

THE RATIONAL SCHUR ALGEBRA

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ABSTRACT. We extend the family of classical Schur algebras in type A , which determine the polynomial representation theory of general linear groups over an infinite field, to a larger family, the rational Schur algebras, which determine the rational representation theory of general linear groups over an infinite field. This makes it possible to study the rational representation theory of such general linear groups directly through finite dimensional algebras. We show that rational Schur algebras are quasihereditary over any field, and thus have finite global dimension.

We obtain explicit cellular bases of a rational Schur algebra by a descent from a certain ordinary Schur algebra. We also obtain a description, by generators and relations, of the rational Schur algebras in characteristic zero.

INTRODUCTION

A theme of contemporary representation theory is to approach representations of infinite groups (e.g. algebraic groups) and related algebraic structures through finite-dimensional algebras. For instance, the polynomial representation theory of general linear groups over an infinite field has been profitably studied from this viewpoint, through the Schur algebras. In that particular case, the efforts of various researchers culminated in [E, Theorem 2.4], which showed that the problem of computing decomposition numbers for such general linear groups is equivalent to the same problem for symmetric groups. (Half of that equivalence was known much earlier; see [Jam, Theorem 3.4].)

The aim of this paper is to extend the classical Schur algebras $S_K(n, r)$ to a larger family $S_K(n; r, s)$ of finite-dimensional algebras, enjoying many similar properties as the Schur algebras, which can be used to directly approach all the rational representations (in the defining characteristic) of the general linear groups over an infinite field, in the same way that the classical Schur algebras approach the polynomial representations.

This paper is organized as follows. In §1 of the paper, some general lemmas, needed later, are proved. The reader should probably skip §1 and refer back as needed. We formulate the main definitions and formalisms, along with some basic results, in §§2 and 3. These two sections may be regarded as an extension of Green's book [G2] to the more general context considered here. One new feature is

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the emergence of an infinite dimensional algebra $S_K(n)_z$ which is useful for grading the rational representations of the general linear group $\mathrm{GL}_n(K)$ (K an infinite field). These infinite dimensional algebras are obtained as an inverse limit of rational Schur algebras, and they may be of independent interest. Moreover, each rational Schur algebra $S_K(n; r, s)$ is a quotient of $S_K(n)_{r-s}$. For fixed $n \geq 2$, as r, s vary over the set \mathbb{N} of nonnegative integers, the family of $S_K(n; r, s)$ -modules is precisely the family of rational $\mathrm{GL}_n(K)$ -modules.

In §4 we give some combinatorial descriptions of the set $\pi = \Lambda^+(n; r, s)$ of dominant weights defining each rational Schur algebra. In §5 we show that the rational Schur algebras are generalized Schur algebras, in the sense of [Do]. It follows that rational Schur algebras are always quasihereditary when taken over a field, so they have finite global dimension. In §6 we describe how to obtain cellular bases for rational Schur algebras, by a descent from an ordinary Schur algebra. In §7 we formulate a presentation of $S_K(n; r, s)$ by generators and relations (when K has characteristic zero) along the same lines as the earlier result in [DG2] for $S_K(n, r)$.

In §8 we consider Schur–Weyl duality and describe the action of a certain algebra $\mathfrak{B}_{r,s}^{(n)}$ centralizing (in characteristic zero) the action of the general linear group $\mathrm{GL}_n(K)$ on mixed tensor space $E_K^{\otimes r} \otimes E_K^*{}^{\otimes s}$. Thus we can regard $S_K(n; r, s)$ as the centralizer algebra

$$S_K(n; r, s) = \mathrm{End}_{\mathfrak{B}_{r,s}^{(n)}}(E_K^{\otimes r} \otimes E_K^*{}^{\otimes s})$$

at least when $n \geq r + s$ and K has characteristic zero. The restrictions (on characteristic and n) are believed unnecessary.

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1. GENERAL LEMMAS

1.1. Let Γ be any semigroup and K an infinite field. Denote by K^Γ the K -algebra of K -valued functions on Γ , with product ff' of elements $f, f' \in K^\Gamma$ given by $s \rightarrow f(s)f'(s)$ for $s \in \Gamma$. Given a representation $\tau : \Gamma \rightarrow \mathrm{End}_K(V)$ in a K vector space V , the *coefficient space* of the representation is the subspace $\mathrm{cf}_\Gamma V$ of K^Γ spanned by the coefficients $\{r_{ab}\}$ of the representation. The coefficients $r_{ab} \in K^\Gamma$ are determined relative to a choice of basis v_a ($a \in I$) for V by

$$(1.1.1) \quad \tau(g)v_b = \sum_{a \in I} r_{ab}(g)v_a$$

for $g \in \Gamma$, $b \in I$. The *envelope* of the representation τ is the subalgebra of $\mathrm{End}_K(V)$ generated by the image of τ . As we shall show below, the notions of coefficient space and envelope, associated to a given representation τ , are dual to one another, at least when V is of finite dimension.

To formulate the result, let $K\Gamma$ be the semigroup algebra of Γ . Elements of $K\Gamma$ are sums of the form $\sum_{g \in \Gamma} a_g g$ ($a_g \in K$) with finitely many $a_g \neq 0$. The group multiplication extends by linearity to $K\Gamma$. Note that $K\Gamma$ is also a coalgebra, with comultiplication given on generators by $x \rightarrow x \otimes x$, and counit by $x \rightarrow 1$, for $x \in \Gamma$. The given representation $\tau : \Gamma \rightarrow \mathrm{End}_K(V)$ extends by linearity to an algebra homomorphism $K\Gamma \rightarrow \mathrm{End}_K(V)$; denote this extended map also by τ . Obviously

the envelope $[V]_\Gamma$ is simply the image $\tau(K\Gamma)$. In other words, the representation τ factors through its envelope: there is a commutative diagram

$$(1.1.2) \quad \begin{array}{ccc} K\Gamma & \xrightarrow{\tau} & \text{End}_K(V) \\ & \searrow & \nearrow \\ & [V]_\Gamma & \end{array}$$

in which the leftmost and rightmost diagonal arrows are a surjection and injection, respectively. Taking linear duals, the above commutative diagram induces another one:

$$(1.1.3) \quad \begin{array}{ccc} (K\Gamma)^* & \xleftarrow{\tau^*} & \text{End}_K(V)^* \\ & \swarrow & \searrow \\ & [V]_\Gamma^* & \end{array}$$

There is a natural isomorphism of algebras $(K\Gamma)^* \simeq K^\Gamma$, given by restricting a linear K -valued map on $K\Gamma$ to Γ ; its inverse is given by the process of linearly extending a K -valued map on Γ to $K\Gamma$. Note that the algebra structure on $(K\Gamma)^*$ comes from dualizing the coalgebra structure on $K\Gamma$. Now we are ready for the aforementioned result.

Lemma 1.2. *The coefficient space $\text{cf}_\Gamma(V)$ may be identified with the image of τ^* , so there is an isomorphism of vector spaces $[V]_\Gamma^* \simeq \text{cf}_\Gamma V$. Moreover, if V has finite dimension, then $[V]_\Gamma \simeq (\text{cf}_\Gamma V)^*$, again a vector space isomorphism. In that case ($\dim_K V < \infty$) the first isomorphism is an isomorphism of coalgebras and the second is an isomorphism of algebras.*

Proof. We prove the first claim. Relative to the basis v_a ($a \in I$) the algebra $\text{End}_K(V)$ has basis e_{ab} ($a, b \in I$), where e_{ab} is the linear endomorphism of V taking v_b to v_a and taking all other v_c , for $c \neq b$, to 0. In terms of this notation, equation (1.1.1) is equivalent with the equality

$$(1.2.1) \quad \tau(g) = \sum_{a,b \in I} r_{ab}(g) e_{ab}.$$

Let e'_{ab} be the basis of $\text{End}_K(V)^*$ dual to the basis e_{ab} , so that e'_{ab} is the linear functional on $\text{End}_K(V)$ taking the value 1 on e_{ab} and taking the value 0 on all other e_{cd} . Then one checks that τ^* carries e'_{ab} onto r_{ab} . This proves that $\text{cf}_\Gamma(V)$ may be identified with the image of τ^* . From the remarks preceding the statement of the lemma, the rest of the claims now follow immediately. Note that the coalgebra structure on $[V]_\Gamma^*$ is that induced by dualizing the algebra structure on $[V]_\Gamma$, and similarly, the algebra structure on $(\text{cf}_\Gamma V)^*$ is induced by dualizing the coalgebra structure on $\text{cf}_\Gamma V$. (See the discussion following Definition 3.1 for more details on the latter point.) \square

Remark 1.3. The first statement of the lemma shows, in particular, that the coefficient space of a representation does not depend on a choice of basis for the representation.

We include the following lemma since we need it later and we are unaware of a reference.

Lemma 1.4. (a) *Suppose Γ is a group. For any finite dimensional $K\Gamma$ -module V we have an isomorphism $(\text{cf}_\Gamma(V^*))^{\text{opp}} \simeq \text{cf}_\Gamma(V)$ as coalgebras.*

(b) *Given a coalgebra C , let C^{opp} be its opposite coalgebra. Then $(C^{\text{opp}})^* \simeq (C^*)^{\text{opp}}$ as algebras. That is, the linear dual of the opposite coalgebra may be identified with the opposite algebra of the linear dual.*

Proof. (a) The isomorphism is given by $\tilde{r}_{ab} \rightarrow r_{ab}$ where the $r_{ab} \in K^\Gamma$ are the coefficients of V . The r_{ab} satisfy the equations

$$(1.4.1) \quad gu_b = \sum_{a \in I} r_{ab}(g)u_a$$

for all $g \in \Gamma$, where (u_b) ($b \in I$) is a chosen basis of V . The \tilde{r}_{ab} are the coefficients of V^* with respect to the dual basis, satisfying $\tilde{r}_{ab}(g) = r_{ab}(g^{-1})$ for all $g \in \Gamma$.

The r_{ab} span $\text{cf}_\Gamma(V)$ and satisfy

$$(1.4.2) \quad \Delta(r_{ab}) = \sum_{c \in I} r_{ac} \otimes r_{cb}$$

for each $a, b \in I$. A calculation shows that the \tilde{r}_{ab} (which span $\text{cf}_\Gamma(V^*)$) satisfy

$$(1.4.3) \quad \Delta(\tilde{r}_{ab}) = \sum_{c \in I} \tilde{r}_{cb} \otimes \tilde{r}_{ac}.$$

The opposite coalgebra structure is the one in which the tensors on the right hand side of the preceding equality are reversed in order. This proves part (a) of the lemma.

(b) This is easily checked and left to the reader. \square

2. THE CATEGORIES $\mathcal{M}_K(n; r, s)$, $\mathcal{M}_K(n)_z$

2.1. Let $n \geq 2$ be an integer and K an arbitrary infinite field. Henceforth we take $\Gamma = \text{GL}_n(K)$, the group of nonsingular $n \times n$ matrices with entries in K . The vector space K^Γ of K -valued functions on Γ is naturally a (commutative, associative) K -algebra with product ff' of elements $f, f' \in K^\Gamma$ given by $s \rightarrow f(s)f'(s)$ for $s \in \Gamma$. The group multiplication $\Gamma \times \Gamma \rightarrow \Gamma$ and unit element $1 \rightarrow \Gamma$ induce the maps

$$(2.1.1) \quad K^\Gamma \xrightarrow{\Delta} K^{\Gamma \times \Gamma}, \quad K^\Gamma \xrightarrow{\varepsilon} K$$

given by

$$(2.1.2) \quad \Delta(f) = [(s, t) \rightarrow f(st), \quad s, t \in \Gamma], \quad \varepsilon(f) = f(1)$$

and one easily checks that both Δ, ε are K -algebra maps. There is a map (tensor products are always taken over K unless we indicate otherwise)

$$(2.1.3) \quad K^\Gamma \otimes K^\Gamma \rightarrow K^{\Gamma \times \Gamma}$$

determined by the condition $f \otimes f' \rightarrow [(s, t) \rightarrow f(s)f'(t), \text{ all } s, t \in \Gamma]$. This map is injective and we use it to identify $K^\Gamma \otimes K^\Gamma$ with its image in $K^{\Gamma \times \Gamma}$. Thus we regard $K^\Gamma \otimes K^\Gamma$ as a subspace of $K^{\Gamma \times \Gamma}$.

