightarrow \qquad u \leqslant_{\mathcal{L},I} v.$$

Proof. For the implication "$\Leftarrow$", see [18, Prop. 9.11]. To prove the implication "$\Rightarrow$", we may assume without loss of generality that $u \neq v$ and $uy \leftarrow_{\mathcal{L},I} vy$, that is, we have $h_{s,vy,uy} \neq 0$ for some $s \in I$. Then we have $sv > v$, $su < u$ and the formula in (2.1) shows that there are two cases: If $svy = uy$, then $u = sv > v$ and so $u \leqslant_{\mathcal{L},I} v$. If $suy < uy < vy < svy$ and $M^s_{uy,vy} \neq 0$, then $su < u < v < sv$ and [18, Lemma 9.10] shows that $M^s_{u,v} = M^s_{uy,vy} \neq 0$. Again, we have $u \leqslant_{\mathcal{L},I} v$. $\square$

Proposition 4.4. *Let $u, v \in W_I$ and $x, y \in Y_I$. Then we have the following implication:*

$$ux \leqslant_{\mathcal{L},I} vy \quad \Rightarrow \quad u \leqslant_{\mathcal{LR},I} v \quad \textit{and} \quad x \leqslant y.$$

In particular, if $ux \sim_{\mathcal{L},I} vy$, then we necessarily have $x = y$ and $u \sim_{\mathcal{L},I} v$.

Proof. We may assume without loss of generality that $ux \leftarrow_{\mathcal{L},I} vy$, that is, $h_{s,vy,ux} \neq 0$ for some $s \in I$. Then the assertion follows from Lemma 4.2. $\square$

Conjecture 4.5 (Relative version of (♠))**.** *Let $u, v \in W_I$ and $x, y \in Y_I$. Then we have the following implication:*

$$ux \leqslant_{\mathcal{L},I} vy \quad \textit{and} \quad u \sim_{\mathcal{LR},I} v \quad \Rightarrow \quad u \sim_{\mathcal{L},I} v \quad \textit{and} \quad x = y.$$

Note that $u \sim_{\mathcal{LR},I} v$ and $u \sim_{\mathcal{L},I} y$ just mean the usual Kazhdan–Lusztig relations inside W_I.

Remark 4.6. Assume that $I = S$; then $W_I = W$ and $Y_I = \{1\}$. In this case, the above conjecture reads:

$$u \leqslant_{\mathcal{L}} v \quad \text{and} \quad u \sim_{\mathcal{LR}} v \quad \Rightarrow \quad u \sim_{\mathcal{L}} v$$

(for any $u, v \in W$). Thus, Conjecture 4.5 can be seen as a generalization of the implication (♠) stated in the introduction. Using computer programs written in the GAP programming language, we have verified that Conjecture 4.5 holds for W of type F_4, all choices of I and all choices of integer-valued weight functions on W (using the techniques in [9]). In Theorem 4.8 we will show that this is also true in the case of equal parameters.

For the remainder of this section, we assume that W is bounded and integral in the sense of [18, 1.11 and 13.2]. Furthermore, we assume that $q_s = q_t$ for all $s, t \in S$ (the "equal parameter" case). Let $q := q_s$ ($s \in S$). Then our hypotheses imply that

$$P^*_{x,y} \in q^{-1}\mathbb{N}[q^{-1}] \qquad \text{and} \qquad h_{x,y,z} \in \mathbb{N}[q, q^{-1}]$$

for all $x, y, z \in W$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. See Lusztig [17], [18, 15.1] and Springer [20]. We shall need some properties of Lusztig's function $\mathbf{a}_I : W_I \to \mathbb{N}$ defined by

$$\mathbf{a}_I(w) = \min\{n \in \mathbb{N} \mid q^n h_{u,v,w} \in \mathbb{Z}[q] \text{ for all } u, v \in W_I\}.$$

Note that $h_{u,v,w} = \overline{h}_{u,v,w}$. So, if $\mathbf{a}_I(w) = n$, then $q^n h_{u,v,w} \in \mathbb{Z}[q]$ and $q^{-n} h_{u,v,w} \in \mathbb{Z}[q^{-1}]$. Furthermore, both q^n and q^{-n} occur with non-zero coefficient in $h_{u,v,w}$. In [18, Chap. 15], the following three properties are established:

- **(P4)** The function $\mathbf{a}_I : W_I \to \mathbb{N}$ is constant on two-sided cells.
- **(P8)** Let $u, v, w \in W_I$ be such that $q^{\mathbf{a}_I(w)} h_{u,v,w}$ has a non-zero constant term. Then $v \sim_{\mathcal{L},I} w$, $u \sim_{\mathcal{R},I} w$ and $u \sim_{\mathcal{L},I} v^{-1}$.
- **(P9)** Let $u, v \in W_I$ be such that $u \leqslant_{\mathcal{L},I} v$ and $\mathbf{a}_I(u) = \mathbf{a}_I(v)$. Then $u \sim_{\mathcal{L},I} v$.

(There is even a list of 15 properties, but we only need the above three.) Note that **(P4), (P9)** together imply that (♠) holds for W_I.

Lemma 4.7. *In the above setting, let $u, v, w \in W_I$ and $x, y \in Y_I$. Then the coefficient $h_{w,vy,ux}$ has the following properties.*

(a) *If $h_{w,vy,ux} \neq 0$, then $u \leqslant_{\mathcal{LR},I} v$ and $x \leqslant y$.*
(b) *If $x = y$, then $h_{w,vy,uy} = h_{w,v,u}$.*
(c) *Assume that $u \sim_{\mathcal{LR},I} v$ and let $n := \mathbf{a}_I(u) = \mathbf{a}_I(v)$; see* **(P4)***. If the coefficient of q^n in $h_{w,vy,ux}$ is non-zero, then $x = y$.*

Proof. If $h_{w,vy,ux} \neq 0$, then $ux \leqslant_{\mathcal{L},I} vy$ and (a) follows from Proposition 4.4. To prove (b) and (c), we use the formula in Lemma 4.2:

$$h_{w,vy,ux} = \sum_{\substack{x_1 \in X_I \\ u',u_1 \in W_I}} a_{u'x_1,vy}\, h_{w,u',u_1}\, b_{ux,u_1x_1},$$

where the sum runs over all x_1, u', u_1 such that

$$(*_1) \qquad x \leqslant x_1 \leqslant y,$$

$$(*_2) \qquad u \leqslant_{\mathcal{LR},I} u_1 \leqslant_{\mathcal{LR},I} u' \leqslant_{\mathcal{LR},I} v.$$

Now, if $x_1 = x$, then $b_{ux,u_1x} = 0$ unless $u = u_1$ (in which case the result is 1; see the conditions in Remark 4.1). Similarly, if $x_1 = y$, then $a_{u'y,vy} = 0$ unless $u' = v$ (in which case the result is 1). Hence, if $x = y$, the above sum reduces to

$$h_{w,vy,ux} = a_{vy,vy}\, h_{w,v,u}\, b_{uy,uy} = h_{w,v,u}.$$

Thus, (b) is proved. Finally, to prove (c), assume that $x < y$ and that the coefficient of q^n in $h_{w,vy,ux}$ is non-zero, where $n = \mathbf{a}_I(u) = \mathbf{a}_I(v)$. We must show that u, v cannot be in the same two-sided cell. Splitting the above sum into three pieces according to $x_1 = x$, $x_1 = y$ and $x < x_1 < y$, we obtain

$$\begin{aligned} h_{w,vy,ux} = {} & \sum_{u' \in W_I} a_{u'x,vy}\, h_{w,u',u} + \sum_{u_1 \in W_I} h_{w,v,u_1}\, b_{ux,u_1y} \\ & + \sum_{u',u_1 \in W_I} \Big(\sum_{\substack{x_1 \in X_I \\ x<x_1<y}} a_{u'x_1,vy}\, b_{ux,u_1x_1} \Big)\, h_{w,u',u_1}. \end{aligned}$$

Note that, since $x < x_1 < y$, all the coefficients $a_{u'x,vy}$, b_{ux,u_1y}, $a_{u'x_1,vy}$ and b_{ux,u_1x_1} occuring in the above expression lie in $q^{-1}\mathbb{Z}[q^{-1}]$; see once more the conditions in Remark 4.1 and recall that $q = q_s$ (for all $s \in S$). Hence we can re-write the above expression as follows:

$$h_{w,vy,ux} = \sum_{u_1,u' \in W_I} f_{u_1,u'}\, h_{w,u',u_1} \qquad \text{where } f_{u_1,u'} \in q^{-1}\mathbb{Z}[q^{-1}],$$

where we can assume that $(*_2)$ holds.

Now, we are assuming that the coefficient of q^n in $h_{w,vy,ux}$ is non-zero. So there exist some $u', u_1 \in W_I$ such that the coefficient of q^n in $f_{u_1,u'}\, h_{w,u',u_1}$ is non-zero. Since $f_{u_1,u'} \in q^{-1}\mathbb{Z}[q^{-1}]$, we deduce that there exists some $m > n$ such q^m has a non-zero coefficient in h_{w,u',u_1}. By the definition of the **a**-function, this means that $\mathbf{a}_I(u_1) \geqslant m > n$. Now, if we had $u \sim_{\mathcal{LR},I} v$, then $(*_2)$ would imply $u \sim_{\mathcal{LR}} u_1 \sim_{\mathcal{LR}} u' \sim_{\mathcal{LR}} v$, yielding the contradiction

$$\mathbf{a}(u_1) = \mathbf{a}(u') = \mathbf{a}(u) = \mathbf{a}(v) = n; \qquad \text{see } \textbf{(P4)}.$$

Consequently, u and v cannot lie in the same two-sided cell. $\square$

Theorem 4.8. *Assume that* W *is bounded, integral in the sense of* [18] *and that* $q_s = q_t$ *for all* $s, t \in S$. *Then Conjecture* 4.5 *holds for all parabolic subgroups* $W_I \subseteq W$.

Proof. Let us fix a subset $I \subseteq S$. Let $u, v \in W_I$ and $x, y \in Y_I$ be such that $ux \leqslant_{\mathcal{L},I} vy$ and $u \sim_{\mathcal{LR},I} v$. We want to show that $x = y$ and $u \sim_{\mathcal{L},I} v$. Suppose we already know that $x = y$. Then, since $uy \leqslant_{\mathcal{L},I} vy$, we can apply Lemma 4.3 and this yields $u \leqslant_{\mathcal{L},I} v$. Thus, we have $u \leqslant_{\mathcal{L},I} v$ and $u \sim_{\mathcal{LR},I} v$. So **(P4), (P9)** imply that $u \sim_{\mathcal{L},I} v$, as desired. Hence, it is sufficient to prove that $x = y$. First of all, using Proposition 4.4, we may assume without loss of generality that $ux \neq vy$ and $ux \leftarrow_{\mathcal{L},I} vy$, that is, $\mathbf{C}_{ux}$ occurs in $\mathbf{C}_s\mathbf{C}_{vy}$ for some $s \in I$ such that $svy > vy$. Since $s \in I$, this implies $sv > v$, and the multiplication rule for the Kazhdan–Lusztig basis (see Section 2) shows that we must have $su < u$ and $u \neq v$. We shall now try to imitate the proof of $(\tilde{P})$ in [18, 15.5].

Since $u \sim_{\mathcal{LR},I} v$, we have $n := \mathbf{a}_I(u) = \mathbf{a}_I(v)$ by **(P4)**. For any Laurent polynomial $f \in \mathbb{Z}[q, q^{-1}]$, we denote by $\pi_n(f)$ the coefficient of q^n in f, where we write $q := q_s$ $(s \in S)$ as above. Now we argue as follows. By the definition of the $\mathbf{a}$-function, there exist some $w, v' \in W_I$ such that $q^n h_{w,v',v}$ has a non-zero constant term. Since $h_{w,v',v} = \overline{h}_{w,v',v}$, this means that the coefficient of q^n in $h_{w,v',v}$ is non-zero. Thus, using **(P8)**, we have

$$\pi_n(h_{w,v',v}) \neq 0 \qquad \text{and} \qquad v' \sim_{\mathcal{L},I} v. \tag{1}$$

We can express the product $\mathbf{C}_s(\mathbf{C}_w\mathbf{C}_{v'y})$ as a linear combination of terms $\mathbf{C}_{wz}$ where $w \in W_I$ and $z \in Y_I$. Denote by κ_{wz} the coefficient of $\mathbf{C}_{wz}$ in that product. We have

$$\kappa_{wz} = \sum_{w_1 \in W_I, z_1 \in Y_I} h_{w,v'y,w_1z_1}\, h_{s,w_1z_1,wz}.$$

In particular,

$$\begin{aligned} \kappa_{ux} &= \sum_{w_1 \in W_I, z_1 \in Y_I} h_{w,v'y,w_1z_1}\, h_{s,w_1z_1,ux} \\ &= h_{w,v'y,vy}\, h_{s,vy,ux} + \sum_{\substack{w_1 \in W_I, z_1 \in Y_I \\ w_1z_1 \neq vy}} h_{w,v'y,w_1z_1}\, h_{s,w_1z_1,ux}. \end{aligned}$$

Since $svy > vy$, the multiplication rule for the Kazhdan–Lusztig basis shows that $h_{s,vy,ux}$ equals 1 or $M^s_{ux,vy}$, and the latter is an integer by [18, 6.5]. Hence we have $h_{s,vy,ux} \in \mathbb{Z}$ in both cases and so

$$\pi_n(h_{w,v'y,vy}\, h_{s,vy,ux}) = \pi_n(h_{w,v'y,vy})\, h_{s,vy,ux} = \pi_n(h_{w,v',v})\, h_{s,vy,ux},$$

where the last equality holds by Lemma 4.7(b). We are assuming that $h_{s,vy,ux} \neq 0$. In combination with (1) and the above identity, we conclude that

$$\pi_n(h_{w,v'y,vy}\, h_{s,vy,ux}) = \pi_n(h_{w,v',v})\, h_{s,vy,ux} \neq 0. \tag{2}$$

Since all polynomials involved in the expression for κ_{ux} have non-negative coefficients (thanks to the assumption that W is integral), the non-zero coefficient of q^n arising from (2) will not cancel out with the coefficients of q^n from the remaining terms in κ_{ux}. So we can conclude, as in the proof of Lusztig [18, 15.5], that

$$\pi_n(\kappa_{ux}) \neq 0.$$

On the other hand, since $\mathbf{C}_s(\mathbf{C}_w\mathbf{C}_{v'y}) = (\mathbf{C}_s\mathbf{C}_w)\mathbf{C}_{v'y}$, we also have the following expression for κ_{ux}:

$$\kappa_{ux} = \sum_{w' \in W_I} h_{s,w,w'} h_{w',v'y,ux}.$$

Since $\pi_n(\kappa_{ux}) \neq 0$, there exists some $w' \in W_I$ such that

$$\pi_n(h_{s,w,w'}\, h_{w',v'y,ux}) \neq 0. \tag{3}$$

By (1), we have $h_{w,v',v} \neq 0$ and so $v \leqslant_{\mathcal{R},I} w$. Hence the left descent set of w is contained in the left descent set of v; see [18, 8.6]. So, since $sv > v$, we also have $sw > w$. Then the multiplication rule for the Kazhdan–Lusztig basis and [18, 6.5] show that $h_{s,w,w'} \in \mathbb{Z}$. Hence (3) implies that

$$\pi_n(h_{w',v'y,ux}) \neq 0 \qquad \text{where} \qquad w' \in W_I.$$

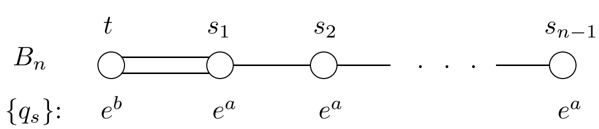
By (1), we also have $v' \sim_{\mathcal{LR},I} v \sim_{\mathcal{LR},I} u$. Hence Lemma 4.7(c) yields $x = y$, as desired. $\square$

Example 4.9. Let $W = \mathfrak{S}_n$ be the symmetric group. Then Conjecture 4.5 holds for all parabolic subgroups $W_I \subseteq W$.