2.2. We will need certain elements of K^Γ . For $1 \leq i, j \leq n$, let $\mathbf{c}_{ij} \in K^\Gamma$ be defined by $\mathbf{c}_{ij}(A) = (i, j)$ of the matrix $A \in \Gamma$. Similarly, let $\tilde{\mathbf{c}}_{ij} \in K^\Gamma$ be defined by $\tilde{\mathbf{c}}_{ij}(A) = \mathbf{c}_{ij}(A^{-1})$ for $A \in \Gamma$. Let $\mathbf{d} \in K^\Gamma$ be given by $\mathbf{d}(A) = \det A$ for $A \in \Gamma$. Note that $\mathbf{d}^{-1} \in K^\Gamma$; that is, \mathbf{d} is an invertible element of K^Γ . The assumption that K is infinite ensures that the $\{\mathbf{c}_{ij}\}$ are algebraically independent amongst themselves; similarly for the $\{\tilde{\mathbf{c}}_{ij}\}$. By elementary calculations with the definitions one verifies:

$$(2.2.1) \quad \Delta(\mathbf{c}_{ij}) = \sum_k \mathbf{c}_{ik} \otimes \mathbf{c}_{kj}, \quad \varepsilon(\mathbf{c}_{ij}) = \delta_{ij},$$

$$(2.2.2) \quad \Delta(\tilde{\mathbf{c}}_{ij}) = \sum_k \tilde{\mathbf{c}}_{kj} \otimes \tilde{\mathbf{c}}_{ik}, \quad \varepsilon(\tilde{\mathbf{c}}_{ij}) = \delta_{ij},$$

$$(2.2.3) \quad \Delta(\mathbf{d}) = \mathbf{d} \otimes \mathbf{d}; \quad \varepsilon(\mathbf{d}) = 1.$$

Here δ is the usual Kronecker delta: δ_{ij} is 1 if $i = j$ and 0 otherwise. By applying \mathbf{c}_{ij} to the usual expression in terms of matrix coordinates for $gg^{-1} = 1 = g^{-1}g$ ($g \in \Gamma$) one verifies that

$$(2.2.4) \quad \sum_k \mathbf{c}_{ik} \tilde{\mathbf{c}}_{kj} = \delta_{ij} = \sum_k \tilde{\mathbf{c}}_{ik} \mathbf{c}_{kj}.$$

The following formal identities also hold:

$$(2.2.5) \quad \mathbf{d} = \sum_{\pi \in \mathfrak{S}_n} \text{sign}(\pi) \mathbf{c}_{1,\pi(1)} \cdots \mathbf{c}_{n,\pi(n)},$$

$$(2.2.6) \quad \mathbf{d}^{-1} = \sum_{\pi \in \mathfrak{S}_n} \text{sign}(\pi) \tilde{\mathbf{c}}_{1,\pi(1)} \cdots \tilde{\mathbf{c}}_{n,\pi(n)}$$

where $\text{sign}(\pi)$ is the signature of a permutation π in the symmetric group \mathfrak{S}_n on n letters.

2.3. By Cramer's rule, each $\tilde{\mathbf{c}}_{ij}$ is expressible as a product of \mathbf{d}^{-1} with a polynomial expression in the variables \mathbf{c}_{ij} ($1 \leq i, j \leq n$). Thus, the subalgebra $\tilde{A}_K(n)$ of K^Γ generated by all \mathbf{c}_{ij} together with \mathbf{d}^{-1} coincides with the subalgebra of K^Γ generated by the all \mathbf{c}_{ij} and all $\tilde{\mathbf{c}}_{ij}$. From formulas (2.2.1) and (2.2.2) it follows that $\Delta \tilde{A}_K(n) \subset \tilde{A}_K(n) \otimes \tilde{A}_K(n)$; hence $\tilde{A}_K(n)$ is a bialgebra with comultiplication Δ and counit ε . Actually, $\tilde{A}_K(n)$ is a Hopf algebra with antipode induced from the inverse map $\Gamma \rightarrow \Gamma$; by (2.2.4) the antipode interchanges \mathbf{c}_{ij} and $\tilde{\mathbf{c}}_{ij}$. As Hopf algebras, we identify $\tilde{A}_K(n)$ with the affine coordinate algebra $K[\Gamma]$. Let $A_K(n)$ be the subalgebra of $\tilde{A}_K(n)$ generated by all \mathbf{c}_{ij} ; this is the subspace spanned by all monomials in \mathbf{c}_{ij} and we identify it with the algebra of polynomial functions on Γ . Clearly, $\tilde{A}_K(n)$ is the localization of $A_K(n)$ at \mathbf{d} .

2.4. Given nonnegative integers r, s let $\tilde{A}_K(n; r, s)$ be the subspace of $\tilde{A}_K(n)$ spanned by all products of the form $\prod_{i,j} (\mathbf{c}_{ij})^{a_{ij}} \prod_{i,j} (\tilde{\mathbf{c}}_{ij})^{b_{ij}}$ such that $\sum a_{ij} = r$ and $\sum b_{ij} = s$. Then $\tilde{A}_K(n; r, s)$ is for each r, s a sub-coalgebra of $\tilde{A}_K(n)$. (Use the fact that Δ is an algebra homomorphism.)

Clearly, $\tilde{A}_K(n) = \sum_{r,s} \tilde{A}_K(n; r, s)$. Because of (2.2.4) this sum is not direct; in fact, for any r, s we have for each $1 \leq i \leq n$ inclusions

$$(2.4.1) \quad \tilde{A}_K(n; r, s) \subset \tilde{A}_K(n; r+1, s+1)$$

since by (2.2.4) any $f \in \tilde{A}_K(n; r, s)$ satisfies $f = f \sum_k \mathbf{c}_{ik} \tilde{\mathbf{c}}_{ki}$ and $f = f \sum_k \tilde{\mathbf{c}}_{ik} \mathbf{c}_{ki}$, and the right hand side of each equality is a sum of members of $\tilde{A}_K(n; r+1, s+1)$.

2.5. Let $\mathbf{I}(n, r)$ denote the set of all multi-indices (i_1, \dots, i_r) such that each i_a lies within the interval $[1, n]$, for $a = 1, \dots, r$. Set $\mathbf{I}(n; r, s) = \mathbf{I}(n, r) \times \mathbf{I}(n, s)$. Given a pair (I, J) of elements of $\mathbf{I}(n; r, s)$, say $I = ((i_1, \dots, i_r), (i'_1, \dots, i'_s))$, $J = ((j_1, \dots, j_r), (j'_1, \dots, j'_s))$, we write

$$(2.5.1) \quad \mathbf{c}_{I,J} = \mathbf{c}_{i_1, j_1} \cdots \mathbf{c}_{i_r, j_r} \tilde{\mathbf{c}}_{i'_1, j'_1} \cdots \tilde{\mathbf{c}}_{i'_s, j'_s}$$

which is an element of $\tilde{A}_K(n; r, s)$. In fact, the set of all such elements $\mathbf{c}_{I,J}$ spans $\tilde{A}_K(n; r, s)$ as (I, J) vary over $\mathbf{I}(n; r, s) \times \mathbf{I}(n; r, s)$. When working with this spanning set, one must take the following equality rule into account:

$$(2.5.2) \quad \mathbf{c}_{I,J} = \mathbf{c}_{L,M} \text{ if } (I, J) \sim (L, M)$$

where we define $(I, J) \sim (L, M)$ if there exists some (σ, τ) in $\mathfrak{S}_r \times \mathfrak{S}_s$ such that $I(\sigma, \tau) = L$, $J(\sigma, \tau) = M$. Here $\mathfrak{S}_r \times \mathfrak{S}_s$ acts on the right on $\mathbf{I}(n; r, s) = \mathbf{I}(n, r) \times \mathbf{I}(n, s)$ by

$$((i_1, \dots, i_r), (i'_1, \dots, i'_s))(\sigma, \tau) = ((i_{\sigma(1)}, \dots, i_{\sigma(r)}), (i'_{\tau(1)}, \dots, i'_{\tau(s)})).$$

The equality rule above states that the symbols $\mathbf{c}_{I,J}$ are constant on $\mathfrak{S}_r \times \mathfrak{S}_s$ -orbits for the above action of $\mathfrak{S}_r \times \mathfrak{S}_s$ on $\mathbf{I}(n; r, s) \times \mathbf{I}(n; r, s)$.

Even after taking this equality rule into account, the spanning set described above is not a basis of $\tilde{A}_K(n; r, s)$, since the set is not linearly independent. One could obtain a basis of “bideterminants” for $\tilde{A}_K(n; r, s)$, by dualizing the procedure in §6.

2.6. **Bidegree and degree.** We let $\mathcal{M}_K(n; r, s)$ denote the category of finite dimensional KT -modules V whose coefficient space $\text{cf}_\Gamma(V)$ (see 1.1) lies in $\tilde{A}_K(n; r, s)$. (The morphisms between two objects V, V' in $\mathcal{M}_K(n; r, s)$ are homomorphisms of KT -modules.) Objects in $\mathcal{M}_K(n; r, s)$ afford rational representations of the algebraic group $\Gamma = \text{GL}_n(K)$. In fact, the rational representations of Γ are precisely the KT -modules V for which $\text{cf}_\Gamma(V)$ is contained in $\tilde{A}_K(n)$. We call objects of $\mathcal{M}_K(n; r, s)$ rational representations of Γ of *bidegree* (r, s) . Note, however, that the bidegree of a rational representation is not well defined, since if V is an object of $\mathcal{M}_K(n; r, s)$, then V is also an object of $\mathcal{M}_K(n; r+1, s+1)$. But it is easy to see that every finite dimensional rational KT -module has a unique *minimal* bidegree; this is the bidegree (r, s) such that V belongs to $\mathcal{M}_K(n; r, s)$, but V does not belong to $\mathcal{M}_K(n; r-1, s-1)$.

Given an integer $z \geq 0$ set $\tilde{A}_K(n)_z = \bigcup_{t \geq 0} \tilde{A}_K(n; z+t, t)$ and set $\tilde{A}_K(n)_{-z} = \bigcup_{t \geq 0} \tilde{A}_K(n; t, z+t)$. Then there is a direct sum decomposition

$$(2.6.1) \quad \tilde{A}_K(n) = \bigoplus_{z \in \mathbb{Z}} \tilde{A}_K(n)_z$$

and each summand in this decomposition is a sub-coalgebra of $\tilde{A}_K(n)$. A finite dimensional rational KT -module is said to be of rational *degree* $z \in \mathbb{Z}$ if its coefficient space lies in $\tilde{A}_K(n)_z$. Note that if V is a homogeneous polynomial module of degree r , then its rational degree is also r . Thus the notion of degree just defined for rational modules extends the corresponding notion as usually defined for polynomial representations.

If V is infinite dimensional such that V is a union of finite dimensional rational submodules each of degree z , then we say that V has degree z .

Let $\mathcal{M}_K(n)_z$ be the category of rational $K\Gamma$ -modules V of (rational) degree z , for any $z \in \mathbb{Z}$. (Again morphisms are just $K\Gamma$ -module homomorphisms.) Obviously $\mathcal{M}_K(n; r, s)$ is a subcategory of $\mathcal{M}_K(n)_{r-s}$ for any r, s .

The equality $\tilde{A}_K(n) = \sum_{r,s} \tilde{A}_K(n; r, s)$ shows that as r, s vary, the categories $\mathcal{M}_K(n; r, s)$ taken together comprise all of the rational representations of Γ . We have the following more precise result (compare with [G2, (2.2c)]).

Theorem 2.7. *Let V be a rational $K\Gamma$ -module.*

(a) *V has a direct sum decomposition of the form $V = \bigoplus_{z \in \mathbb{Z}} V_z$, where V_z is a rational $K\Gamma$ -submodule of V of degree z .*

(b) *For each $z \in \mathbb{Z}$, V_z is expressible as a union of an ascending chain of rational $K\Gamma$ -submodules as follows:*

- (i) *If $z \geq 0$, then V_z is the union of a chain of the form $V_{z,0} \subset V_{z+1,1} \subset \cdots \subset V_{z+t,t} \subset \cdots$ where each $V_{z+t,t}$ is a rational $K\Gamma$ -module of bidegree $(z+t, t)$.*
- (ii) *If $z \leq 0$, then V_z is the union of a chain of the form $V_{0,-z} \subset V_{1,-z+1} \subset \cdots \subset V_{t,-z+t} \subset \cdots$ where each $V_{t,-z+t}$ is a rational $K\Gamma$ -module of bidegree $(t, -z+t)$.*

Proof. (a) The decomposition in (a) follows from a general theorem on comodules; see [G1, (1.6c)].