Indeed, $\mathfrak{S}_n$ is finite, hence bounded. Since the product of any two generators has order 2 or 3, the group is integral. Furthermore, since all generators are conjugate, all the parameters are equal. Hence the hypotheses of Theorem 4.8 are satisfied.

5. On the left pre-order $\leqslant_{\mathcal{L}}$ in type B_n

In this and the subsequent sections, we let $W = W_n$ be a Coxeter group of type B_n ($n \geqslant 2$). We assume that the generators, relations and the weight function $L \colon W_n \to \Gamma$ are given by the following diagram:



where $a, b \in \Gamma$ are such that $a > 0$ and $b > 0$. Let $\mathcal{H}_n$ be the corresponding Iwahori–Hecke algebra over $A = \mathbb{Z}[\Gamma]$, where we set

$$Q := q_t = e^b \qquad \text{and} \qquad q := q_{s_1} = \cdots = q_{s_{n-1}} = e^a.$$

Let K be the field of fractions of A and set $\mathcal{H}_{n,K} = K \otimes_A \mathcal{H}_n$. Throughout this and the subsequent sections, we assume that b/a is "large" with respect to n, more precisely:

$$\boxed{b > (n-1)a}$$

(Here, $(n-1)a$ means $a + \cdots + a$ in Γ, with $n-1$ summands.) We refer to this hypothesis as the **"asymptotic case"** in type B_n.

The main results of this section are:

- Theorem 5.11, which gives a strengthening of the results of Bonnafé and Iancu [3] concerning the left cells of W_n (and, as a bi-product, also yields a new proof of Bonnafé's result [4] on the two-sided cells);
- Theorem 5.13, which shows that (♠) holds in W_n.

Remark 5.1. Let us consider the abelian group $\Gamma^\circ = \mathbb{Z}^2$ and let $\leqslant$ be the usual lexicographic order on Γ°. Thus, we have $(i,j) < (i',j')$ if $i < i'$ or if $i = i'$ and $j < j'$. Let $L^\circ \colon W_n \to \mathbb{Z}^2$ be the weight function such that

$$L(t) = (1,0) \qquad \text{and} \qquad L(s_1) = \cdots = L(s_{n-1}) = (0,1).$$

Then $A^\circ = \mathbb{Z}[\Gamma^\circ]$ is nothing but the ring of Laurent polynomials in two independent indeterminates $V = e^{(1,0)}$ and $v = e^{(0,1)}$. This is the "asymptotic case" originally considered by Bonnafé and Iancu [3]. We may refer to this case as the **"generic asymptotic case"** in type B_n. Let us denote the corresponding Iwahori–Hecke algebra by $\mathcal{H}_n^\circ$; let $\{\mathbf{C}_w^\circ \mid w \in W_n\}$ be the Kazhdan–Lusztig basis of $\mathcal{H}_n^\circ$ and write

$$\mathbf{C}_x^\circ\, \mathbf{C}_y^\circ = \sum_{z \in W_n} h_{x,y,z}^\circ\, \mathbf{C}_z^\circ \quad \text{where} \quad h_{x,y,z}^\circ \in A^\circ = \mathbb{Z}[V^{\pm 1}, v^{\pm 1}].$$

Now, given an abelian group Γ as above and two elements $a, b > 0$, we have a unique ring homomorphism

$$\theta \colon A^\circ \to A, \qquad V^i v^j \mapsto e^{ib+ja}.$$

Bonnafé [4, §5] has shown that, if $b > (n-1)a$, then the Kazhdan–Lusztig basis of $\mathcal{H}_n$ (with respect to $L \colon W_n \to \Gamma$) is obtained by "specialisation" from the Kazhdan–Lusztig of $\mathcal{H}_n^\circ$ and that we have

(a) $$h_{x,y,z} = \theta(h_{x,y,z}^\circ) \qquad \text{for all } x, y, z \in W_n.$$

In particular, denoting by $\leqslant_{\mathcal{L}}^\circ$, $\sim_{\mathcal{L}}^\circ$, $\leqslant_{\mathcal{R}}^\circ$, $\sim_{\mathcal{R}}^\circ$, $\leqslant_{\mathcal{LR}}^\circ$, $\sim_{\mathcal{LR}}^\circ$ the pre-order relations on W_n with respect to L°, we have the implications:

(b) $$x \leqslant_{\mathcal{L}} y \Rightarrow x \leqslant_{\mathcal{L}}^\circ y, \qquad x \leqslant_{\mathcal{R}} y \Rightarrow x \leqslant_{\mathcal{R}}^\circ y, \qquad x \leqslant_{\mathcal{LR}} y \Rightarrow x \leqslant_{\mathcal{LR}}^\circ y.$$

These results show that it is usually sufficient to prove identities concerning the Kazhdan–Lusztig basis in the "generic asymptotic case"; the analogous identity in the general "asymptotic case" then follows by specialisation, assuming that $b > (n-1)a$. (In this and the following sections, we make an explicit remark at places where we use this kind of argument.)

We shall need some notation from [3]. Given $w \in W_n$, we denote by $l_t(w)$ the number of occurences of the generator t in a reduced expression for w, and call this the "t-length" of w.

The parabolic subgroup $\mathfrak{S}_n := \langle s_1, \ldots, s_{n-1}\rangle$ is naturally isomorphic to the symmetric group on $\{1, \ldots, n\}$, where s_i corresponds to the basic transposition $(i, i+1)$. Let $1 \leqslant l \leqslant n-1$. Then we set $\Sigma_{l,n-l} := \{s_1, \ldots, s_{n-1}\} \setminus \{s_l\}$. For $l = 0$ or $l = n$, we also set $\Sigma_n := \Sigma_{0,n} = \Sigma_{n,0} = \{s_1, \ldots, s_{n-1}\}$. Let $X_{l,n-l}$ be the set of distinguished left coset representatives of the Young subgroup $\mathfrak{S}_{l,n-l} := \langle \Sigma_{l,n-l}\rangle$ in $\mathfrak{S}_n$. We have the parabolic subalgebra $\mathcal{H}_{l,n-l} = \langle T_\sigma \mid \sigma \in \mathfrak{S}_{l,n-l}\rangle_A \subseteq \mathcal{H}_n$. Given $x, y \in W_n$, we write

$$x \leqslant_{\mathcal{L},l} y \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad x \leqslant_{\mathcal{L},\Sigma_{l,n-l}} y \qquad \text{(see Section 3).}$$

Furthermore, as in [3, §4], we set $a_0 = 1$ and

$$a_l := t(s_1 t)(s_2 s_1 t) \cdots (s_{l-1} s_{l-2} \cdots s_1 t) \qquad \text{for } l > 0.$$

Then, by [3, Prop. 4.4], the set $X_{l,n-l} a_l$ is precisely the set of distinguished left coset representatives of $\mathfrak{S}_n$ in W_n whose t-length equals l. Furthermore, every element $w \in W_n$ has a unique decomposition

$$w = a_w a_l \sigma_w b_w^{-1} \qquad \text{where } l = l_t(w),\ \sigma_w \in \mathfrak{S}_{l,n-l} \text{ and } a_w, b_w \in X_{l,n-1};$$

see [3, 4.6]. On a combinatorial level, Bonnafé and Iancu [3, §3] define a generalized Robinson–Schensted correspondence which associates with each element $w \in W_n$ a pair of n-standard bi-tableaux $(A(w), B(w))$ such that $A(w)$ and $B(w)$ have the same shape. Here, a standard n-bitableau is a pair of standard tableaux with a total number of n boxes (filled with the numbers $1, \ldots, n$), and the shape of such a bitableau is a pair of partitions $\lambda = (\lambda_1, \lambda_2)$ such that $n = |\lambda_1| + |\lambda_2|$. With this notation, we have the following result.

Theorem 5.2 (Bonnafé and Iancu [3] and Bonnafé [4, §5])**.** *In the above setting, let $x, y \in W_n$. Then the following conditions are equivalent:*

- (a$_1$) *$x \sim_{\mathcal{L}} y$;*
- (a$_2$) *$x \sim_{\mathcal{L}}^{\circ} y$ (see Remark* 5.1*);*
- (b) *$l := l_t(x) = l_t(y)$, $b_x = b_y$ and $\sigma_x \sim_{\mathcal{L},l} \sigma_y$;*
- (c) *$B(x) = B(y)$.*

(This is the first example where the discussion in Remark 5.1 applies: the equivalences between (a$_2$), (b) and (c) are proved in [3, Theorem 7.7]; the equivalence between (a$_1$) and (a$_2$) is proved in [4, Cor. 5.2].)

Note that the equivalence "(a$_1$) $\Leftrightarrow$ (c)" is in complete formal analogy to the situation in the symmetric group $\mathfrak{S}_n$; see Example 2.6(a).

Let Λ_n be the set of all pairs of partitions of total size n. We set

$$\mathfrak{R}_\lambda := \{w \in W_n \mid A(w), B(w) \text{ have shape } \lambda\} \qquad \text{for } \lambda \in \Lambda_n.$$

Thus, we have a partition $W_n = \coprod_{\lambda \in \Lambda_n} \mathfrak{R}_\lambda$. The above result and the properties of the generalized Robinson–Schensted correspondence in [3, §3] immediately imply the following statement:

Corollary 5.3 (Bonnafé and Iancu [3])**.** *In the above setting, let $\lambda \in \Lambda_n$ and denote by $\mathbb{T}_\lambda$ the set of n-standard bitableaux of shape λ. Then the generalized Robinson–Schensted correspondence defines a bijection*

$$w_\lambda \colon \mathbb{T}_\lambda \times \mathbb{T}_\lambda \xrightarrow{\sim} \mathfrak{R}_\lambda, \qquad (T, T') \mapsto w_\lambda(T, T'),$$

with the following property:

- (a) *For a fixed T', the elements $\{w_\lambda(T, T') \mid T \in \mathbb{T}_\lambda\}$ form a left cell.*
- (b) *For a fixed T, the elements $\{w_\lambda(T, T') \mid T' \in \mathbb{T}_\lambda\}$ form a right cell.*
- (c) *We have $w_\lambda(T, T')^{-1} = w_\lambda(T', T)$ for all $T, T' \in \mathbb{T}_\lambda$.*

In particular, any left cell contained in $\mathfrak{R}_\lambda$ meets any right cell contained in $\mathfrak{R}_\lambda$ in exactly one element. Furthermore, every left cell contains a unique element of the set $\mathcal{D}_n := \{z \in W_n \mid z^2 = 1\}$.

In order to prove the main results of this section, we need a number of preliminary steps. We shall frequently use the following result.

Proposition 5.4 (Bonnafé and Iancu [3, Cor. 6.7] and Bonnafé [4, §5])**.** *In the above setting, let $x, y \in W_n$ be such that $x \leqslant_{\mathcal{LR}} y$. Then $l_t(y) \leqslant l_t(x)$. In particular, if $x \sim_{\mathcal{LR}} y$, then $l_t(x) = l_t(y)$.*

(The above result was first proved in [3] for the weight function $L^\circ \colon W_n \to \mathbb{Z}^2$; then Remark 5.1(b) immediately yields the analogous statement in the general "asymptotic case".) The following two results give some information about certain elements of the Kazhdan–Lusztig basis of $\mathcal{H}_n$.

Lemma 5.5 (Bonnafé [4, §2]). *For any $\sigma \in \mathfrak{S}_n$ and any $0 \leqslant l \leqslant n$, we have*

$$\mathbf{C}_\sigma \, \mathbf{C}_{a_l} = \mathbf{C}_{\sigma a_l} \qquad \text{and} \qquad \mathbf{C}_{a_l} \, \mathbf{C}_\sigma = \mathbf{C}_{a_l \sigma}.$$

Furthermore, if $\sigma \in \mathfrak{S}_{l,n-l}$, then

$$\mathbf{C}_\sigma \, \mathbf{C}_{a_l} = \mathbf{C}_{\sigma a_l} = \mathbf{C}_{a_l} \mathbf{C}_{a_l \sigma a_l} \qquad \text{where } a_l \sigma a_l \in \mathfrak{S}_{l,n-l}.$$

Proof. By Remark 5.1, it is sufficient to prove the equality $\mathbf{C}_\sigma \, \mathbf{C}_{a_l} = \mathbf{C}_{\sigma a_l}$ (for $\sigma \in \mathfrak{S}_n$) in the original setting of [3] where we consider the weight function $L^\circ \colon W_n \to \mathbb{Z}^2$. In this case, the statement is proved in [4, Prop. 2.3]. The equality $\mathbf{C}_{a_l} \, \mathbf{C}_\sigma = \mathbf{C}_{a_l \sigma}$ is proved similarly.

Finally, since $a_l = a_l^{-1}$ stabilizes $\Sigma_{l,n-l}$, we have $a_l \sigma a_l \in \mathfrak{S}_{l,n-l}$ for any $\sigma \in \mathfrak{S}_{l,n-l}$, which yields the second statement. □

The following result plays an essential role in the proof of Lemma 5.10.

Lemma 5.6. *For any $0 \leqslant l \leqslant n-1$, we have*

$$T_{ts_1 \cdots s_l} \, \mathbf{C}_{a_l} = \mathbf{C}_{a_{l+1}} + h(a_l) \, \mathbf{C}_{a_l}$$

where $h(a_l) \in \mathcal{H}_n$ is an A-linear combination of basis elements T_w with $w \leqslant s_1 s_2 \cdots s_l$. (For $l = 0$, we have $a_0 = 1$, $a_1 = t$ and $h(a_0) = -Q^{-1} T_1$.)

Proof. Following Dipper and James [5, 3.2], we define

$$u_k^+ = (T_{t_1} + Q^{-1} T_1)(T_{t_2} + Q^{-1} T_1) \cdots (T_{t_k} + Q^{-1} T_1)$$

for any $1 \leqslant k \leqslant n$, where $t_1 = t$ and $t_{i+1} = s_i t_i s_i$ for $i \geqslant 1$. The factors in the definition of u_k^+ commute with each other and we have

$$u_k^+ \, T_{s_i} = T_{s_i} \, u_k^+ \quad \text{for } 1 \leqslant i \leqslant k-1;$$

see [5, §3]. By Bonnafé [4, Prop. 2.5], we have

$$\mathbf{C}_{a_k} = u_k^+ \, T_{\sigma_k}^{-1} = T_{\sigma_k}^{-1} \, u_k^+,$$

where σ_k is the longest element in $\mathfrak{S}_k$. (Again, this is first proved in the "generic asymptotic case"; the general case follows from the argument in Remark 5.1.) Now let $k = l+1$ and note that

$$T_{\sigma_{l+1}} = T_{\sigma_l} T_{s_l \cdots s_2 s_1} \qquad \text{and} \qquad u_{l+1}^+ = (T_{t_{l+1}} + Q^{-1} T_1) \, u_l^+.$$

Since $T_{t_{l+1}}$ commutes with T_{s_i} for $1 \leqslant i \leqslant l$, we conclude that

$$\begin{aligned} \mathbf{C}_{a_{l+1}} &= T_{s_l \cdots s_2 s_1}^{-1} \, T_{\sigma_l}^{-1} \, (T_{t_{l+1}} + Q^{-1} T_1) \, u_l^+ \\ &= T_{s_l \cdots s_2 s_1}^{-1} \, (T_{t_{l+1}} + Q^{-1} T_1) \, T_{\sigma_l}^{-1} \, u_l^+ \\ &= T_{s_l \cdots s_2 s_1}^{-1} \, (T_{t_{l+1}} + Q^{-1} T_1) \, \mathbf{C}_{a_l} \end{aligned}$$

and so $T_{ts_1 s_2 \cdots s_l} \mathbf{C}_{a_l} = \mathbf{C}_{a_{l+1}} - Q^{-1} T_{s_l \cdots s_1}^{-1} \mathbf{C}_{a_l}$, as required. □

The following definitions are inspired by Bonnafé's construction in [4, §3]. Let $w \in W_n$ and write $w = a_w a_l \sigma_w b_w^{-1}$ as usual, where $l := l_t(w)$. We set

$$E_w := T_{a_w} \, \mathbf{C}_{a_l} \, \mathbf{C}_{\sigma_w b_w^{-1}} = T_{a_w} \, \mathbf{C}_{a_l \sigma_w b_w^{-1}},$$

where the second equality holds by Lemma 5.5. One easily shows that the elements $\{E_w \mid w \in W_n\}$ form a basis of $\mathcal{H}_n$. We will be interested in the base change from the Kazhdan–Lusztig basis to this new basis.