(b) Let $V_{r,s}$ in case $r - s = z$ be the unique maximal $K\Gamma$ -submodule of V_z such that $\text{cf}_\Gamma V_{r,s} \subset \tilde{A}_K(n; r, s)$. □

In other words, part (a) says that every rational representation of $\text{GL}_n(K)$ is a direct sum of homogeneous ones, where we call objects of $\mathcal{M}_K(n)_z$ ($z \in \mathbb{Z}$) *homogeneous* of (rational) degree z .

3. THE RATIONAL SCHUR ALGEBRA $S_K(n; r, s)$

Theorem 2.7 shows that each indecomposable rational $K\Gamma$ -module V is homogeneous; i.e., $V \in \mathcal{M}_K(n)_z$ for some $z \in \mathbb{Z}$. Thus one may as well confine one's attention to homogeneous modules. Set $S_K(n)_z = (\tilde{A}_K(n)_z)^*$. Since $\tilde{A}_K(n)_z$ is a K -coalgebra, $S_K(n)_z$ is a K -algebra in the standard way. The algebra $S_K(n)_z$ is in general infinite dimensional. Since our main goal is to approach the rational representations of Γ through finite dimensional algebras, we make the following definition.

Definition 3.1. The *rational Schur algebra*, denoted by $S_K(n; r, s)$, is the K -algebra $\tilde{A}_K(n; r, s)^* = \text{Hom}_K(\tilde{A}_K(n; r, s), K)$, with multiplication induced from the coproduct Δ on $\tilde{A}_K(n; r, s)$.

The product of two elements ξ, ξ' of $S_K(n; r, s)$ is computed by first taking the tensor product $\xi \otimes \xi'$ of the two maps. This gives a linear map from $\tilde{A}_K(n; r, s) \otimes \tilde{A}_K(n; r, s)$ to $K \otimes K$. By identifying $K \otimes K$ with K we regard the map $\xi \otimes \xi'$ as taking values in the field K ; it is customary to write $\xi \overline{\otimes} \xi'$ for this slightly altered map. Finally, by composing with Δ we obtain a linear map from $\tilde{A}_K(n; r, s)$ to K ; i.e., an element of $S_K(n; r, s)$. So the product $\xi \xi'$ is given by

$$(3.1.1) \quad \xi \xi' = (\xi \overline{\otimes} \xi') \circ \Delta.$$

Products in $S_K(n)_z$ are computed in a similar way.

Since $\tilde{A}_K(n; r, s)$ is defined by a finite spanning set, it and its dual $S_K(n; r, s)$ are finite dimensional, for any r, s . Note that the restriction of ε to $\tilde{A}_K(n; r, s)$ is the identity element of $S_K(n; r, s)$.

For each r, s the inclusions $\tilde{A}_K(n; r, s) \subset \tilde{A}_K(n)_{r-s}$ and (2.4.1) induce surjective algebra maps

$$(3.1.2) \quad S_K(n)_{r-s} \rightarrow S_K(n; r, s), \quad S_K(n; r+1, s+1) \rightarrow S_K(n; r, s).$$

In fact, for $z \in \mathbb{Z}$ it is clear that $\tilde{A}_K(n)_z$ is the union of one of the two chains

$$(3.1.3) \quad \begin{aligned} \tilde{A}_K(n; z, 0) \subset \tilde{A}_K(n; z+1, 1) \subset \cdots, \\ \tilde{A}_K(n; 0, -z) \subset \tilde{A}_K(n; 1, 1-z) \subset \cdots \end{aligned}$$

according to whether $z \geq 0$ or $z \leq 0$. It follows that $S_K(n)_z$ is the inverse limit of the corresponding dual chain of rational Schur algebras

$$(3.1.4) \quad S_K(n)_z \simeq \begin{cases} \lim_{\leftarrow t} S_K(n; z+t, t) & \text{if } z \geq 0, \\ \lim_{\leftarrow t} S_K(n; t, t-z) & \text{if } z \leq 0. \end{cases}$$

Since the rational Schur algebras determine the algebras $S_K(n)_z$ in this simple way, it makes sense to focus our attention on them instead of the more complicated infinite dimensional $S_K(n)_z$.

3.2. J.A. Green [G2] formulated the notion of a (classical) Schur algebra $S_K(n, r)$ for $\Gamma = \mathrm{GL}_n(K)$, extending some results of Schur's dissertation from characteristic zero to arbitrary characteristic. By definition, $S_K(n, r)$ is the linear dual of a coalgebra $A_K(n, r)$; here $A_K(n, r)$ is simply the span of all monomials in the \mathbf{c}_{ij} of total degree r . Note that the classical Schur algebras in Green's sense are included among the rational Schur algebras; the classical Schur algebra $S_K(n, r)$ is just the rational Schur algebra $S_K(n; r, 0)$. This follows immediately from the equality $A_K(n, r) = \tilde{A}_K(n; r, 0)$.

3.3. Following [G2, 2.4], for each $g \in \Gamma$ let $e_g \in S_K(n; r, s)$ be determined by $e_g(c) = c(g)$ for any $c \in \tilde{A}_K(n; r, s)$. We have $e_g e_{g'} = e_{gg'}$ for any $g, g' \in \Gamma$; moreover, $e_1 = \varepsilon$. Thus by extending the map $g \rightarrow e_g$ linearly one obtains a map $e : K\Gamma \rightarrow S_K(n; r, s)$, a homomorphism of K -algebras.

Any $f \in K^\Gamma$ has a unique extension to a linear map $f : K\Gamma \rightarrow K$. As discussed in §1, this gives an identification $K^\Gamma \simeq (K\Gamma)^*$. With this identification, the image of an element $a = \sum a_g g$ of $K\Gamma$ under e is simply evaluation at a : $e(a)$ takes c to $c(a)$ for all $c \in \tilde{A}_K(n; r, s)$.

Proposition 3.4. (a) *The map $e : K\Gamma \rightarrow S_K(n; r, s)$ is surjective.*

(b) *Let $f \in K^\Gamma$. Then $f \in \tilde{A}_K(n; r, s)$ if and only if $f(\ker e) = 0$.*

(c) *Let V be a finite dimensional $K\Gamma$ -module. Then V belongs to $\mathcal{M}_K(n; r, s)$ if and only if $(\ker e)V = 0$.*

Proof. See [G2, 2.4b,c]. The arguments given there are also valid in the current context. \square

The proposition shows that the category $\mathcal{M}_K(n; r, s)$ is equivalent to the category of finite dimensional $S_K(n; r, s)$ -modules. An object V in either category is transformed into an object of the other, using the rule:

$$(3.4.1) \quad av = e(a)v, \quad (a \in K\Gamma, v \in V).$$

Since both actions determine the same algebra of linear transformations on V , the concepts of submodule, module homomorphism, etc., coincide in the two categories.

3.5. Let \mathbf{E}_K be the vector space K^n , regarded as column vectors. The group $\Gamma = \mathrm{GL}_n(K)$ acts on \mathbf{E}_K , on the left, by matrix multiplication; this action makes \mathbf{E}_K into a rational $K\Gamma$ -module. The linear dual space $\mathbf{E}_K^* = \mathrm{Hom}_K(\mathbf{E}_K, K)$ is also a rational $K\Gamma$ -module, with $g \in \Gamma$ acting via $(gf) = [v \rightarrow f(g^{-1}v), v \in \mathbf{E}_K]$. Now if \mathbf{v}_i ($1 \leq i \leq n$) is the standard basis of \mathbf{E}_K , i.e. \mathbf{v}_i is the vector with a 1 in the i th position and 0 elsewhere, and if \mathbf{v}'_i ($1 \leq i \leq n$) is the basis dual to the \mathbf{v}_i , then we have

$$(3.5.1) \quad g\mathbf{v}_j = \sum_i \mathbf{c}_{ij}(g)\mathbf{v}_i, \quad g\mathbf{v}'_i = \sum_j \tilde{\mathbf{c}}_{ij}(g)\mathbf{v}'_j$$

which shows that $\mathrm{cf}_\Gamma(\mathbf{E}_K)$ is the subspace of K^Γ spanned by the \mathbf{c}_{ij} and $\mathrm{cf}_\Gamma(\mathbf{E}_K^*)$ is the subspace of K^Γ spanned by the $\tilde{\mathbf{c}}_{ij}$. Thus $\mathrm{cf}_\Gamma(\mathbf{E}_K) = \tilde{A}_K(n; 1, 0)$ and $\mathrm{cf}_\Gamma(\mathbf{E}_K^*) = \tilde{A}_K(n; 0, 1)$.

It is well known that *coefficient spaces are multiplicative*, in the following sense: If V, W are two representations of Γ , then $\mathrm{cf}_\Gamma(V \otimes W) = (\mathrm{cf}_\Gamma V)(\mathrm{cf}_\Gamma W)$, where the product is taken in the algebra K^Γ . From this and the preceding remarks it follows immediately that

$$(3.5.2) \quad \tilde{A}_K(n; r, s) = \mathrm{cf}_\Gamma(\mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{*\otimes s}).$$

Now by Lemma 1.2 it follows immediately that

$$(3.5.3) \quad S_K(n; r, s) \simeq [\mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{*\otimes s}]_\Gamma;$$

i.e., the rational Schur algebra in bidegree r, s may be identified with the envelope of the representation of Γ on mixed tensor space $\mathbf{E}_K^{r,s} := \mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{*\otimes s}$. So we *may identify* $S_K(n; r, s)$ with the image of the representation

$$(3.5.4) \quad \rho_K : K\Gamma \rightarrow \mathrm{End}_K(\mathbf{E}_K^{r,s}).$$

3.6. Next we obtain an alternative description of the rational Schur algebras, as a quotient of the hyperalgebra \mathfrak{U}_K . This will ultimately lead to a proof that rational Schur algebras are generalized Schur algebras.

Set $\mathfrak{U}_\mathbb{Q} = \mathfrak{U}_\mathbb{Q}(\mathfrak{gl}_n)$, the universal enveloping algebra of the Lie algebra \mathfrak{gl}_n , over the rational field \mathbb{Q} . The space $\mathbf{E}_\mathbb{Q}$ is naturally a $\mathfrak{U}_\mathbb{Q}$ -module, with action induced from the Γ -action (the action of \mathfrak{gl}_n is given by left matrix multiplication if we view elements of $\mathbf{E}_\mathbb{Q}$ as column vectors). The dual space $\mathbf{E}_\mathbb{Q}^*$ is also a $\mathfrak{U}_\mathbb{Q}$ -module in the usual way, by regarding it as the dual module for the Lie algebra \mathfrak{gl}_n , with action

$$(x \cdot f)(v) = -f(xv), \quad \text{for all } x \in \mathfrak{gl}_n, f \in \mathbf{E}_\mathbb{Q}^*, v \in \mathbf{E}_\mathbb{Q};$$

(this is the action induced from the Γ -action), and so $\mathbf{E}_\mathbb{Q}^{r,s} = \mathbf{E}_\mathbb{Q}^{\otimes r} \otimes \mathbf{E}_\mathbb{Q}^{*\otimes s}$ is naturally a $\mathfrak{U}_\mathbb{Q}$ -module.

It follows from the Poincaré–Birkhoff–Witt theorem that $\mathfrak{U}_\mathbb{Q}$ is generated as an algebra by elements (which we shall denote by the same symbols) corresponding to the matrices X_{ij} ($1 \leq i, j \leq n$) in \mathfrak{gl}_n . Here, for any $1 \leq i, j \leq n$, $X_{ij} = (\delta_{ik}\delta_{jl})_{1 \leq k, l \leq n}$ is the $n \times n$ matrix with a unique 1 in the i th row and j th column,

and 0 elsewhere. We set $H_i := X_{ii}$ for each $i = 1, \dots, n$. We shall need the Kostant \mathbb{Z} -form $\mathfrak{U}_{\mathbb{Z}}$, the subring (with 1) of $\mathfrak{U}_{\mathbb{Q}}$ generated by all

$$(3.6.1) \quad \frac{X_{ij}^p}{p!} \quad (1 \leq i \neq j \leq n); \quad \left(\begin{matrix} H_i \\ q \end{matrix} \right) \quad (1 \leq i \leq n)$$

for any integers $p, q \geq 0$.