For $y, w \in W_n$, we write $y \preceq w$ if the following conditions are satisfied:

(1) $l := l_t(y) = l_t(w)$,

(2) $\sigma_y b_y^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$, and
(3) $l(y) < l(w)$ or $y = w$.

We write $y \prec w$ if $y \preceq w$ and $y \neq w$. Since $\{E_w\}$ is a basis of $\mathcal{H}_n$, we can write uniquely

$$\overline{E}_w = T_{a_w^{-1}}^{-1}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}} = \sum_{y \in W_n} \overline{\lambda}_{y,w}\, E_y \qquad \text{where } \lambda_{y,w} \in A.$$

Lemma 5.7. *We have $\lambda_{w,w} = 1$ and $\lambda_{y,w} = 0$ unless $y \preceq w$. Furthermore, we have $\lambda_{y,w} \in \mathbb{Z}[q, q^{-1}]$.*

Proof. We argue as in the proof of Lemma 3.3. Let $w \in W_n$ and $l = l_t(w)$. We have

$$T_{a_w^{-1}}^{-1} = \sum_{z \in W_n} \overline{R}^*_{z,a_w}\, T_z$$

where $R^*_{z,a_w} \in A$ are the "absolute" R-polynomials defined in [15, §1]. We have $R^*_{a_w,a_w} = 1$ and $R^*_{z,a_w} = 0$ unless $z \leqslant a_w$. Since $a_w \in \mathfrak{S}_n$, we have $R^*_{z,a_w} \in \mathbb{Z}[q, q^{-1}]$.

Now let $z \in W_n$ be such that T_z occurs in the above expression. Then we can write $z = c\sigma$ where $c \in X_{l,n-l}$ and $\sigma \in \mathfrak{S}_{l,n-l}$. Since $l(c\sigma) = l(c) + l(\sigma)$, we have $T_z = T_c\, T_\sigma$ and so

$$\overline{E}_w = \sum_{c,\sigma} \overline{R}^*_{c\sigma,a_w}\, T_c\, T_\sigma\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}},$$

where the sum runs over all $c \in X_{l,n-l}$ and $\sigma \in \mathfrak{S}_{l,n-l}$. Now we can also write $T_\sigma = \sum_{\sigma'} \tilde{p}_{\sigma',\sigma} \mathbf{C}_{\sigma'}$ where $\tilde{p}_{\sigma',\sigma} \in \mathbb{Z}[q, q^{-1}]$ and the sum runs over all $\sigma' \in \mathfrak{S}_{l,n-l}$. Note that $\tilde{p}_{\sigma,\sigma} = 1$ and $\tilde{p}_{\sigma',\sigma} = 0$ unless $\sigma' \leqslant \sigma$. Thus, we have

$$\overline{E}_w = \sum_{c,\sigma,\sigma'} \overline{R}^*_{c\sigma,a_w} \tilde{p}_{\sigma',\sigma}\, T_c\, \mathbf{C}_{\sigma'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}},$$

where the sum runs over all $c \in X_{l,n-l}$ and all $\sigma, \sigma' \in \mathfrak{S}_{l,n-l}$. Now Lemma 5.5 shows that

$$\mathbf{C}_{\sigma'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}} = \mathbf{C}_{a_l}\, \mathbf{C}_{a_l \sigma' a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}}.$$

Since $a_l \sigma' a_l \in \mathfrak{S}_{l,n-l}$, we can write

$$\mathbf{C}_{a_l\sigma' a_l} \mathbf{C}_{\sigma_w b_w^{-1}} = \sum_{\sigma'' \in \mathfrak{S}_{l,n-l}} h_{a_l\sigma' a_l, \sigma_w b_w^{-1}, \sigma''}\, \mathbf{C}_{\sigma''},$$

where $h_{a_l\sigma' a_l, \sigma_w b_w^{-1}, \sigma''} \in \mathbb{Z}[q, q^{-1}]$. So we conclude that

$$\overline{E}_w = \sum_{c,\sigma,\sigma',\sigma''} \overline{R}^*_{c\sigma,a_w} \tilde{p}_{\sigma',\sigma} h_{a_l\sigma' a_l, \sigma_w b_w^{-1}, \sigma''}\, T_c\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma''},$$

where the sum runs over all $c \in X_{l,n-l}$ and all $\sigma, \sigma', \sigma'' \in \mathfrak{S}_{l,n-l}$. Now every term $T_c \mathbf{C}_{a_l} \mathbf{C}_{\sigma''}$ in the above sum is of the form E_y for a unique $y \in W_n$ where $l = l_t(y)$, $a_y = c$, $\sigma_y b_y^{-1} = \sigma''$. So we can re-write the above expression as

$$\overline{E}_w = \sum_{\substack{y \in W_n \\ l_t(y) = l}} \overline{\lambda}_{y,w}\, E_y$$

where

$$\lambda_{y,w} = \sum_{\sigma,\sigma' \in \mathfrak{S}_{l,n-l}} R^*_{a_y\sigma,a_w}\, \overline{\tilde{p}}_{\sigma',\sigma}\, h_{a_l\sigma' a_l, \sigma_w b_w^{-1}, \sigma_y b_y^{-1}} \in \mathbb{Z}[q, q^{-1}].$$

Assume that $\lambda_{y,w} \neq 0$. We must show that $y \preceq w$. First of all, we certainly have $l = l_t(w) = l_t(y)$. Furthermore, there exist $\sigma, \sigma' \in \mathfrak{S}_{l,n-l}$ such that

$$R^*_{a_y\sigma,a_w} \neq 0, \qquad \tilde{p}_{\sigma',\sigma} \neq 0, \qquad h_{a_l\sigma' a_l, \sigma_w b_w^{-1}, \sigma_y b_y^{-1}} \neq 0.$$

The first condition implies $a_y\sigma \leqslant a_w$ and so $l(a_y\sigma) \leqslant l(a_w)$. The second condition implies $l(\sigma') \leqslant l(\sigma)$, while the third condition implies that $\sigma_y b_y^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$ and $l(\sigma_y b_y^{-1}) \leqslant l(\sigma') + l(\sigma_w b_w^{-1})$. (See (2.1) and note that $l(a_l\sigma' a_l) = l(\sigma')$.) Hence we also have $l(y) \leqslant l(w)$. Altogther, this means that $y \preceq w$. Finally, if $y = w$, it is readily checked that $\lambda_{w,w} = 1$. $\square$

The above result shows that, for any $w \in W_n$, we have

$$\overline{E}_w = E_w + \sum_{\substack{y \in W_n \\ y \prec w}} \overline{\lambda}_{y,w}\, E_y \qquad \text{where } \lambda_{y,w} \in \mathbb{Z}[q, q^{-1}].$$

We can now use exactly the same arguments as in the proofs of Lemma 3.2 and Proposition 3.3 in [8] (which themselves are an adaptation of the proof of Lusztig [15, Prop. 2]) to conclude that

$$\mathbf{C}_w = E_w + \sum_{\substack{y \in W_n \\ y \prec w}} \pi_{y,w}\, E_y$$

where $\pi_{y,w} \in q^{-1}\mathbb{Z}[q^{-1}]$ for any $y \prec w$. Indeed, the family of elements

$$\{\pi_{y,w} \mid y, w \in W_n, y \preceq w\}$$

is uniquely determined by the following three conditions:

(KL1') $$\pi_{w,w} = 1,$$

(KL2') $$\pi_{y,w} \in A_{<0} \qquad \text{if } y \prec w,$$

(KL3') $$\overline{\pi}_{y,w} - \pi_{y,w} = \sum_{\substack{z \in W_n \\ y \prec z \preceq w}} \lambda_{y,z}\, \pi_{z,w} \qquad \text{if } y \prec w.$$

Since $\lambda_{y,w} \in \mathbb{Z}[q, q^{-1}]$, it then follows that $\pi_{y,w} \in q^{-1}\mathbb{Z}[q^{-1}]$ if $y \prec w$.

Corollary 5.8. *Let $w \in W_n$.*

(a) *$\mathbf{C}_w$ can be written as an A-linear combination of E_w and terms E_y where $y \prec w$.*
(b) *E_w can be written as an A-linear combination of $\mathbf{C}_w$ and terms $\mathbf{C}_y$ where $y \prec w$.*

Proof. (a) See the above expression for $\mathbf{C}_w$. (b) Argue as in the proof of Corollary 3.5. $\square$

The next two results describe the action of $\mathbf{C}_t$ and $\mathbf{C}_{s_i}$ on E_w.

Lemma 5.9. *Let $w \in W_n$ and $s = s_i$ for some $1 \leqslant i \leqslant n-1$. Then $\mathbf{C}_s E_w$ is an A-linear combination of terms E_z where $l := l_t(z) = l_t(w)$ and $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$.*

Proof. Recall that $E_w = T_{a_w}\,\mathbf{C}_{a_l}\,\mathbf{C}_{\sigma_w b_w^{-1}}$. Now $\mathbf{C}_s = T_s + q^{-1}T_1$ and so

$$\mathbf{C}_s E_w = T_s E_w + q^{-1} E_w.$$

By Deodhar's Lemma (see [11, 2.1.2]), there are three cases to consider.

(i) $sa_w \in X_{l,n-l}$ and $l(sa_w) > l(a_w)$. Then

$$T_s\, E_w = T_s\, T_{a_w}\,\mathbf{C}_{a_l}\,\mathbf{C}_{\sigma_w b_w^{-1}} = T_{sa_w}\,\mathbf{C}_{a_l}\,\mathbf{C}_{\sigma_w b_w^{-1}} = E_{sw}$$

and so $\mathbf{C}_s E_w = E_{sw} + q^{-1} E_w$. Since $sw = (sa_w) a_l \sigma_w b_w^{-1}$, the required conditions are satisfied.

(ii) $sa_w \in X_{l,n-l}$ and $l(sa_w) < l(a_w)$. Then $T_s T_{a_w} = T_{sa_w} + (q - q^{-1}) T_{a_w}$ and so

$$T_s E_w = E_{sw} + (q - q^{-1}) E_w.$$

This yields $\mathbf{C}_s E_w = E_{sw} + q E_w$. Since, again, $sw = (sa_w) a_l \sigma_w b_w^{-1}$, the required conditions are satisfied.

(iii) $sa_w = a_w s'$ for some $s' \in \Sigma_{l,n-l}$. Then $l(sa_w) = l(a_w) + 1 = l(a_w s')$ and so $T_s T_{a_w} = T_{sa_w} = T_{a_w s'} = T_{a_w} T_{s'}$. This yields

$$T_s E_w = T_{a_w}\, T_{s'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}} = T_{a_w}\, \mathbf{C}_{s'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}} - q^{-1} E_w$$

and so

$$\mathbf{C}_s E_w = T_{a_w}\, \mathbf{C}_{s'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}}.$$

Now Lemma 5.5 shows that

$$\mathbf{C}_{s'}\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}} = \mathbf{C}_{a_l}\, \mathbf{C}_{a_l s' a_l} \mathbf{C}_{\sigma_w b_w^{-1}}.$$

Since $a_l s' a_l \in \mathfrak{S}_{l,n-l}$, we can express $\mathbf{C}_{a_l s' a_l} \mathbf{C}_{\sigma_w b_w^{-1}}$ as an A-linear combination of terms $\mathbf{C}_{\rho b^{-1}}$ where $\rho \in \mathfrak{S}_{l,n-l}$ and $b \in X_{l,n-l}$ are such that $\rho b^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$. We conclude that $\mathbf{C}_s E_w$ is an A-linear combination of terms E_y where $a_y = a_w$, $l := l_t(y) = l_t(w)$ and $\sigma_y b_y^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$. $\square$

Lemma 5.10. *Let $w \in W_n$ and $l = l_t(w)$. Then $\mathbf{C}_t E_w$ is an A-linear combination of terms E_z where $l_t(z) > l$ or where $l_t(z) = l$ and $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$.*

Proof. The following argument is inspired from the proof of Dipper, James, and Murphy [6, Lemma 4.9]. Write $w = a_w a_l \sigma_w b_w^{-1}$. We distinguish three cases.

Case 1. We have $l = 0$. Then $a_w = 1$ and so $E_w = \mathbf{C}_{\sigma_w b_w^{-1}}$. By Proposition 5.4, $\mathbf{C}_t E_w$ is a linear combination of terms $\mathbf{C}_z$ where $l_t(z) \geqslant 1$. Using Corollary 5.8(a), we see that $\mathbf{C}_t E_w$ can also be written as a linear combination of term $E_{z'}$ where $l_t(z') \geqslant 1$.

Case 2. We have $l \geqslant 1$ and the element a_w fixes the number 1. (Here, we regard a_w as an element of $\mathfrak{S}_n$.) Then T_t commutes with T_{a_w}. Since $l(ta_l) < l(a_l)$, we have $T_t \mathbf{C}_{a_l} = -Q^{-1} \mathbf{C}_{a_l}$. So $\mathbf{C}_t E_w$ is a multiple of E_w and we are done in this case.

Case 3. We have $l \geqslant 1$ and the element a_w does not fix the number 1. Then we consider the Young subgroup $\mathfrak{S}_{1,n-1} \subset \mathfrak{S}_n$. We can write $a_w \in \mathfrak{S}_n$ as a product of an element of $\mathfrak{S}_{1,n-1}$ times a distinguished right coset representative of $\mathfrak{S}_{1,n-1}$ in $\mathfrak{S}_n$. These coset representatives are given by

$$\{1, \quad s_1, \quad s_1 s_2, \quad s_1 s_2 s_3, \quad \ldots, \quad s_1 s_2 s_3 \cdots s_{n-1}\}.$$

Thus, we have $a_w = \sigma s_1 s_2 \cdots s_m$ for some $m \in \{0, 1, \ldots, n-1\}$ where $l(a_w) = m + l(\sigma)$. Now, the fact that $a_w \in X_{l,n-l}$ implies that we must have $m = l$ and so

$$a_w = \sigma s_1 s_2 \cdots s_l \quad \text{for some } \sigma \in \mathfrak{S}_{1,n-1} \text{ such that } l(a_w) = l + l(\sigma).$$

This yields

$$T_t T_{a_w} = T_t T_\sigma T_{s_1 s_2 \cdots s_l} = T_\sigma T_t T_{s_1 s_2 \cdots s_l} = T_\sigma T_{t s_1 s_2 \cdots s_l}.$$

Using the expression in Lemma 5.6, we obtain

$$T_t T_{a_w} \mathbf{C}_{a_l} = T_\sigma\, T_{t s_1 s_2 \cdots s_l}\, \mathbf{C}_{a_l} = T_\sigma\, \mathbf{C}_{a_{l+1}} + T_\sigma\, h(a_l)\, \mathbf{C}_{a_l},$$

where $h(a_l)$ is an A-linear combination of basis elements T_π with $\pi \leqslant s_1 \cdots s_l$. This yields

$$T_t E_w = T_\sigma\, \mathbf{C}_{a_{l+1}} \mathbf{C}_{\sigma_w b_w^{-1}} + T_\sigma\, h(a_l)\, \mathbf{C}_{a_l} \mathbf{C}_{\sigma_w b_w^{-1}}.$$

Now Lemma 5.9 shows that $T_\sigma\, h(a_l)\, \mathbf{C}_{a_l}\, \mathbf{C}_{\sigma_w b_w^{-1}}$ is a linear combination of terms E_z where $l = l_t(z)$ and $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$. On the other hand, by Proposition 5.4, $T_\sigma \mathbf{C}_{a_{l+1}} \mathbf{C}_{\sigma_w b_w^{-1}}$ is a linear combination of terms $\mathbf{C}_{w'}$ where $l_t(w') \geqslant l+1$. Hence this is also a linear combination of terms $E_{z'}$ where $l_t(z') \geqslant l+1$. □

Theorem 5.11. *Let $x, y \in W_n$ be such that $l := l_t(x) = l_t(y)$. Then we have $x \leqslant_{\mathcal{L}} y$ if and only if $\sigma_x b_x^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$.*

Proof. First assume that $x \leqslant_{\mathcal{L}} y$. We must show that $\sigma_x b_x^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$. Now, by definition, there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that $x_{i-1} \leftarrow_{\mathcal{L}} x_i$ for all i. By Proposition 5.4, we have $l_t(x_{i-1}) \geqslant l_t(x_i)$ for all i. Since $l_t(x) = l_t(y)$, we conclude that all x_i have the same t-length. Thus, it is enough to consider the case where $x \leftarrow_{\mathcal{L}} y$, that is, we have that $\mathbf{C}_x$ occurs in $\mathbf{C}_s \mathbf{C}_y$, for some $s \in \{t, s_1, \ldots, s_{n-1}\}$.

Assume first that $s = s_i$ for some $i \in \{1, \ldots, n-1\}$. By Corollary 5.8(a), we can write $\mathbf{C}_y$ as an A-linear combination of E_w where $w \preceq y$. So $\mathbf{C}_s \mathbf{C}_y$ is an A-linear combination of terms of the form $\mathbf{C}_s E_w$ where $w \preceq y$. Now consider such a term. By Lemma 5.9, $\mathbf{C}_s E_w$ is a linear combination of terms E_z where $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$. Consequently, by Corollary 5.8(b), $\mathbf{C}_s E_w$ is a linear combination of terms $\mathbf{C}_z$ where $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$, as required.

Now assume that $s = t$. By Corollary 5.8(a), we can write $\mathbf{C}_y$ as an A-linear combination of E_w where $w \preceq y$. So $\mathbf{C}_t \mathbf{C}_y$ is an A-linear combination of terms of the form $\mathbf{C}_t E_w$ where $w \preceq y$. By Lemma 5.10 and Corollary 5.8(b), we can write any such term as a linear combination of terms $\mathbf{C}_z$ where $l_t(z) > l$ or $l_t(z) = l$ and $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_w b_w^{-1}$.

Summarizing, we have shown that $\mathbf{C}_t \mathbf{C}_y$ is a linear combination of terms $\mathbf{C}_z$ where $l = l_t(z) = l_t(y)$ and $\sigma_z b_z^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$, and terms $\mathbf{C}_{w'}$ where $l_t(w') > l$. Hence, since $l_t(x) = l$, we must have $\sigma_x b_x^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$, as required.

Conversely, let us assume that $\sigma_x b_x^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$. We must show that $x \leqslant_{\mathcal{L}} y$. Again, it is enough to consider the case where $\sigma_x b_x^{-1} \leftarrow_{\mathcal{L},l} \sigma_y b_y^{-1}$, that is, $\mathbf{C}_{\sigma_x b_x^{-1}}$ occurs in $\mathbf{C}_s \mathbf{C}_{\sigma_y b_y^{-1}}$ for some $s = s_i$ where $i \neq l$. Thus, writing

$$\mathbf{C}_{s_i} \mathbf{C}_{\sigma_y b_y^{-1}} = \sum_{\pi \in \mathfrak{S}_{l,n-l}} \sum_{z \in X_{l,n-l}} h_{s_i, \sigma_y b_y^{-1}, \pi z^{-1}} \mathbf{C}_{\pi z^{-1}},$$

we have $h_{s_i, \sigma_y b_y^{-1}, \sigma_x b_x^{-1}} \neq 0$. Multiplying the above equation on the left by $\mathbf{C}_{a_l}$ and using Lemma 5.5, we conclude that

$$\begin{aligned}
\mathbf{C}_{s'} \mathbf{C}_{a_l \sigma_y b_y^{-1}} &= \mathbf{C}_{s'} \mathbf{C}_{a_l} \mathbf{C}_{\sigma_y b_y^{-1}} = \mathbf{C}_{a_l} \mathbf{C}_{s_i} \mathbf{C}_{\sigma_y b_y^{-1}} \\
&= \sum_{\pi \in \mathfrak{S}_{l,n-l}} \sum_{z \in X_{l,n-l}} h_{s_i, \sigma_y b_y^{-1}, \pi z^{-1}} \mathbf{C}_{a_l \pi z^{-1}},
\end{aligned}$$

where $s' = a_l s_i a_l \in \mathfrak{S}_{l,n-l}$. Considering the term corresponding to $\pi = \sigma_x$ and $z = b_x$, we see that $a_l \sigma_x b_x^{-1} \leqslant_{\mathcal{L}} a_l \sigma_y b_y^{-1}$. Finally, this yields

$$x = a_x a_l \sigma_x b_x^{-1} \sim_{\mathcal{L}} a_l \sigma_x b_x^{-1} \leqslant_{\mathcal{L}} a_l \sigma_y b_y^{-1} \sim_{\mathcal{L}} a_x a_l \sigma_x b_x^{-1} = y,$$

by Theorem 5.2. □

The above result has two immediate applications.

First, it provides a refinement of Theorem 5.2. Indeed, if we have $x \sim_{\mathcal{L}} y$, then Theorem 5.11 shows that $\sigma_x b_x^{-1} \sim_{\mathcal{L},l} \sigma_y b_y^{-1}$ and, hence, $b_x = b_y$ and $\sigma_x \sim_{\mathcal{L},l} \sigma_y$ (by Proposition 4.4 and Lemma 4.3).

Second, it refines the methods that Bonnafé used in [4]. Indeed, we obtain a new proof of the following statement concerning the two-sided Kazhdan–Lusztig pre-order.

Corollary 5.12 (See Bonnafé [4])**.** *Let $x, y \in W_n$. Then the following hold.*

(a) *If $l := l_t(x) = l_t(y)$ and $x \leqslant_{\mathcal{LR}} y$, then $\sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$.*

(b) *If $x \sim_{\mathcal{LR}} y$, then $l := l_t(x) = l_t(y)$ and $\sigma_x \sim_{\mathcal{LR},l} \sigma_y$.*

Proof. (a) Assume that $l := l_t(x) = l_t(y)$. To prove the implication "$x \leqslant_{\mathcal{LR}} y \Rightarrow \sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$", we may assume without loss of generality that $x \leqslant_{\mathcal{L}} y$ or $x^{-1} \leqslant_{\mathcal{L}} y^{-1}$ (since these are the elementary steps in the definition of $\leqslant_{\mathcal{LR}}$.) If $x \leqslant_{\mathcal{L}} y$, then Theorem 5.11 and Proposition 4.4 immediately yield $\sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$, as required. Assume now that $x^{-1} \leqslant_{\mathcal{L}} y^{-1}$. We have

$$x^{-1} = (a_x a_l \sigma_x b_x^{-1})^{-1} = b_x a_l (a_l \sigma_x^{-1} a_l) a_x^{-1}$$

and so

$$a_{x^{-1}} = b_x, \quad \sigma_{x^{-1}} = \sigma_l \sigma_x^{-1} \sigma_l, \quad b_{x^{-1}} = a_x$$

where σ_l is the longest element of $\mathfrak{S}_l$. Note that $a_l = w_l \sigma_l$ where w_l is the longest element in W_l, and that w_l commutes with all elements of $\mathfrak{S}_{l,n-l}$; see [3, §4]. A similar remark applies to $y = a_y a_l \sigma_y b_y^{-1}$. Now Theorem 5.11 and Proposition 4.4 imply $\sigma_l \sigma_x^{-1} \sigma_l \leqslant_{\mathcal{LR},l} \sigma_l \sigma_y^{-1} \sigma_l$. Furthermore, conjugation with σ_l defines a Coxeter group automorphism of $\mathfrak{S}_{l,n-l}$ and, hence, preserves the Kazhdan–Lusztig pre-order relations $\leqslant_{\mathcal{L},l}$, $\leqslant_{\mathcal{R},l}$ and $\leqslant_{\mathcal{LR},l}$; see [18, Cor. 11.7]. Consequently, we have $\sigma_x^{-1} \leqslant_{\mathcal{LR},l} \sigma_y^{-1}$. Finally, note that inversion certainly preserves the two-sided pre-order $\leqslant_{\mathcal{LR},l}$. Hence we have $\sigma_x \leqslant_{\mathcal{LR},l} \sigma_y$, as desired.

(b) If $x \leqslant_{\mathcal{LR}} y$, then $l_t(y) \leqslant l_t(x)$ by Proposition 5.4. Hence, if $x \sim_{\mathcal{LR}} y$, then we automatically have $l := l_t(x) = l_t(y)$ and (a) yields $\sigma_x \sim_{\mathcal{LR},l} \sigma_y$. ☐

Now our efforts will be rewarded. Combining Example 4.9 with Theorem 5.2, Theorem 5.11 and Corollary 5.12, we obtain:

Theorem 5.13. *Recall that we are in the "asymptotic case" in type B_n. Then the following implication holds for all $x, y \in W_n$:*

$$(\spadesuit) \qquad x \leqslant_{\mathcal{L}} y \quad \text{and} \quad x \sim_{\mathcal{LR}} y \quad \Rightarrow \quad x \sim_{\mathcal{L}} y.$$

Proof. Let $x, y \in W_n$ be such that $x \leqslant_{\mathcal{L}} y$ and $x \sim_{\mathcal{LR}} y$. First of all, Corollary 5.12 implies that $l := l_t(x) = l_t(y)$ and $\sigma_x \sim_{\mathcal{LR},l} \sigma_y$. Furthermore, Theorem 5.11 implies that $\sigma_x b_x^{-1} \leqslant_{\mathcal{L},l} \sigma_y b_y^{-1}$. Thus, the hypotheses of Conjecture 4.5 are satisfied for the elements $\sigma_x b_x^{-1}$ and $\sigma_y b_y^{-1}$ in the symmetric group $\mathfrak{S}_n$, where we consider the parabolic subgroup $\mathfrak{S}_{l,n-l}$. Hence Example 4.9 implies that $b_x = b_y$ and $\sigma_x \sim_{\mathcal{L},l} \sigma_y$. Then Theorem 5.2 yields $x \sim_{\mathcal{L}} y$, as desired. ☐

Corollary 5.14. *The sets $\{\mathfrak{R}_\lambda \mid \lambda \in \Lambda_n\}$ are precisely the two-sided cells of W_n.*

Proof. Once ($\spadesuit$) is known to hold, two elements $x, y \in W_n$ lie in the same two-sided cell if and only if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ of elements in W_n such that, for each i, we have $x_{i-1} \sim_{\mathcal{L}} x_i$ or $x_{i-1} \sim_{\mathcal{R}} x_i$. Hence the assertion is an immediate consequence of Corollary 5.3. ☐

6. On the left cell representations in type B_n

We keep the set-up of the previous section, where W_n is a Coxeter group of type B_n and where we consider the Kazhdan–Lusztig cells in the "asymptotic case". Recall the partition

$$W_n = \coprod_{\lambda \in \Lambda_n} \mathfrak{R}_\lambda,$$

where Λ_n is set of all pairs of partitions of total size n. An element $w \in W_n$ belongs to $\mathfrak{R}_\lambda$ if and only if w corresponds to a pair of bitableaux of shape λ under the generalized Robinson–Schensted correspondence. By Corollary 5.14, each set $\mathfrak{R}_\lambda$ is a two-sided cell.

Recall that we denote by $\mathrm{Irr}(\mathcal{H}_{n,K})$ the set of irreducible characters of $\mathcal{H}_{n,K}$. For any left cell $\mathfrak{C}$, we denote by $\chi_{\mathfrak{C}}$ the character afforded by the $\mathcal{H}_{n,K}$-module $[\mathfrak{C}]_K = K \otimes_A [\mathfrak{C}]_A$.

Theorem 6.1 (Bonnafé and Iancu [3] and Bonnafé [4, §5]). *In the above setting, we have $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_{n,K})$ for any left cell $\mathfrak{C}$ in W_n. Furthermore, let $\mathfrak{C}, \mathfrak{C}_1$ be left cells and assume that $\mathfrak{C} \subseteq \mathfrak{R}_\lambda$, $\mathfrak{C}_1 \subseteq \mathfrak{R}_\mu$ where $\lambda, \mu \in \Lambda_n$. Then the characters $\chi_{\mathfrak{C}}$ and $\chi_{\mathfrak{C}_1}$ are equal if and only if $\lambda = \mu$.*

(This is another example where the discussion in Remark 5.1 applies: the above statements were first proved in [3, §7] for the weight function $L^\circ \colon W_n \to \mathbb{Z}^2$. Using Remark 5.1(a), one easily shows that $[\mathfrak{C}]_A = A \otimes_{A^\circ} [\mathfrak{C}]_{A^\circ}$ where A is regarded as an A°-module via the map $\theta \colon A^\circ \to A$.)

The main result of this section is Theorem 6.3 which shows that we even have $\mathfrak{C} \approx \mathfrak{C}_1$ for any two left cells $\mathfrak{C}, \mathfrak{C}_1 \subseteq \mathfrak{R}_\lambda$, where "$\approx$" is the relation introduced in Definition 2.5.

Let us fix a pair of partitions $\lambda = (\lambda_1, \lambda_2) \in \Lambda_n$ and let $\mathfrak{C} \subseteq \mathfrak{R}_\lambda$ be a left cell. We set $l := |\lambda_2|$. By [3, Prop. 4.8], we have $l_t(w) = l$ for all $w \in \mathfrak{R}_\lambda$. In particular, we have $l_t(w) = l$ for all $w \in \mathfrak{C}$. Now recall the decomposition $w = a_w a_l \sigma_w b_w^{-1}$ for any element $w \in W_n$, where $l = l_t(w)$. We set

$$\overline{\mathfrak{C}} := \{\sigma \in \mathfrak{S}_{l,n-l} \mid \sigma = \sigma_w \text{ for some } w \in \mathfrak{C}\}.$$

By Theorem 5.2, $\overline{\mathfrak{C}}$ is a left cell in $\mathfrak{S}_{l,n-l}$. Next recall that $\mathfrak{S}_{l,n-l} = \mathfrak{S}_l \times \mathfrak{S}_{[l+1,n]}$ where $\mathfrak{S}_{[l+1,n]} \cong \mathfrak{S}_{n-l}$. It is well-known and easy to check that the Kazhdan–Lusztig pre-order relations are compatible with direct products; in particular, every left cell in $\mathfrak{S}_{l,n-l}$ is a product of a left cell in $\mathfrak{S}_l$ and a left cell in $\mathfrak{S}_{[l+1,n]}$. Thus, we can write

$$\overline{\mathfrak{C}} = \overline{\mathfrak{C}}^{(l)} \cdot \overline{\mathfrak{C}}^{(n-l)}$$

where $\overline{\mathfrak{C}}^{(l)}$ is a left cell in $\mathfrak{S}_l$ and $\overline{\mathfrak{C}}^{(n-l)}$ is a left cell in $\mathfrak{S}_{[l+1,n]}$. We use the explicit dot to indicate that the lengths of elements add up in this product: we have $l(\sigma\tau) = l(\sigma) + l(\tau)$ for $\sigma \in \overline{\mathfrak{C}}^{(l)}$ and $\tau \in \overline{\mathfrak{C}}^{(n-l)}$. By Theorem 5.2, we have $b_x = b_y$ for all $x, y \in \mathfrak{C}$. Let us denote $b = b_w$ for $w \in \mathfrak{C}$. Then we have

$$\mathfrak{C} = X_{l,n-l} \cdot a_l \cdot \overline{\mathfrak{C}} \cdot b^{-1} = \{c a_l \sigma b^{-1} \mid c \in X_{l,n-l}, \sigma \in \overline{\mathfrak{C}}\}.$$

A first reduction is provided by the following result:

Lemma 6.2 (Bonnafé and Iancu [3, Prop. 7.2] and Remark 5.1). *In the above setting, $\mathfrak{C}b$ is a left cell and we have*

$$\mathfrak{C} \approx \mathfrak{C}b = X_{l,n-l} \cdot a_l \cdot \overline{\mathfrak{C}}.$$

Now we can state the main result of this section. Again, this is in complete formal analogy to the situation in the symmetric group $\mathfrak{S}_n$; see Example 2.6(b).