Set $\mathbf{E}_{\mathbb{Z}} = \mathbb{Z}^n = \mathfrak{U}_{\mathbb{Z}}\mathbf{v}_1$ (note that $\mathbf{v}_1 \in \mathbf{E}_{\mathbb{Q}}$ is a highest weight vector). The \mathbb{Z} -module $\mathbf{E}_{\mathbb{Z}}$ is an admissible lattice in $\mathbf{E}_{\mathbb{Q}}$; similarly, its dual space $\mathbf{E}_{\mathbb{Z}}^* = \text{Hom}_{\mathbb{Z}}(\mathbf{E}_{\mathbb{Z}}, \mathbb{Z})$ is an admissible lattice in $\mathbf{E}_{\mathbb{Q}}^*$. We have the equalities $\mathbf{E}_{\mathbb{Z}} = \sum \mathbb{Z}\mathbf{v}_i$ and $\mathbf{E}_{\mathbb{Z}}^* = \sum \mathbb{Z}\mathbf{v}'_i$. It is clear that there are isomorphisms of vector spaces

$$(3.6.2) \quad \mathbf{E}_K \simeq \mathbf{E}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K; \quad \mathbf{E}_K^* \simeq \mathbf{E}_{\mathbb{Z}}^* \otimes_{\mathbb{Z}} K$$

for any field K . This induces an isomorphism of vector spaces

$$(3.6.3) \quad \mathbf{E}_K^{r,s} \simeq \mathbf{E}_{\mathbb{Z}}^{r,s} \otimes_{\mathbb{Z}} K$$

where $\mathbf{E}_{\mathbb{Z}}^{r,s} = \mathbf{E}_{\mathbb{Z}}^{\otimes r} \otimes_{\mathbb{Z}} \mathbf{E}_{\mathbb{Z}}^{*\otimes s}$ for any r, s . For any field K , set $\mathfrak{U}_K = \mathfrak{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. We have actions of \mathfrak{U}_K on $\mathbf{E}_K, \mathbf{E}_K^*, \mathbf{E}_K^{r,s}$ induced from the action of $\mathfrak{U}_{\mathbb{Z}}$ on $\mathbf{E}_{\mathbb{Z}}, \mathbf{E}_{\mathbb{Z}}^*, \mathbf{E}_{\mathbb{Z}}^{r,s}$ by a change of base ring. Let

$$(3.6.4) \quad \varphi_K : \mathfrak{U}_K \rightarrow \text{End}_K(\mathbf{E}_K^{r,s})$$

be the representation affording the \mathfrak{U}_K -module structure on $\mathbf{E}_K^{r,s}$.

Proposition 3.7. *For any infinite field K , the rational Schur algebra $S_K(n; r, s)$ may be identified with the image of the representation $\varphi_K : \mathfrak{U}_K \rightarrow \text{End}_K(\mathbf{E}_K^{r,s})$.*

Proof. This is essentially the Chevalley group construction. Each $\frac{X_{ij}^m}{m!} \in \mathfrak{U}_{\mathbb{Z}}$ ($1 \leq i \neq j \leq n$) induces a corresponding element $\frac{X_{ij}^m}{m!} \otimes 1$ of \mathfrak{U}_K . For $t \in K$ and each $i \neq j$ let $x_{ij}(t)$ be the element $I + tX_{ij}$ of $\text{SL}_n(K)$, where I is the $n \times n$ identity matrix. The action of $\Gamma = \text{GL}_n(K)$ on $\mathbf{E}_K^{r,s}$ is given by the representation ρ_K of (3.5.4). The element $x_{ij}(t)$ acts on $\mathbf{E}_K^{r,s}$ as the K -linear endomorphism $\rho_K(x_{ij}(t))$, where

$$(3.7.1) \quad \rho_K(x_{ij}(t)) = \sum_{m \geq 0} t^m \varphi_K\left(\frac{X_{ij}^m}{m!} \otimes 1\right).$$

The sum is finite because X_{ij} acts nilpotently on $\mathbf{E}_{\mathbb{Z}}^{r,s}$. Hence for each fixed $i \neq j$ there exists a natural number N such that

$$(3.7.2) \quad \rho_K(x_{ij}(t)) = 1 + t\varphi_K(X_{ij} \otimes 1) + \dots + t^N \varphi_K\left(\frac{X_{ij}^N}{N!} \otimes 1\right)$$

for all $t \in K$. Choosing $N + 1$ distinct values t_0, t_1, \dots, t_N for t in K (which is always possible since K is infinite) we can solve the resulting linear system, by inverting the Vandermonde matrix of coefficients, to obtain equalities

$$(3.7.3) \quad \varphi_K\left(\frac{X_{ij}^m}{m!} \otimes 1\right) = \sum_k a_{mk} \rho_K(x_{ij}(t_k))$$

for certain $a_{mk} \in K$.

The elements $x_{ij}(t)$ ($1 \leq i \neq j \leq n, t \in K$) generate the group $\text{SL}_n(K)$. Equations (3.7.1) show that the image $\rho_K(K\text{SL}_n(K))$ is contained in $\varphi_K(\mathfrak{U}'_K)$ where

\mathfrak{U}'_K is the K -subalgebra of \mathfrak{U}_K generated by all $\frac{X_{ij}^m}{m!} \otimes 1$. On the other hand, equations (3.7.3) justify the opposite inclusion. Thus we have proved that

$$(3.7.4) \quad \rho_K(K\mathrm{SL}_n(K)) = \varphi_K(\mathfrak{U}'_K).$$

For the algebraic closure \overline{K} of K the group $\mathrm{GL}_n(\overline{K})$ is generated by $\mathrm{SL}_n(\overline{K})$ together with the scalar matrices. Since the scalar matrices act as scalars on $\mathbf{E}_{\overline{K}}^{r,s}$ we have the equality $\rho_{\overline{K}}(\overline{K}\mathrm{GL}_n(\overline{K})) = \rho_{\overline{K}}(\overline{K}\mathrm{SL}_n(\overline{K}))$, as the scalar operators are clearly already present in $\rho_{\overline{K}}(\overline{K}\mathrm{SL}_n(\overline{K}))$. For general (infinite) K we have by construction equalities of dimension:

$$\begin{aligned} \dim_K \rho_K(K\mathrm{SL}_n(K)) &= \dim_{\overline{K}} \rho_{\overline{K}}(\overline{K}\mathrm{SL}_n(\overline{K})), \\ \dim_K \rho_K(K\mathrm{GL}_n(K)) &= \dim_{\overline{K}} \rho_{\overline{K}}(\overline{K}\mathrm{GL}_n(\overline{K})), \end{aligned}$$

and hence the natural inclusion $\rho_K(K\mathrm{SL}_n(K)) \subseteq \rho_K(K\mathrm{GL}_n(K))$ must be an equality.

Moreover, \mathfrak{U}_K is generated by \mathfrak{U}'_K along with the elements $\binom{H_i}{m} \otimes 1$. By restricting φ_K to the isomorphic copy of $\mathfrak{U}(\mathfrak{gl}_2)$ generated by X_{ij} , X_{ji} , H_i , and H_j (for $i \neq j$) one can show by calculations similar to those in [DG1] that the $\varphi_K(\binom{H_i}{m} \otimes 1)$ are already present in $\varphi_K(\mathfrak{U}'_K)$. (See also Proposition 6.4; this justifies the applicability of [DG1].) It follows that $\varphi_K(\mathfrak{U}'_K) = \varphi_K(\mathfrak{U}_K)$, and the proof is complete. \square

We note that the algebra \mathfrak{U}'_K appearing in the proof is simply the hyperalgebra of \mathfrak{sl}_n , obtained from the Kostant \mathbb{Z} -form of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{sl}_n)$ by change of base ring. The above proof also reveals the following.

Corollary 3.8. *For any infinite field K , $S_K(n; r, s)$ is the image of the restricted representation $K\mathrm{SL}_n(K) \rightarrow \mathrm{End}_K(\mathbf{E}_K^{r,s})$. This may be identified with the image of the restricted representation $\mathfrak{U}'_K \rightarrow \mathrm{End}_K(\mathbf{E}_K^{r,s})$.*

The inverse anti-automorphism $g \rightarrow g^{-1}$ on Γ induces the following result, which, in particular, says that $S_K(n; 0, s)$ is isomorphic with the opposite algebra of the ordinary Schur algebra $S_K(n, s)$.

Proposition 3.9. *Let K be an infinite field. For any n, r, s we have an isomorphism $S_K(n; r, s)^{\mathrm{opp}} \simeq S_K(n; s, r)$.*

Proof. Clearly, we have an isomorphism $(\mathbf{E}_K^{s,r})^* \simeq \mathbf{E}_K^{r,s}$, as rational $K\Gamma$ -modules. Applying Lemma 1.4(a) to $V = \mathbf{E}_K^{s,r}$ we obtain an isomorphism of coalgebras $(\mathrm{cf}_{\Gamma}(\mathbf{E}_K^{r,s}))^{\mathrm{opp}} \simeq \mathrm{cf}_{\Gamma} \mathbf{E}_K^{s,r}$. Thus we have an isomorphism of coalgebras $\tilde{A}_K(n; r, s)^{\mathrm{opp}} \simeq \tilde{A}_K(n; s, r)$, and the result follows by taking linear duals, using Lemma 1.4(b). \square

4. WEIGHTS OF $\mathcal{M}_K(n; r, s)$

4.1. Elements of the diagonal torus $\mathbb{T}_n = \{\mathrm{diag}(t_1, \dots, t_n) \in \mathrm{GL}_n(K)\}$ act semisimply on any object of the category $\mathcal{M}_K(n; r, s)$, where the action is the one obtained from the action of $\Gamma = \mathrm{GL}_n(K)$ by restriction. Given an object V in $\mathcal{M}_K(n; r, s)$, in particular, V is a rational $K\Gamma$ -module, so V is a direct sum of its weight spaces,

$$(4.1.1) \quad V = \bigoplus_{\lambda \in \mathbb{Z}^n} V_{\lambda}$$

where $V_\lambda = \{v \in V : \text{diag}(t_1, \dots, t_n)v = t_1^{\lambda_1} \cdots t_n^{\lambda_n} v\}$. Here we write λ for the vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. The set of λ for which $V_\lambda \neq 0$ is the set of weights of V .

Alternatively, the weights may be computed by regarding V as a \mathfrak{U}_K -module. Here the zero part \mathfrak{U}_K^0 acts semisimply on V . Moreover, $\mathfrak{U}_K^0 = \mathfrak{U}_{\mathbb{Z}}^0 \otimes_{\mathbb{Z}} K$ and $\mathfrak{U}_{\mathbb{Z}}^0$ is generated by all $\binom{H_i}{m}$ ($m \geq 0$), so the action of \mathfrak{U}_K^0 on any V is determined by the action of $H_i \otimes 1$, for $i = 1, \dots, n$. So $V = \bigoplus_{\lambda} V_\lambda$ where $V_\lambda = \{v \in V \mid (H_i \otimes 1)v = \lambda_i v, \text{ for all } i = 1, \dots, n\}$. One easily checks that the weights (and weight spaces) in this sense coincide with the weights in the sense of the preceding paragraph.

A weight $\lambda \in \mathbb{Z}^n$ is *dominant* (relative to the Borel subgroup of upper triangular matrices in Γ) if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Weyl group \mathfrak{S}_n acts on \mathbb{Z}^n by index permutation, i.e. $\sigma \lambda = (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(n)})$, and the set of dominant weights is a set of representatives for the orbits.

Not every element of \mathbb{Z}^n can appear in the set of weights of an object of $\mathcal{M}_K(n; r, s)$. The following result describes those weights that can appear.