Theorem 6.3. *Let $\lambda \in \Lambda_n$. Then we have $\mathfrak{C} \approx \mathfrak{C}_1$ for all left cells $\mathfrak{C}, \mathfrak{C}_1 \subseteq \mathfrak{R}_\lambda$. Recall that this means that there exists a bijection $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, such that $h_{w,x,y} = h_{w,x_1,y_1}$ for all $w \in W_n$ and all $x, y \in \mathfrak{C}$.*

The bijection $x \mapsto x_1$ is uniquely determined by the condition that $x_1 \in \mathfrak{C}_1$ is the unique element in the same right cell as $x \in \mathfrak{C}$.

Proof. First note that the second statement (concerning the uniqueness of the bijection) is a consequence of the first. Indeed, if there exists a bijection $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}_1$, $x \mapsto x_1$, satisfying ($\heartsuit$), then Proposition 2.13 shows that $x \sim_{\mathcal{R}} x_1$ for any $x \in \mathfrak{C}$. But Corollary 5.3 shows that two elements which are in the same right cell and in the same left cell are equal. Hence the element x_1 is uniquely determined by the condition that $x_1 \sim_{\mathcal{R}} x$.

To establish the existence of such a bijection, let $\lambda = (\lambda_1, \lambda_2)$ and set $l := |\lambda_2|$. Let $\mathfrak{C}, \mathfrak{C}_1 \subseteq \mathfrak{R}_\lambda$ be two left cells. We set

$$\overline{\mathfrak{C}} := \{\sigma \in \mathfrak{S}_{l,n-l} \mid \sigma = \sigma_w \text{ for some } w \in \mathfrak{C}\},$$
$$\overline{\mathfrak{C}}_1 := \{\sigma \in \mathfrak{S}_{l,n-l} \mid \sigma = \sigma_w \text{ for some } w \in \mathfrak{C}_1\};$$

by the above discussion, these are left cells in $\mathfrak{S}_{l,n-l}$. Furthermore, we can write

$$\overline{\mathfrak{C}} = \overline{\mathfrak{C}}^{(l)} \cdot \overline{\mathfrak{C}}^{(n-l)} \qquad \text{and} \qquad \overline{\mathfrak{C}}_1 = \overline{\mathfrak{C}}_1^{(l)} \cdot \overline{\mathfrak{C}}_1^{(n-l)}$$

where $\overline{\mathfrak{C}}^{(l)}$, $\overline{\mathfrak{C}}_1^{(l)}$ are left cells in $\mathfrak{S}_l$ and $\overline{\mathfrak{C}}^{(n-l)}$, $\overline{\mathfrak{C}}_1^{(n-l)}$ are left cells in $\mathfrak{S}_{[l+1,n]}$. We claim that

$$(*) \qquad \overline{\mathfrak{C}}^{(l)} \approx \overline{\mathfrak{C}}_1^{(l)}, \qquad \overline{\mathfrak{C}}^{(n-l)} \approx \overline{\mathfrak{C}}_1^{(n-l)}, \qquad \overline{\mathfrak{C}} \approx \overline{\mathfrak{C}}_1.$$

Indeed, by Example 2.6, the classical Robinson–Schensted correspondence associates to a left cell of $\mathfrak{S}_l$ a partition of l and to a left cell in $\mathfrak{S}_{[l+1,n]}$ a partition of $n-l$. Thus, we can associate a pair of partitions to $\overline{\mathfrak{C}}$. By [3, 4.7], that pair of partitions is given by (λ_2, λ_1). A similar remark applies to $\overline{\mathfrak{C}}_1$, where we obtain the same pair of partitions. Now $(*)$ follows from Example 2.6(b) and the compatibility of left cells with direct products.

To continue the proof it is sufficient, by Lemma 6.2, to consider the case where

$$\mathfrak{C} = X_{l,n-l} \cdot a_l\overline{\mathfrak{C}} \qquad \text{and} \qquad \mathfrak{C}_1 = X_{l,n-l} \cdot a_l\overline{\mathfrak{C}}_1.$$

In this situation, we note that the sets $a_l\overline{\mathfrak{C}}$ and $a_l\overline{\mathfrak{C}}_1$ are contained in the parabolic subgroup

$$W_{l,n-l} = W_l \times \mathfrak{S}_{[l+1,n]} \quad \text{where} \quad W_l = \langle t, s_1, \ldots, s_{l-1}\rangle \quad (\text{type } B_l).$$

By [3, 4.1], we have $a_l = w_l\sigma_l$ where w_l is the longest element in W_l and σ_l is the longest element in $\mathfrak{S}_l$. Since multiplication with the longest element preserves left cells, the sets $\sigma_l\overline{\mathfrak{C}}^{(l)}$ and $\sigma_l\overline{\mathfrak{C}}_1^{(l)}$ are left cells in $\mathfrak{S}_l$. Hence $(*)$ and Lemma 2.8 show that

$$\sigma_l\overline{\mathfrak{C}}^{(l)} \approx \sigma_l\overline{\mathfrak{C}}_1^{(l)}.$$

Applying Theorem 5.2 to the group W_l, we notice that every left cell in $\mathfrak{S}_l$ also is a left cell in W_l. Hence the sets $\sigma_l\overline{\mathfrak{C}}^{(l)}$ and $\sigma_l\overline{\mathfrak{C}}_1^{(l)}$ are left cells in W_l. Then multiplication with the longest element $w_l \in W_l$ and Lemma 2.8 yield that

$$a_l\overline{\mathfrak{C}}^{(l)} = w_l(\sigma_l\overline{\mathfrak{C}}^{(l)}) \approx w_l(\sigma_l\overline{\mathfrak{C}}_1^{(l)}) = a_l\overline{\mathfrak{C}}_1^{(l)},$$

where the above sets are left cells in W_l. Using the compatibility of left cells with direct products, we obtain

$$a_l\overline{\mathfrak{C}} = (a_l\overline{\mathfrak{C}}^{(l)}) \cdot \overline{\mathfrak{C}}^{(n-l)} \approx (a_l\overline{\mathfrak{C}}_1^{(l)}) \cdot \overline{\mathfrak{C}}_1^{(n-l)} = a_l\overline{\mathfrak{C}}_1,$$

where the above sets are left cells in $W_{l,n-l}$.

Thus, we have two left cells in the parabolic subgroup $W_{l,n-l}$ which are related by "$\approx$". Now let $\hat{X}_{l,n-l}$ be the set of distinguished left coset representatives of $W_{l,n-l}$ in W_n. We certainly have $X_{l,n-l} \subseteq \hat{X}_{l,n-l}$ and so

$$\begin{aligned} \mathfrak{C} = X_{l,n-l} \cdot a_l\overline{\mathfrak{C}} \quad &\subseteq \quad \hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}, \\ \mathfrak{C}_1 = X_{l,n-l} \cdot a_l\overline{\mathfrak{C}}_1 \quad &\subseteq \quad \hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}_1. \end{aligned}$$

By Theorem 3.6, the sets $\hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}$ and $\hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}_1$ are both unions of left cells in W_n and we have

$$\hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}} \approx \hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}_1;$$

see Proposition 3.9. Furthermore, by Corollary 3.10, there is a left cell

$$\tilde{\mathfrak{C}} \subseteq \hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}} \qquad \text{such that} \qquad \tilde{\mathfrak{C}} \approx \mathfrak{C}_1.$$

It remains to show that $\mathfrak{C} = \tilde{\mathfrak{C}}$. This can be seen as follows. Since $\tilde{\mathfrak{C}} \approx \mathfrak{C}_1$, we have $\tilde{\mathfrak{C}} \subseteq \mathfrak{R}_\lambda$. In particular, all elements in $\tilde{\mathfrak{C}}$ must have t-length l. Now we leave it as an exercice to the reader to check that

$$X_{l,n-l} \cdot W_{l,n-l} = \{w \in W_n \mid l_t(w) \leqslant l\}.$$

Hence we must have $\tilde{\mathfrak{C}} \subseteq X_{l,n-l} \cdot W_{l,n-l}$. On the other hand, we also have $\tilde{\mathfrak{C}} \subseteq \hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}$. Since $X_{l,n-l} \subseteq \hat{X}_{l,n-l}$ and $a_l\overline{\mathfrak{C}} \subseteq W_{l,n-l}$, we conclude that

$$\tilde{\mathfrak{C}} \subseteq \big(X_{l,n-l} \cdot W_{l,n-l}\big) \cap \big(\hat{X}_{l,n-l} \cdot a_l\overline{\mathfrak{C}}\big) = X_{l,n-l} \cdot a_l\overline{\mathfrak{C}} = \mathfrak{C}$$

and so $\tilde{\mathfrak{C}} = \mathfrak{C}$, as required. $\square$

Following Graham and Lehrer [12, Definition 1.1], a quadruple $(\Lambda, M, C, *)$ is called a "cell datum" for $\mathcal{H}_n$ if the following conditions are satisfied.

(C1) Λ is a partially ordered set, $\{M(\lambda) \mid \lambda \in \Lambda\}$ is a collection of finite sets and

$$C\colon \coprod_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to \mathcal{H}_n$$

is an injective map whose image is an A-basis of $\mathcal{H}_n$;

(C2) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, write $C(S,T) = C_{S,T}^\lambda \in \mathcal{H}_n$. Then $*\colon \mathcal{H}_n \to \mathcal{H}_n$ is an A-linear anti-involution such that $(C_{S,T}^\lambda)^* = C_{T,S}^\lambda$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any element $h \in \mathcal{H}_n$ we have

$$hC_{S,T}^\lambda \equiv \sum_{S' \in M(\lambda)} r_h(S', S)\, C_{S',T}^\lambda \quad \mod \mathcal{H}_n(<\lambda),$$

where $r_h(S', S) \in A$ is independent of T and where $\mathcal{H}_n(<\lambda)$ is the A-submodule of $\mathcal{H}_n$ generated by $\{C_{S'',T''}^\mu \mid \mu < \lambda; S'', T'' \in M(\mu)\}$.

In this case, we call the basis $\{C_{S,T}^\lambda\}$ a **"cellular basis"** of $\mathcal{H}_n$.

One reason for the importance of a cellular structure lies in the fact that it leads to a general theory of "Specht modules" and various applications concerning modular representations; see [12] for more details. Graham and Lehrer [12, §5] already showed that $\mathcal{H}_n$ has a cellular structure, where they use a mixture of the Kazhdan–Lusztig basis and the standard basis. The point of the following result is that the Kazhdan–Lusztig basis in the "asymptotic case" directly gives a cellular structure. (The relation between the two structures will be discussed elsewhere.)

Corollary 6.4. *Recall that we are in the "asymptotic case" in type B_n. Then the Kazhdan–Lusztig basis $\{\mathbf{C}_w \mid w \in W_n\}$ is a "cellular basis" of $\mathcal{H}_n$.*

Proof. We specify a "cell datum" as follows. First of all, let $\Lambda := \Lambda_n$, the set of all pairs of partitions of total size n. By Corollary 5.14, these parametrize the two-sided cells of W_n. Hence we can define a partial order "$\leqslant$" on Λ by

$$\lambda \leqslant \mu \quad \text{if} \quad x \leqslant_{\mathcal{LR}} y \text{ for some } x \in \mathfrak{R}_\lambda,\, y \in \mathfrak{R}_\mu.$$

(More explicitly, we could use the dominance order on bipartitions; see [10, Prop. 5.3].) Next, for each $\lambda \in \Lambda_n$, let $M(\lambda) := \mathbb{T}_\lambda$, the set of n-standard bitableaux of shape λ. By Corollary 5.3, we have a bijection

$$\mathbb{T}_\lambda \times \mathbb{T}_\lambda \to \mathfrak{R}_\lambda, \qquad (T, T') \mapsto w_\lambda(T, T')$$

such that $w_\lambda(T, T')^{-1} = w_\lambda(T', T)$ for all $T, T' \in \mathbb{T}_\lambda$. We set

$$C^\lambda_{S,T} := \mathbf{C}_{w_\lambda(S,T)} \qquad \text{for } \lambda \in \Lambda_n \text{ and } S, T \in \mathbb{T}_\lambda.$$

Then the map

$$C\colon \coprod_{\lambda \in \Lambda_n} \mathbb{T}_\lambda \times \mathbb{T}_\lambda \to \mathcal{H}_n, \qquad (S,T) \mapsto C^\lambda_{S,T} \qquad (S, T \in \mathbb{T}(\lambda)),$$

satisfies the requirements in (C1).

We define $*\colon \mathcal{H}_n \to \mathcal{H}_n$ by $T_w^* = T_w^\flat = T_{w^{-1}}$ for all $w \in W_n$. This is an A-linear anti-involution such that $\mathbf{C}_w^* = \mathbf{C}_{w^{-1}}$ for all $w \in W_n$; see the remarks in (2.2). Thus, we have

$$(C^\lambda_{S,T})^* = \mathbf{C}^*_{w_\lambda(S,T)} = \mathbf{C}_{w_\lambda(S,T)^{-1}} = \mathbf{C}_{w_\lambda(T,S)} = C^\lambda_{T,S}$$

for all $\lambda \in \Lambda_n$ and $S, T \in \mathbb{T}_\lambda$. Hence condition (C2) is satisfied.

In order to check (C3), it is sufficient to assume that $h = \mathbf{C}_w$ for some $w \in W_n$. Let $\lambda \in \Lambda_n$ and $T \in \mathbb{T}_\lambda$. For any $S, S' \in \mathbb{T}_\lambda$, we define

$$r_w(S', S) := h_{w,x,x'} \quad \text{where} \quad \begin{cases} x := w_\lambda(S,T), \\ x' := w_\lambda(S',T). \end{cases}$$

Now consider the product

$$\mathbf{C}_w C^\lambda_{S,T} = \mathbf{C}_w \mathbf{C}_x = \sum_{y \in W_n} h_{w,x,y} \mathbf{C}_y \qquad \text{where } x = w_\lambda(S,T).$$

If $h_{w,x,y} \neq 0$, then $y \leqslant_{\mathcal{L}} x$. Hence Theorem 5.13 shows that either $x \sim_{\mathcal{L}} y$ or $x <_{\mathcal{LR}} y$. So we can write

$$\mathbf{C}_w C^\lambda_{S,T} = \sum_{\substack{y \in W_n \\ x \sim_{\mathcal{L}} y}} h_{w,x,y} \mathbf{C}_y \quad \bmod \mathcal{H}_n(< \lambda).$$

Using Corollary 5.3, every $y \in W_n$ such that $x \sim_{\mathcal{L}} y$ has the form $y = w_\lambda(S', T)$ for some $S' \in \mathbb{T}_\lambda$. So we can rewrite the above relation as follows:

$$\mathbf{C}_w C^\lambda_{S,T} = \sum_{S' \in \mathbb{T}_\lambda} r_w(S', S) C^\lambda_{S',T} \quad \bmod \mathcal{H}_n(< \lambda).$$

Finally, we must check that $r_w(S', S)$ is independent of T. To see this, let $T_1 \in \mathbb{T}_\lambda$ and define $r_w^1(S', S) := h_{w,x_1,x_1'}$ where $x_1 = w_\lambda(S, T_1)$ and $x_1' = w_\lambda(S', T_1)$. Arguing as above, we see that

$$\mathbf{C}_w C^\lambda_{S,T_1} = \sum_{S' \in \mathbb{T}_\lambda} r_w^1(S', S) C^\lambda_{S',T_1} \quad \mod \mathcal{H}_n(< \lambda).$$

Hence we have

$$r_w(S', S) = r_w^1(S, S') \qquad \Leftrightarrow \qquad h_{w,x,x'} = h_{w,x_1,x_1'}.$$

Now, Corollary 5.3 shows that $x \sim_{\mathcal{R}} x_1$, $x' \sim_{\mathcal{R}} x_1'$, $x \sim_{\mathcal{L}} x'$ and $x_1 \sim_{\mathcal{L}} x_1'$. Hence the desired equality follows from Theorem 6.3. $\square$

Remark 6.5. The above proof is modeled on the discussion of the Iwahori–Hecke algebra of the symmetric group $\mathfrak{S}_n$ in [12, Example 1.2]. In that case, Graham and Lehrer state that (C3) is already implicit in Kazhdan–Lusztig [13] (or the work of Barbasch and Vogan and Vogan), which is not really the case. In fact, as the above proof shows, (C3) relies on the validity of both (♡) and (♠), and the latter was first proved by Lusztig [14] (even for the symmetric group $\mathfrak{S}_n$).

7. Lusztig's homomorphism from $\mathcal{H}_n$ to the ring J

As a further application of the results of the previous section, we will now construct a new basis of $\mathcal{H}_n$ with integral structure constants. First of all, this will lead to an analogue of Lusztig's "canonical" isomorphism from $\mathcal{H}_{n,K}$ onto the group algebra KW_n; see Theorem 7.8. At the end of this section, we will see that the subring generated by that new basis is nothing but Lusztig's ring J. To establish that identification, we will rely on the recent results of Iancu and the author [10] concerning Lusztig's **a**-function.

Recall the basic set-up from the previous sections. In particular, recall the partition

$$W_n = \coprod_{\lambda \in \Lambda_n} \mathfrak{R}_\lambda,$$

where Λ_n is the set of all pairs of partitions of total size n. Let us fix $\lambda \in \Lambda_n$. In the following discussion, we will make repeated use of the bijection

$$\mathbb{T}_\lambda \times \mathbb{T}_\lambda \xrightarrow{\sim} \mathfrak{R}_\lambda, \qquad (T, T') \mapsto w_\lambda(T, T');$$

see Corollary 5.3. Recall that this implies, in particular, that every left cell contains a unique element from the set

$$\mathcal{D}_n := \{z \in W_n \mid z^2 = 1\}.$$

For $z \in W_n$, we denote by d_z the unique element in $\mathcal{D}_n$ such that $z \sim_{\mathcal{L}} d_z$.

By Theorem 6.1, we have $\chi_{\mathfrak{C}} \in \mathrm{Irr}(\mathcal{H}_{n,K})$ for all left cells in W_n. This allows us to make the following construction. Let $\lambda \in \Lambda_n$. We fix one left cell in $\mathfrak{R}_\lambda$ and denote its elements by $\{x_1, \ldots, x_{d_\lambda}\}$. We have a corresponding matrix representation

$$\mathfrak{X}_\lambda \colon \mathcal{H}_{n,K} \to M_{d_\lambda}(K), \quad \text{where} \quad \mathfrak{X}_\lambda^{ij}(\mathbf{C}_w) = h_{w,x_j,x_i}$$

for $1 \leqslant i, j \leqslant d_\lambda$; see (2.3). Let $\chi_\lambda \in \mathrm{Irr}(\mathcal{H}_{n,K})$ be the character afforded by $\mathfrak{X}_\lambda$. Now, if we vary λ, we get all irreducible characters of $\mathcal{H}_{n,K}$ exactly once. Thus, we have a labelling

$$\mathrm{Irr}(\mathcal{H}_{n,K}) = \{\chi_\lambda \mid \lambda \in \Lambda_n\}.$$

As in Section 2, denote by $\{\mathbf{D}_w \mid w \in W_n\}$ the basis which is dual to the basis $\{\mathbf{C}_w \mid w \in W_n\}$ with respect to the symmetrizing trace τ. We have the following formula:

$$\tau = \sum_{\lambda \in \Lambda_n} \frac{1}{c_\lambda}\,\chi_\lambda, \qquad \text{where } c_\lambda \in A \text{ for all } \lambda \in \Lambda_n.$$

(In the present case, we do have $c_\lambda \in A$; see [11, Theorem 9.3.5].) The main idea in this section is to apply Neunhöffer's results from the end of Section 2, concerning the explicit Wedderburn decomposition of $\mathcal{H}_{n,K}$ in terms of the products $\mathbf{C}_x\mathbf{D}_{y^{-1}}$ where $x \sim_{\mathcal{L}} y$.

The following result is inspired by an analogous result for the symmetric group; see Neunhöffer [19, Kap. VI, §4]. It crucially relies on Theorem 6.3.

Proposition 7.1. *The elements $\{\mathbf{C}_z\mathbf{D}_{d_z} \mid z \in W_n\}$ form a K-basis of $\mathcal{H}_{n,K}$. We have the following identity for any $w \in W$:*

$$\mathbf{C}_w = \sum_{z \in W} h_{w,d_z,z}\, c_{\lambda_z}^{-1}\, \mathbf{C}_z\mathbf{D}_{d_z},$$

where $\lambda_z \in \Lambda_n$ is defined by the condition that $z \in \mathfrak{R}_{\lambda_z}$. Furthermore, we have the following multiplication rule: If $w, z \in W_n$ satisfy $w \sim_{\mathcal{R}} z^{-1}$, then

$$\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w} = c_{\lambda_z}\, \mathbf{C}_u\mathbf{D}_{d_u},$$

where $u \in W_n$ is the unique element such that $z \sim_{\mathcal{R}} u \sim_{\mathcal{L}} w$ (see Corollary 5.3). Otherwise, we have $\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w} = 0$.

Proof. First we prove the multiplication rule, by using a representation-theoretic argument. Let $w, z \in W_n$ and suppose that $\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w} \neq 0$. Since $\mathcal{H}_{n,K}$ is split semisimple, we have

$$\mathfrak{X}_\lambda(\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w}) \neq 0 \qquad \text{for some } \lambda \in \Lambda_n.$$

Let $\mathfrak{C}$ be the left cell containing z and let $\mathfrak{C}_1$ be the left cell containing w. We claim that

$$(*) \qquad \mathfrak{C}, \mathfrak{C}_1 \subseteq \mathfrak{R}_\lambda \qquad \text{and} \qquad \mathfrak{C} \approx \mathfrak{C}_1 \approx \{x_1, \ldots, x_{d_\lambda}\},$$

where $\{x_1, \ldots, x_{d_\lambda}\}$ is our chosen left cell in $\mathfrak{R}_\lambda$. Indeed, since $\mathfrak{X}_{\mathfrak{C}}$ is irreducible, there exists some $\mu \in \Lambda$ such that $\mathfrak{X}_{\mathfrak{C}}$ is equivalent to $\mathfrak{X}_\mu$. If we had $\lambda \neq \mu$, then Lemma 2.11 would imply $\mathfrak{X}_\mu(\mathbf{C}_z\mathbf{D}_{d_z}) = 0$, contradicting the choice of λ. Thus, we have $\mu = \lambda$ and so $\chi_{\mathfrak{C}} = \chi_\lambda$. A similar argument shows that we also have $\chi_{\mathfrak{C}_1} = \chi_\lambda$. But then Theorem 6.1 implies that $\mathfrak{C}, \mathfrak{C}_1 \subseteq \mathfrak{R}_\lambda$, and Theorem 6.3 yields the second statement in $(*)$.

Let $i, j \in \{1, \ldots, d_\lambda\}$ be such that $z \sim_{\mathcal{R}} x_i$ and $d_z \sim_{\mathcal{R}} x_j$. (These indices exist and are unique by Corollary 5.3.) Then Theorem 6.3 and Lemma 2.12 imply that

$$\mathbf{C}_z\mathbf{D}_{d_z} = \mathbf{C}_{x_i}\mathbf{D}_{x_j^{-1}}.$$

Similarly, if $k, l \in \{1, \ldots, d_\lambda\}$ are such that $w \sim_{\mathcal{R}} x_k$ and $d_w \sim_{\mathcal{R}} x_l$, then

$$\mathbf{C}_w\mathbf{D}_{d_w} = \mathbf{C}_{x_k}\mathbf{D}_{x_l^{-1}}.$$

Now, by Lemma 2.11, the above elements are multiples of matrix units with respect to the representation $\mathfrak{X}_\lambda$. (Recall that this is the representation afforded by the left cell $\{x_1, \ldots, x_{d_\lambda}\}$.) Hence the usual multiplication rules for matrix units imply that $j = k$ and

$$\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w} = \mathbf{C}_{x_i}\mathbf{D}_{x_j^{-1}} \cdot \mathbf{C}_{x_j}\mathbf{D}_{x_l^{-1}} = c_\lambda\, \mathbf{C}_{x_i}\mathbf{D}_{x_l^{-1}}.$$

Finally, let $d \in \mathcal{D}_n$ be the unique element such that $d \sim_{\mathcal{R}} x_l$. Then, by Corollary 5.3, there is a unique element $u \in \mathfrak{R}_\lambda$ such that $d = d_u$ and $u \sim_{\mathcal{R}} x_i$. In particular, this means that

$$u \sim_{\mathcal{R}} x_i \sim_{\mathcal{R}} z \quad \text{and} \quad u \sim_{\mathcal{L}} d_u = d_u^{-1} \sim_{\mathcal{L}} x_l^{-1} \sim_{\mathcal{L}} d_w \sim_{\mathcal{L}} w.$$

Furthermore, the condition $j = k$ means that $w \sim_{\mathcal{R}} x_k = x_j \sim_{\mathcal{R}} d_z = d_z^{-1} \sim_{\mathcal{R}} z^{-1}$. Thus, if $\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w} \neq 0$, we have established the desired multiplication rule. Conversely, by following the above arguments backwards, one readily checks that $\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_w\mathbf{D}_{d_w}$ has the desired result if $w, z \in \mathfrak{R}_\lambda$ for some $\lambda \in \Lambda_n$, $w \sim_{\mathcal{R}} z^{-1}$ and $z \sim_{\mathcal{R}} u \sim_{\mathcal{L}} w$ where $u \in \mathfrak{R}_\lambda$. Thus, the multiplication rule is proved.

Now let $x \in W_n$. Then we have

$$\mathbf{C}_{d_x}\mathbf{D}_{x^{-1}} = \mathbf{C}_w\mathbf{D}_{d_w} \qquad \text{where } d_x \sim_{\mathcal{R}} w \text{ and } x \sim_{\mathcal{R}} d_w.$$

Hence, for any $z \in W_n$, we obtain

$$\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_{d_x}\mathbf{D}_{x^{-1}} = \begin{cases} c_{\lambda_z}\, \mathbf{C}_u\mathbf{D}_{d_u} & \text{if } z \sim_{\mathcal{R}} u \sim_{\mathcal{L}} w \sim_{\mathcal{R}} z^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that we are dealing with a pair of dual basis, this yields

$$\tau(\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_{d_x}\mathbf{D}_{x^{-1}}) = \begin{cases} c_{\lambda_z} & \text{if } z \sim_{\mathcal{R}} u = d_u \sim_{\mathcal{L}} w \sim_{\mathcal{R}} z^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, if the above condition on u, z, w is satisfied, then we have $w \sim_{\mathcal{R}} z^{-1}$ and $w \sim_{\mathcal{L}} d_u = d_u^{-1} \sim_{\mathcal{L}} z^{-1}$; so we must have $w = z^{-1}$ by Corollary 5.3. But then we have $x \sim_{\mathcal{L}} d_x = d_x^{-1} \sim_{\mathcal{L}} w^{-1} = z$ and $x^{-1} \sim_{\mathcal{L}} d_w^{-1} = d_w \sim_{\mathcal{L}} w^{-1} = z$, which yields $x = z$. Thus, we have shown that

$$\tau(\mathbf{C}_z\mathbf{D}_{d_z} \cdot \mathbf{C}_{d_x}\mathbf{D}_{x^{-1}}) = \begin{cases} c_{\lambda_z} & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove the identity $\mathbf{C}_w = \sum_{z \in W} h_{w,d_z,z}\, c_{\lambda_z}^{-1}\, \mathbf{C}_z\mathbf{D}_{d_z}$, we just multiply both sides by $\mathbf{C}_{d_x}D_{x^{-1}}$ and note that, upon applying τ, we obtain the same result. Once this identity is established, it follows that the elements $\{\mathbf{C}_z D_{d_z} \mid z \in W_n\}$ generate $\mathcal{H}_{n,K}$. Since this generating set has the correct cardinality, it forms a basis. ☐

Corollary 7.2. *The matrix* $\left(h_{w,d_z,z}\right)_{w,z \in W_n}$ *is invertible over* K.

Proof. By Proposition 7.1, the above matrix describes the base change between two basis of $\mathcal{H}_{n,K}$. ☐

Definition 7.3. In the above setting, we consider the $\mathbb{Z}$-submodule

$$J_n := \langle \hat{t}_w \mid w \in W_n \rangle_{\mathbb{Z}} \subseteq \mathcal{H}_{n,K},$$

where we set $\hat{t}_w := c_{\lambda_w}^{-1}\, \mathbf{C}_w\mathbf{D}_{d_w}$ for any $w \in W_n$. The multiplication rules in Proposition 7.1 immediately imply that J_n is a subring of $\mathcal{H}_{n,K}$; indeed, we have

$$\hat{t}_x\hat{t}_y = \begin{cases} \hat{t}_z & \text{if } x \sim_{\mathcal{L}} y^{-1} \text{ and } x \sim_{\mathcal{R}} z \sim_{\mathcal{L}} y, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we have a decomposition

$$J_n = \bigoplus_{\lambda \in \Lambda_n} J_{n,\lambda} \qquad \text{(direct sum of two-sided ideals)},$$

where $J_{n,\lambda} := \langle \hat{t}_w \mid w \in \mathfrak{R}_\lambda \rangle_{\mathbb{Z}}$ for every $\lambda \in \Lambda_n$.

We will see at the end of this section that J_n actually is the ring J introduced by Lusztig in [18, Chap. 18]. However, our basis elements $\hat{t}_w$ will not correspond directly to Lusztig's basis elements. We have to perform a transformation of the following type.

Let $w \mapsto \hat{n}_w$ be an integer-valued function on W_n satisfying the following two properties:

(N1) we have $\hat{n}_w = \pm 1$ for all $w \in W_n$;
(N2) the function $w \mapsto \hat{n}_w$ is constant on right cells.

Having fixed a function as above, we set $t_w := \hat{n}_w \hat{t}_w$ for all $w \in W_n$. By **(N1)**, the elements $\{t_w \mid w \in W_n\}$ form a new $\mathbb{Z}$-basis of the ring J_n. Writing

$$t_x\, t_y = \sum_{z \in W_n} \hat{\gamma}_{x,y,z^{-1}}\, t_z \qquad (x, y \in W_n),$$

the structure constants $\hat{\gamma}_{x,y,z^{-1}}$ are given by

$$\textbf{(N3)} \qquad \hat{\gamma}_{x,y,z^{-1}} = \begin{cases} \hat{n}_y = \pm 1 & \text{if } x \sim_{\mathcal{L}} y^{-1} \text{ and } x \sim_{\mathcal{R}} z \sim_{\mathcal{L}} y, \\ 0 & \text{otherwise;} \end{cases}$$

see Proposition 7.1. The following results will all be formulated in terms of the basis $\{t_w \mid w \in W_n\}$ of J_n, where we assume throughout that a function satisfying **(N1)**, **(N2)** has been fixed. An obvious example is given by the function $\hat{n}_w = 1$ for all $w \in W_n$. (As we will see at the end of this section, we have to take a different function in order to identify t_w with the corresponding element in Lusztig's construction.)

Corollary 7.4. *The ring J_n introduced above has unit element*

$$T_1 = \sum_{z \in \mathcal{D}_n} \hat{n}_z\, t_z \qquad \textit{(where } T_1 \textit{ is the unit element in } \mathcal{H}_n\textit{)}.$$

For every $\lambda \in \Lambda_n$, we have $J_{n,\lambda} \cong M_{d_\lambda}(\mathbb{Z})$.