Lemma 4.2. *Let $\Lambda(n; r, s)$ be the set of all $\lambda \in \mathbb{Z}^n$ such that $\sum\{\lambda_i : \lambda_i > 0\} = r - t$ and $\sum\{\lambda_i : \lambda_i < 0\} = t - s$ for some t , $0 \leq t \leq \min(r, s)$. Let $\Lambda^+(n; r, s)$ be the set of dominant weights in $\Lambda(n; r, s)$.*

(a) *Let V be an object of $\mathcal{M}_K(n; r, s)$. The set of weights of V is contained in $\Lambda(n; r, s)$. The set of weights of $\mathbf{E}_K^{r,s}$ is $\Lambda(n; r, s)$.*

(b) *Let π be the set of dominant weights of $\mathbf{E}_K^{r,s}$, as in the preceding section. Then $\pi = \Lambda^+(n; r, s)$.*

Proof. Let $\varepsilon_1, \dots, \varepsilon_n$ be the standard basis of \mathbb{Z}^n . The weight of v_i is ε_i , so the weight of v'_i is $-\varepsilon_i$. Thus, in the notation of 2.5, for a pair $(I, J) \in \mathbf{I}(n; r, s) = \mathbf{I}(n, r) \times \mathbf{I}(n, s)$ the weight of

$$\mathbf{v}_{I,J} := v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v'_{j_1} \otimes \cdots \otimes v'_{j_s}$$

is $\lambda \in \mathbb{Z}^n$ such that λ_i is the difference between the number of occurrences of v_i and the number of occurrences of v'_i in the tensor $\mathbf{v}_{I,J}$. So such tensors form a basis of weight vectors for $\mathbf{E}_K^{r,s}$. The second claim in part (a) follows immediately, and part (b) follows from the second claim in (a).

Now let V be an object of $\mathcal{M}_K(n; r, s)$. The weights of V are independent of characteristic, so without loss of generality we may assume that $K = \mathbb{C}$. Then V is completely reducible (all rational $\mathbf{GL}_n(\mathbb{C})$ -modules are completely reducible).

The simple $S_{\mathbb{C}}(n; r, s)$ -modules are the simple factors of $\mathbf{E}_{\mathbb{C}}^{r,s}$, so their highest weights lie in $\pi = \Lambda^+(n; r, s)$ and their weights lie in $\Lambda(n; r, s)$ since $\Lambda(n; r, s)$ is stable under the Weyl group. Thus the weights of V lie in $\Lambda(n; r, s)$. This proves the first claim in (a). \square

Remark 4.3. The description of $\pi = \Lambda^+(n; r, s)$ in the preceding lemma can be used to give a combinatorial proof that the set π is saturated, in the sense used by Donkin [Do]. This means that if $\mu \in \pi$ and $\lambda \leq \mu$ (in the dominance order on \mathbb{Z}^n ; see §6.2) for some dominant λ , then $\lambda \in \pi$. That this property holds for the set π is obvious, however. Indeed, π is the set of weights of the module $\mathbf{E}_{\mathbb{C}}^{r,s}$ in $\mathcal{M}_{\mathbb{C}}(n; r, s)$, and thus is a union of sets of weights of its simple factors. That the set of weights of a finite dimensional simple rational $\mathbf{C}\Gamma$ -module is saturated is well known, and unions of saturated sets are clearly saturated.

There is an alternative description of the set $\Lambda(n; r, s)$. Let λ be an element of \mathbb{Z}^n . A *proper partial sum* of λ_i 's is a sum of the form $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, where i_1, i_2, \dots, i_k are distinct integers in the interval $[1, n]$, $0 < k < n$.

Lemma 4.4. $\Lambda(n; r, s)$ is the set of all $\lambda \in \mathbb{Z}^n$ satisfying the conditions

- (a) $\sum \lambda_i = r - s$;
- (b) $P \in [-s, r]$, for all proper partial sums P of λ_i 's.

Proof. Suppose $\lambda \in \Lambda(n; r, s)$. Then $\lambda \in \mathbb{Z}^n$ and $\sum\{\lambda_i : \lambda_i > 0\} = r - t$, $\sum\{\lambda_i : \lambda_i < 0\} = t - s$, for some $0 \leq t \leq \min(r, s)$. In particular, $\sum \lambda_i = r - s$. Let P be a proper partial sum of λ_i 's. Write $P = P^+ + P^-$ where P^+ is the sum of the positive contributions to the sum P and P^- is the sum of the negative contributions. If there are no positive (resp., negative) summands in P , then P^+ (resp., P^-) is defined to be 0. Then one easily sees that $P^+ \in [0, r]$, $P^- \in [-s, 0]$, and so $P \in [-s, r]$.

On the other hand, if λ satisfies conditions (a), (b) above, then in particular $\sum\{\lambda_i : \lambda_i > 0\} \in [0, r]$ and $\sum\{\lambda_i : \lambda_i < 0\} \in [-s, 0]$. Consequently, $\sum\{\lambda_i : \lambda_i > 0\} = r - t$ for some $0 \leq t \leq r$ and $\sum\{\lambda_i : \lambda_i < 0\} = t' - s$ for some $0 \leq t' \leq s$. But putting the two sums together must give $r - s$, so $t = t'$ and $0 \leq t \leq \min(r, s)$. The proof is complete. \square

4.5. There is a bijection between dominant weights and certain pairs of partitions. Given a dominant weight $\lambda \in \mathbb{Z}^n$, let a pair (λ^+, λ^-) of partitions be determined as follows:

- (1) $\lambda^+ = (\lambda_1, \dots, \lambda_i)$ where λ_i is the rightmost positive entry of λ ; if λ has no positive entries, then λ^+ is the empty partition.
- (2) $\lambda^- = (-\lambda_n, \dots, -\lambda_j)$ where λ_j is the leftmost negative entry of λ ; if λ has no negative entries, then λ^- is the empty partition.

In other words, λ^+ is the partition of the positive entries in λ and λ^- is the partition obtained from the negative entries by writing their absolute values in reverse order. The reader may easily check that this procedure defines a bijection between the set of dominant weights and the set of pairs of partitions of total length not exceeding n . The element of \mathbb{Z}^n corresponding to a given pair of partitions (of total length not exceeding n) is obtained by following the first partition with the negative reverse of the second, inserting as many zeros in between as needed to make an n -tuple.

Under this bijection, the set $\pi = \Lambda^+(n; r, s)$ corresponds with the set of all pairs of partitions such that for some $0 \leq t \leq \min(r, s)$ the first member of the pair is a partition of $r - t$, the second is a partition of $s - t$, and the combined number of parts in the pair does not exceed n .

5. $S_K(n; r, s)$ IS A GENERALIZED SCHUR ALGEBRA

In this section, we prove that the rational Schur algebra $S_K(n; r, s)$ may be identified with Donkin's generalized Schur algebra $S_K(\pi)$, where $\pi = \Lambda^+(n; r, s)$ is the (saturated) set of dominant weights occurring in $\mathbf{E}_K^{r,s} = \mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{\otimes s}$.

5.1. We recall from [Do, 3.2] that $S_R(\pi)$, for any integral domain R , is constructed as \mathfrak{U}_R/I_R where $\mathfrak{U}_R = \mathfrak{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ is the hyperalgebra taken over R and where I_R is the ideal of \mathfrak{U}_R consisting of all elements of \mathfrak{U}_R that annihilate every admissible \mathfrak{U}_R -module belonging to π . A \mathfrak{U}_R -module is admissible if it is finitely generated and free over R ; such modules are direct sums of their weight spaces. (Weight spaces

are computed using the action of the zero part $\mathfrak{U}_R^0 = \mathfrak{U}_{\mathbb{Z}}^0 \otimes_{\mathbb{Z}} R$ of \mathfrak{U}_R , which is a set of commuting semisimple operators.) A \mathfrak{U}_R -module belongs to π if every dominant weight of the module lies in the set π .

In particular, taking $\pi = \Lambda^+(n; r, s)$, the set of dominant weights of $\mathbf{E}_{\mathbb{Q}}^{r,s}$, we have $S_R(\pi) = \mathfrak{U}_R/I_R$. In case $R = \mathbb{Z}$ we denote this algebra by $S_{\mathbb{Z}}(n; r, s)$. That it is an integral form for the rational Schur algebra is the content of part (b) of the next result.

Theorem 5.2. *Fix $n \geq 2$, $r, s \geq 0$ and let $\pi = \Lambda^+(n; r, s)$ be the set of dominant weights occurring in $\mathbf{E}_{\mathbb{Q}}^{r,s}$. Let K be an infinite field.*

- (a) $S_K(n; r, s) = S_K(\pi)$.
- (b) $S_K(n; r, s) \simeq S_{\mathbb{Z}}(n; r, s) \otimes_{\mathbb{Z}} K$.

Proof. (a) Let I'_K be the kernel of the representation $\varphi_K : \mathfrak{U}_K \rightarrow \text{End}_K(\mathbf{E}_K^{r,s})$ (see (3.6.4)). Since $\mathbf{E}_K^{r,s}$ belongs to π , I'_K is contained in I'_K .

Since $\mathbf{E}_K, \mathbf{E}_K^*$ are irreducible Weyl modules, they are tilting modules. Thus $\mathbf{E}_K^{r,s}$ is a tilting module. (The category of tilting modules is closed under tensor products; see [Jan, Part II, E.7].) Hence $\mathbf{E}_K^{r,s}$ has a Δ -filtration; that is, a series of submodules

$$0 = F_0 \subset F_1 \subset \cdots \subset F_M = \mathbf{E}_K^{r,s}$$

such that $F_{i+1}/F_i \simeq \Delta(\lambda_i)$ for each i . Here $\Delta(\lambda_i)$ is the Weyl module of highest weight $\lambda_i \in \pi$.

In characteristic zero, every $\Delta(\lambda)$ for any $\lambda \in \pi$ occurs as a filtration quotient (in fact, as a direct summand) of a Δ -filtration; see [St]. Thus for any K and any $\lambda \in \pi$ there must be some index i such that $\lambda = \lambda_i$; in other words, $\Delta(\lambda)$ is a subquotient of $\mathbf{E}_K^{r,s}$. It follows that the irreducible module $L(\lambda)$ of highest weight λ is a subquotient of $\mathbf{E}_K^{r,s}$, and that any element of I'_K annihilates $L(\lambda)$.

Hence any element of I'_K must annihilate every irreducible \mathfrak{U}_K -module belonging to π . This shows the opposite inclusion $I'_K \subset I_K$. Thus $I_K = I'_K$ and $S_K(n; r, s) = S_K(\pi)$.

(b) This follows from [Do, (3.2b)]. Although in that reference the algebraic group is semisimple, the results are equally valid in the reductive case. Alternatively, one can use Corollary 3.8 to reduce the question to the semisimple case, and then apply [Do, (3.2b)]. \square

5.3. Note that Donkin [Do, (3.2b)] showed that the natural map $I_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \rightarrow I_R$ induces an isomorphism $S_{\mathbb{Z}}(\pi) \otimes_{\mathbb{Z}} R \rightarrow S_R(\pi)$, for any integral domain R . Thus it makes sense to define $S_R(n; r, s)$ for any integral domain R by $S_R(n; r, s) = S_{\mathbb{Z}}(n; r, s) \otimes_{\mathbb{Z}} R$. In particular, $S_K(n; r, s)$ is now defined for any field K , not just for infinite fields.

As an immediate consequence of the preceding theorem, we obtain the result that rational Schur algebras are quasihereditary, in the sense defined by Cline, Parshall, and Scott [CPS]. In particular, this means that rational Schur algebras have finite global dimension.

Corollary 5.4. *$S_K(n; r, s)$ is quasihereditary, for any field K .*

Proof. It easily follows from results of [Do] that any generalized Schur algebra (over a field) is quasihereditary. \square

An alternate proof of the preceding result may be given based on the cellular structure described in the next section.

6. CELLULAR BASES FOR $S_K(n; r, s)$

The purpose of this section is to show that every rational Schur algebra $S_K(n; r, s)$ inherits a cellular basis from a certain ordinary Schur algebra, of which it is a quotient.

6.1. Cellular algebras. For the reader's convenience, we recall the original definition of cellular algebra from Graham and Lehrer [GL].

Let A be an associative algebra over a commutative ring R , free and of finite rank as an R -module. The algebra A is *cellular* with cell datum (π, M, C, ι) if the following conditions are satisfied:

(C1) The finite set π is a partially ordered set, and for each $\lambda \in \pi$ there is a finite set $M(\lambda)$, such that the algebra A has an R -basis $\{C_{S,T}^\lambda\}$, where (S, T) runs over all elements of $M(\lambda) \times M(\lambda)$ and λ runs over π .

(C2) The map ι is an R -linear anti-involution of A , interchanging $C_{S,T}^\lambda$ and $C_{T,S}^\lambda$.

(C3) For each $\lambda \in \pi$, $a \in A$, and $S, T \in M(\lambda)$ we have

$$aC_{S,T}^\lambda = \sum_{U \in M(\lambda)} r_a(U, S)C_{U,T}^\lambda \quad (r_a(U, S) \in R)$$

modulo a linear combination of basis elements with upper index μ strictly less than λ in the given partial order.