Proof. Let $z \in \mathcal{D}_n$ and assume that $z \in \mathfrak{R}_\lambda$. Since $d_z = z$ is an involution, the argument in the proof of Proposition 7.1 now shows that

$$\mathbf{C}_z \mathbf{D}_z = \mathbf{C}_{x_i} \mathbf{D}_{x_i^{-1}} \quad \text{for some } 1 \leqslant i \leqslant d_\lambda,$$

where $\{x_1, \ldots, x_{d_\lambda}\} \subseteq \mathfrak{R}_\lambda$ is our chosen left cell. Furthermore, we have

$$\sum_{z \in \mathcal{D}_n \cap \mathfrak{R}_\lambda} \mathbf{C}_z \mathbf{D}_z = \sum_{i=1}^{d_\lambda} \mathbf{C}_{x_i} \mathbf{D}_{x_i^{-1}}.$$

By Lemma 2.11, the image of the above element under $\mathfrak{X}_\mu$ is 0 if $\lambda \neq \mu$, and c_λ times the identity matrix if $\lambda = \mu$.

Hence we conclude that the image of $\varepsilon := \sum_{z \in \mathcal{D}_n} \hat{t}_z \in J_n$ under $\mathfrak{X}_\lambda$ (for any λ) is the identity matrix. Since $\mathcal{H}_{n,K}$ is split semisimple, this implies that ε is the identity element in $\mathcal{H}_{n,K}$, that is, we have $\varepsilon = T_1$. $\square$

We can now establish the following result which is in complete analogy to Lusztig [18, Theorem 18.9].

Corollary 7.5. *Let $J_{n,A} = A \otimes_{\mathbb{Z}} J_n = \langle t_w \mid w \in W_n \rangle_A \subseteq \mathcal{H}_{n,K}$. The A-linear map $\phi \colon \mathcal{H}_n \to J_{n,A}$ defined by*

$$\phi(\mathbf{C}_w^\delta) = \sum_{z \in W_n} h_{w,d_z,z}\, \hat{n}_z\, t_z \qquad (w \in W_n)$$

is a homomorphism of A-algebras respecting the unit elements.

Proof. Since $J_{n,A} \hookrightarrow \mathcal{H}_{n,K}$, the above formula actually defines a K-linear map $\phi_K \colon \mathcal{H}_{n,K} \to \mathcal{H}_{n,K}$ whose restriction to $\mathcal{H}_n$ is ϕ. By Proposition 7.1, the set $\{\hat{t}_w \mid w \in W_n\}$ is a basis of $\mathcal{H}_{n,K}$. Furthermore, the formula in that proposition shows that $\phi_K(\hat{t}_w^\delta) = \hat{t}_w$ for all $w \in W_n$. Thus, we have $\phi_K \circ \delta = \mathrm{id}$ on $\mathcal{H}_{n,K}$ and, consequently, ϕ_K is a K-algebra homomorphism respecting the unit elements. $\square$

The next result can be regarded as a weak version of property (P15) in Lusztig's list of conjectures in [18, Chap. 14].

Proposition 7.6 (Compare Lusztig [18, 18.9(b)])**.** *Let $x, x', y, z, w \in W_n$ be such that $y \sim_{\mathcal{L}} x' \sim_{\mathcal{R}} x^{-1}$. Then we have*

$$\sum_{z \in W_n} h_{w,z,y}\, \hat{\gamma}_{x,x',z^{-1}} = \sum_{z \in W_n} h_{w,x,z}\, \hat{\gamma}_{z,x',y^{-1}}.$$

Proof. By assumption, we have $x \sim_{\mathcal{L}} x'^{-1}$ and $x' \sim_{\mathcal{L}} y$. Hence we have $x, x', y \in \mathfrak{R}_\lambda$ for some $\lambda \in \mathfrak{R}_n$. So, by Corollary 5.3, there exist unique elements $z_0, z_1 \in \mathfrak{R}_\lambda$ such that

$$x \sim_{\mathcal{R}} z_0 \sim_{\mathcal{L}} x' \qquad \text{and} \qquad x'^{-1} \sim_{\mathcal{L}} z_1 \sim_{\mathcal{R}} y.$$

Now Proposition 7.1 shows that $\hat{\gamma}_{x,x',z^{-1}} = 0$ unless $z = z_0$, in which case the result is $\hat{\gamma}_{x,x',z_0^{-1}} = \hat{n}_{x'}$. Similarly, we have $\hat{\gamma}_{z,x',y^{-1}} = 0$ unless $z = z_1$, in which case the result is $\hat{\gamma}_{z_1,x',y^{-1}} = \hat{n}_{x'}$. Hence the desired equality is equivalent to the identity

$$(*) \qquad h_{w,z_0,y} = h_{w,x,z_1}.$$

Suppose that $h_{w,z_0,y} \neq 0$. Then $y \leqslant_{\mathcal{L}} z_0$. We conclude that

$$x' \sim_{\mathcal{L}} y \leqslant_{\mathcal{L}} z_0 \sim_{\mathcal{L}} x'$$

and so $y \sim_{\mathcal{L}} z_0$. On the other hand, we have $z_0 \sim_{\mathcal{R}} x$ and, hence,

$$z_1 \sim_{\mathcal{L}} x'^{-1} \sim_{\mathcal{L}} x \sim_{\mathcal{L}} z_0.$$

So we can apply Theorem 6.3 and this yields $h_{w,z_0,y} = h_{w,x,z_1}$. Thus, $(*)$ holds in this case. Conversely, if $h_{w,x,z_1} \neq 0$, then a similar argument shows that, again, $(*)$ holds. Finally, this also yields that $h_{w,z_0,y} = 0$ if and only if $h_{w,x,z_1} = 0$. Thus, $(*)$ holds in all cases. $\square$

Let $\mathcal{E}$ be the free A-module with basis $\{\varepsilon_x \mid x \in W_n\}$. Identifying $\mathcal{H}_n$ and $\mathcal{E}$ via $\mathbf{C}_w \mapsto \varepsilon_w$, the obvious $\mathcal{H}_n$-module on $\mathcal{H}_n$ (given by left multiplication) becomes the $\mathcal{H}_n$-module on $\mathcal{E}$ given by

$$\mathbf{C}_w.\varepsilon_x = \sum_{y \in W_n} h_{w,x,y}\, \varepsilon_y \qquad (w, x \in W_n).$$

On the other hand, we can also identify $\mathcal{E}$ with $J_{n,A}$, via $\varepsilon_w \mapsto \hat{n}_w t_w$. Then the obvious $J_{n,A}$-module structure on $J_{n,A}$ (given by left multiplication) becomes the $J_{n,A}$-module structure on $\mathcal{E}$ given by

$$t_w * \varepsilon_x = \sum_{y \in W_n} \hat{\gamma}_{w,x,y^{-1}}\, \hat{n}_x \hat{n}_y\, \varepsilon_y \qquad (w, x \in W_n).$$

Now we can state the following result.

Corollary 7.7 (Compare Lusztig [18, 18.10]). *For any $h \in \mathcal{H}_n$ and any $x \in W_n$, the difference $h.\varepsilon_x - \phi(h^\delta) \star \varepsilon_x$ is an A-linear combination of elements ε_y where $y <_{\mathcal{LR}} x$ (that is, we have $y \leqslant_{\mathcal{LR}} x$ but $y \not\sim_{\mathcal{LR}} x$).*

Proof. It is enough to prove this for $h = \mathbf{C}_w$ where $w \in W_n$. Then we have

$$\begin{aligned}\phi(\mathbf{C}_w^\delta) \star \varepsilon_x &= \sum_{z \in W_n} h_{w,d_z,z}\, \hat{n}_z\, t_z \star \varepsilon_x \\ &= \sum_{z \in W_n} \sum_{y \in W_n} h_{w,d_z,z}\, \hat{\gamma}_{z,x,y^{-1}}\, \hat{n}_z \hat{n}_x \hat{n}_y\, \varepsilon_y.\end{aligned}$$

Now, if the term correponding to y, z is non-zero, then we have $\gamma_{z,x,y^{-1}} \neq 0$ and so $z \sim_{\mathcal{R}} y$. Hence we also have $\hat{n}_z = \hat{n}_y$ and so $\hat{n}_z \hat{n}_y = 1$. By a similar argument, we can also assume that $x \sim_{\mathcal{L}} y$ and $x^{-1} \sim_{\mathcal{L}} z \sim_{\mathcal{L}} d_z$. In particular, we have $d_z = d_{x^{-1}}$. Consequently, we can rewrite the above sum as follows:

$$\begin{aligned}\phi(\mathbf{C}_w^\delta) \star \varepsilon_x &= \sum_{\substack{y \in W_n \\ x \sim_{\mathcal{L}} y}} \Big(\sum_{z \in W_n} h_{w,d_{x^{-1}},z}\, \hat{\gamma}_{z,x,y^{-1}} \Big)\, \hat{n}_x\, \varepsilon_y \\ &= \sum_{\substack{y \in W_n \\ x \sim_{\mathcal{L}} y}} \Big(\sum_{z \in W_n} h_{w,z,y}\, \hat{\gamma}_{d_{x^{-1}},x,z^{-1}} \Big)\, \hat{n}_x\, \varepsilon_y\end{aligned}$$

where the second equality holds by Proposition 7.6. Now, by **(N3)**, we have $\gamma_{d_{x^{-1}},x,z^{-1}} = 0$ unless $x = z$ in which case the result equals $\hat{n}_x$. Hence the above sum reduces to:

$$\phi(\mathbf{C}_w^\delta) \star \varepsilon_x = \sum_{\substack{y \in W_n \\ x \sim_{\mathcal{L}} y}} h_{w,x,y}\, \varepsilon_y.$$

On the other hand, we know that (♠) holds by Theorem 5.13. So, for any $y' \in W_n$, we have $h_{w,x,y'} = 0$ unless $y' \sim_{\mathcal{L}} x$ or $y' <_{\mathcal{LR}} x$. Hence we see that, indeed, the difference $h.\varepsilon - \phi(\mathbf{C}_w^\delta) \star \varepsilon_x$ has the required form. $\square$

We now apply the above results to construct a "canonical" algebra isomorphism from $\mathcal{H}_{n,K}$ onto KW_n, the group algebra of W_n over K. Let $R = \mathbb{Q}[\Gamma] = \mathbb{Q} \otimes_{\mathbb{Z}} A$ and set $\mathcal{H}_{n,R} = R \otimes_A \mathcal{H}_n$, $J_{n,R} := R \otimes_A J_A$. The previously defined modules structures of $\mathcal{H}_n$ and $J_{n,A}$ on $\mathcal{E}$ naturally extend to module structures of $\mathcal{H}_{n,R}$ and $J_{n,R}$, respectively, on $\mathcal{E}_R = R \otimes_A \mathcal{E}$. Now we also describe an RW_n-module structure on $\mathcal{E}_R = R \otimes_A \mathcal{E}$, as follows. We have a ring homomorphism

$$\theta\colon R \to R, \qquad e^\gamma \mapsto 1 \qquad (\gamma \in \Gamma).$$

We can regard R as an R-module via θ; then we obtain $R \otimes_R \mathcal{H}_n = RW_n$. We denote $c_w = 1 \otimes \mathbf{C}_w \in RW_n$ for any $w \in W_n$. Hence, we may also regard $\mathcal{E}_R$ as an RW_n-module, where c_w ($w \in W_n$) acts by

$$c_w \diamond \varepsilon_x = \sum_{y \in W_n} \theta(h_{w,x,y})\, \varepsilon_y \qquad \text{for any } x \in W_n.$$

Note that this RW_n-module structure on $\mathcal{E}_R$ coincides with the obvious structure (given by left multiplication), where we identity RW_n and $\mathcal{E}_R$ via $c_w \mapsto \varepsilon_w$.

Theorem 7.8 (See Lusztig [14, Theorem 3.1] in the case of equal parameters)**.** *There is a unique homomorphism of R-algebras $\Phi\colon \mathcal{H}_{n,R} \to RW_n$ such that, for any $h \in \mathcal{H}_{n,R}$ and any $x \in W_n$, the difference $h.\varepsilon_x - \Phi(h) \diamond \varepsilon_x$ is a linear combination of elements ε_y with $y <_{\mathcal{LR}} x$. Furthermore, writing*

$$\Phi(\mathbf{C}_w) = \sum_{z \in W_n} \Phi_{w,z}\, z \qquad \textit{where } \Phi_{w,z} \in R,$$

we have $\Phi_{w,z} = \overline{\Phi}_{w,z}$ and $\theta(\Phi_{w,z}) = \delta_{wz}$ for all $w, z \in W_n$. Finally, the induced map $\Phi_K\colon \mathcal{H}_{n,K} \to KW_n$ is an isomorphism.

(Here, δ_{wz} denotes the Kronecker delta, and $r \mapsto \bar{r}$ is the ring involution such that $e^{\gamma} \mapsto e^{-\gamma}$ for all $\gamma \in \Gamma$).

Proof. First we show the uniqueness statement. Let $\Phi_i\colon \mathcal{H}_{n,R} \to RW_n$ $(i = 1, 2)$ be two homomorphisms such that, for any $h \in \mathcal{H}_{n,R}$ and any $x \in W_n$, the difference $h.\varepsilon_x - \Phi_i(h) \diamond \varepsilon_x$ is a linear combination of elements ε_y with $y <_{\mathcal{LR}} x$. Then the difference $(\Phi_1(h) - \Phi_2(h)) \diamond \varepsilon_x$ is a linear combination of elements ε_y with $y <_{\mathcal{LR}} y$. Consequently, $\Phi_1(h) - \Phi_2(h) \in RW_n \subseteq KW_n$ acts as a nilpotent operator on $\mathcal{E}_K = K \otimes_R \mathcal{E}$. But, as we already noted above, $\mathcal{E}_K$ is the left regular KW_n-module, hence we must have $\Phi_1(h) - \Phi_2(h) = 0$.

So it remains to show that an R-algebra homomorphism Φ with the required properties does exist. In Corollary 7.5, we extend scalars from A to R and obtain a homomorphism of R-algebras

$$\alpha\colon \mathcal{H}_{n,R} \to J_R = R \otimes_A J_A, \qquad \mathbf{C}_w \mapsto \phi(\mathbf{C}_w^{\delta}).$$

Explicitly, α is given by the formula

$$\alpha(\mathbf{C}_w) = \sum_{z \in W_n} h_{w,d_z,z}\, \hat{n}_z\, t_z \qquad \text{for any } w \in W_n.$$

(We can take $\hat{n}_z = 1$ for all $z \in W_n$.) By Corollary 7.7, the above homomorphism has the property that, for any $h \in \mathcal{H}_{n,R}$ and any $x \in W_n$, the difference $h.\varepsilon_x - \alpha(h) \star \varepsilon_x$ is an R-linear combination of elements ε_y where $y <_{\mathcal{LR}} x$.

Now, as before, we regard R as an R-module via θ and extend scalars. Since the structure constants of J_n with respect to the basis $\{t_w\}$ are integers, they are not affected by θ. Hence we obtain an induced homomorphism of R-algebras

$$\beta\colon RW_n \to J_R$$

such that

$$\beta(c_w) = \sum_{z \in W_n} \theta(h_{w,d_z,z})\, \hat{n}_z\, t_z \qquad \text{for any } w \in W_n.$$

Now the identity in Proposition 7.1 "specializes" to an analogous identity in RW_n. (Note that $\theta(c_\lambda) = |W_n|/d_\lambda \neq 0$ for each $\lambda \in \Lambda_n$; see [11, §8.1].) We deduce from this that the matrix

$$\big(\theta(h_{w,d_z,z})\big)_{w,z \in W_n}$$

is invertible over K. Since the coefficients of that matrix lie in $\mathbb{Q}$, so do the coefficients of its inverse. Consequently, β is an isomorphism of R-algebras. Furthermore, a computation analogous to that in the proof of Corollary 7.7 shows that we have

$$\beta(c_w) \star \varepsilon_x = \sum_{\substack{y \in W_n \\ x \sim_{\mathcal{L}} y}} \theta(h_{w,x,y})\, \varepsilon_y \qquad \text{for any } x, w \in W_n,$$

and that $\beta(c_w)\star\varepsilon_x - c_w \diamond \varepsilon_x$ is an R-linear combination of elements ε_y where $y <_{\mathcal{LR}} x$. Consequently, since β is an isomorphism, we also have that, for any $\iota \in J_{n,R}$ and any $x \in W_n$, the difference $\iota\star\varepsilon_x - \beta^{-1}(\iota)\diamond\varepsilon_x$ is an R-linear combination of elements ε_y where $y <_{\mathcal{LR}} x$. Now we set

$$\Phi := \beta^{-1} \circ \alpha \colon \mathcal{H}_{n,R} \to RW_n.$$

Let $h \in \mathcal{H}_{n,R}$ and $x \in W_n$. Setting $\iota := \alpha(h) \in J_{n,R}$, we obtain that

$$\begin{aligned} h.\varepsilon_x - \Phi(h) \diamond \varepsilon_x &= h.\varepsilon_x - \alpha(h) \star \varepsilon_x + \alpha(h) \star \varepsilon_x - \Phi(h) \diamond \varepsilon_x \\ &= (h.\varepsilon_x - \alpha(h) \star \varepsilon_x) + (\iota \star \varepsilon_x - \beta^{-1}(\iota) \diamond \varepsilon_x) \end{aligned}$$

is an R-linear combination of elements ε_y where $y <_{\mathcal{LR}} x$, as required.