Note that the coefficients $r_a(U, S)$ in the expression in (C3) do not depend on $T \in M(\lambda)$.

Suppose that A is cellular. For each $\lambda \in \pi$, let $A[\leq \lambda]$ (respectively, $A[< \lambda]$) be the R -submodule of A spanned by all $C_{S,T}^\mu$ such that $\mu \leq \lambda$ (respectively, $\mu < \lambda$) and $S, T \in M(\mu)$. It is easy to see that $A[\leq \lambda]$ and $A[< \lambda]$ are two-sided ideals of A , for each $\lambda \in \pi$. If we choose some total ordering for the elements of π , say $\pi = \{\lambda^{(1)}, \dots, \lambda^{(t)}\}$, refining the given partial order \leq on π , setting $A_i = A[\leq \lambda^{(i)}]$ gives a chain of two-sided ideals of A ,

$$(6.1.1) \quad \{0\} \subset A_1 \subset A_2 \subset \dots \subset A_t = A$$

such that $\iota(A_i) = A_i$ for each $1 \leq i \leq t$.

Axiom (C3) guarantees that, for any $T \in M(\lambda)$, the R -submodule of $A[\leq \lambda]/A[< \lambda]$ spanned by all

$$(6.1.2) \quad C_{S,T}^\lambda + A[< \lambda] \quad (S \in M(\lambda))$$

is a left A -module. We denote this A -module by Δ_T^λ . The elements in (6.1.2) form an R -basis of this module, so Δ_T^λ is free over R of R -rank $|M(\lambda)|$. Thus $A[\leq \lambda]/A[< \lambda]$ is the direct sum of Δ_T^λ as T runs over $M(\lambda)$; all these left A -modules are isomorphic to the abstract left A -module $\Delta(\lambda)$ with basis C_S ($S \in M(\lambda)$) and action given by $aC_S = \sum_{U \in M(\lambda)} r_a(U, S)C_U$ for any $a \in A$. The module $\Delta(\lambda)$ is called a cell module.

By applying the involution ι we may decompose $A[\leq \lambda]/A[< \lambda]$ into a similar direct sum of $|M(\lambda)|$ right A -modules, all of which are isomorphic with $\iota(\Delta(\lambda))$. In fact, the A -bimodule $A[\leq \lambda]/A[< \lambda]$ is isomorphic with $\Delta(\lambda) \otimes_R \iota(\Delta(\lambda))$. Note that [KX] gives an alternative approach to the theory of cellular algebras based on these bimodules.

6.2. Cellular bases of $S_K(n, r)$. There are two well known cellular bases of a classical Schur algebra $S_K(n, r)$: the codeterminant basis of J.A. Green [G3] and the canonical basis (see [BLM]; [Du]). The former is compatible with the Murphy basis [M1, M2] and the latter with the Kazhdan–Lusztig basis [KL] (for the symmetric group algebra or the Hecke algebra in type A). Both bases of $S_K(n, r)$ may be obtained by lifting the corresponding basis from the group algebra of the symmetric group. Alternatively, the latter basis may be realized by a descent from Lusztig’s modified form (see [Lu]) of \mathfrak{U}_Q . (See [D1, §6.14] for details.)

In both cases, one takes the set π to be $\Lambda^+(n, r)$; this may be identified with the set of partitions of r into not more than n parts. The partial order on π is the *reverse* dominance order; i.e., we have to read \leq as \supseteq . Here $\mu \supseteq \lambda$ (for any $\mu = (\mu_1, \dots, \mu_n)$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$) if $\sum_{1 \leq j \leq i} \mu_j \geq \sum_{1 \leq j \leq i} \lambda_j$ for all $i = 1, \dots, n$. This is the partial order (with respect to the upper triangular Borel subgroup) on the set of weights \mathbb{Z}^n defined by $\mu \supseteq \lambda$ if $\mu - \lambda$ is a sum of positive roots. For each $\lambda \in \pi$, the set $M(\lambda)$ is the set of row semistandard λ -tableaux, i.e., Young diagrams of shape λ with entries from the set $\{1, \dots, n\}$ such that entries strictly increase down columns and weakly increase along rows. In both cases there is a cellular basis $\{C_{S, T}^\lambda\}$ of $S_K(n, r)$, and in both cases the cell modules $\Delta(\lambda)$ are isomorphic with the Weyl modules of highest weight $\lambda \in \pi$.

The codeterminant basis of $S_K(n, r)$ is quite simple to describe. Let $\xi_{I, J}$ for $I, J \in \mathbf{I}(n, r)$ be the elements of $S_K(n, r)$ described in [G2, §2.3]. For each pair S, T of row semistandard λ -tableaux, for $\lambda \in \Lambda^+(n, r)$, one obtains corresponding elements $I, J \in \mathbf{I}(n, r)$ by reading the entries in left-to-right order across the rows of each tableau, in order from the top row to the bottom row. Then $C_{S, T}^\lambda = \xi_{I, \ell} \xi_{\ell, J}$ where $\ell = \ell(\lambda)$ is the element of $\mathbf{I}(n, r)$ corresponding to the λ -tableau with all entries in the i th row equal to i , as i varies through the rows. That this is in fact a cellular basis follows easily from the straightening algorithm of Woodcock [Wo]. A q -analogue of this straightening algorithm was given in [RMG1] and a proof of the cellularity of the codeterminant basis can be found in [RMG2, Proposition 6.2.1]; that same argument is valid in the $q = 1$ case.

6.3. The quotient map $S_K(n, r + (n - 1)s) \rightarrow S_K(n; r, s)$. Let \mathcal{C} be the category of rational $K\Gamma$ -modules. There is a functor Ψ from \mathcal{C} to \mathcal{C} , sending V to $V \otimes \det$. This is clearly an invertible functor. For any $s \in \mathbb{Z}$ we have a corresponding functor Ψ^s sending V to $V \otimes \det^{\otimes s}$ if $s \geq 0$, and sending V to $V \otimes (\det^{-1})^{\otimes |s|}$ in case $s < 0$. Clearly, Ψ^{-s} is inverse to Ψ^s for any $s \in \mathbb{Z}$. In fact, $\Psi^r \circ \Psi^s = \Psi^{r+s}$ for any $r, s \in \mathbb{Z}$, and Ψ^0 is the identity functor.

Set $X = \mathbb{Z}^n$, regarded as an additive abelian group, and consider its group ring $\mathbb{Z}[X]$, with basis $\{e(\lambda) \mid \lambda \in X\}$ and product determined by $e(\lambda) \cdot e(\mu) = e(\lambda + \mu)$ for $\lambda, \mu \in X$. The formal character of V is the element of $\mathbb{Z}[X]$ given by $\text{ch } V = \sum_{\lambda \in X} (\dim V_\lambda) e(\lambda)$, where V_λ is the λ -weight space as in §4.1. Clearly we have

$$(6.3.1) \quad \text{ch}(\Psi^s V) = \sum_{\lambda \in X} (\dim V_\lambda) e(\lambda + s\omega)$$

for any V and any s . Here $\omega := (1^n) = (1, \dots, 1) \in \mathbb{Z}^n$. In particular, if V is a highest weight module of highest weight λ , then $\Psi^s V$ is a highest weight module of highest weight $\lambda + s\omega$ (for any $s \in \mathbb{Z}$).

The existence of the quotient map is motivated by the observation that $\Psi \mathbf{E}_K^* = \mathbf{E}_K^* \otimes \det \simeq \Lambda^{n-1} \mathbf{E}_K$ (as $K\Gamma$ -modules or \mathfrak{U}_K -modules). The existence of this isomorphism is clear from comparing the highest weight of each module (both are irreducible in any characteristic). Thus we have an isomorphism of rational $K\Gamma$ -modules

$$(6.3.2) \quad \Psi^s(\mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{*\otimes s}) \simeq \mathbf{E}_K^{\otimes r} \otimes (\Lambda^{n-1} \mathbf{E}_K)^{\otimes s}$$

for any $r, s \geq 0$. The $K\Gamma$ -module $\Lambda^{n-1} \mathbf{E}_K$ is the Weyl module of highest weight $(1, \dots, 1, 0) \in \mathbb{Z}^n$, and may be realized as a submodule of tensor space $\mathbf{E}_K^{\otimes(n-1)}$ by means of the Carter–Lusztig construction, as the submodule spanned by all antisymmetric tensors of the form

$$\sum_{\sigma \in \mathfrak{S}_{n-1}} (-1)^{\text{sign}(\sigma)} \mathbf{v}_{\sigma(i_1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(i_{n-1})}$$

for any $I = (i_1, \dots, i_{n-1}) \in \mathbf{I}(n, n-1)$. See [G2, 5.2, Example 2] for details.

The above embedding $\gamma : \Lambda^{n-1} \mathbf{E}_K \rightarrow \mathbf{E}_K^{\otimes(n-1)}$ induces a corresponding embedding

$$(6.3.3) \quad \mathbf{E}_K^{\otimes r} \otimes (\Lambda^{n-1} \mathbf{E}_K)^{\otimes s} \rightarrow \mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{\otimes(n-1)s} = \mathbf{E}_K^{\otimes(r+(n-1)s)}$$

obtained by using the identity map in the first r tensor positions and repeating γ in the rest of the factors s times.

By results in §2 we have an injective map $\tilde{A}_K(n; r, s) \rightarrow A_K(n, r+(n-1)s)$ given by $c \rightarrow c \cdot \mathbf{d}^s$ for any $c \in \tilde{A}_K(n; r, s)$. Since \mathbf{d} is a group-like element, this map is a coalgebra morphism. By dualizing we therefore obtain a surjective algebra map

$$(6.3.4) \quad S_K(r+(n-1)s) \rightarrow S_K(n; r, s)$$

for any $r, s \geq 0$ and any field K .

When $n = 2$ the rational Schur algebras are not new. More precisely, we have the following result.

Proposition 6.4. *The quotient map $S_K(2, r+s) \rightarrow S_K(2; r, s)$ is an isomorphism of algebras, for any $r, s \geq 0$.*

Proof. This follows from the isomorphism $\mathbf{E}_K^* \otimes \det \simeq \mathbf{E}_K$ (when $n = 2$). This implies that $\Psi^s \mathbf{E}_K^{r,s} \simeq \mathbf{E}_K^{\otimes(r+s)}$, and it follows by taking coefficient spaces that $\tilde{A}_K(2; r, s) \rightarrow A_K(2, r+s)$ is an isomorphism of coalgebras. The result follows by dualizing. \square

Let us now return to general $n \geq 2$, and let $r, s \geq 0$ be given. Given any $S_K(n; r, s)$ -module V , we may regard V as an $S_K(n, r+(n-1)s)$ -module by composing the action with the quotient map $S_K(n, r+(n-1)s) \rightarrow S_K(n; r, s)$. Denote by V' the resulting $S_K(n, r+(n-1)s)$ -module. It is easy to see that V' is isomorphic with $\Psi^s V = V \otimes \det^{\otimes s}$, as $S_K(n, r+(n-1)s)$ -modules. Since the weights $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ of V satisfy the condition $\lambda_i \in [-s, r]$ for each $i = 1, \dots, n$ it is clear that the weights $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ of $\Psi^s V$ satisfy the condition $\mu_i \in [0, r+s]$ for each $i = 1, \dots, n$.

Note that if V is a highest weight module for $S_K(n, r+(n-1)s)$ of highest weight λ , then $\Psi^{-s} V$ is a highest weight module for $S_K(n; r, s)$ if and only if the weight λ satisfies the condition $\lambda_i \in [0, r+s]$ for each $i = 1, \dots, n$.

Theorem 6.5. *Let $\{C_{S,T}^\lambda\}$ be any cellular basis of the ordinary Schur algebra $S_K(n, r+(n-1)s)$ such that $\pi = \Lambda^+(n, r+(n-1)s)$ ordered by the reverse dominance order \supseteq . Then the kernel of the quotient map $S_K(n, r+(n-1)s) \rightarrow S_K(n; r, s)$ is spanned by the set of all $C_{S,T}^\mu$ such that μ has at least one part exceeding $r+s$.*

Proof. Set $A = S_K(n, r+(n-1)s)$. Set J equal to the sum of all ideals $A[\supseteq\mu]$ for $\mu \in \pi$ satisfying $\mu_i > r+s$ for some index i . The set of such μ is a saturated subset π' of π under the dominance order, and the two-sided ideal J , regarded as a left A -module, is filtered by $\Delta(\nu)$ for $\nu \in \pi'$. Thus J is contained in the kernel, so $S_K(n; r, s)$ is a quotient of A/J .

The quotient A/J has a filtration by $\Delta(\lambda)$ with $\lambda \in \pi - \pi'$, and its dimension is $\sum_\lambda (\dim \Delta(\lambda))^2$ as λ runs over the set $\pi - \pi'$. But $\dim \Delta(\lambda) = \dim \Psi^{-s} \Delta(\lambda) = \dim \Delta(\lambda - s\omega)$, so $\dim A/J = \sum_\mu (\dim \Delta(\mu))^2$ as μ runs over the set $\Lambda^+(n; r, s)$. Thus A/J has the same dimension as $S_K(n; r, s)$. The proof is complete. \square

Corollary 6.6. *Under the quotient map $S_K(n, r+(n-1)s) \rightarrow S_K(n; r, s)$, the images of all $C_{S,T}^\lambda$, for λ satisfying the condition $\lambda_i \leq r+s$ (for $i = 1, \dots, n$) form a cellular basis of the rational Schur algebra $S_K(n; r, s)$.*

To multiply two basis elements obtained by the procedure in the corollary, one computes their product in $A = S_K(n, r+(n-1)s)$ expressed as a linear combination of the other cellular basis elements, omitting any terms belonging to the ideal $J = \sum_{\mu \in \pi'} A[\supseteq\mu]$ (in the notation of the proof of Theorem 6.5).

Note that if the cellular basis of $S_K(n, r+(n-1)s)$ is defined over \mathbb{Z} , then the same is true of the basis inherited by the quotient $S_K(n; r, s)$. Since the codeterminant and the canonical bases of Schur algebras both have this property, we see that in fact the above procedure gives cellular bases for $S_{\mathbb{Z}}(n; r, s)$ which induce via change of base ring the corresponding cellular basis of $S_K(n; r, s)$ for each field K .

6.7. Examples. As previously mentioned, $S_K(n; r, 0) = S_K(n, r)$ for any r . By Corollary 3.9 we have an isomorphism $S_K(n; 0, s) \simeq S_K(n, s)^{\text{opp}}$ for any s . Moreover, by Proposition 6.4 we have $S_K(2; r, s) \simeq S_K(2, r+s)$ for any r, s .

Thus the smallest interesting new example is $S_K(3; 1, 1)$. According to Theorem 6.5, this is a quotient of $S_K(3, 3)$. The Weyl modules for $S_K(3, 3)$ are

$$\begin{aligned} \Delta(3, 0, 0) & \dim = 10, \\ \Delta(2, 1, 0) & \dim = 8, \\ \Delta(1, 1, 1) & \dim = 1. \end{aligned}$$

(The dimension of $\Delta(\lambda)$, for a partition $\lambda \in \Lambda^+(n, r)$, is the number of semistandard λ -tableaux.) So $S_K(3, 3)$ has dimension $165 = 10^2 + 8^2 + 1^2$. The quotient map $A = S_K(3, 3) \rightarrow S_K(3; 1, 1)$ has kernel $A[\supseteq(3, 0, 0)]$; the kernel has dimension 100 and is spanned by all $C_{S,T}^\lambda$ for $\lambda = (3, 0, 0)$ as S, T varies over all pairs of λ -tableaux. Thus $S_K(3; 1, 1)$ has dimension 65, and has two Weyl modules

$$\begin{aligned} \Delta(1, 0, -1) &= \Psi^{-1} \Delta(2, 1, 0) & \dim = 8, \\ \Delta(0, 0, 0) &= \Psi^{-1} \Delta(1, 1, 1) & \dim = 1. \end{aligned}$$

One could at this point write out the elements of a cellular basis for $S_K(3; 1, 1)$, indexed by pairs of semistandard μ -tableaux for $\mu = (2, 1, 0)$ and $(1, 1, 1)$, and compute the structure constants with respect to the basis.

The next simplest cases are $S_K(3; 2, 1)$ or $S_K(3; 1, 2)$. These are related by $S_K(3; 2, 1) \simeq S_K(3; 1, 2)^{\text{opp}}$; both algebras have dimension 270. Note that $S_K(3, 2, 1)$ is a quotient of $S_K(3, 4)$ while $S_K(3; 1, 2)$ is a quotient of $S_K(3, 5)$, according to Theorem 6.5.

Now $\Lambda^+(3, 4) = \{(4, 0, 0), (3, 1, 0), (2, 2, 0), (2, 1, 1)\}$. The Weyl modules for $S_K(3, 4)$ are

$$\begin{aligned} \Delta(4, 0, 0) & \dim = 15, \\ \Delta(3, 1, 0) & \dim = 15, \\ \Delta(2, 2, 0) & \dim = 6, \\ \Delta(2, 1, 1) & \dim = 3. \end{aligned}$$

The dimension of $S_K(3, 4)$ is $495 = 15^2 + 15^2 + 6^2 + 3^2$. The kernel of the quotient map $A = S_K(3, 4) \rightarrow S_K(3; 2, 1)$ is $A[\succeq(4, 0, 0)]$ of dimension 225. Thus the Weyl modules for $S_K(3; 2, 1)$ are

$$\begin{aligned} \Delta(2, 0, -1) &= \Psi^{-1}\Delta(3, 1, 0) & \dim = 15, \\ \Delta(1, 1, -1) &= \Psi^{-1}\Delta(2, 2, 0) & \dim = 6, \\ \Delta(1, 0, 0) &= \Psi^{-1}\Delta(2, 1, 1) & \dim = 3. \end{aligned}$$

Again, one could write out a cellular basis at this point.

Similarly, $\Lambda^+(3, 5) = \{(5, 0, 0), (4, 1, 0), (3, 2, 0), (3, 1, 1), (2, 2, 1)\}$ and the Weyl modules for $S_K(3, 5)$ are

$$\begin{aligned} \Delta(5, 0, 0) & \dim = 21, \\ \Delta(4, 1, 0) & \dim = 24, \\ \Delta(3, 2, 0) & \dim = 15, \\ \Delta(3, 1, 1) & \dim = 6, \\ \Delta(2, 2, 1) & \dim = 3. \end{aligned}$$

In this case $\dim S_K(3, 5)$ is $1287 = 21^2 + 24^2 + 15^2 + 6^2 + 3^2$ and the kernel of the quotient map $A = S_K(3, 5) \rightarrow S_K(3; 1, 2)$ is $A[\succeq(4, 1, 0)]$ of dimension $1017 = 21^2 + 24^2$. Note that the kernel contains $A[\succeq(5, 0, 0)]$. The Weyl modules for $S_K(3; 1, 2)$ are

$$\begin{aligned} \Delta(1, 0, -2) &= \Psi^{-2}\Delta(3, 2, 0) & \dim = 15, \\ \Delta(1, -1, -1) &= \Psi^{-2}\Delta(3, 1, 1) & \dim = 6, \\ \Delta(0, 0, -1) &= \Psi^{-2}\Delta(2, 2, 1) & \dim = 3. \end{aligned}$$

As above, one could write out a cellular basis at this point.

The next example would be $S_K(3; 2, 2)$, which has dimension 994 and is realized as a quotient of $S_K(3, 6)$ (of dimension 3003). We leave the details to the reader.

7. GENERATORS AND RELATIONS

Another consequence of the existence of the quotient map (6.3.4) is a simple description of $S_{\mathbb{Q}}(n; r, s)$ by generators and relations, generalizing the presentation of $S_{\mathbb{Q}}(n, d)$ obtained in [DG2]. We recall the elements $\varepsilon_1, \dots, \varepsilon_n$ from the proof of Lemma 4.2; these elements form a natural basis for the set \mathbb{Z}^n of weights of $\Gamma = \text{GL}_n(K)$. We set $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $1 \leq j \leq n-1$. These are the usual simple roots in type A_{n-1} . We have an inner product on \mathbb{Z}^n given on generators by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

In the following we work over \mathbb{Q} , but in fact \mathbb{Q} may be replaced by any field of characteristic zero.

7.1. We recall from [DG2, Theorem 1.1] that the algebra $S_{\mathbb{Q}}(n, d)$ is isomorphic with the associative algebra with 1 generated by symbols e_i, f_i ($1 \leq i \leq n-1$) and H_i ($1 \leq i \leq n$) subject to the relations

- (a) $H_i H_j = H_j H_i$,
- (b) $e_i f_j - f_j e_i = \delta_{ij}(H_j - H_{j+1})$,
- (c) $H_i e_j - e_j H_i = (\varepsilon_i, \alpha_j) e_j$, $H_i f_j - f_j H_i = -(\varepsilon_i, \alpha_j) f_j$,
- (d) $e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0$, $f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0$ ($|i-j|=1$),
- (e) $e_i e_j = e_j e_i$, $f_i f_j = f_j f_i$ ($|i-j| \neq 1$),
- (f) $H_1 + H_2 + \cdots + H_n = d$,
- (g) $H_i(H_i - 1) \cdots (H_i - d + 1)(H_i - d) = 0$.

We also recall that Serre proved that the enveloping algebra $\mathfrak{U} = \mathfrak{U}_{\mathbb{Q}}(\mathfrak{gl}_n)$ is the \mathbb{Q} -algebra on the same generators but subject only to relations (a)–(e). Thus the algebra $S_{\mathbb{Q}}(n, d)$ is a homomorphic image of \mathfrak{U} , by the map given on generators by $e_i \rightarrow e_i, f_i \rightarrow f_i, H_i \rightarrow H_i$.

Proposition 7.2. *Assume that $n \geq 2$ and set $A = S_{\mathbb{Q}}(n, d)$ with $d = r + (n-1)s$. Let $A[\pi'] = \bigcup_{\mu \in \pi'} A[\geq \mu]$ where π' is the set of all μ in $\Lambda^+(n, d)$ such that at least one part μ_i exceeds $r+s$. Then the quotient algebra $A/A[\pi']$ is isomorphic with the algebra given by the same generators as in §7.1 subject to the relations 7.1(a)–(f) along with*

$$(g') \quad H_i(H_i - 1) \cdots (H_i - r - s + 1)(H_i - r - s) = 0.$$

Proof. Since relation 7.1(g) is a consequence of relation (g') above, it follows that the algebra A' given by the generators and relations of the proposition is a quotient of A . We let I be the kernel so that we have $A' \simeq A/I$. Clearly, I is generated by all $H_i(H_i - 1) \cdots (H_i - r - s + 1)(H_i - r - s)$. To prove the proposition we must show that I coincides with the cell ideal $A[\pi']$.

To see this we first observe that the idempotents $1_{\mu} \in A$ (for $\mu \in \pi'$) all map to zero in the quotient A' . These idempotents were defined in [DG2] by $1_{\mu} = \prod_i \binom{H_i}{\mu_i}$, but one easily sees as in [DEH, Lemma 5.3] that they coincide with Green's idempotents ξ_{μ} defined in [G2, §3.2]. These are in fact codeterminants: $\xi_{\mu} = \xi_{\ell, \ell} \xi_{\ell, I}$ where I is an appropriate element of $\mathbf{I}(n, r)$ (ℓ is the particular element of $\mathbf{I}(n, r)$ defined in 6.2). This is an easy consequence of the multiplication rule [G2, (2.3b)]. Let $\xi = \sum \xi_{\mu}$ where the sum is taken over all $\mu \in \pi'$. Then $\xi C = C$ for all codeterminants C in $A[\pi']$, so it follows that C maps to zero in the quotient A' . Thus $A[\pi'] \subseteq I$.