Finally, ϕ_K is an isomorphism since α_K is invertible (see Corollary 7.2) and β is an isomorphism. Furthermore, the coefficients $\Phi_{w,z}$ have the stated properties, since Φ is defined as the composition of α (whose matrix is given by the coefficients $h_{w,d_z,z}$) and the inverse of β (whose matrix is given by the inverse of the matrix with coefficients $\theta(h_{w,d_z,z})$). □

Note that the above proof relies on the existence of the homomorphism $\phi\colon \mathcal{H}_n \to J_n$ and Corollaries 7.2, 7.7. We could not follow Lusztig's original proof in [14] since, in the present case, the constants $M_{y,w}^s$ appearing in the multiplication formula for the Kazhdan–Lusztig basis are no longer integers.

Example 7.9. Let us consider the case $n = 2$, where $W_2 = \langle t, s_1 \rangle$ is the dihedral group of order 8. We set $s_0 = t$. The coefficients $h_{s,y,z}$ (for $s = s_0, s_1$) and the left cells have already been determined by an explicit computation in [15, §6]. The left cells are

$$\{1\}, \quad \{s_1\}, \quad \{s_0, s_1s_0\}, \quad \{s_1s_0s_1, s_0s_1\}, \quad \{s_0s_1s_0\}, \quad \{w_0\}$$

where $w_0 = s_1s_0s_1s_0$ is the unique element of maximal length. For each left cell, the first element listed is the unique element from $\mathcal{D}_2$ in that left cell. From the information in [15, §6], we know $h_{w,d_z,z}$ for $w \in \{s_0, s_1\}$. This yields the following formulas for the homomorphism $\phi\colon \mathcal{H}_2 \to J_{2,A}$:

$$\begin{aligned} \phi(\mathbf{C}_{s_0}) &= (Q{+}Q^{-1})t_{s_0} + (Qq^{-1}{+}Q^{-1}q)t_{s_0s_1} \\ &\qquad + (Q{+}Q^{-1})t_{s_0s_1s_0} + (Q{+}Q^{-1})t_{w_0}, \\ \phi(\mathbf{C}_{s_1}) &= (q{+}q^{-1})t_{s_1} + t_{s_1s_0} + (q{+}q^{-1})t_{s_1s_0s_1} + (q{+}q^{-1})t_{w_0}, \end{aligned}$$

where we take the function $\hat{n}_w = 1$ for all $w \in W_2$. Using the multiplication formula for the Kazhdan–Lusztig basis, we can deduce explicit expressions of $\phi(\mathbf{C}_w)$, for any $w \in W_2$; this yields the whole matrix of coefficients $(h_{w,d_z,z})_{w,z}$. In order to construct $\Phi\colon \mathcal{H}_{2,R} \to RW_2$, we follow the proof of Theorem 7.8. First, we apply the ring homomorphism $\theta\colon R \to R$, that is, we specialise $Q, q \mapsto 1$. The matrix of all coefficients $\theta(h_{w,d_z,z})$ is given in Table 1. Composing the matrix of ϕ with the inverse of the matrix in Table 1 and expressing the basis $\{c_w\}$ of RW_2 in terms

TABLE 1. The coefficients $\theta(h_{w,d_z,z})$ in type B_2

$\theta(h_{w,d_z,z})$	1	s_1	s_0	s_1s_0	$s_1s_0s_1$	s_0s_1	$s_0s_1s_0$	w_0
1	1	1	1	0	1	0	1	1
s_1	0	2	0	1	2	0	0	2
s_0	0	0	2	0	0	2	2	2
s_1s_0	0	0	0	2	2	0	0	4
$s_1s_0s_1$	0	0	0	2	4	0	0	8
s_0s_1	0	0	2	0	0	4	0	4
$s_0s_1s_0$	0	0	0	0	0	0	-4	4
w_0	0	0	0	0	0	0	0	8

of the standard basis consisting of group elements, we obtain the following explicit description of the homomorphism $\Phi\colon \mathcal{H}_{2,R} \to RW_2$:

$$\begin{aligned}\Phi(T_{s_0}) &= \frac{1}{2}(Q-Q^{-1})\cdot 1 + \frac{1}{2}(Q+Q^{-1})\cdot s_0\\ &\quad + \frac{1}{4}(Q-Qq^{-1}-Q^{-1}q+Q^{-1})\cdot(-s_1+s_1s_0-s_0s_1+s_0s_1s_0),\\ \Phi(T_{s_1}) &= \frac{1}{2}(q-q^{-1})\cdot 1+\frac{1}{2}(q+q^{-1})\cdot s_1\\ &\quad + \frac{1}{4}(q-2+q^{-1})\cdot(-s_0-s_1s_0+s_0s_1+s_1s_0s_1).\end{aligned}$$

Note that the formulas do not make any reference to the Kazhdan–Lusztig basis. Further note that the above formulas specialise to $T_{s_0} \mapsto s_0$, $T_{s_1} \mapsto s_1$ when we set $Q, q \mapsto 1$. Also, if we consider the images of $\mathbf{C}_{s_0} = T_{s_0} + Q^{-1}T_1$ and $\mathbf{C}_{s_1} = T_{s_1} + q^{-1}T_1$, then all the coefficients are seen to be fixed by the involution $r \mapsto \bar{r}$, as stated in Theorem 7.8.

To close this section, we explain how to identify J_n with Lusztig's ring J. First we need some definitions.

Let $z \in W_n$. Following Lusztig [18, 14.1], we define an element $\mathbf{\Delta}_n(z) \in \Gamma$ and an integer $0 \neq n_z \in \mathbb{Z}$ by the condition

$$e^{\mathbf{\Delta}_n(z)}\, P_{1,z}^* \equiv n_z \bmod A_{<0};$$

note that $\mathbf{\Delta}_n(z) \geqslant 0$. Following [18, 13.6], we define a function $\mathbf{a}_n\colon W_n \to \Gamma$ as follows. Let $z \in W_n$. Then we set

$$\mathbf{a}_n(z) := \min\{\gamma \geqslant 0 \mid e^{\gamma}\, h_{x,y,z} \in A_{\geqslant 0} \text{ for all } x, y \in W_n\}.$$

Furthermore, for any $x, y, z \in W_n$, we set

$$\gamma_{x,y,z^{-1}} = \text{ constant term of } e^{\mathbf{a}_n(z)}\, h_{x,y,z} \in A_{\geqslant 0}.$$

Following [18, Chap. 18], we use the constants $\gamma_{x,y,z}$ to define a new bilinear pairing on our free abelian group J_n with basis $\{t_w \mid w \in W_n\}$ by

$$t_x \bullet t_y := \sum_{z\in W} \gamma_{x,y,z^{-1}}\, t_z \qquad \text{for all } x, y \in W.$$

As explained in [18, Chap. 18], one can show that $(J_n, \bullet)$ is an associative ring with unit element $1_J = \sum_{d\in\mathcal{D}_n} n_d t_d$, if the following properties from Lusztig's list of conjectures in [18, Chap. 14] hold, where

$$\mathcal{D} := \{z \in W_n \mid \mathbf{a}_n(z) = \mathbf{\Delta}_n(z)\}.$$

- **(P1)** For any $z \in W_n$ we have $\mathbf{a}_n(z) \leqslant \mathbf{\Delta}_n(z)$.
- **(P2)** If $d \in \mathcal{D}_n$ and $x, y \in W_n$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.
- **(P3)** If $y \in W_n$, there exists a unique $d \in \mathcal{D}_n$ such that $\gamma_{y^{-1},y,d} \neq 0$.
- **(P4)** If $z' \leqslant_{\mathcal{LR}} z$, then $\mathbf{a}_n(z') \geqslant \mathbf{a}_n(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$, then $\mathbf{a}_n(z) = \mathbf{a}_n(z')$.
- **(P5)** If $d \in \mathcal{D}_n$, $y \in W_n$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.
- **(P6)** If $d \in \mathcal{D}_n$, then $d^2 = 1$.
- **(P7)** For any $x, y, z \in W_n$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.
- **(P8)** Let $x, y, z \in W_n$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.

Now we have the following result:

Theorem 7.10 (Geck and Iancu [10, Theorem 1.3])**.** *In the "asymptotic case" in type B_n, all the properties* **(P1)**–**(P15)** *from Lusztig's list of conjectures in* [18, Chap. 14] *hold, except possibly* **(P9)***,* **(P10)** *and* **(P15)***. Furthermore, we have $\mathcal{D} = \mathcal{D}_n := \{z \in W_n \mid z^2 = 1\}$.*

In particular, **(P1)**–**(P8)** hold and so we do have an associative ring $(J_n, \bullet)$ with unit element.

Following [18, 18.8], we set $\hat{n}_w := n_d$, where $d \in \mathcal{D}_n$ is the unique element such that $d \sim_{\mathcal{L}} w^{-1}$ (which exists and is unique by Corollary 5.3). By construction, the function $w \mapsto \hat{n}_w$ is constant on the right cells of W_n. Thus, $w \mapsto \hat{n}_w$ is an integer-valued function on W_n which satisfies **(N2)**. By **(P5)**, we see that **(N1)** also holds.

We shall now take this function in the above discussion. That is, the formula **(N3)** reads:

$$\hat{\gamma}_{x,y,z^{-1}} = \begin{cases} n_d & \text{if } x \sim_{\mathcal{L}} y^{-1},\, y \sim_{\mathcal{L}} z,\, z \sim_{\mathcal{R}} x,\, d = d^{-1} \sim_{\mathcal{L}} y^{-1}, \\ 0 & \text{otherwise;} \end{cases}$$

Now we can state the following result.

Proposition 7.11. *For any $x, y, z \in W_n$, we have $\hat{\gamma}_{x,y,z^{-1}} = \gamma_{x,y,z^{-1}}$.*

Proof. Let $x, y, z \in W_n$ be such that $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z$ and $z^{-1} \sim_{\mathcal{L}} x^{-1}$. By **(P3)**, there exists a unique $d \in \mathcal{D}_n$ such that $\gamma_{x^{-1},x,d} \neq 0$. We have $d^2 = 1$. Hence, by **(P8)**, we obtain $d \sim_{\mathcal{L}} x \sim_{\mathcal{L}} y^{-1}$ and so

$$\hat{\gamma}_{x,y,z^{-1}} = \hat{n}_y = n_d;$$

see the formula in Definition 7.3. This yields the identity

$$\hat{\gamma}_{x,y,z^{-1}} = n_d = \gamma_{x^{-1},x,d} = \gamma_{x,d,x^{-1}};$$

see **(P5)**, **(P7)**. Now, since $d = d^{-1} \sim_{\mathcal{R}} y$, $x \sim_{\mathcal{R}} z$ and $y \sim_{\mathcal{L}} z$, we have

$$h_{x,d,x} = h_{x,y,z};$$

see Theorem 6.3. By **(P4)**, we have $\mathbf{a}_n(x) = \mathbf{a}_n(z)$. Hence the above identity implies that $\mathbf{a}_n(x)h_{x,d,x}$ and $\mathbf{a}(z)_n h_{x,y,z}$ have the same constant term and so

$$\hat{\gamma}_{x,y,z^{-1}} = \gamma_{x,d,x^{-1}} = \gamma_{x,y,z^{-1}},$$

as required. It remains to consider the case where x, y, z do not satisfy the conditions $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z$, $z^{-1} \sim_{\mathcal{L}} x^{-1}$. But then we have $\hat{\gamma}_{x,y,z^{-1}} = 0$ by

Proposition 7.1 and $\gamma_{x,y,z^{-1}} = 0$ by **(P8)**. Hence we have $\gamma_{x,y,z^{-1}} = \hat{\gamma}_{x,y,z^{-1}}$ in all cases. $\square$

Combining Theorem 7.10 with the results in this paper, we can summarize the situation as follows.

Corollary 7.12. *In the "asymptotic case" in type B_n, the properties* **(P1)–(P14)** *from Lusztig's list in* [18, Chap. 14] *hold. Furthermore, we have the weak version of* **(P15)** *in Proposition* 7.6. *The ring J_n with its ring structure given by $J_n \subseteq \mathcal{H}_{n,K}$ as in Definition* 7.3 *is Lusztig's ring* $(J_n, \bullet)$.

Proof. The statement concerning J_n follows from Proposition 7.11. Furthermore, taking into account Theorem 7.10, it only remains to consider:

(P9) $\qquad x \leqslant_{\mathcal{L}} y \quad \text{and} \quad \mathbf{a}_n(x) = \mathbf{a}_n(y) \quad \Rightarrow \quad x \sim_{\mathcal{L}} y,$

(P10) $\qquad x \leqslant_{\mathcal{R}} y \quad \text{and} \quad \mathbf{a}_n(x) = \mathbf{a}_n(y) \quad \Rightarrow \quad x \sim_{\mathcal{R}} y.$

Now, by [18, 14.10], property **(P10)** is a formal consequence of **(P9)**. To prove **(P9)**, let $x, y \in W_n$ be such that $x \leqslant_{\mathcal{L}} y$ and $\mathbf{a}_n(x) = \mathbf{a}_n(y)$. By Theorem 7.10, we know that the following holds:

(P11) $\qquad x \leqslant_{\mathcal{LR}} y \quad \text{and} \quad \mathbf{a}_n(x) = \mathbf{a}_n(y) \quad \Rightarrow \quad x \sim_{\mathcal{LR}} y.$

Hence we conclude that $x \sim_{\mathcal{LR}} y$, and ($\spadesuit$) yields $x \sim_{\mathcal{L}} y$, as desired. $\square$

Remark 7.13. The defining formula for ϕ in [18, 18.9] reads

$$\phi(\mathbf{C}_w^{\delta}) = \sum_{\substack{z \in W_n, d \in \mathcal{D}_n \\ \mathbf{a}_n(z) = \mathbf{a}_n(d)}} h_{w,d,z}\, \hat{n}_z\, t_z.$$

But, once **(P9)** is known to hold, the above formula reduces to the one in Corollary 7.5. Indeed, assume that $z \in W_n$ and $d \in \mathcal{D}_n$ are such that $\mathbf{a}_n(z) = \mathbf{a}_n(d)$ and $h_{w,d,z} \neq 0$. Then $z \leqslant_{\mathcal{L}} d$ and **(P9)** implies that $z \sim_{\mathcal{L}} d$. Thus, we must have $d = d_z$.

Remark 7.14. In [7], it is shown that the existence of Lusztig's homomorphism into the ring J has various applications in the modular representation theory of Iwahori–Hecke algebras, most notably the fact that there is a natural "unitriangular" structure on the decomposition matrix associated with a non-semisimple specialisation. Given Corollary 7.12 and the results in this paper, we can now apply the theory developed in [7] to $\mathcal{H}_n$ as well. This should lead to new proofs of some results by Dipper, James, and Murphy [6].

Note added in proof (November 2006)

Recently, L. Iancu and C. Pallikaros have shown that the cell modules in the asymptotic case in type B_n are canonically isomorphic to the Specht modules defined in [6]. This will be published elsewhere.

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