On the other hand, every other idempotent maps to a nonzero idempotent in the quotient A' , and thus it follows that every cell module $\Delta(\lambda)$ indexed by some $\lambda \in \Lambda^+(n, d) - \pi'$ is an irreducible A' -module. Since A' is a semisimple algebra, it follows that

$$\dim_{\mathbb{Q}} A' \geq \sum_{\lambda \in \Lambda^+(n, d) - \pi'} (\dim_{\mathbb{Q}} \Delta(\lambda))^2.$$

But $\dim_{\mathbb{Q}} A' = \dim_{\mathbb{Q}} A - \dim_{\mathbb{Q}} I$ and $\dim_{\mathbb{Q}} A = \sum_{\lambda \in \Lambda^+(n, d)} (\dim_{\mathbb{Q}} \Delta(\lambda))^2$, so we get

$$\dim_{\mathbb{Q}} I \leq \sum_{\mu \in \pi'} (\dim_{\mathbb{Q}} \Delta(\mu))^2 = \dim_{\mathbb{Q}} A[\pi'].$$

Since $A[\pi'] \subseteq I$, we conclude that $I = A[\pi']$ by dimension comparison. The proof is complete. \square

7.3. The proof of the preceding proposition reveals that the irreducible representations of $A' = A/A[\pi']$ are precisely the $\Delta(\lambda)$ with $\lambda \in \Lambda^+(n, d)$ such that each part $\lambda_i \in [0, r + s]$. As already noted in §6, this set of modules is in bijective correspondence with the set of irreducible $S_{\mathbb{Q}}(n; r, s)$ -modules. Now we consider the effect of replacing the generators H_i in A' with new generators $H'_i = H_i - s$. This induces an algebra automorphism of A' , and has the effect of changing the defining relations slightly, to the following:

- (a) $H'_i H'_j = H'_j H'_i$,
- (b) $e_i f_j - f_j e_i = \delta_{ij}(H'_j - H'_{j+1})$,
- (c) $H'_i e_j - e_j H'_i = (\varepsilon_i, \alpha_j) e_j$, $H'_i f_j - f_j H'_i = -(\varepsilon_i, \alpha_j) f_j$,
- (d) $e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0$, $f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0$ ($|i - j| = 1$),
- (e) $e_i e_j = e_j e_i$, $f_i f_j = f_j f_i$ ($|i - j| \neq 1$),
- (f) $H'_1 + H'_2 + \cdots + H'_n = r - s$,
- (g') $(H'_i + s)(H'_i + s - 1) \cdots (H'_i - r + 1)(H'_i - r) = 0$.

This algebra should be regarded as a quotient of \mathfrak{U} via the map given on generators by $e_i \rightarrow e_i$, $f_i \rightarrow f_i$, $H_i \rightarrow H'_i + s$. By the remarks preceding Theorem 6.5, the irreducible modules for A' now become precisely the set of irreducible \mathfrak{U} -modules whose highest weights belong to the set $\Lambda^+(n; r, s)$. This gives the following result.

Theorem 7.4. *For $n \geq 2$, the algebra $S_{\mathbb{Q}}(n; r, s)$ is isomorphic with the associative algebra with 1 given by the generators e_i , f_i ($1 \leq i \leq n - 1$) and H'_i ($1 \leq i \leq n$) subject to the relations 7.3(a)–(g').*

Of course, this algebra is isomorphic with A' . By the theory of semisimple algebras, it makes no difference whether we regard the irreducible representations for the algebra as the $\Delta(\lambda)$ for $\lambda \in \pi'$ or the $\Delta(\lambda)$ for $\lambda \in \Lambda^+(n; r, s)$.

Remark 7.5. In light of results of [DG2] it is now clear how one might define a natural candidate for the rational q -Schur algebra. It is the $\mathbb{Q}(v)$ -algebra (v an indeterminate) given by generators E_i , F_i ($1 \leq i \leq n - 1$) and K'_i ($1 \leq i \leq n$) with the relations

- (a) $K'_i K'_j = K'_j K'_i$,
- (b) $E_i F_j - F_j E_i = \delta_{ij} \frac{K'_j K'_{j+1}{}^{-1} - K'_j{}^{-1} K'_{j+1}}{v - v^{-1}}$,
- (c) $K'_i E_j - E_j K'_i = v^{(\varepsilon_i, \alpha_j)} E_j$, $K'_i F_j - F_j K'_i = v^{-(\varepsilon_i, \alpha_j)} F_j$,
- (d) $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$, $F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ ($|i - j| = 1$),
- (e) $E_i E_j = E_j E_i$, $F_i F_j = F_j F_i$ ($|i - j| \neq 1$),
- (f) $K'_1 K'_2 \cdots K'_n = v^{r-s}$,
- (g') $(K'_i - v^{-s})(K'_i - v^{-s+1}) \cdots (K'_i - v^{r-1})(K'_i - v^r) = 0$.

8. SCHUR–WEYL DUALITY

Let K be a field of characteristic zero. Classical Schur–Weyl duality, based on tensor space $\mathbf{E}_K^{r,0} = \mathbf{E}_K^{\otimes r}$, has been extended in [B] to mixed tensor space $\mathbf{E}_K^{r,s} = \mathbf{E}_K^{\otimes r} \otimes \mathbf{E}_K^{* \otimes s}$. Thus we may regard the rational Schur algebra as a centralizer algebra for the action of a certain algebra $\mathfrak{B}_{r,s}^{(n)}$, the so-called *walled* or *rational* Brauer algebra. This diagram algebra is cellular [GM].

8.1. The Brauer algebra $\mathfrak{B}_r^{(x)}$. This algebra was introduced by Brauer in [Br]. Let R be a commutative ring, let $R[x]$ be the ring of polynomials in an indeterminate

x , and consider the free $R[x]$ -module $\mathfrak{B}_r^{(x)}$ with basis consisting of the set of r -diagrams. An r -diagram is an (undirected) graph on $2r$ vertices and r edges such that each vertex is incident to precisely one edge. One usually thinks of the vertices as arranged in two rows of r each, which are then called the top and bottom rows. Edges connecting two vertices in the same row (different rows) are called *horizontal* (resp., *vertical*). We can compose two such diagrams D_1, D_2 by identifying the bottom row of vertices in the first diagram with the top row of vertices in the second diagram. The result is a graph with a certain number, $\delta(D_1, D_2)$, of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new r -diagram $D_1 \circ D_2$, the *composite* diagram. Multiplication of r -diagrams is then defined by the rule

$$(8.1.1) \quad D_1 \cdot D_2 = x^{\delta(D_1, D_2)}(D_1 \circ D_2).$$

One can check that this multiplication makes $\mathfrak{B}_r^{(x)}$ into an associative algebra. Note that the subalgebra of $\mathfrak{B}_r^{(x)}$ spanned by the diagrams containing just vertical edges may be identified with the group algebra $R[x]\mathfrak{S}_r$.

8.2. The walled Brauer algebra. The walled Brauer algebra (also called the rational Brauer algebra) is a certain subalgebra of $\mathfrak{B}_{r+s}^{(x)}$ which first appeared in [B]. Some examples and further details may be found in [D2].

By definition, an (r, s) -diagram is an $(r+s)$ -diagram in which we imagine a wall separating the first r columns of vertices from the last s columns of vertices, such that:

- (a) all horizontal edges cross the wall,
- (b) no vertical edges cross the wall.

Let $\mathfrak{B}_{r,s}^{(x)}$ be the subalgebra of $\mathfrak{B}_{r+s}^{(x)}$ spanned by the set of (r, s) -diagrams. This is a subalgebra of $\mathfrak{B}_{r+s}^{(x)}$, with multiplication in $\mathfrak{B}_{r,s}^{(x)}$ the same as that of $\mathfrak{B}_{r+s}^{(x)}$. Given an (r, s) -diagram, interchanging the vertices in each column on one side of the wall results in a diagram with no horizontal edges. Thus there is a bijection between the set of (r, s) -diagrams and \mathfrak{S}_{r+s} . It follows that $\mathfrak{B}_{r,s}^{(x)}$ is a free R -module of rank $(r+s)!$.

Label the vertices on the top and bottom rows of an (r, s) -diagram by the numbers $1, \dots, r$ to the left of the wall and $-1, \dots, -s$ to the right of the wall. Let $c_{i,-j}$ ($1 \leq i \leq r; 1 \leq j \leq s$) be the diagram with a horizontal edge connecting vertices i and $-j$ on the top row and the same on the bottom row, and with all other edges connecting vertex k ($k \neq i, -j$) in the top and bottom rows. Given $\sigma \in \mathfrak{S}_r$ let t_σ denote the (r, s) -diagram with the diagram corresponding to σ appearing to the left of the wall and the identity to the right. Similarly, given $\tau \in \mathfrak{S}_s$ let t'_τ denote the (r, s) -diagram with the diagram corresponding to τ appearing to the right of the wall and the identity to the left. It is easy to see that $\mathfrak{B}_{r,s}^{(x)}$ is generated by the permutations $\{t_\sigma, t'_\tau\}$ along with just one of the $c_{i,-j}$.

8.3. Now we specialize to $R = K$, a field, and set $x = n \cdot 1_K \in K$, where as usual $n = \dim_K E_K$. The resulting K -algebra $\mathfrak{B}_{r,s}^{(n)}$ acts on $E_K^{r,s}$, on the right, as follows. A permutation diagram t_σ ($\sigma \in \mathfrak{S}_r$) acts by place permutation on the first r factors of $E_K^{r,s} = E_K^{\otimes r} \otimes E_K^{*\otimes s}$. Similarly, a permutation diagram t'_τ ($\tau \in \mathfrak{S}_s$) acts by place permutation on the last s factors of $E_K^{r,s}$. Finally, the element $c_{r,-1}$ acts as

a contraction in tensor positions $(r, -1)$; i.e.,

$$(8.3.1) \quad \begin{aligned} & (\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_r} \otimes \mathbf{v}'_{j_1} \otimes \cdots \otimes \mathbf{v}'_{j_s}) c_{r,-1} \\ &= \delta_{i_r, j_1} \mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_{r-1}} \otimes \left(\sum_{k=1}^n \mathbf{v}_k \otimes \mathbf{v}'_k \right) \otimes \mathbf{v}'_{j_2} \otimes \cdots \otimes \mathbf{v}'_{j_s}. \end{aligned}$$

The other $c_{i,-j}$ similarly contract in the $(i, -j)$ th tensor positions.

One checks that the above action of $\mathfrak{B}_{r,s}^{(n)}$ commutes with the natural action of $\Gamma = \mathrm{GL}_n(K)$.

8.4. Double centralizer property. Assuming that K is a field of characteristic zero and $n \geq r + s$, it has been shown in [B] (see also [K]) that the two commuting actions on $\mathbf{E}_K^{r,s}$ mutually centralize one another. In other words,

$$(8.4.1) \quad \begin{aligned} \rho_K(K\Gamma) &= \mathrm{End}_{\mathfrak{B}_{r,s}^{(n)}}(\mathbf{E}_K^{r,s}), \\ \tau_K(\mathfrak{B}_{r,s}^{(n)}) &= \mathrm{End}_{\Gamma}(\mathbf{E}_K^{r,s}) \end{aligned}$$

where τ_K is the representation of $\mathfrak{B}_{r,s}^{(n)}$ affording the previously defined action of $\mathfrak{B}_{r,s}^{(n)}$ on $\mathbf{E}_K^{r,s}$.

When K is any infinite field, we have the equality $S_K(n; r, s) = \rho_K(K\Gamma)$, by (3.5.4). It follows that the dimension of $\rho_K(K\Gamma)$ is independent of the (infinite) field K . Moreover, still for K infinite, we also know that the dimension of $\mathrm{End}_{\Gamma}(\mathbf{E}_K^{r,s})$ is independent of K , since the module $\mathbf{E}_K^{r,s}$ is tilting. (This follows by a simple induction using filtrations, and the well-known fact that $\mathrm{Hom}_{\Gamma}(\Delta(\lambda), \nabla(\mu))$ is K if $\lambda = \mu$ and 0 otherwise.)

Thus, in order to show that (8.4.1) holds for any infinite field K , one need only prove that the dimensions of $\tau_K(\mathfrak{B}_{r,s}^{(n)})$ and $\mathrm{End}_{\mathfrak{B}_{r,s}^{(n)}}(\mathbf{E}_K^{r,s})$ are independent of K . In other words, the question amounts to knowing something about the representation theory of $\mathfrak{B}_{r,s}^{(n)}$. In case $n \geq r + s$ the authors have shown, by an argument similar to one in [DDH, Theorem 3.4], that τ_K is injective in any characteristic, i.e., the algebra $\mathfrak{B}_{r,s}^{(n)}$ acts faithfully, so in that case the independence of $\dim_K \tau_K(\mathfrak{B}_{r,s}^{(n)})$ can be established. The details will appear elsewhere.

Since $S_K(n; r, s) = \rho_K(K\Gamma)$, we have in characteristic zero for $n \geq r + s$ the equality

$$(8.4.2) \quad S_K(n; r, s) = \mathrm{End}_{\mathfrak{B}_{r,s}^{(n)}}(\mathbf{E}_K^{r,s}).$$

If (8.4.1) can be established for all K , and for all n , then (8.4.2) will be true without restriction on n or characteristic.

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